Strong Detectability and Observers*

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ABSTRACT

Conditions are given for the existence of observers that estimate unmeasured outputs on the basis of partial information on the input and the state. The concepts of strong detectability and strong observability, introduced before in the literature for discrete-time systems only, are defined and studied for continuous-time systems. It is shown that there are two different concepts of strong detectability, which coincide for discrete-time systems. Algebraic conditions for either concept are given. It is shown that these concepts are intimately related to the existence of strong observers, i.e. observers that only use the output of the plant.

INTRODUCTION

The classical theory of observers (see [13]) was concerned with the problem of reconstructing the state (or estimating the state) from the input and output of the system. This problem can be and has been generalized in various ways. We will be interested in the situation where the input is not completely available for measurement and where one is not interested in the reconstruction for the whole state vector, but in an output different from the measured output. These generalizations were considered in e.g. [1], [3], [9], [10]. Much attention was paid to the algorithms to compute such observers, but not often were existence conditions given comparable to results in classical observer theory (detectability).

The objective of this paper is the investigation of the existence of observers for continuous-time systems in the situation where the system Σ has

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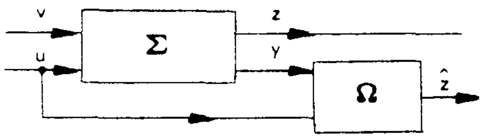
two types of input, a measured input u and an unmeasured input v, and two types of output, a measured output y and an output z which is to be estimated:



The observer has to yield an output $\hat{z}$ for which the error estimate $\hat{z} - z$ tends to zero as $t \to \infty$ for arbitrary initial states of plant Σ and observer Ω and for arbitrary inputs u and v.

Existence conditions for such observers have been given in terms of (C, A)-detectable subspaces in [10]. These conditions may be regarded as generalizations of the condition for detectability that the unstable subspace must be contained in the kernel of the output map ($\mathcal{X}^+(A) \subseteq \ker D$ in the notation of [11]). Here we want to give conditions in the frequency-domain formulation, which may be considered generalizations of the condition $\operatorname{rank}[sI - A', C'] = n$ for $\operatorname{Re} s \geqslant 0$ (see [14]). Our setup enables us to deal, without any added effort, with systems with direct connection terms between input and output. The conditions become particularly transparent in the case where the output z to be estimated is the state. The concepts of strong and strong* detectability to be introduced in Section 1 play a role (compare the corresponding discrete-time notion [6, 7]).

A conceptually appealing and generally valid condition—which, however, is not constructive—is that the variable to be estimated should tend to zero whenever the measured variables do. Curiously enough, the condition is not valid for discrete-time systems (see [12]).

We will simplify the formulas considerably by assuming that the measured input u does not appear in the problem. This is no loss of generality, since the general case can be reduced to this particular case by considering the measured input as part of the measured output, i.e. by introducing a new output $y = [u', y']'$ which contains the information about u and by viewing $\mathbf{u} = [u', v']'$ as one input with no part being measured (for details see [3]).

In Section 1 we introduce the concepts of strong observability, strong detectability, and strong* detectability for continuous-time systems, and we give (without proof) existence conditions for state observers and delayed state observers. In Section 2 some preliminary results are given, based on [2], and in Section 3 the general results as well as proofs are given.

1. STATE OBSERVERS

The objective of this section is to investigate the relation between strong detectability and the existence of what we will call strong observers, i.e. observers using only the output and not the input of the plant.

Strong detectability has been introduced in [6] for discrete-time systems. The definition given in [6] is: *A system is strongly detectable if* $y(t) \to 0$ $(t \to \infty)$ *implies* $x(t) \to 0$ $(t \to \infty)$ *irrespective of the input and the initial state.* An algebraic criterion for strong detectability (also given in [6]) is the minimum-phase condition: *All zeros of the system should lie within the unit circle.* Strong detectability is a weaker property than strong observability, which can be defined either by the condition that $y(t) = 0$ $(t > 0)$ implies $x(t) = 0$ $(t > 0)$ for any input and initial state, or by the condition that the system has no zeros at all (see [4], [6], [7]). From the latter condition it follows that strong observability implies strong detectability.

In the continuous-time case, strong observability can be defined in exactly the same fashion, and again the characterizations given above for the discrete-time case are equivalent. It turns out, however, that the two characterizations for strong detectability as stated above in the discrete-time case are no longer equivalent in the continuous-time case (of course, here we say that a system is minimum-phase if its zeros are in the left half plane, $\operatorname{Re} s < 0$). As a consequence, we introduce two concepts of strong detectability of the continuous-time system Σ:

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du. \tag{1.1}$$

DEFINITION 1.2. The system (1.1) is *strongly detectable* if

$$y(t) = 0 \quad \text{for } t > 0 \quad \text{implies} \quad x(t) \to 0 \quad (t \to \infty),$$

for all inputs and initial states.

DEFINITION 1.3. The system (1.1) is *strong* detectable* if

$$y(t) \to 0 \quad (t \to \infty) \quad \text{implies} \quad x(t) \to 0 \quad (t \to \infty).$$

Our first objective will be the derivation of algebraic conditions for strong and strong* detectability. We will see that strong detectability corresponds to the minimum-phase condition. For this, we define the zeros of the system (1.1) to be the values of $s \in \mathbb{C}$ (the complex plane) for which

$$\operatorname{rank}\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} < n + \operatorname{rank}\begin{bmatrix} -B \\ D \end{bmatrix}, \tag{1.4}$$

where n denotes the dimension of the state space. (Compare [8]; the definition of a zero given here is slightly different from the one given in [8], but the two definitions coincide in the case where the system is left invertible.) Our first result will be

THEOREM 1.5. *The system* (1.1) *is strongly detectable if and only if all its zeros* s *satisfy* $\operatorname{Re} s < 0$.

The result will be proven in Section 3.

It follows immediately from Definitions 1.2 and 1.3 that strong* detectability implies strong detectability. More specifically:

THEOREM 1.6. *The system* (1.1) *is strong* detectable if and only if it is strongly detectable and in addition*

$$\operatorname{rank}\begin{bmatrix} CB & D \\ D & 0 \end{bmatrix} = \operatorname{rank} D + \operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix}. \tag{1.7}$$

This will also be shown in Section 3.

With regard to strong observability, we have the following result:

THEOREM 1.8. *The system* (1.1) *is strongly observable if and only if it has no zeros.*

As a consequence (and also directly from the definition), strong observability implies strong detectability, as it does in the discrete-time case, but it does not imply strong* detectability. The latter statement will be exemplified as follows. Consider the single-input single-output system with coefficient matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ -a_n & \cdot & \cdot & \cdot & \cdot & \cdot & -a_1 \end{bmatrix}, \tag{1.9}$$

$$b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

The relation between input and output is given by

$$y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}\dot{y}+a_ny=u, \tag{1.10}$$

and the state is given by $(y,\dot{y},\dots,y^{(n-1)})'$. Now, if for example $y(t)=t^{-1}\sin t^2$, we have $y(t)\to 0$, but $\dot{y}(t)\not\to 0$, and hence, if we take u to be the input defined by substituting $y=t^{-1}\sin t^2$ into (1.10), we have that $y\to 0$ and $x(t)\not\to 0$. On the other hand, the system (1.1) with A,b,c defined by (1.9) is strongly observable, since if $y=0$ for $t>0$, then $y^{(k)}(t)=0$ ($k=1,2,\dots$) for $t>0$ and hence $x(t)=0$ for $t>0$.

Next we discuss the relevance of the concepts of strong and strong* detectability for the construction of strong observers. Let us first give a definition.

DEFINITION 1.11. A system Ω with input y and output $\hat{x}$ is called a *strong observer* for Σ [i.e., the system (1.1)] if for each input and for each initial state of the original system Σ as well as of the system Ω, we have

$$x(t)-\hat{x}(t)\to 0 \qquad (t\to\infty).$$

The observer is called strong because it gives an asymptotically correct estimate of the state of Σ based on the output of Σ alone. A necessary and sufficient condition for the existence of such an observer (which will be proved in Section 3) is the following:

THEOREM 1.12. *The system Σ has a strong observer if and only if it is strong* detectable.*

Thus, Theorem 1.12 gives a condition for the existence of a strong observer. Explicitly, for such an observer to exist it is necessary and sufficient that (1.7) and (1.4) hold for $\operatorname{Re} s\geqslant 0$. Using Hermite, Smith, or Kronecker normal forms, the condition (1.4) can easily be seen to be constructively verifiable.

The foregoing shows that strong detectability or even strong observability is not sufficient for the existence of an observer. In discrete-time systems strong and strong* detectability can be shown to be equivalent. [An explanation of this discrepancy between the continuous-time and discrete-time cases is that $y(t)\to 0$ does not imply $\dot{y}(t)\to 0$ (see the example given before), but $y(t)\to 0$ does imply $y(t-1)\to 0$. Hence a discrete counterpart of the example (1.9) does not exist.] A consequence of this equivalence is that strong* detectability, as defined in Definition 1.3, is not sufficient for the existence of a strong observer. For the existence of such an observer one needs strong (or equivalently strong*) detectability and the condition (1.7). If the discrete-time system is strongly detectable, but does not satisfy (1.7), a *delayed strong observer* can be constructed, i.e., a system Ω with input y (the output of the original system) and output $\hat{x}(t)$ which satisfies $\hat{x}(t)-x(t-T)\to 0$ for $t\to\infty$ for some nonnegative integer T, irrespective of initial states and controls (compare the related concept of delayed inverse; see [5]). Such a delayed observer is satisfactory if one is interested in the values of the state. However, it cannot always be used for the stabilization of the system (e.g., if we have the single variable system $x(t+1)=2x(t)+u(t)$, and $\hat{x}(t)=x(t-1)$ for all t, it is not possible to find a "feedback" of the form $u(t)=a\hat{x}(t)$ that stabilizes the system). In the continuous-time case there is no such natural concept as the delayed observer. One can, of course, introduce integrating observers. However, since an integrator is not stable, this concept is not compatible with the stability requirements we always impose on observers. Instead we introduce for continuous-time systems:

DEFINITION 1.13. A system Ω with input y and output $\hat{x}$ is called a *strong stable integrating observer* if there exists a real $a>0$ and a nonnegative integer k such that any input and initial value of Σ and Ω we have

$$(1.14)\qquad \left(\frac{d}{dt}+a\right)^k \hat{x}(t)-x(t)\to 0 \qquad (t\to\infty).$$

Then we have the following result:

THEOREM 1.15. *There exists a strong stable integrating observer for Σ if and only if Σ is strongly detectable.*

For a proof of a more general result see Section 3.

In most situations, it is reasonable to assume that $\operatorname{rank}[B', D']=m$, which means that there are no superfluous input channels. In this case, the conditions (1.4) and (1.7) can be somewhat simplified. In particular, it turns out that (1.7) means that the system has a 1-integrating left inverse (see [5]).

2. ASYMPTOTIC PROPERTIES OF MATRIX DIFFERENTIAL OPERATORS

Our proofs will be based on results obtained in [2]. The basic tool will be a simplified version of Theorem 2 in [2], which we formulate now:

THEOREM 2.1. *Let $P(s)$, $Q(s)$, and $R(s)$ be matrix polynomials of dimensions $p \times l$, $q \times l$, $r \times l$, respectively. Then the following statements are equivalent*:

(i)

$$\left.\begin{array}{l} P(s)w(t)=0 \\ Q(s)w(t)\to 0 \end{array}\right\} \quad \Rightarrow \quad R(s)w(t)\to 0 \qquad (t\to\infty).$$

Here s stands for the differentiation operator: $sw = \dot{w}$, $s^2 w = \ddot{w}$, etc. Furthermore, for any function $z(t)\to 0$ means $\lim_{t\to\infty} z(t)=0$.

(ii) *The equation*

$$M(s)P(s)+N(s)Q(s)=R(s) \tag{2.2}$$

in the rational matrices M and N has a solution (M, N) which is stable (i.e. no poles in $\mathrm{Re}\, s \geqslant 0$) and such that $N(s)$ is in addition proper.

(iii) *The following two conditions are satisfied*:

(a) *Equation* (2.2) *has a solution (M_∞, N_∞) with $N_\infty(s)$ proper.*

(b) *For every s_0 in* $\mathrm{Re}\, s_0 \geqslant 0$, *Equation* (2.2) *has a (complex) solution M_{s_0}, N_{s_0} without a pole in s_0.*

One obtains this result by substituting $A=P$, $B=Q$, $C=0$, and $D=R$ into Theorem 2 of [2]. Let us point out that (because $C=0$) for the proof of Theorem 2.1 it is not necessary to go through all the technical details of the proof of [2, Theorem 2]. A proof very similar to the proof of [2, Theorem 1] can be given, and this is much simpler, but we will not work this out here. In Theorem 2.1, it is obvious that (ii) $\Rightarrow$ (iii). The proof of (iii) $\Rightarrow$ (ii) depends on the following algebraic lemma, which also will be used elsewhere in this paper.

LEMMA 2.3. *Let R be a principal-ideal domain with quotient field Q, and assume that $R \neq Q$. Let $A \in R^{n\times m}$ (i.e., A is an $n\times m$ matrix with entries in R) and $B \in R^{r\times m}$. Then there exists a matrix $M \in R^{r\times n}$ such that $MA = B$ if and only if for each $p \in Q^n$ we have*

$$Ap \in R^n \quad \Rightarrow \quad Bp \in R^r.$$

The proof of this result relies on the Smith canonical form for matrices over a principal-ideal domain (see [2, Lemma 3.7]).

When applying this lemma, we use the fact that the set of polynomials and the set of proper rational functions are principal-ideal domains.

3. OBSERVERS ESTIMATING AN OUTPUT

We consider a system with two types of output:

$$\dot{x} = Ax + Bu,\quad y = Cx + Du,\quad z = Ex + Fu. \tag{3.1}$$

We assume that y is an output which is available for measurement and that z is an output the value of which we are interested in. Thus we want to construct an observer which, using y as input, gives us an estimate $\hat{z}$ of z in the sense that always $z - \hat{z} \to 0$. Such a system we call a *z-observer based* on y. For the existence of such an observer we have the following result.

THEOREM 3.2. *The following statements are equivalent:*

(i) $y \to 0$ *implies* $z \to 0$ *for every control and initial state.*

(ii) *There exists a z-observer based on* y.

(iii) *The equation*

$$\begin{bmatrix} M(s) & N(s) \end{bmatrix} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} E & F \end{bmatrix} \tag{*}$$

has a stable solution (M, N) *such that* $N(s)$ *is proper.*

(iv)

(a) *Equation* (*) *has a solution* (M_∞, N_∞) *such that* N_∞ *is proper.*
(b) *For every* s_0 *with* $\operatorname{Re} s_0 \geqslant 0$, *Equation* (*) *has a solution with no pole in* s_0.

Proof. The equivalences (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follow from Theorem 2.1 by setting $w := [x', u']'$, $P := [sI - A, -B]'$, $Q := [C, D]$, $R := [E, F]$. It suffices to show that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (ii): The first two equations of (3.1) can be written in Laplace transform as

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x_0 \\ y \end{bmatrix},$$

where x_0 is the initial state. Multiplying from the left by $[M(s), N(s)]$, we obtain

$$z = Ex + Fu = M(s)x_0 + N(s)y.$$

The matrix $N(s)$ is the transfer matrix of a stable system. Thus if we define the system Ω by its transfer matrix:

$$\hat{z} = N(s)y,$$

then $z - \hat{z} = M(s)x_0 \to 0$, since $M(s)$ is stable. Hence, Ω is a z-observer, based on y. (Notice that $M(s)$ can be written as $M_p + M_{sc}$, where M_p is polynomial and M_{sc} is strictly proper and hence a stable transfer-function matrix. Then $M_p x_0$ is an impulse at $t = 0$, only setting up the initial values, and $M_{sc}x_0 \to 0$. This is most easily formulated in terms of distribution theory; see [2].)

(ii) $\Rightarrow$ (i): In the first place we remark that an observer necessarily must have a stable transfer function. For if $u = 0$, identically, and $x_0 = 0$, then we have $y = 0$ and $z = 0$, identically. The solution of the observer equation is then completely determined by the observer's initial state. If the transfer function of the observer is not stable, there exists an initial state w_0 of the observer such that the corresponding output $\hat{z}$ (with input $y = 0$) does not tend to zero. Then $\hat{z} - z \to 0$ is impossible, which is in contradiction with the definition of the observer. From the stability of the observer it follows that if the initial state of the observer is zero, we have that $y \to 0$ implies $\hat{z} \to 0$ and hence $z = \hat{z} + (z - \hat{z}) \to 0$. ■

We want to specialize our results to the situation where $z = x$, the state, but first we make a remark that pertains to the general situation.

In the first place, the observer constructed in Theorem 3.2 may be complex although the original system is real. However, it is easily seen that, if $N(s)$ is the transfer matrix of a complex observer, then the system N_1 obtained by defining $N_1(s) := \operatorname{Re} N(s)$, for real s, is an observer too.

Let us now restrict to the case where $z = x$, i.e. $E = I$, $F = 0$. In this case, the equation (*) in $M(s)$, $N(s)$ of Theorem 3.2(iii) reduces to

$$[M(s), N(s)]\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = [I, 0] \tag{3.3}$$

Let $\operatorname{rank}[-B', D'] = m_1$, and choose an $m_1 \times m$ matrix T (of rank m_1) such that

$$\begin{bmatrix} -B \\ D \end{bmatrix} = \begin{bmatrix} -B_1 \\ D_1 \end{bmatrix} T,$$

where $[-B_1', D_1']'$ has full column rank. Then we write

$$\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} sI-A & -B_1 \\ C & D_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}.$$

Choose constant matrices M_1, N_1 such that $-M_1B_1 + N_1D_1 = I$, which is possible because $[-B_1', D_1']'$ has full column rank. If (3.3) has a solution $[M_\alpha, N_\alpha]$ with no pole at α, it follows that

$$\begin{bmatrix} M_\alpha & N_\alpha \\ M_1 & N_1 \end{bmatrix} \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ Q(s) & T \end{bmatrix}$$

for some polynomial matrix $Q(s)$. Consequently the matrix

$$P(s) := \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}$$

must have rank $n + m_1$ at such a point. Conversely, if $P(\alpha)$ has rank $n + m_1$, then

$$P_1(s) = \begin{bmatrix} sI-A & -B_1 \\ C & D_1 \end{bmatrix}$$

has full column rank for $s = \alpha$. Therefore, $P_1(s)$ has a left inverse $[\tilde{M}_\alpha(s), \tilde{N}_\alpha(s)]$ that has no pole at $s = \alpha$. It follows that

$$[\tilde{M}_\alpha, \tilde{N}_\alpha]P = \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix},$$

and hence the first n rows of $(\tilde{M}_\alpha, \tilde{N}_\alpha)$ form a solution of (3.3) with no pole at α. Thus we obtain:

PROPOSITION 3.4. *Condition* (iv)(b) *of Theorem* 3.2 *is satisfied for the case* $[E, F] = [I, 0]$ *iff* (1.4) *implies* $\operatorname{Re} s < 0$.

Now, if we show that (iv)(a) is equivalent to (1.7), we have proved Theorem 1.6 and Theorem 1.12.

PROPOSITION 3.5. *For the case* $[E, F] = [I, 0]$, *condition* (iv)(a) *of Theorem* 3.2 *is equivalent to* (1.7).

Proof. We use Lemma 2.3 applied to Equation (3.3) over the ring of proper rational functions. In the first place we remark that if (M_∞, N_∞) is a solution of (3.3) such that N_∞ is proper, then M_∞ also must be proper, since

$$M_\infty(s) = [I - N(s)C](sI - A)^{-1}.$$

Hence, (iv)(a) is equivalent to the following: For each pair of rational vectors $p(s), q(s)$ we have (see Lemma 2.3)

$$\text{(3.6)}\quad \text{If}\quad \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} p(s) \\ q(s) \end{bmatrix} \text{ is proper}\quad \text{then}\quad p(s) \text{ is proper.}$$

First suppose that (1.7) is satisfied, and let $p(s), q(s)$ satisfy the premise of (3.6). We show that $p(s)$ is proper. Write down the expansion of p and q in powers of s^{-1}:

$$p(s) = p_k s^k + p_{k-1}s^{k-1} + \cdots,$$

$$q(s) = q_l s^l + \cdots.$$

We assume that $p_k \neq 0$ and $k \geqslant 1$, and we have to show that this leads to a contradiction. The properness of

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix}$$

yields the equations

$$\text{(3.7)}\qquad p_k - Bq_{k+1} = 0, \qquad Cp_k + Dq_k = 0, \qquad Dq_{k+1} = 0$$

The rank condition (1.7) can be rewritten as

$$\operatorname{rank}\begin{bmatrix} D & CB \\ 0 & D \end{bmatrix} = \operatorname{rank}\begin{bmatrix} D & 0 \\ 0 & B \\ 0 & D \end{bmatrix}.$$

Since

$$\begin{bmatrix} D & CB \\ 0 & D \end{bmatrix} = \begin{bmatrix} I & C & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} D & 0 \\ 0 & B \\ 0 & D \end{bmatrix},$$

this is equivalent to

$$\begin{bmatrix} D & CB \\ 0 & D \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} D & 0 \\ 0 & B \\ 0 & D \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0,$$

or, equivalently,

$$\begin{bmatrix} D & CB \\ 0 & D \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0 \quad \Rightarrow \quad Bv = 0.$$

This can be applied to $u = q_k$, $v = q_{k+1}$, since (3.7) implies that

$$\begin{bmatrix} D & CB \\ 0 & D \end{bmatrix}\begin{bmatrix} q_k \\ q_{k+1} \end{bmatrix} = 0.$$

It follows that $p_k = Bq_{k+1} = 0$.

Conversely, assume that

$$\begin{bmatrix} D & CB \\ 0 & D \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

We have to prove, assuming that (3.6) holds, that $Bv = 0$. To this extent, we choose polynomials

$$p(s) := sBv + ABv + Bu,$$

$$q(s) := s^2 v + su.$$

A short computation yields that

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} p(s) \\ q(s) \end{bmatrix} = \begin{bmatrix} -A^2Bv - Abu \\ CABv + CBu \end{bmatrix},$$

where we have used that $Dv = 0$ and $Du + CBv = 0$. Now (3.6) implies that $p(s)$ is proper; hence $Bv = 0$. ■

A characterization of strong detectability can also be expressed in terms of the equation (3.3). A straightforward application of Theorem 2.1 with

$$P(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}, \qquad Q(s) = 0, \qquad R(s) = [I \quad 0]$$

yields that the system (1.1) is strongly detectable iff (3.3) has a stable solution

$[M(s), N(s)]$, or equivalently, if for every α in $\operatorname{Re}\alpha \geq 0$, Equation (3.3) has a solution M_α, N_α with no pole in α. Now, Proposition 3.4 immediately yields Theorem 1.5.

Now consider the situation where in Theorem 3.2 only (iv)(b) is satisfied and not (iv)(a). Then a solution $[M, N]$ of (*) exists which is known to be stable, but N is not necessarily proper. Choose any $k \in \mathbb{N}$ such that $N_1(s) := (s+a)^{-k}N(s)$ is proper, where a is a positive real number. Then N_1 is a stable proper transfer function, and we may consider a system Ω with transfer matrix $N_1(s)$. Let $\hat{z}$ be the output of this system, i.e., let $\hat{z}$ be given by

$$\hat{z} = N_1(s)y + N_2(s)w_0,$$

where $N_2(s)w_0$ is the zero-input response due to the initial state w_0 of the observer, which can be chosen to be stable. Then

$$(s+a)^k\hat{z} - z = (s+a)^k\{N_2(s)w_0 + M(s)x_0\} \to 0,$$

so that $N_1(s)$ defines a stable integrating z-observer based on y of the system (3.1). (The definition of such an observer is obtained by the obvious modification of Definition 1.13.) Conversely, completely analogously to the proof of Theorem 3.2, one shows that condition (iv)(b) is also necessary for the existence of a strong integrating z-observer based on y. Thus, one obtains

THEOREM 3.8. *The following statements in connection with the system* (3.1) *are equivalent*:

(i) $y = 0$ *implies* $z \to 0$ *for every control and initial state.*
(ii) *There exists a stable integrating z-observer based on y.*
(iii) *Equation* (*) *of Theorem* 3.2 *has a stable solution* $[M(s), N(s)]$.
(iv) *For every s_0 in* $\operatorname{Re} s_0 \geq 0$, *Equation* (*) *has a solution with no pole in* s_0.

The proofs of these equivalences, not given above, are completely analogous to the proof of Theorem 3.2 and will be omitted. Theorem 1.15 follows from Theorem 3.8 by the specialization $E = I$, $F = 0$ and with the aid of Proposition 3.4.

We conclude this section with the proof of Theorem 1.8.

If there are no zeros of Σ, then, by applying Lemma 2.3 to the ring of polynomials, we obtain that there exist polynomial matrices $M(s), N(s)$

solving (3.3). [As in the proof of Proposition 3.4, one has to work with $P_1(s)$ rather than $P(s)$.] If we consider differentiation to be understood in the distribution sense, we have (s denoting the differentiation operator d/dt)

$$[M(s), N(s)]\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = x$$

and on the other hand

$$\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} = 0 \qquad \text{for} \quad t>0$$

(if $y=0$ for $t>0$). Hence $x=0$ for $t>0$. Conversely, suppose that

$$P(s) := \begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}$$

has rank less than $n+\operatorname{rank}[-B', D']$ for $s=s_0$. Then there exists (x_0, u_0) with $x_0 \neq 0$ such that

$$\begin{bmatrix} s_0 I-A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0.$$

Since

$$P(s)-P(s_0) = (s-s_0)\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

we have, upon defining $x(s) := (s-s_0)^{-1}x_0$, $u(s) := (s-s_0)^{-1}u_0$, that

$$\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}\begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

By inverse Laplace transformation we obtain that $\tilde{x}(t), \tilde{u}(t)$ satisfy the equation (1.1) and that the corresponding output equals 0. Hence Σ is not strongly observable.

4. CONCLUSION

The concepts of strong and strong* detectability have been defined for continuous-time systems. Algebraic conditions for these system properties have been given. It has been shown that strong* detectability is equivalent to the existence of a strong state observer, i.e. an observer estimating the state based on data about the output (and not about the input). Strong detectability turns out to be a weaker property, only guaranteeing the existence of what we have called a strong stable integrating observer. Similar, but less explicit, results have been given for the existence of a strong observer estimating an output rather than the state. The method given uses frequency-domain descriptions and is largely based on a paper of J. J. A. M. Brands and the author.

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