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SPECTRAL CATEGORIES AND VARIETIES OF PREADDITIVE CATEGORIES

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In this paper we reinterpret the main results about the spectral categories by making use of the theory of sheaves and introducing the notion of 'variety of preadditive categories'. This approach allows us to visualize the structure of the spectral categories better, to explain the different decompositions of a spectral category (discrete and continuous part [4], type I, II and III [14]) and to give an incisive interpretation to the dimension theory for the objects of a spectral category [7]. Moreover, we give an explicit description of the Grothendieck groups of the dense subcategories of a spectral category.

Spectral categories, that is abelian categories with exact direct limits in which every exact sequence splits, naturally arise in the study of injective modules (or, more generally, in the study of the injective objects of any Grothendieck category) and their Krull–Remak–Schmidt–Gabriel decompositions [12]. They were introduced by Gabriel and Oberst [4], who discovered that any spectral category is the product of a discrete spectral category and a continuous one. Later, on the lines of Kaplansky's theory of types for AW^*-algebras [8], Roos [14] discovered that every spectral category can be decomposed into a product of categories of three distinct types (type I, II and III). Finally Goodearl and Boyle [7] constructed a complete and beautiful dimension theory for the objects of a spectral category. The directly finite case of that theory is partially based on ideas of Von Neumann and Loomis.

We study the spectral categories by means of the varieties of preadditive categories (their definition is given in Section 1). Essentially a variety of preadditive categories is for a preadditive category [12] what a ringed space is for a ring. Ringed spaces have been extensively used by Dauns and Hofmann [1] and Pierce [11] in the study of Von Neumann (bi)regular rings. Here we study the spectral categories by means of varieties of categories. Given any abelian category $\mathscr{C}$, we construct an associated variety of preadditive categories having the spectrum $\mathfrak{X}[\mathscr{C}]$ of the Boolean algebra of all idempotents of the center of $\mathscr{C}$ as a basis and suitable quotient

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categories $\mathscr{C}/\mathscr{A}_M$, $M\in\mathfrak{X}[\mathscr{C}]$, of $\mathscr{C}$ as stalks (Sections 3 and 4). Here 'quotient category' means 'quotient category in the sense of Gabriel' [3].

Vice versa, given any variety of preadditive categories, we may construct its category of global sections. If we limit our attention to the spectral categories, we obtain a one-to-one correspondence (up to isomorphism) between spectral categories and reduced spectral varieties of categories (Section 6). Here 'reduced' means a 'variety having a Boolean space as a basis and indecomposable categories as stalks'. We can thus begin our study of the variety associated to a spectral category (Section 7). The stalk $\mathscr{C}/\mathscr{A}_M$ of this variety has a very simple structure and it is possible to completely determine the class $\mathscr{I}_M$ of all the isomorphism classes of the objects of the stalk $\mathscr{C}/\mathscr{A}_M$. The class $\mathscr{I}_M$ turns out to be totally ordered (in the order induced by the relation 'to be isomorphic to a subobject of') and it is partitioned into a directly finite part and a purely infinite part. The directly finite part of $\mathscr{I}_M$ is the positive cone of a totally ordered abelian group, and Goodearl and Boyle's finite dimensions d_M [7] are the real valuations of this group. Their infinite dimensions μ_M also are defined on the stalks $\mathscr{C}/\mathscr{A}_M$ of the associated variety.

We then turn to the study of the Grothendieck groups of the dense subcategories (o, Serre subcategories) of a spectral category $\mathscr{C}$ (Section 8). These groups turn out to be groups of global sections of sheaves which have $\mathfrak{X}[\mathscr{C}]$ as a basis and totally ordered abelian groups as stalks. These totally ordered abelian groups are convex subgroups of the ordered group whose positive cone is the directly finite part of $\mathscr{I}_M$.

Finally, in the last section we study some examples by means of the associated variety of categories. In particular we reinterpret Gabriel and Oberst's decomposition into discrete and continuous part [4] and Roos's decomposition into types [14], [7].

1. Notation and definitions

We want to define a 'sheaf of preadditive categories' or, to be more precise, the analogue for a preadditive category of what a ringed space is for a ring. The definition we give in this section avoids all set-theoretic problems.

A *variety of preadditive categories* $(X, \mathscr{V})$ consists of:

(a) a topological space X, called the *basis* of $(X, \mathscr{V})$;

(b) a class $\mathrm{Ob}(\mathscr{V})$, whose elements are called *objects* of $(X, \mathscr{V})$;

(c) a sheaf of abelian groups $\mathscr{H}om_{\mathscr{V}}(A,B)$ over X for each ordered pair (A,B) of objects of $(X, \mathscr{V})$;

(d) for each $x\in X$ a preadditive category $\mathscr{V}_x$, called the *stalk* of $(X, \mathscr{V})$ at x, such that $\mathrm{Ob}(\mathscr{V}_x)=\mathrm{Ob}(\mathscr{V})$ and $\mathrm{Hom}_{\mathscr{V}_x}(A, B)=\mathscr{H}om_{\mathscr{V}}(A, B)_x$ for all $x\in X$, $A, B\in\mathrm{Ob}(\mathscr{V})$.

Before stating the axioms a variety of preadditive categories must satisfy, let us recall that there are two formally different (but equivalent) definitions of a sheaf of abelian groups. The one we shall make use of is due essentially to Leray: a sheaf

$\mathcal{S}$ of abelian groups over X consists of a topological space $\mathcal{S}$ and a local homeomorphism $\pi: \mathcal{S} \to X$ such that $\pi^{-1}(x) = \mathcal{S}_x$ is an abelian group for all $x \in X$ and the mappings induced by the operations are continuous (see [15]).

The composition functions on the stalks $\mathcal{V}_x$ of a variety of preadditive categories $(X, \mathcal{V})$ are subject to two axioms:

(i) for each triple (A, B, C) of objects of $(X, \mathcal{V})$, let $\mathcal{H}om_{\mathcal{V}}(A,B) + \mathcal{H}om_{\mathcal{V}}(B,C)$ denote the disjoint union $\bigcup_{x \in X}(\mathcal{H}om_{\mathcal{V}}(A,B)_x \times \mathcal{H}om_{\mathcal{V}}(B,C)_x)$ considered as a topological subspace of $\mathcal{H}om_{\mathcal{V}}(A,B) \times \mathcal{H}om_{\mathcal{V}}(B,C)$ endowed with the product topology; then the mapping $\mathcal{H}om_{\mathcal{V}}(A,B) + \mathcal{H}om_{\mathcal{V}}(B,C) \to \mathcal{H}om_{\mathcal{V}}(A,C)$ induced by the composition functions on the stalks, $(f_x, g_x) \to g_x \circ f_x$, is continuous;

(ii) for each $A \in \mathrm{Ob}(\mathcal{V})$, the mapping $X \to \mathcal{H}om_{\mathcal{V}}(A,A)$, $x \to 1_A \in \mathrm{Hom}_{\mathcal{V}_x}(A,A)$ is continuous, i.e. it is a global section of the sheaf of abelian groups $\mathcal{H}om_{\mathcal{V}}(A,A)$.

An example of a variety of preadditive categories is given by the *variety* $(X, \mathcal{V})$ *over* X *with constant stalk* $\mathscr{C}$. Here X is a topological space and $\mathscr{C}$ is a preadditive category [12]; set $\mathrm{Ob}(\mathcal{V}) = \mathrm{Ob}(\mathscr{C})$, $\mathcal{V}_x = \mathscr{C}$ for all $x \in X$ and let $\mathcal{H}om_{\mathcal{V}}(A,B)$ be the constant sheaf of abelian groups over X with stalk $\mathrm{Hom}_{\mathscr{C}}(A,B)$ for all $A, B \in \mathrm{Ob}(\mathcal{V})$.

If $(X, \mathcal{V})$ is a variety of preadditive categories and A, B are objects of $(X, \mathcal{V})$, a *morphism* σ of A into B, denoted by $\sigma: A \to B$, is a global section $\sigma \in \Gamma(X, \mathcal{H}om_{\mathcal{V}}(A,B))$. A morphism $\sigma: A \to B$ in a variety $(X, \mathcal{V})$ is an *isomorphism* if there exists $\tau: B \to A$ such that $\tau \circ \sigma = 1_A$ and $\sigma \circ \tau = 1_B$ (here the composition $\circ$ is componentwise). A and B are then *isomorphic* objects in $(X, \mathcal{V})$.

Recall that a *Boolean space* is a totally disconnected compact Hausdorff space and a *complete Boolean space* is an extremally disconnected compact Hausdorff space (i.e. the closure of any open set is open.) In the sequel we shall generally consider varieties of preadditive categories over Boolean spaces. Boolean spaces have the following property (*partition property* [11, p. 12]): if X is a Boolean space and $\{N_i \mid i \in I\}$ is a covering of X by open sets, there exists a partition $\{M_1, M_2, \ldots, M_r\}$ of X, such that every element M_j of the partition is a clopen subset of X contained in N_i for some $i \in I$ (depending on j).

1.1. Lemma. *Let* $(X, \mathcal{V})$ *be a variety of preadditive categories. Suppose* X *is a Boolean space. Then two objects* A, B *of* $(X, \mathcal{V})$ *are isomorphic in* $(X, \mathcal{V})$ *if and only if they are isomorphic in* $\mathcal{V}_x$ *for all* $x \in X$.

Proof. Suppose that A and B are isomorphic in $\mathcal{V}_x$ for all $x \in X$. Then for each $x \in X$ there exist sections $\sigma^{(x)}, \tau^{(x)}$ of $\mathcal{H}om_{\mathcal{V}}(A,B)$, $\mathcal{H}om_{\mathcal{V}}(B,A)$ resp., defined in a neighborhood of x, such that $\tau^{(x)}(x) \circ \sigma^{(x)}(x) = 1_A$ and $\sigma^{(x)}(x) \circ \tau^{(x)}(x) = 1_B$ in $\mathcal{V}_x$. But then $\tau^{(x)} \circ \sigma^{(x)} = 1_A$ and $\sigma^{(x)} \circ \tau^{(x)} = 1_B$ in a neighborhood of x. By the partition property it is possible to construct two global sections $\sigma \in \Gamma(X, \mathcal{H}om_{\mathcal{V}}(A,B))$, $\tau \in \Gamma(X, \mathcal{H}om_{\mathcal{V}}(B,A))$ such that $\tau \circ \sigma = 1_A$, $\sigma \circ \tau = 1_B$. Hence A and B are isomorphic in $(X, \mathcal{V})$. The converse is obvious. □

If A is an object of a variety of preadditive categories $(X, \mathscr{V})$, we define the *zero-set* $z(A)$ of A as the subset $z(A)=\{x\in X \mid A$ is a zero object in $\mathscr{V}_x\}$ of X.

1.2. Lemma. *Let $(X, \mathscr{V})$ be a variety of preadditive categories and let A be an object of $(X, \mathscr{V})$. Then $z(A)$ is an open set in X. If X is Boolean, then $z(A)$ is a clopen set in X.*

Proof. $z(A)$ is the subset of X on which the two global sections 0 and 1 of $\mathscr{H}om_{\mathscr{V}}(A, A)$ coincide. □

Finally, two varieties $(X, \mathscr{V}), (X', \mathscr{V}')$ of preadditive categories are *isomorphic* if there exist (i) a homeomorphism $\phi: X\to X'$, (ii) a one-to-one correspondence $\Phi: \mathrm{Ob}(\mathscr{V})\to \mathrm{Ob}(\mathscr{V}')$ and (iii) a homeomorphism $\Phi(A,B)$ for each ordered pair (A,B) of objects of $(X, \mathscr{V}), \Phi(A,B): \mathscr{H}om_{\mathscr{V}}(A,B)\to \mathscr{H}om_{\mathscr{V}'}(\Phi(A), \Phi(B))$, which maps $\mathscr{H}om_{\mathscr{V}}(A,B)_x$ isomorphically onto $\mathscr{H}om_{\mathscr{V}'}(\Phi(A),\Phi(B))_{\phi(x)}$ for all $x\in X$ and induces a functor $\Phi_x: \mathscr{V}_x\to \mathscr{V}'_{\phi(x)}$ for all $x\in X$.

2. Abelian varieties of categories

Let $(X, \mathscr{V})$ be a variety of preadditive categories and let Y be a subset of X. We define the *category of sections of* $(X, \mathscr{V})$ *over* Y, denoted by $\Gamma(Y, \mathscr{V})$, as the category whose objects are the objects of $(X, \mathscr{V})$ and whose morphisms are defined as $\mathrm{Hom}_{\Gamma(Y,\mathscr{V})}(A,B)=\Gamma(Y, \mathscr{H}om_{\mathscr{V}}(A,B))$ for every ordered pair of objects A, B. Thus the morphisms of A into B in $\Gamma(Y, \mathscr{V})$ are the sections of $\mathscr{H}om_{\mathscr{V}}(A,B)$ over Y. The category $\Gamma(Y, \mathscr{V})$ is clearly a preadditive category. In particular $\Gamma(X, \mathscr{V})$ is the *category of global sections* of $(X, \mathscr{V})$. A variety of preadditive categories $(X, \mathscr{V})$ is an *abelian variety of categories* if $\Gamma(X, \mathscr{V})$ is an abelian category. Note that the morphisms of the category $\Gamma(X, \mathscr{V})$ are exactly what we had defined to be the morphisms of the variety $(X, \mathscr{V})$ in Section 1.

Recall that a *dense* subcategory $\mathscr{A}$ of an abelian category $\mathscr{C}$ is a full subcategory of $\mathscr{C}$ such that for every exact sequence $0\to A'\to A\to A''\to 0$ in $\mathscr{C}$, A is in $\mathscr{A}$ if and only if both A' and A'' are in $\mathscr{A}$[3]. If $\mathscr{A}$ is a dense subcategory of an abelian category $\mathscr{C}$, it is possible to construct the quotient category $\mathscr{C}/\mathscr{A}$ (in the sense of Gabriel) and $\mathscr{C}/\mathscr{A}$ turns out to be an abelian category. If $(X, \mathscr{V})$ is an abelian variety of categories and Y is a closed subset of X, let $\mathscr{A}_Y$ be the full subcategory of $\Gamma(X, \mathscr{V})$ whose objects are all the objects A of $(X, \mathscr{V})$ with $z(A)\supseteq Y$. Clearly $\mathscr{A}_Y$ is a dense subcategory of $\Gamma(X, \mathscr{V})$. Hence it is possible to construct the quotient category $\Gamma(X, \mathscr{V})/\mathscr{A}_Y$. We denote the canonical functor $\Gamma(X, \mathscr{V})\to \Gamma(X, \mathscr{V})/\mathscr{A}_Y$ by T_Y.

Now, we may define a functor $T'_Y: \Gamma(X, \mathscr{V})\to \Gamma(Y, \mathscr{V})$ in the following way: $T'_Y(A)=A$ for all $A\in \mathrm{Ob}(\Gamma(X, \mathscr{V}))$ and if $f\in \mathrm{Hom}_{\Gamma(X,\mathscr{V})}(A,B)=\Gamma(X, \mathscr{H}om_{\mathscr{V}}(A,B))$, $T'_Y(f)\in \mathrm{Hom}_{\Gamma(Y,\mathscr{V})}(A,B)=\Gamma(Y, \mathscr{H}om_{\mathscr{V}}(A,B))$ is the restriction of the global section f to the closed subset Y of X.

2.1. Proposition. *Suppose that X is a Boolean space. Then there exists a unique functor $F_Y : \Gamma(X, \mathscr{V})/\mathscr{A}_Y \to \Gamma(Y, \mathscr{V})$ such that $F_Y T_Y = T'_Y$. Moreover F_Y is an isomorphism of categories. In particular $\Gamma(Y, \mathscr{V})$ is an abelian category for all closed subsets Y of X.*

Proof. Recall that $\Gamma(X, \mathscr{V})/\mathscr{A}_Y$ is defined as the category whose objects are the objects of $\Gamma(X, \mathscr{V})$ and whose morphisms are defined by

$$\mathrm{Hom}_{\Gamma(X, \mathscr{V})/\mathscr{A}_Y}(A, B) = \varinjlim \mathrm{Hom}_{\Gamma(X, \mathscr{V})}(A', B/B')$$

where A' ranges in the set of the subobjects of A in $\Gamma(X, \mathscr{V})$ such that $A/A' \in \mathscr{A}_Y$ and B' ranges in the set of the subobjects B' of B in $\Gamma(X, \mathscr{V})$ such that $B' \in \mathscr{A}_Y$. Since Y is a closed subset of X, the restriction $\Gamma(X, \mathscr{H}om_{\mathscr{V}}(A, B)) \to \Gamma(Y, \mathscr{H}om_{\mathscr{V}}(A, B))$ is a surjective homomorphism whose kernel consists of all the global sections of $\mathscr{H}om_{\mathscr{V}}(A, B)$ which have zero restriction to Y ([11, Lemma 3.3]). Hence we only have to prove that the canonical homomorphism

$$\mathrm{Hom}_{\Gamma(X, \mathscr{V})}(A, B) \to \varinjlim \mathrm{Hom}_{\Gamma(X, \mathscr{V})}(A', B/B') \tag{1}$$

is surjective and that its kernel consists of all the elements $f \in \mathrm{Hom}_{\Gamma(X, \mathscr{V})}(A, B) = \Gamma(X, \mathscr{H}om_{\mathscr{V}}(A, B))$ whose restriction to Y is the zero section.

Let $f \in \mathrm{Hom}_{\Gamma(X, \mathscr{V})}(A, B)$. Then f is in the kernel of the canonical morphism if and only if there exist A', B' subobjects of A, B in $\Gamma(X, \mathscr{V})$ with $A/A', B' \in \mathscr{A}_Y$ such that if $i : A' \to A$ and $p : B \to B/B'$ are the injection and the projection in $\Gamma(X, \mathscr{V})$, then $p \circ f \circ i = 0$. But if $A/A', B' \in \mathscr{A}_Y$, then $i(y) : A' \to A$ and $p(y) : B \to B/B'$ are isomorphisms in $\mathscr{V}_y$ for all $y \in Y$, so that $p \circ f \circ i = 0$ implies $f(y) = 0$, i.e. the restriction of f to Y is the zero section. Conversely if the restriction of f to Y is the zero section and if $i : A' \to A$ is the kernel of f in $\Gamma(X, \mathscr{V})$ then it is easy to check that $i(x) : A' \to A$ is the kernel of $f(x)$ in $\mathscr{V}_x$ for all $x \in X$. Hence $i(y)$ is an isomorphism for all $y \in Y$ so that A/A', the target of the cokernel of $i : A \to A'$ in $\Gamma(X, \mathscr{V})$, is a zero object in $\mathscr{V}_y$ for all $y \in Y$. Hence $A/A' \in \mathscr{A}_Y$. But then it is clear that f is in the kernel of the canonical morphism (1). We have thus proved that the kernel of the canonical morphism (1) consists of all the global sections of $\mathscr{H}om_{\mathscr{V}}(A, B)$ whose restriction to Y is zero. Let us now prove that the canonical morphism (1) is surjective.

Let A', B' be subobjects of A, B resp. in $\Gamma(X, \mathscr{V})$. Suppose that A/A' and B' are in $\mathscr{A}_Y$. Then the inclusion $A' \to A$ and the projection $B \to B/B'$ are morphisms in $\Gamma(X, \mathscr{V})$, i.e. global sections, and their multiplication induces a sheaf morphism $\mathscr{H}om_{\mathscr{V}}(A, B) \to \mathscr{H}om_{\mathscr{V}}(A', B/B')$. If $x \in z(A/A') \cap z(B')$ this sheaf morphism induces a group isomorphism between the two stalks over x. Hence the sheaf morphism $\mathscr{H}om_{\mathscr{V}}(A, B) \to \mathscr{H}om_{\mathscr{V}}(A', B/B')$ induces a group isomorphism $\Gamma(Y, \mathscr{H}om_{\mathscr{V}}(A, B)) \to \Gamma(Y, \mathscr{H}om_{\mathscr{V}}(A', B/B'))$. By composing with the restriction $\Gamma(X, \mathscr{H}om_{\mathscr{V}}(A, B)) \to \Gamma(Y, \mathscr{H}om_{\mathscr{V}}(A, B))$ we get an epimorphism $\Gamma(X, \mathscr{H}om_{\mathscr{V}}(A, B)) \to \Gamma(Y, \mathscr{H}om_{\mathscr{V}}(A', B/B'))$. Hence for every element f of $\mathrm{Hom}_{\Gamma(X, \mathscr{V})}(A', B/B')$ there exists an element g of $\mathrm{Hom}_{\Gamma(X, \mathscr{V})}(A, B)$ such that f and g have the same restriction to Y. Therefore (1) is surjective. □

2.2. Corollary. *If $(X, \mathscr{V})$ is an abelian variety of categories and its basis X is a Boolean space, then $\mathscr{V}_x$ is an abelian category for all $x \in X$.* □

Let us conclude this section with another definition. An abelian variety $(X, \mathscr{V})$ of categories is said to be a *spectral variety* if the category $\Gamma(X, \mathscr{V})$ is a spectral category [4], i.e. an abelian category with exact direct limits and a generator in which every exact sequence splits.

3. The decomposition space of a preadditive category

In Section 4 we shall associate a variety of preadditive categories to each abelian category. We need a topological space, which will serve as a basis for the associated variety of preadditive categories. Our aim in this section is to construct that topological space.

Recall that the *center* $Z[\mathscr{C}]$ of a preadditive category $\mathscr{C}$ is the ring of all functorial morphisms of the identity functor of $\mathscr{C}$ into itself [12]. It is a commutative ring; an element u of $Z[\mathscr{C}]$ consists of a morphism $u_A : A \to A$ for every object A of $\mathscr{C}$ such that $fu_A = u_B f$ for all morphisms $f: A \to B$ in $\mathscr{C}$. Consider the Boolean algebra $B[\mathscr{C}]$ of all idempotent elements of $Z[\mathscr{C}]$: if $u, v \in B[\mathscr{C}]$, their sum in $B[\mathscr{C}]$ corresponds to the endomorphism $u_A + v_A - u_A v_A$ of A for every object A of $\mathscr{C}$, and their product corresponds to the endomorphism $u_A v_A$. The *decomposition space* $\mathfrak{X}[\mathscr{C}]$ of $\mathscr{C}$ is the topological space $\operatorname{Spec} B[\mathscr{C}]$ with the Zariski topology. It is a Boolean space and the sets $\mathfrak{X}[\mathscr{C}]_u = \{M \in \mathfrak{X}[\mathscr{C}] \mid u \notin M\}$, $u \in B[\mathscr{C}]$, are the clopen sets in $\mathfrak{X}[\mathscr{C}]$ and form a basis of open sets for the topology of $\mathfrak{X}[\mathscr{C}]$. If $\mathscr{C}$ is a Grothendieck category and U is a generator of $\mathscr{C}$, the center $Z[\mathscr{C}]$ of $\mathscr{C}$ and the center of the ring $\operatorname{Hom}_{\mathscr{C}}(U, U)$ are isomorphic [10], so that the decomposition space $\mathfrak{X}[\mathscr{C}]$ of $\mathscr{C}$ is homeomorphic to the decomposition space of the ring $\operatorname{Hom}_{\mathscr{C}}(U, U)$ [11, p. 8].

A category $\mathscr{C}$ is a *null category* if every object of $\mathscr{C}$ is a zero object. A preadditive category $\mathscr{C}$ is *indecomposable* if it is not a null category and for all preadditive categories $\mathscr{C}_1, \mathscr{C}_2$ with $\mathscr{C}_1 \times \mathscr{C}_2$ equivalent to $\mathscr{C}$, either $\mathscr{C}_1$ or $\mathscr{C}_2$ is a null category. Note that if $\mathscr{C}_1, \mathscr{C}_2$ are preadditive categories, $Z[\mathscr{C}_1 \times \mathscr{C}_2]$ is canonically isomorphic to $Z[\mathscr{C}_1] \times Z[\mathscr{C}_2]$, so that $\mathfrak{X}[\mathscr{C}_1 \times \mathscr{C}_2]$ is the disjoint union of two clopen subsets canonically homeomorphic to $\mathfrak{X}[\mathscr{C}_1]$ and $\mathfrak{X}[\mathscr{C}_2]$ respectively. Also note that equivalent categories have isomorphic centers and homeomorphic decomposition spaces. It is not difficult to check that a preadditive category $\mathscr{C}$ is indecomposable if and only if $B[\mathscr{C}] = \{0, 1\}$, i.e. if and only if $\mathfrak{X}[\mathscr{C}]$ has a unique point.

Let us conclude this section by relating the decomposition space to the spectrum of a Grothendieck category.

Recall that if $\mathscr{C}$ is a Grothendieck category, the *spectrum* $\operatorname{Sp}(\mathscr{C})$ of $\mathscr{C}$ is the set of all types of injective indecomposable objects [3]. If $\mathscr{C}$ is spectral, $\operatorname{Sp}(\mathscr{C})$ is then the set of all isomorphism classes of simple objects. In this case if $\operatorname{Dis}(\mathscr{C})$ is the full subcategory of $\mathscr{C}$ generated by all objects of $\mathscr{C}$ which are direct sums of simple objects and Cont($\mathscr{C}$) is the full subcategory of $\mathscr{C}$ generated by all the objects of $\mathscr{C}$ which have no simple subobjects, then $\mathscr{C}$ is equivalent to Dis($\mathscr{C}$)×Cont($\mathscr{C}$) [4].

3.1. Lemma. *Let $\mathscr{C}$ be a spectral category. Then there is a canonical injection of* Sp($\mathscr{C}$) *into* $\mathfrak{X}[\mathscr{C}]$. *The image of every element of* Sp($\mathscr{C}$) *is an isolated point of* $\mathfrak{X}[\mathscr{C}]$, *and the closure of the image of* Sp($\mathscr{C}$) *in* $\mathfrak{X}[\mathscr{C}]$ *is homeomorphic to* $\mathfrak{X}[\mathrm{Dis}(\mathscr{C})]$ *and is the Stone–Čech compactification of the image.*

Proof. Let U be a generator of $\mathscr{C}$. Then there is a one-to-one correspondence between Sp($\mathscr{C}$) and the set of isomorphism classes of simple subobjects of U. Moreover U is isomorphic to the direct sum $(\bigoplus_{i\in \mathrm{Sp}(\mathscr{C})} U_i)\oplus U_{\mathrm{cont}}$, where for every $i\in \mathrm{Sp}(\mathscr{C})$, U_i is a direct sum of simple objects of type i and U_{cont} has no simple submodules. Note that the U_i's and U_{cont} are fully invariant in U. Thus

$$\mathrm{End}_{\mathscr{C}}(U)\cong\Big(\prod_{i\in \mathrm{Sp}(\mathscr{C})}\mathrm{End}_{\mathscr{C}}(U_i)\Big)\times \mathrm{End}_{\mathscr{C}}(U_{\mathrm{cont}}),$$

so that

$$B[\mathscr{C}]\cong\Big(\prod_{i\in \mathrm{Sp}(\mathscr{C})}B(\mathrm{End}_{\mathscr{C}}(U_i))\Big)\times B(\mathrm{End}_{\mathscr{C}}(U_{\mathrm{cont}})).$$

Now $B(\mathrm{End}_{\mathscr{C}}(U_i))=\{0,1\}$ for all $i\in \mathrm{Sp}(\mathscr{C})$. Therefore $\mathfrak{X}[\mathscr{C}]$ is homeomorphic to $\mathfrak{X}[\mathrm{Dis}(\mathscr{C})]\cup\mathfrak{X}[\mathrm{Cont}(\mathscr{C})]$ and $\mathfrak{X}[\mathrm{Dis}(\mathscr{C})]$ is the spectrum of the ring $\{0,1\}^{\mathrm{Sp}(\mathscr{C})}$, i.e. it is the Stone–Čech compactification of Sp($\mathscr{C}$) with the discrete topology. □

4. The variety of preadditive categories associated to an abelian category

In the previous section we have associated the Boolean algebra $B[\mathscr{C}]$ and its spectrum $\mathfrak{X}[\mathscr{C}]$ to the preadditive category $\mathscr{C}$. Now suppose that $\mathscr{C}$ is abelian, and fix a maximal ideal M of $B[\mathscr{C}]$, i.e. a point M of $\mathfrak{X}[\mathscr{C}]$. Consider the full subcategory $\mathscr{A}_M$ of $\mathscr{C}$ whose objects are the objects A of $\mathscr{C}$ such that $u_A=0$ for some $u\in B[\mathscr{C}]$, $u\notin M$. Let us show that $\mathscr{A}_M$ is a dense subcategory of $\mathscr{C}$.

4.1. Lemma. *If $\mathscr{C}$ is an abelian category and M is a maximal ideal in $B[\mathscr{C}]$, $\mathscr{A}_M$ is a dense subcategory of $\mathscr{C}$.*

Proof. For every exact sequence $0\to A'\to A\to A''\to 0$ in $\mathscr{C}$, the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle u_{A'}} & & \downarrow{\scriptstyle u_A} & & \downarrow{\scriptstyle u_{A''}} & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0
\end{array} \qquad (2)$$

is commutative. Hence if A is in $\mathscr{A}_M$ and $u_A=0$, then $u_{A'}=0$ and $u_{A''}=0$, so that

A' and A'' are in $\mathscr{A}_M$. Conversely if A' and A'' are in $\mathscr{A}_M$ and $u', u'' \in B[\mathscr{C}]\backslash M$ with $u'_{A'} = 0$, $u''_{A''} = 0$, then $u = u'u'' \in B[\mathscr{C}]\backslash M$ and $u_{A'} = 0$, $u_{A''} = 0$. The commutativity of (2) gives $u_A = 0$, i.e. $A \in \mathscr{A}_M$. □

Since $\mathscr{A}_M$ is a dense subcategory of $\mathscr{C}$, we may construct the quotient category $\mathscr{C}/\mathscr{A}_M$ [3]: the objects of $\mathscr{C}/\mathscr{A}_M$ are the objects of $\mathscr{C}$ and if A, B are objects of $\mathscr{C}/\mathscr{A}_M$

$$\mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A, B) = \varinjlim \mathrm{Hom}_{\mathscr{C}}(A', B/B'),$$

where A' ranges in the set of the subobjects of A such that $A/A' \in \mathscr{A}_M$ and B' ranges in the set of the subobjects of B such that $B' \in \mathscr{A}_M$. The category $\mathscr{C}/\mathscr{A}_M$ is an abelian category [3, Prop. III.1.1]. We shall denote the canonical functor $\mathscr{C} \to \mathscr{C}/\mathscr{A}_M$ by T_M; it maps the object A of $\mathscr{C}$ into the object A of $\mathscr{C}/\mathscr{A}_M$ and the morphism $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ into its canonical image

$$T_M(f) \in \varinjlim \mathrm{Hom}_{\mathscr{C}}(A', B/B') = \mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A, B).$$

Our next proposition gives another description of the groups of morphisms in $\mathscr{C}/\mathscr{A}_M$.

4.2. Proposition. *Let $\mathscr{C}$ be an abelian category, $Z[\mathscr{C}]$ be its center and $B[\mathscr{C}]$ be the Boolean algebra of all idempotents of $Z[\mathscr{C}]$. Let A, B be objects of $\mathscr{C}$. Then* $\mathrm{Hom}_{\mathscr{C}}(A, B)$ *has a natural structure of $Z[\mathscr{C}]$-module. If M is a maximal ideal in $B[\mathscr{C}]$ and $T_M(A, B): \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A, B)$ is the group homomorphism induced by the functor T_M, then $T_M(A, B)$ is surjective and its kernel is* $M\mathrm{Hom}_{\mathscr{C}}(A, B)$. *Hence* $\mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A, B)$ *is canonically isomorphic to* $\mathrm{Hom}_{\mathscr{C}}(A, B)/M\mathrm{Hom}_{\mathscr{C}}(A, B)$.

(Recall that if R is a ring, $B(R)$ is the Boolean algebra of all central idempotents of R, M is a maximal ideal of $B(R)$ and C is an R-module, then $MC = \{ex \mid e \in M, x \in C\}$. It is an R-submodule of C [11, p. 7].)

Proof. If $u \in Z[\mathscr{C}]$ and $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, define their product as $uf = u_B f$. It is clear that in this way $\mathrm{Hom}_{\mathscr{C}}(A, B)$ becomes a $Z[\mathscr{C}]$-module. Let us prove that $T_M(A, B): \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A, B)$ is surjective. Let $\bar{h}$ be an element of $\mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A, B)$. Then $\bar{h}$ is the image of an element $h \in \mathrm{Hom}_{\mathscr{C}}(A', B/B')$, where A', B' are subobjects of A, B resp. with A/A' and B' in $\mathscr{A}_M$. There exist $u, v \in B[\mathscr{C}]\backslash M$ such that $u_{A/A'} = 0$ and $v_{B'} = 0$. Since $u_A: A \to A$ is idempotent, there exist a subobject A'' of A and an epimorphism $p: A \to A''$ such that if $i: A'' \to A$ is the inclusion then $ip = u_A$ and $pi = 1_{A''}$ [12, ex. 2.1.5]. Similarly there exist a subobject B'' of B and a monomorphism $j: B/B'' \to B$ such that if $q: B \to B/B''$ is the projection then $jq = v_B$ and $qj = 1_{B/B''}$. Let $r: A \to A/A'$ be the canonical projection; then $ru_A = u_{A/A'}r = 0$, i.e. $A'' = \mathrm{im}\, u_A \le \ker r = A'$ [12, p. 34]. Similarly let $k: B' \to B$ be the inclusion; then $v_B k = k v_{B'} = 0$, i.e. $B' = \mathrm{im}\, k \le \ker v_B = B''$. Finally let $i': A' \to A$ and $i'': A'' \to A'$ be the inclusions and $q': B \to B/B'$, $q'': B/B' \to B/B''$ be the canonical projections, so that $i=i'i''$, $q=q''q'$. We shall prove that $T_M(A,B)$ is surjective by showing that $T_M(A,B)(jq''hi''p)=\bar{h}$, i.e. by showing that the image of $jq''hi''p\in\mathrm{Hom}_{\mathscr{C}}(A,B)$ in $\mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A,B)$ is equal to the image of $h\in\mathrm{Hom}_{\mathscr{C}}(A',B/B')$ in $\mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A,B)$. Now if $t:A\to A/A''$ is the canonical projection, i.e. t is the cokernel of $i:A''\to A$, then $u_{A/A''}t=tu_A=tip=0$ and therefore $u_{A/A''}=0$ (t is an epimorphism). Hence A/A'' is in $\mathscr{A}_M$. Similarly B'' is in $\mathscr{A}_M$. Hence it is sufficient to prove that the image $q(jq''hi''p)i$ of $jq''hi''p\in\mathrm{Hom}_{\mathscr{C}}(A,B)$ into $\mathrm{Hom}_{\mathscr{C}}(A'',B/B'')$ is equal to the image $q''hi''$ of $h\in\mathrm{Hom}_{\mathscr{C}}(A',B/B')$ into $\mathrm{Hom}_{\mathscr{C}}(A'',B/B'')$. By computing: $q(jq''hi''p)i=(qj)q''hi''(pi)=1_{B/B'}q''hi''1_{A''}=q''hi''$. This shows that $T_M(A,B)$ is surjective. Let us determine its kernel. Let $f\in\mathrm{Hom}_{\mathscr{C}}(C_1,C_2)$. Then $T_M(A,B)(f)=0$ if and only if $\mathrm{im}\,f$ is in $\mathscr{A}_M$[3], i.e. if and only if there exists $u\in B[\mathscr{C}]$, $u\notin M$, such that $u_Bf=0$. Now if $u_Bf=0$ with $u\in B[\mathscr{C}]$, $u\notin M$, then $f=(1-u)f\in M\mathrm{Hom}_{\mathscr{C}}(A,B)$. Conversely if $f\in M\mathrm{Hom}_{\mathscr{C}}(A,B)$, then $f=u_Bg$ with $u\in M$ and $g\in\mathrm{Hom}_{\mathscr{C}}(A,B)$, so that $1-u\notin M$ and $(1-u)f=(1-u)_Bu_Bg=0$. Hence $M\mathrm{Hom}_{\mathscr{C}}(A,B)$ is the kernel of $T_M(A,B)$. □

Let us fix two objects A and B of an abelian category $\mathscr{C}$. Then it is possible to associate a sheaf of abelian groups over $\mathfrak{X}[C]$ to the $Z[\mathscr{C}]$-module $\mathrm{Hom}_{\mathscr{C}}(A,B)$ (see Pierce [11, p. 18]). If $M\in\mathfrak{X}[\mathscr{C}]$, the stalk over M of the sheaf associated to $\mathrm{Hom}_{\mathscr{C}}(A,B)$ is simply $\mathrm{Hom}_{\mathscr{C}}(A,B)/M\mathrm{Hom}_{\mathscr{C}}(A,B)$. But in our case $\mathrm{Hom}_{\mathscr{C}}(A,B)/M\mathrm{Hom}_{\mathscr{C}}(A,B)$ is canonically isomorphic to $\mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A,B)$. Via these canonical isomorphisms it is possible to give the following description of the sheaf over $\mathfrak{X}[\mathscr{C}]$ associated to $\mathrm{Hom}_{\mathscr{C}}(A,B)$ (we shall denote this sheaf by $\mathscr{H}om_{\mathscr{C}}(A,B)$): for $M\in\mathfrak{X}[\mathscr{C}]$, set

$$\mathscr{H}om_{\mathscr{C}}(A,B)_M=\mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A,B)$$

and

$$\mathscr{H}om_{\mathscr{C}}(A,B)=\bigcup_{M\in\mathfrak{X}[\mathscr{C}]}\mathscr{H}om_{\mathscr{C}}(A,B)_M.$$

Let $\pi:\mathscr{H}om_{\mathscr{C}}(A,B)\to\mathfrak{X}[\mathscr{C}]$ be given by $\pi^{-1}(M)=\mathscr{H}om_{\mathscr{C}}(A,B)_M$. Let $T_M:\mathscr{C}\to\mathscr{C}/\mathscr{A}_M$ be the canonical functor and let $T_M(A,B):\mathrm{Hom}_{\mathscr{C}}(A,B)\to\mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A,B)$ be the induced group homomorphism. For $f\in\mathrm{Hom}_{\mathscr{C}}(A,B)$ and $M\in\mathfrak{X}[\mathscr{C}]$ set $\tau_f(A,B)(M)=T_M(A,B)(f)$ so that $\tau_f(A,B)$ is a mapping $\mathfrak{X}[\mathscr{C}]\to\mathscr{H}om_{\mathscr{C}}(A,B)$. Topologize $\mathscr{H}om_{\mathscr{C}}(A,B)$ by taking all sets $\tau_f(A,B)(\mathfrak{X}[\mathscr{C}]_e)$, with $f\in\mathrm{Hom}_{\mathscr{C}}(A,B)$, $e\in B[\mathscr{C}]$, as a basis for the open sets. In this way we have constructed a sheaf of abelian groups $\mathscr{H}om_{\mathscr{C}}(A,B)$ over $\mathfrak{X}[\mathscr{C}]$ [11] for each ordered pair (A,B) of objects of $\mathscr{C}$.

4.3. Theorem. *If $\mathscr{C}$ is an abelian category, the topological space $\mathfrak{X}[\mathscr{C}]$, the class $\mathrm{Ob}(\bar{\mathscr{C}})=\mathrm{Ob}(\mathscr{C})$, the sheaves of abelian groups $\mathscr{H}om_{\mathscr{C}}(A,B)$ ($A,B\in\mathrm{Ob}(\mathscr{C})$) and the categories $\mathscr{C}/\mathscr{A}_M$ ($M\in\mathfrak{X}[\mathscr{C}]$) form a variety of preadditive categories.*

Proof. We must show that if A,B,C are objects of $\mathscr{C}$, the mapping $c:\mathscr{H}om_{\mathscr{C}}(A,B)+\mathscr{H}om_{\mathscr{C}}(B,C)\to\mathscr{H}om_{\mathscr{C}}(A,C)$ defined by

$$c(T_M(A,B)(f),T_M(B,C)(g))=T_M(A,C)(g\circ f)$$

for $M\in\mathfrak{X}[\mathscr{C}]$, $f\in\mathrm{Hom}_{\mathscr{C}}(A,B)$, $g\in\mathrm{Hom}_{\mathscr{C}}(B,C)$ is continuous. Let us fix $M_0\in\mathfrak{X}[\mathscr{C}]$, $f_0\in\mathrm{Hom}_{\mathscr{C}}(A,B)$, $g_0\in\mathrm{Hom}_{\mathscr{C}}(B,C)$ and an open set $\tau_{h_0}(A,C)(\mathfrak{X}[\mathscr{C}]_e)$ of $\mathscr{H}om_{\mathscr{C}}(A,C)$ containing

$$c(T_{M_0}(A,B)(f_0),T_{M_0}(B,C)(g_0))=T_{M_0}(A,C)(g_0\circ f_0),$$

with $h_0\in\mathrm{Hom}_{\mathscr{C}}(A,C)$, $e\in B[\mathscr{C}]$. Since $T_{M_0}(A,C)(g_0\circ f_0)\in\tau_{h_0}(A,C)(\mathfrak{X}[\mathscr{C}]_e)$ we have that $T_{M_0}(A,C)(g_0\circ f_0)=\tau_{h_0}(A,C)(M_0)$, i.e. $T_{M_0}(A,C)(g_0\circ f_0)=T_{M_0}(A,C)(h_0)$. By Proposition 4.2 it follows that $g_0\circ f_0-h_0\in M_0\mathrm{Hom}_{\mathscr{C}}(A,C)$, i.e. $g_0\circ f_0-h_0=u_0l_0$ for some $u_0\in M_0$, $l_0\in\mathrm{Hom}_{\mathscr{C}}(A,C)$. Now $\tau_{f_0}(A,B)(\mathfrak{X}[\mathscr{C}]_{e(1-u_0)})$ is a neighborhood of $T_{M_0}(A,B)(f_0)$ in $\mathscr{H}om_{\mathscr{C}}(A,B)$, and $\tau_{g_0}(A,B)(\mathfrak{X}[\mathscr{C}]_{e(1-u_0)})$ is a neighborhood of $T_{M_0}(A,B)(g_0)$ in $\mathscr{H}om_{\mathscr{C}}(B,C)$ and it easy to verify that

$$c[(\tau_{f_0}(A,B)(\mathfrak{X}[\mathscr{C}]_{e(1-u_0)})\times\tau_{g_0}(A,B)(\mathfrak{X}[\mathscr{C}]_{e(1-u_0)}))\cap(\mathscr{H}om_{\mathscr{C}}(A,B)+\mathscr{H}om_{\mathscr{C}}(B,C))]$$
$$\subseteq\tau_{h_0}(A,C)(\mathfrak{X}[\mathscr{C}]_e).$$

This proves that c is continuous.

Moreover, according to [11], for each $A\in\mathrm{Ob}(\mathscr{C})$ the mapping $\mathfrak{X}[\mathscr{C}]\to\mathscr{H}om_{\mathscr{C}}(A,A)$, $M\mapsto T_M(A,A)(1_A)$, is continuous. □

Thus we have associated a variety of preadditive categories $(\mathfrak{X}[\mathscr{C}],\tilde{\mathscr{C}})$ to every abelian category $\mathscr{C}$.

Let us conclude this section with a corollary which will be useful in the sequel.

4.4. Corollary. *Let A,B be objects of an abelian category $\mathscr{C}$. Then the set $\{M\in\mathfrak{X}[\mathscr{C}]\mid A$ and B are isomorphic objects of $\mathscr{C}/\mathscr{A}_M\}$ is an open set in $\mathfrak{X}[\mathscr{C}]$.*

Proof. Let S be the set in question. If $M\in S$, by Proposition 4.2 there exists $f\in\mathrm{Hom}_{\mathscr{C}}(A,B)$ such that $T_M(A,B)(f)\in\mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A,B)$ is an isomorphism. By [3] the objects $\ker f$ and $\operatorname{coker} f$ of $\mathscr{C}$ are in $\mathscr{A}_M$, i.e. they are zero objects in $\mathscr{C}/\mathscr{A}_M$. Thus $M\in z(\ker f)\cap z(\operatorname{coker} f)\subseteq S$. By Theorem 4.3 and Lemma 1.2 $z(\ker f)\cap z(\operatorname{coker} f)$ is open in $\mathfrak{X}[\mathscr{C}]$. Hence S is an open set. □

5. Reduced varieties of preadditive categories

In Section 4 we associated a variety of preadditive categories $(\mathfrak{X}[\mathscr{C}],\tilde{\mathscr{C}})$ to every abelian category $\mathscr{C}$. The stalks of $(\mathfrak{X}[\mathscr{C}],\tilde{\mathscr{C}})$ were the quotient categories $\mathscr{C}/\mathscr{A}_M$, $M\in\mathfrak{X}[\mathscr{C}]$. Our aim is to study *spectral* categories [4], i.e. Grothendieck categories in which every exact sequence splits, and in Section 7 we shall study the categories $\mathscr{C}/\mathscr{A}_M$, with $\mathscr{C}$ spectral, in detail. For the time being we only need the following result.

5.1. Lemma. *Assume that $\mathscr{C}$ is a spectral category and M is a point of $\mathfrak{X}[\mathscr{C}]$. If A,B*

are objects of $\mathscr{C}$, then in $\mathscr{C}/\mathscr{A}_M$ either A is isomorphic to a subobject of B or B is isomorphic to a subobject of A.

Proof. By [7, Theorem 1.14] we may suppose that $\mathscr{C}$ is the category of all non-singular injective right modules over a regular right self-injective ring R. By [7, Theorem 3.3] there exists a central idempotent $e \in R$ such that $Ae \lesssim Be$ and $B(1-e) \lesssim A(1-e)$. Without loss of generality we may suppose that M is a maximal ideal of the Boolean algebra of all central idempotents of R. Then either $e \in M$ or $(1-e) \in M$. If $e \in M$ then the two objects A and $A(1-e)$ of $\mathscr{C}/\mathscr{A}_M$ are isomorphic. Similarly for B and $B(1-e)$. Thus B is isomorphic to a subobject of A in $\mathscr{C}/\mathscr{A}_M$. On the other hand, if $(1-e) \in M$, A is isomorphic to a subobject of B. □

5.2. Corollary. *If $\mathscr{C}$ if a spectral category and M is a point of $\mathfrak{X}[\mathscr{C}]$, then $\mathscr{C}/\mathscr{A}_M$ is an indecomposable category.*

Proof. Every decomposable category $\mathscr{D}$ contains two non-zero objects A, B with $\mathrm{Hom}_{\mathscr{D}}(A, B) = 0$. □

We say that a variety of preadditive categories $(X, \mathscr{V})$ is a *reduced* variety if X is a Boolean space and $\mathscr{V}_x$ is an indecomposable category for all $x \in X$.

5.3. Corollary. *If $\mathscr{C}$ is a spectral category, the variety of preadditive categories $(\mathfrak{X}[\mathscr{C}], \tilde{\mathscr{C}})$ associated to $\mathscr{C}$ is a reduced variety.* □

Note that if $\mathscr{C}$ is a spectral category, the categories $\mathscr{C}/\mathscr{A}_M$, $M \in \mathfrak{X}[\mathscr{C}]$, are abelian categories in which every exact sequence splits [3; Cor. III.1.1], but they are not necessarily spectral categories (see Example 9.1).

6. The isomorphism theorems

The object of this section is to prove that for spectral categories and reduced spectral varieties of preadditive categories the correspondences $\mathscr{C} \mapsto (\mathfrak{X}[\mathscr{C}], \tilde{\mathscr{C}})$ and $(X, \mathscr{V}) \mapsto \Gamma(X, \mathscr{V})$ are inverses of each other up to isomorphism.

6.1. Theorem. *Let $\mathscr{C}$ be an abelian category, let $(\mathfrak{X}[\mathscr{C}], \tilde{\mathscr{C}})$ be the variety of preadditive categories associated to $\mathscr{C}$, and let $\Gamma(\mathfrak{X}[\mathscr{C}], \tilde{\mathscr{C}})$ be the category of global sections of $(\mathfrak{X}[\mathscr{C}], \tilde{\mathscr{C}})$. Then $\mathscr{C}$ is canonically isomorphic to $\Gamma(\mathfrak{X}[\mathscr{C}], \tilde{\mathscr{C}})$.*

Proof. Let us define a canonical functor $I: \mathscr{C} \to \Gamma(\mathfrak{X}[\mathscr{C}], \tilde{\mathscr{C}})$ in the following way: if A is an object of $\mathscr{C}$, set $I(A) = A$ and if $f: A \to B$ is a morphism in $\mathscr{C}$ let $I(f) \in \Gamma(\mathfrak{X}[\mathscr{C}], \mathscr{H}om_{\mathscr{C}}(A, B))$ be the global section of the sheaf $\mathscr{H}om_{\mathscr{C}}(A, B)$ over $\mathfrak{X}[\mathscr{C}]$, for which $I(f)(M)$ is the image of $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ into $\mathrm{Hom}_{\mathscr{C}/\mathscr{A}_M}(A, B)$.

I is clearly a functor and induces a bijection between the classes $\mathrm{Ob}(\mathscr{C})$ and $\mathrm{Ob}(\Gamma(\mathfrak{X}[\mathscr{C}],\mathscr{C}))$. In order to prove that I is an isomorphism of categories we have to show that it induces a group isomorphism $\mathrm{Hom}_{\mathscr{C}}(A,B)\to\Gamma(\mathfrak{X}[\mathscr{C}],\mathscr{H}om_{\mathscr{C}}(A,B))$ for all $A,B\in\mathrm{Ob}(\mathscr{C})$. But this is exactly the isomorphism of $Z[\mathscr{C}]$-modules of [11, Theorem 4.5] (see Proposition 4.2). □

6.2. Corollary. *If $\mathscr{C}$ is an abelian (resp. spectral) category, then $(\mathfrak{X}[\mathscr{C}],\mathscr{C})$ is an abelian (resp. spectral) variety of categories.* □

6.3. Theorem. *Let $(X,\mathscr{V})$ be a reduced variety of preadditive categories. If $x\in X$, the set $M_x=\{u\in B[\Gamma(X,\mathscr{V})]\mid u_A(x)=0$ for all $A\in\mathrm{Ob}(\mathscr{V})\}$ is a maximal ideal of $B[\Gamma(X,\mathscr{V})]$ and the mapping $\phi:X\to\mathfrak{X}[\Gamma(X,\mathscr{V})]$, $\phi(x)=M_x$, is a homeomorphism.*

Proof. Consider the ring morphism $B[\Gamma(X,\mathscr{V})]\to B[\mathscr{V}_x]$ such that $u_A\mapsto u_A(x)$ for every object A. It is surjective and $B[\mathscr{V}_x]=\{0,1\}$ because $\mathscr{V}_x$ is indecomposable. Since M_x is its kernel, M_x is a maximal ideal of $B[\Gamma(X,\mathscr{V})]$.

Let x,y be two distinct points of X. Since X is Boolean, there exists a clopen set Y of X with $x\notin Y$ and $y\in Y$. If $A\in\mathrm{Ob}(\mathscr{V})$ define $u_A\in\mathrm{Hom}_{\Gamma(X,\mathscr{V})}(A,A)=\Gamma(X,\mathscr{H}om_{\mathscr{V}}(A,A))$ by $u_A(t)=1_A$ if $t\in Y$, and $u_A(t)=0_A$ if $t\in X\setminus Y$. Then $u\in B[\Gamma(X,\mathscr{V})]$, $u(x)=0$ and $u(y)\neq 0$. Thus $M_x\neq M_y$ and the mapping ϕ is injective.

A basis for the open sets of $\mathfrak{X}[\Gamma(X,\mathscr{V})]$ is given by the sets

$$\mathfrak{X}_e=\{M\in\mathfrak{X}[\Gamma(X,\mathscr{V})]\mid e\notin M\},$$

$e\in B[\Gamma(X,\mathscr{V})]$, and

$$\phi^{-1}(\mathfrak{X}_e)=\{x\in X\mid e\notin M_x\}=\{x\in X\mid e(x)\neq 0\}$$
$$=\bigcup_{A\in\mathrm{Ob}(\mathscr{V})}\{x\in X\mid e_A(x)\neq 0\}.$$

But since $\mathscr{V}_x$ is an indecomposable category, either $e_A(x)=0_A$ or $e_A(x)=1_A$. Hence

$$\phi^{-1}(\mathfrak{X}_e)=\bigcup_{A\in\mathrm{Ob}(\mathscr{V})}\{x\in X\mid e_A(x)=1_A\}.$$

But $\{x\in X\mid e_A(x)=1_A\}$ is the set on which the two global sections 1_A and e_A of $\Gamma(X,\mathscr{H}om_{\mathscr{V}}(A,A))$ coincide and it is open. Hence $\phi^{-1}(\mathfrak{X}_e)$ is open and ϕ is continuous. Note that if $\mathfrak{X}_e\neq\emptyset$, i.e. if $e\neq 0$, then there exists $x\in X$ with $e(x)\neq 0$. But $e(x)\in B[\mathscr{V}_x]=\{0,1\}$. Hence $e(x)=1$ and $e\notin M_x$. Thus $\phi(X)\cap\mathfrak{X}_e\neq\emptyset$ and the image of the continuous injection ϕ is dense in $\mathfrak{X}[\Gamma(X,\mathscr{V})]$. Since X is compact and $\mathfrak{X}[\Gamma(X,\mathscr{V})]$ is Hausdorff, ϕ is a homeomorphism. □

Remark. By Theorem 6.3 and by [7, Prop. 4.1] if $(X,\mathscr{V})$ is a reduced spectral variety of categories, then X is a *complete* Boolean space.

6.4. Theorem. *Let $(X,\mathscr{V})$ be a reduced abelian variety of categories and let*

$(\mathfrak{X}[\Gamma(X,\mathscr{V})],\overline{\Gamma(X,\mathscr{V})})$ be the abelian variety of categories associated to the abelian category $\Gamma(X,\mathscr{V})$. Then the varieties $(X,\mathscr{V})$ and $(\mathfrak{X}[\Gamma(X,\mathscr{V})],\overline{\Gamma(X,\mathscr{V})})$ are canonically isomorphic.

Proof. Let $\phi: X\to\mathfrak{X}[\Gamma(X,\mathscr{V})]$ be the homeomorphism defined in Theorem 6.3 and $\Phi:\mathrm{Ob}(\mathscr{V})\to\mathrm{Ob}(\overline{\Gamma(X,\mathscr{V})})$ be the identity mapping. We have to define a homeomorphism $\Phi(A,B)$ for each ordered pair (A,B) of objects of $(X,\mathscr{V})$, $\Phi(A,B):\mathscr{H}om_{\mathscr{V}}(A,B)\to\mathscr{H}om_{\overline{\Gamma(X,\mathscr{V})}}(A,B)$, which maps $\mathscr{H}om_{\mathscr{V}}(A,B)_x$ isomorphically onto $\mathscr{H}om_{\overline{\Gamma(X,\mathscr{V})}}(A,B)_{\phi(x)}$ for all $x\in X$ and induces a functor $\Phi_x:\mathscr{V}_x\to\Gamma(X,\mathscr{V})/\mathscr{A}_{\phi(x)}$ for all $x\in X$, where $\mathscr{A}_{\phi(x)}$ is the full subcategory of $\Gamma(X,\mathscr{V})$ whose objects are the objects A of $\Gamma(X,\mathscr{V})$ such that $u_A=0$ for some $u\in B[\Gamma(X,\mathscr{V})]$, $u\notin M_x$. Note that if $u\in B[\Gamma(X,\mathscr{V})]$, then $u\notin M_x$ if and only if $u(x)\neq 0$. But $(X,\mathscr{V})$ is reduced and $u(x)\in B[\mathscr{V}_x]$, so that either $u(x)=1$ or $u(x)=0$. Therefore an object A of $\Gamma(X,\mathscr{V})$ is in $\mathscr{A}_{\phi(x)}$ if and only if $u_A=0$ for some $u\in B[\Gamma(X,\mathscr{V})]$ with $u(x)=1$. By Lemma 1.2 it easily follows that the objects of $\mathscr{A}_{\phi(x)}$ are exactly the zero objects of $\mathscr{V}_x$. Let

$$\Phi(A,B):\mathscr{H}om_{\mathscr{V}}(A,B)_x\to\mathrm{Hom}_{\Gamma(X,\mathscr{V})/\mathscr{A}_{\phi(x)}}(A,B)$$
$$=\Gamma(X,\mathscr{H}om_{\mathscr{V}}(A,B))/M_x\Gamma(X,\mathscr{H}om_{\mathscr{V}}(A,B))$$

be the canonical isomorphism of [11, Lemma 5.2(c)]. Then $\Phi(A,B)$ has the required properties. □

We have thus proved our main theorem:

6.5. Theorem. *If we associate the reduced spectral variety of categories $(\mathfrak{X}[\mathscr{C}],\tilde{\mathscr{C}})$ to the spectral category $\mathscr{C}$ and the spectral category $\Gamma(X,\mathscr{V})$ to the reduced spectral variety of categories $(X,\mathscr{V})$, we get a one-to-one correspondence between the isomorphism classes of reduced spectral varieties of categories and the isomorphism classes of spectral categories.* □

Now that we have proved Theorem 6.5 we may begin our study of spectral categories. We shall investigate the properties of spectral categories by analyzing the structure of the associated reduced spectral variety of categories.

7. The stalks of the variety associated to a spectral category

In Section 4 we have associated an abelian variety of categories $(\mathfrak{X}[\mathscr{C}],\tilde{\mathscr{C}})$ to every abelian category $\mathscr{C}$. The stalks of $(\mathfrak{X}[\mathscr{C}],\tilde{\mathscr{C}})$ are the quotient categories $\mathscr{C}/\mathscr{A}_M$, $M\in\mathfrak{X}[\mathscr{C}]$. In this section we study the categories $\mathscr{C}/\mathscr{A}_M$, $M\in\mathfrak{X}[\mathscr{C}]$, when $\mathscr{C}$ is a spectral category. We shall make use of the results of [7]. We are primarily interested in the class of all the isomorphism classes of objects of $\mathscr{C}/\mathscr{A}_M$.

In this section $\mathscr{C}$ denotes a spectral category. If M is a point of $\mathfrak{X}[\mathscr{C}]$, we shall denote by $\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$ the class of the isomorphism classes of $\mathscr{C}/\mathscr{A}_M$. If A is an object of $\mathscr{C}/\mathscr{A}_M$, $[A]$ will denote the class of all the objects isomorphic to A in $\mathscr{C}/\mathscr{A}_M$. Thus $\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$ is the class whose elements are the classes $[A]$, where A ranges in $\mathrm{Ob}(\mathscr{C}/\mathscr{A}_M)$. By [3, Prop. III.1.1] $\mathscr{C}/\mathscr{A}_M$ is an abelian category and by [3, Cor. III.1.1] every exact sequence in $\mathscr{C}/\mathscr{A}_M$ splits. Thus in $\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$ we may define a composition law $+$ by $[A]+[B]=[A\oplus B]$ and a relation $\leq$ by setting $[A]\leq[B]$ if A is isomorphic to a subobject of B in $\mathscr{C}/\mathscr{A}_M$. The composition law $+$ and the relation $\leq$ have the following properties:

(i) $+$ is associative and commutative;

(ii) if 0 is a zero object in $\mathscr{C}/\mathscr{A}_M$, then $[0]+[A]=[A]$ for every $[A]\in\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$;

(iii) for every $[A],[B]\in\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$, $[A]\leq[B]$ if and only if there exists $[C]\in\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$ with $[A]+[C]=[B]$;

(iv) $\leq$ is reflexive and transitive;

(v) $\leq$ is antisymmetric: if $[A],[B]\in\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$, $[A]\leq[B]$ and $[B]\leq[A]$, then $[A]=[B]$;

(vi) if $[A],[B]\in\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$, either $[A]\leq[B]$ or $[B]\leq[A]$;

(vii) if $[A],[B],[C]\in\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$ and $[A]\leq[B]$, then $[A]+[C]\leq[B]+[C]$.

The proof of (i)–(iv) is trivial and (vi) follows from Lemma 5.1. Let us prove (v). If $[A]\leq[B]$ and $[B]\leq[A]$, by (iii) there exist C and C' with $A\oplus C\cong B$ and $B\oplus C'\cong A$ ($\cong$ in $\mathscr{C}/\mathscr{A}_M$). Let f,f' be elements of $\mathrm{Hom}_{\mathscr{C}}(A\oplus C,B)$, $\mathrm{Hom}_{\mathscr{C}}(B\oplus C',A)$ resp. such that $T_M(f)$ and $T_M(f')$ are isomorphisms. Then there exist $u,u'\in B[\mathscr{C}]\backslash M$ such that f and f' induce isomorphisms $u_{A\oplus C}(A\oplus C)\to u_B(B)$, $u'_{B\oplus C'}(B\oplus C')\to u'_A(A)$. But then $u_Au'_A(A)$ is isomorphic to a subobject of $u_Bu'_B(B)$ in $\mathscr{C}$ and similarly $u_Bu'_B(B)$ is isomorphic to a subobject of $u_Au'_A(A)$. By Bumby's Theorem [7, Theorem 1.2], (Schroeder–Bernstein theorem for the injective modules), $u_Au'_A(A)$ and $u_Bu'_B(B)$ are isomorphic in $\mathscr{C}$. Then $[A]=[u_Au'_A(A)]=[u_Bu'_B(B)]=[B]$ in $\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$. □

If A is an object of an abelian category, A is *purely infinite* if $A\oplus A\cong A$, and A is *directly finite* if the zero subobject is the unique purely infinite subobject of A. Note that an object is purely infinite and directly finite at the same time if and only if it is a zero object.

7.1. Lemma. *If $\mathscr{C}$ is a spectral category, every object in $\mathscr{C}/\mathscr{A}_M$ is either directly finite or purely infinite.*

Proof. We have to prove that if an object contains a non-zero purely infinite subobject then the object itself is purely infinite. Let A be an object of $\mathscr{C}/\mathscr{A}_M$ and suppose that A contains a non-zero purely infinite subobject B. Then there exists $u\in B[\mathscr{C}]\backslash M$ such that $u_A(A)$ contains $u_B(B)$ as a subobject in $\mathscr{C}$ and $u_B(B)$ is a non-zero object in $\mathscr{C}$. Furthermore since $B\cong B\oplus B$ in $\mathscr{C}/\mathscr{A}_M$ we may suppose that $u_B(B)$ and $u_B(B)\oplus u_B(B)$ are isomorphic in $\mathscr{C}$. Therefore $u_B(B)$ is purely infinite in

$\mathscr{C}$. By [7, Prop. 7.4 and Cor. 7.7], there exist orthogonal idempotents u', u'' of $B[\mathscr{C}]$ such that $u'+u''=1$, $u'_A(A) \oplus u''_A(A) = A$, $u'_A(A)$ is purely infinite in $\mathscr{C}$ and $u''_A(A)$ is directly finite in $\mathscr{C}$. Now $u_B u''_B(B)$ is purely infinite and is contained in $u''_A(A)$ which is directly finite. Hence $u_B u''_B(B)$ is a zero object in $\mathscr{C}$. Since B is a non-zero object in $\mathscr{C}/\mathscr{A}_M$, it follows that $uu'' \in M$, so that $u'' \in M$ and therefore $u' \notin M$. Hence A is isomorphic to $u'_A(A)$ in $\mathscr{C}/\mathscr{A}_M$. It follows that A is purely infinite in $\mathscr{C}/\mathscr{A}_M$. □

By Lemma 7.1, the direct sum in $\mathscr{C}/\mathscr{A}_M$ of a directly finite object and a non-zero purely infinite object is purely infinite. It is clear that the sum of two purely infinite objects is purely infinite. By [7, Theorem 3.6] and by the next lemma, the sum of two directly finite objects is directly finite.

7.2. Lemma. *Every directly finite object of $\mathscr{C}/\mathscr{A}_M$ is of the form $T_M(A)$ for a directly finite object A of $\mathscr{C}$. Every purely infinite object of $\mathscr{C}/\mathscr{A}_M$ is of the form $T_M(A)$ for a purely infinite object A of $\mathscr{C}$.*

Proof. Any object of $\mathscr{C}/\mathscr{A}_M$ is of the form $T_M(B)$ for some object B of $\mathscr{C}$. By [7, Theorem 7.4 and Cor. 7.7] there exist $u, u' \in B[\mathscr{C}]$ with $u_B(B)$ directly finite in $\mathscr{C}$, $u'_B(B)$ purely infinite in $\mathscr{C}$ and $u+u'=1$. Hence either $u \in B[\mathscr{C}]\setminus M$ or $u' \in B[\mathscr{C}]\setminus M$, so that either $u_B(B)$ or $u'_B(B)$ is isomorphic to B in $\mathscr{C}/\mathscr{A}_M$. Hence every object of $\mathscr{C}/\mathscr{A}_M$ is of the form $T_M(A)$, where A is either directly finite or purely infinite in $\mathscr{C}$. If A is purely infinite in $\mathscr{C}$, $T_M(A)$ is purely infinite in $\mathscr{C}/\mathscr{A}_M$. Suppose that A is directly finite in $\mathscr{C}$; if $u \in B[\mathscr{C}]\setminus M$ and $u_A(A)$ is purely infinite in $\mathscr{C}$, then $u_A(A)=0$. From this and from Lemma 7.1 it easily follows that $T_M(A)$ is directly finite. □

7.3. Proposition. *If A is a directly finite object and B is a non-zero purely infinite object in $\mathscr{C}/\mathscr{A}_M$, then $[A]+[B]=[B]$ in $\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$.*

Proof. If $[B] \le [A]$, then $[B]=0$ (because A is directly finite and B is purely infinite), contradiction. Hence $[A] \le [B]$ and $[B] \le [A]+[B] \le [B]+[B]=[B]$. □

We may now give a first description of $(\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}, +, \le)$. The class $\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$ is the disjoint union of two classes, the class $\mathscr{I}_f\{\mathscr{C}/\mathscr{A}_M\}$ consisting of all the isomorphism classes of directly finite objects of $\mathscr{C}/\mathscr{A}_M$ and the class $\mathscr{I}_\infty\{\mathscr{C}/\mathscr{A}_M\}$ consisting of the isomorphism classes of all non-zero purely infinite objects of $\mathscr{C}/\mathscr{A}_M$. Any element of $\mathscr{I}_f\{\mathscr{C}/\mathscr{A}_M\}$ is $<$ than any element of $\mathscr{I}_\infty\{\mathscr{C}/\mathscr{A}_M\}$. Furthermore $\mathscr{I}_f\{\mathscr{C}/\mathscr{A}_M\}$ and $\mathscr{I}_\infty\{\mathscr{C}/\mathscr{A}_M\}$ are both closed with respect to the operation $+$, and if $[A] \in \mathscr{I}_f\{\mathscr{C}/\mathscr{A}_M\}$ and $[B] \in \mathscr{I}_\infty\{\mathscr{C}/\mathscr{A}_M\}$ then $[A]+[B]=[B]$. Therefore in order to study the behavior of $\mathscr{I}\{\mathscr{C}/\mathscr{A}_M\}$ with respect to $+$ and $\le$, it is clear that we only have to study the behavior of $\mathscr{I}_f\{\mathscr{C}/\mathscr{A}_M\}$ and $\mathscr{I}_\infty\{\mathscr{C}/\mathscr{A}_M\}$ separately.

Let us begin with $\mathscr{I}_f\{\mathscr{C}/\mathscr{A}_M\}$. It is clear that $\mathscr{I}_f\{\mathscr{C}/\mathscr{A}_M\}$ is a set. Furthermore if A, B, C are directly finite objects of $\mathscr{C}$ and $A \oplus C \cong B \oplus C$ then $A \cong B$ [7, Theorem

3.8]. It follows from Lemma 7.2 that $(\mathcal{I}_f\{\mathscr{C}/\mathscr{A}_M\},+)$ is a commutative monoid with the cancellation property which is totally ordered in its natural order [2, Cor. X.6]. It follows that $(\mathcal{I}_f\{\mathscr{C}/\mathscr{A}_M\},+)$ may be embedded as the positive cone of a totally ordered abelian group G. By Hahn's Embedding Theorem [2, Theorem IV.16], G is isomorphic to an ordered subgroup of a lexicographic product L of copies of the group of the real numbers $\mathbb{R}$. The lexicographic product L is constructed over the set $\{\langle g\rangle^* \mid g\in G, g>0\}$ of all non-zero principal convex subgroups of G inversely ordered (here if $h\in G$, $\langle h\rangle^*$ denote the convex subgroup generated by h.) Thus L is a subgroup of $\prod_{\langle g\rangle^*\leq G}\mathbb{R}$. In order to study the embedding of G into $L\subseteq\prod_{\langle g\rangle^*\leq G}\mathbb{R}$ we may compose this embedding with the canonical projections $\pi_h:\prod_{\langle g\rangle^*\leq G}\mathbb{R}\to\mathbb{R}$, $h\in G$, $h>0$. The maps we obtain in this way are group homomorphisms $\phi_{\langle h\rangle^*}:G\to\mathbb{R}$ constructed by extending the ordered homomorphisms $\langle h\rangle^*\to\mathbb{R}$, defined by mapping h into 1. Unfortunately the mappings $\phi_{\langle h\rangle^*}$ are not ordered morphisms and are not uniquely determined on the elements of G which do not belong to $\langle h\rangle^*$. We may avoid these ambiguities by giving the following definition.

Let G be a totally ordered abelian group. A *real valuation* of G is a mapping $v:G\to\mathbb{R}\cup\{\infty\}$ such that (i) there exists an element $h\in G$, $h>0$, with $v(h)=1$, (ii) for all $g\in G$, $v(g)=\infty$ if and only if g does not belong to the convex subgroup of G generated by h; (iii) the restriction of v to the convex subgroup of G generated by h is a homomorphism of ordered groups.

Note that Hahn's Embedding Theorem has naturally let us to this definition of real valuation. Clearly any real valuation of G is uniquely determined by an element $h>0$ of G. We shall call the real valuation of G corresponding to $h\in G$, $h>0$, the *real valuation of G centred in h*, and we shall denote it by v_h.

Let us go back to the commutative monoid $\mathcal{I}_f\{\mathscr{C}/\mathscr{A}_M\}$. Recall that if $\mathscr{C}$ is a spectral category, Goodearl and Boyle [7] have defined the *relative dimension* $d_M(A:C)$ of A with respect to C in M (here $A,B\in\mathrm{Ob}(\mathscr{C})$, $M\in\mathfrak{X}[\mathscr{C}]$), in the following way: if $e\in B[\mathscr{C}]$, $d_e(A:C)$ is the infimum of all rational numbers m/n such that $m,n>0$ and $e_A(A)^n$ is isomorphic to a subobject of $e_C(C)^m$ (if no such m/n exist, then $d_e(A:C)=\infty$) and $d_M(A:C)=\inf\{d_e(A:C)\mid e\in B[\mathscr{C}]\setminus M\}$.

7.4. Theorem. *Let $\mathscr{C}$ be a spectral category, $M\in\mathfrak{X}[\mathscr{C}]$. Let G be a totally ordered abelian group containing $\mathcal{I}_f\{\mathscr{C}/\mathscr{A}_M\}$ as a positive cone. Then for all directly finite objects A,C of $\mathscr{C}/\mathscr{A}_M$, if $C\neq 0$, then $v_{[C]}([A])=d_M(A:C)$.*

Proof. We have that $d_M(A:C)=\infty$ if and only if $d_e(A:C)=\infty$ for all $e\in B[\mathscr{C}]\setminus M$, i.e. if and only if $e_A(A)$ is not isomorphic to a subobject of $e_C(C)^n$ for all $n\geq 0$ and for all e. This happens if and only if $[A]\not\leq n[C]$ in $\mathcal{I}_f\{\mathscr{C}/\mathscr{A}_M\}$ for all n, i.e. $[A]$ does not belong to the convex subgroup of G generated by $[C]$. But this is equivalent to $v_{[C]}([A])=\infty$. Since $d_M(C:C)=1$ [7, Prop. 10.1(c)], it remains to prove that $d_M(-:C)$ extends (uniquely) to a homomorphism from the convex subgroup of G generated by $[C]$ into $\mathbb{R}$. This follows from [7, Theorem 9.5]. □

We shall return to the study of $\mathscr{I}_f\{\mathscr{C}/\mathscr{A}_M\}$ later (Section 8). Let us pass now to $\mathscr{I}_\infty\{\mathscr{C}/\mathscr{A}_M\}$, i.e. to the non-zero purely infinite objects of $\mathscr{C}/\mathscr{A}_M$. The analogue of the dimensions d_M for the purely infinite objects of $\mathscr{C}$ are the dimensions μ_M defined by Goodearl and Boyle [7]:

Let $\mathscr{C}$ be a spectral category, let $M \in \mathfrak{X}[\mathscr{C}]$ and let A be an object of $\mathscr{C}$. If $e_A = 0$ for some $e \in B[\mathscr{C}]\setminus M$, define $\mu_M(A) = 0$. If $e_A \neq 0$ for all $e \in B[\mathscr{C}]\setminus M$, define $\mu_M(A)$ to be the smallest infinite cardinal α such that for some $e \in B[\mathscr{C}]\setminus M$, $e_A(A)$ does not contain a direct sum of α non-zero pairwise isomorphic subobjects [6, Chap. 12].

It is immediately possible to reinterpret Prop. 12.2 and 12.4 of [6]: $\mu_M(A) = 0$ if and only if $T_M(A)$ is a zero object in $\mathscr{C}/\mathscr{A}_M$, $\mu_M(A) = \aleph_0$ if and only if $T_M(A)$ is a non-zero directly finite object in $\mathscr{C}/\mathscr{A}_M$. For non-zero purely infinite objects the definition of μ_M now takes the following easy form:

7.5. Proposition. *Let A be an object of $\mathscr{C}$ and suppose that $T_M(A)$ is a non-zero purely infinite object of $\mathscr{C}/\mathscr{A}_M$. Then $\mu_M(A)$ is the cardinal characterized by the following property: for every non-zero cardinal α, $\mu_M(A) > \alpha$ if and only if the objects A and αA of $\mathscr{C}$ are isomorphic in $\mathscr{C}/\mathscr{A}_M$.*

(Recall that if A is an object of $\mathscr{C}$, αA is the *direct sum* of α copies of A [7].)

Proof. If A and αA are isomorphic in $\mathscr{C}/\mathscr{A}_M$, there exists $e \in B[\mathscr{C}]\setminus M$ such that $\alpha e_A(A) \cong e_A(A) \leq A$, so that $\mu_M(A) > \alpha$. Conversely if α is a non-zero cardinal and α is infinite the conclusion follows from [6, Prop. 12.9]; if α is finite $A \cong \alpha A$ in $\mathscr{C}/\mathscr{A}_M$ because $T_M(A)$ is purely infinite. Therefore $\mu_M(A)$ has the property stated in the proposition. Moreover there cannot exist two cardinals with this property. □

7.6. Corollary. *Let $\mathscr{C}$ be a spectral category, let α be a cardinal number, and let A be an object of $\mathscr{C}$. Then the set $\{M \in B[\mathscr{C}] \mid \mu_M(A) > \alpha\}$ is an open set in $\mathfrak{X}[\mathscr{C}]$.*

Proof. Proposition 7.5 and Corollary 4.4. □

Let us consider the class of all mappings $f: \mathfrak{X}[\mathscr{C}] \to \mathrm{Card}$, where Card denotes the class of all cardinal numbers. The image of f is a set of cardinals, and therefore for every $f: \mathfrak{X}[\mathscr{C}] \to \mathrm{Card}$ there is a cardinal ξ such that $f(M) < \xi$ for all $M \in \mathfrak{X}[\mathscr{C}]$. Thus f may be viewed as a function of $\mathfrak{X}[\mathscr{C}]$ into the interval $[0, \xi[$ of all the cardinals less than ξ and we shall say that $f: \mathfrak{X}[\mathscr{C}] \to \mathrm{Card}$ is continuous if $f: \mathfrak{X}[\mathscr{C}] \to [0, \xi[$ is continuous when $[0, \xi[$ has the topology whose open sets are the set $[0, \xi[$ and the sets $]\alpha, \xi[$, where α ranges in $[0, \xi]$. By Corollary 7.6 the mappings $\mathfrak{X}[\mathscr{C}] \to \mathrm{Card}$ defined by $M \mapsto \mu_M(A)$ are continuous for every object A of $\mathscr{C}$. In this way by [7, Cor. 13.11] it is possible to embed the class of all the isomorphism classes of the purely infinite objects of $\mathscr{C}$ into the class of all continuous functions of $\mathfrak{X}[\mathscr{C}]$ into Card. Note that if f and g are continuous functions of $\mathfrak{X}[\mathscr{C}]$ into Card, then there is a partition of $\mathfrak{X}[\mathscr{C}]$ in two clopen sets such that $f \geq g$ on one of these sets and $g \geq f$ on the other set. It follows that if we fix a point $M \in \mathfrak{X}[\mathscr{C}]$ the class of all germs at M of continuous functions of $\mathfrak{X}[\mathscr{C}]$ into Card is totally ordered. And it is clear that $\mathscr{I}_\infty\{\mathscr{C}/\mathscr{A}_M\}$ embeds as a totally ordered class into the class of all germs at M of continuous functions of $\mathfrak{X}[\mathscr{C}]$ into Card.

8. Grothendieck groups of dense subcategories of spectral categories

We want to apply our results to the study of the Grothendieck groups of dense subcategories of a spectral category.

Let $\mathscr{C}$ be a spectral category and let $(\mathfrak{X}[\mathscr{C}], \tilde{\mathscr{C}})$ be the associated spectral variety of categories. Let $\mathscr{S}$ be a dense subcategory of $\mathscr{C}$ and suppose that the isomorphism classes of the elements of $\mathscr{S}$ form a set. Under this hypothesis we can construct the Grothendieck group $K_0(\mathscr{S})$ of $\mathscr{S}$, i.e. the abelian group with one generator $[A]$ for each $A \in \mathscr{S}$ and with relations $[A]+[B]-[C]=0$ for all $A, B, C \in \mathscr{S}$ such that $A \oplus B \cong C$.

If $T_M : \mathscr{C} \to \mathscr{C}/\mathscr{A}_M$ is the canonical functor, we may consider the image $\mathscr{S}_M$ of $\mathscr{S}$ with respect to T_M. Clearly $\mathscr{S}_M$ is a dense subcategory of $\mathscr{C}/\mathscr{A}_M$ and it is possible to construct its Grothendieck group $K_0(\mathscr{S}_M)$. Hence for every $M \in \mathfrak{X}[\mathscr{C}]$ we have an abelian group $K_0(\mathscr{S}_M)$. Let $\mathscr{K}_0(\mathscr{S})$ be the disjoint union of the sets $K_0(\mathscr{S}_M)$, $M \in \mathfrak{X}[\mathscr{C}]$. For every $A, B \in \mathscr{S}$, $u \in B[\mathscr{C}]$, consider the subset of $\mathscr{K}_0(\mathscr{S})$

$$[A, B, u] = \{[T_M(A)] - [T_M(B)] \mid M \in \mathfrak{X}[\mathscr{C}], u \notin M\}.$$

Note that if $M \in \mathfrak{X}[\mathscr{C}]$, then $T_M(A), T_M(B) \in \mathscr{S}_M$ so that $[T_M(A)] - [T_M(B)] \in K_0(\mathscr{S}_M)$. It is not difficult to verify that the sets $[A, B, u]$, when A, B range in $\mathscr{S}$ and u ranges in $B[\mathscr{C}]$, are a basis of open sets for a topology on $\mathscr{K}_0(\mathscr{S})$ and that $\mathscr{K}_0(\mathscr{S})$ with this topology is a sheaf of abelian groups over $\mathfrak{X}[\mathscr{C}]$.

Consider the group homomorphism $i : K_0(\mathscr{S}) \to \Gamma(\mathfrak{X}[\mathscr{C}], \mathscr{K}_0(\mathscr{S}))$ defined by $i([A])(M) = [T_M(A)]$ for all $A \in \mathscr{S}$.

8.1. Theorem. *The mapping i is a group isomorphism.*

Proof. The group $K_0(\mathscr{S})$ has a canonical structure of $B[\mathscr{C}]$-module: if $A \in \mathscr{S}$ and $e \in B[\mathscr{C}]$, then $e_A(A)$ is a submodule of A and therefore it is in $\mathscr{S}$ and we may define $e[A] = [e_A(A)]$. The functor $T_M : \mathscr{C} \to \mathscr{C}/\mathscr{A}_M$ induces a surjective group morphism $K_0(\mathscr{S}) \to K_0(\mathscr{S}_M)$ and it is not difficult to verify that the kernel of this morphism is $MK_0(\mathscr{S})$. Thus $K_0(\mathscr{S}_M) \cong K_0(\mathscr{S})/MK_0(\mathscr{S})$.

It is now easy to check that $\mathscr{K}_0(\mathscr{S})$ is the sheaf of abelian groups over $\mathfrak{X}[\mathscr{C}]$ associated to the $B[\mathscr{C}]$-module $K_0(\mathscr{S})$ (see Pierce [11, p. 18]). Now i is an isomorphism by [11, Theorem 4.5]. □

By Theorem 8.1, the study of $K_0(\mathcal{S})$, when $\mathcal{S}$ is a dense subcategory of a spectral category, is essentially a 'local' problem: $K_0(\mathcal{S})$ is the group of global sections of a sheaf of abelian groups whose stalks are the groups $K_0(\mathcal{S}_M)$, $M \in \mathfrak{X}[\mathcal{C}]$. Let us study the groups $K_0(\mathcal{S}_M)$.

8.2. Lemma. *Let $\mathcal{S}$ be a dense subcategory of a spectral category $\mathcal{C}$ and let $M \in \mathfrak{X}[\mathcal{C}]$. If $\mathcal{S}_M$ contains a purely infinite object of $\mathcal{C}/\mathcal{A}_M$, then $K_0(\mathcal{S}_M)=0$. Otherwise $K_0(\mathcal{S}_M)$ is a convex subgroup of the totally ordered abelian group whose positive cone is $\mathcal{I}_f\{\mathcal{C}/\mathcal{A}_M\}$.*

Proof. If A is a purely infinite object of $\mathcal{C}/\mathcal{A}_M$, then for any directly finite object B of $\mathcal{S}_M$ $A \oplus B \cong A$, so that $[B]=0$ in $K_0(\mathcal{S}_M)$; and for any purely infinite object C of $\mathcal{S}_M$ $C \cong C \oplus C$, so that $[C]=0$ in $K_0(\mathcal{S}_M)$. Thus if $\mathcal{S}_M$ contains a purely infinite object of $\mathcal{C}/\mathcal{A}_M$, $K_0(\mathcal{S}_M)=0$.

On the other hand if every object of $\mathcal{S}_M$ is a directly finite object of $\mathcal{C}/\mathcal{A}_M$, the cancellation property holds in $\mathcal{S}_M$ and therefore $K_0(\mathcal{S}_M)$ is a convex subgroup of the totally ordered abelian group whose positive cone is $\mathcal{I}_f\{\mathcal{C}/\mathcal{A}_M\}$. □

In particular if $\mathcal{C}$ is a spectral category, let $\mathcal{F}$ be the full subcategory of $\mathcal{C}$ generated by all the directly finite objects of $\mathcal{C}$. Then $\mathcal{F}$ is a dense subcategory of $\mathcal{C}$, so that $K_0(\mathcal{F})$ is the group of global sections of the sheaf $\mathcal{K}_0(\mathcal{F})$ over $\mathfrak{X}[\mathcal{C}]$. The stalks of $\mathcal{K}_0(\mathcal{F})$ are the totally ordered abelian groups $K_0(\mathcal{F}_M)$; moreover $K_0(\mathcal{F}_M)$ is canonically isomorphic to the totally ordered abelian group whose positive cone is $\mathcal{I}_f\{\mathcal{C}/\mathcal{A}_M\}$.

Now suppose that $\mathcal{S}, \mathcal{S}'$ are two dense subcategories of a spectral category $\mathcal{C}$. If $\mathcal{S} \subseteq \mathcal{S}'$, the inclusion $\mathcal{S} \to \mathcal{S}'$ induces a morphism of sheaves of abelian groups $\mathcal{K}_0(\mathcal{S}) \to \mathcal{K}_0(\mathcal{S}')$. In particular, if $\mathcal{S}$ is a dense category, its subcategory $\mathcal{S} \cap \mathcal{F}$ is a dense subcategory, so that the inclusion $\mathcal{S} \cap \mathcal{F} \to \mathcal{S}$ induces a sheaf morphism $\mathcal{K}_0(\mathcal{S} \cap \mathcal{F}) \to \mathcal{K}_0(\mathcal{S})$. By Lemma 8.2 this sheaf morphism is surjective and its kernel is the restriction $\mathcal{K}_0(\mathcal{S} \cap \mathcal{F})_U$ of the sheaf $\mathcal{K}_0(\mathcal{S} \cap \mathcal{F})$ to the open subset U of $\mathfrak{X}[\mathcal{C}]$, where U denotes the set of all $M \in \mathfrak{X}[\mathcal{C}]$ such that $\mathcal{S}$ contains a purely infinite object A with $T_M(A)$ non-zero in $\mathcal{C}/\mathcal{A}_M$ [5, II.2.9]. Furthermore the inclusion $\mathcal{S} \cap \mathcal{F} \to \mathcal{F}$ induces a sheaf morphism $\mathcal{K}_0(\mathcal{S} \cap \mathcal{F}) \to \mathcal{K}_0(\mathcal{F})$. This sheaf morphism is clearly injective. Note that $\mathcal{K}_0(\mathcal{S} \cap \mathcal{F})_U = \mathcal{K}_0(\mathcal{F})_U$. We have thus proved the following proposition.

8.3. Proposition. *Let $\mathcal{S}$ be a dense subcategory of a spectral category $\mathcal{C}$, let U be the open set consisting of all $M \in \mathfrak{X}[\mathcal{C}]$ such that $\mathcal{S}$ contains a purely infinite object A with $T_M(A)$ non-zero in $\mathcal{C}/\mathcal{A}_M$ and let $\mathcal{F}$ be the dense subcategory of $\mathcal{C}$ consisting of all directly finite objects of $\mathcal{C}$. Then $\mathcal{K}_0(\mathcal{F})_U \subseteq \mathcal{K}_0(\mathcal{F} \cap \mathcal{S}) \subseteq \mathcal{K}_0(\mathcal{F})$ and $\mathcal{K}_0(\mathcal{S}) \cong \mathcal{K}_0(\mathcal{F} \cap \mathcal{S})/\mathcal{K}_0(\mathcal{F})_U$.* □

In particular $\Gamma(\mathfrak{X}[\mathcal{C}], \mathcal{K}_0(\mathcal{F})_U) \leq K_0(\mathcal{F} \cap \mathcal{S}) \leq K_0(\mathcal{F})$ and $K_0(\mathcal{S}) \cong K_0(\mathcal{F} \cap \mathcal{S})/$

$\Gamma(\mathfrak{X}[\mathscr{C}], \mathscr{K}_0(\mathscr{F})_U)$. By [5, Th. II.2.9.3] $\mathscr{K}_0(\mathscr{S})$ is isomorphic to a subsheaf of $\mathscr{K}_0(\mathscr{F})_{\mathfrak{X}[\mathscr{C}]\setminus U}$. Also note that $K_0(\mathscr{S}) \cong \Gamma(\mathfrak{X}[\mathscr{C}], \mathscr{K}_0(\mathscr{S}))$ is a lattice ordered group.

9. Examples

9.1. Example. *Discrete categories.* It is well known that every discrete spectral category is equivalent to a category $\mathscr{C} = \prod_{i \in I} \text{Mod-}K_i$, where the K_i's are division rings and I is the spectrum of $\mathscr{C}$ (see Section 3 and [4]). In this case $\mathfrak{X}[\mathscr{C}]$ is homeomorphic to $\beta(I)$, the Stone–Čech compactification of I with the discrete topology (Lemma 3.1). Recall that $\beta(I)$, as a set, may be viewed as the set of all ultrafilters over I. Clearly $K_0(\mathscr{F}) \cong \mathbb{Z}^I = \mathbb{Z}^{\mathrm{Sp}(\mathscr{C})}$. If $\mathscr{U} \in \beta(I)$, i.e. $\mathscr{U}$ is an ultrafilter on I, then the stalk $K_0(\mathscr{F}_{\mathscr{U}})$ of $\mathscr{K}_0(\mathscr{F})$ at $\mathscr{U}$ is the ultrapower $\mathbb{Z}^I/\mathscr{U}$. Note that if $\mathscr{U}$ is a free ultrafilter, $\mathbb{Z}^I/\mathscr{U}$ is not Archimedean, so that the category $\mathscr{C}/\mathscr{A}_{\mathscr{U}}$, which is an indecomposable abelian category in which every exact sequence splits, is not a spectral category. In fact it easily follows from [7] that the groups $K_0(\mathscr{F})$ of the indecomposable spectral categories are Archimedean groups.

9.2. Example. *Continuous categories.* A spectral category is continuous if and only if $K_0(\mathscr{F})$ has no atoms (as a partially ordered set), i.e. if and only if for every isolated point M of $\mathfrak{X}[\mathscr{C}]$, $K_0(\mathscr{F}_M)$ is not isomorphic to $\mathbb{Z}$. In particular $\mathfrak{X}[\mathrm{Dis}(\mathscr{C})]$ is the closure of the set of all the isolated points M of $\mathfrak{X}[\mathscr{C}]$ with $K_0(\mathscr{F}_M) \cong \mathbb{Z}$.

9.3. Example. *Spectral categories of type* I, II, *and* III. Recall that any spectral category is the direct product of three categories of type I, II and III respectively [14].

Since a spectral category is of type III if and only if it has no directly finite objects, it is clear that a spectral category $\mathscr{C}$ is of type III if and only if $\mathscr{K}_0(\mathscr{F}) = 0$.

By [7, Theorem 10.8] and by Theorem 7.5 a spectral category is of type II if and only if the stalks of $\mathscr{K}_0(\mathscr{F})$ are *continuous* groups (a totally ordered abelian group G is continuous if $v(G) \supseteq \mathbb{R}$ for every real valuation v of G). Similarly, a spectral category is of type I if and only if the set of all positive elements of $K_0(\mathscr{F}_M)$ has a minimum for every stalk $K_0(\mathscr{F}_M)$ of $\mathscr{K}_0(\mathscr{F})$.

It is possible to give a better description of the monoid $\mathscr{I}_f\{\mathscr{C}/\mathscr{A}_M\}$ when $\mathscr{C}$ is a spectral category of type I or II. (When $\mathscr{C}$ is of type III $\mathscr{I}_f\{\mathscr{C}/\mathscr{A}_M\} = 0$.) Denote by P the monoid of all non-negative integers if $\mathscr{C}$ is of type I or the monoid of all non-negative reals if $\mathscr{C}$ is of type II. Let P have the topology induced by the topology of the real numbers and let $P^* = P \cup \{\infty\}$ be the compactification of P with one point. For every open neighborhood U of M in $\mathfrak{X}[\mathscr{C}]$ let P_U^* be the monoid of all the continuous functions $f: U \to P^*$ such that the open set $f^{-1}(P)$ is dense in U and with the sum in P_U^* defined by components. If $U_1 \subseteq U_2$ there is a canonical morphism $P_{U_2}^* \to P_{U_1}^*$ given by the restriction. Then $\mathscr{I}_f\{\mathscr{C}/\mathscr{A}_M\}$ is isomorphic to the direct limit $\lim P_U^*$ where U ranges in the filter of all the open neighborhoods of M. This follows from [7, Theorem 14.2].

References

[1] J. Dauns and K.H. Hofmann, The representation of biregular rings by sheaves, Math. Z. 91 (1966) 103–123.

[2] L. Fuchs, Partially ordered algebraic systems (Pergamon, Oxford, 1963).

[3] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90 (1962) 323–448.

[4] P. Gabriel and U. Oberst, Spektralkategorien und reguläre Ringe im von Neumannschen Sinn, Math. Z. 92 (1966) 389–395.

[5] R. Godement, Topologie algébrique et théorie des faisceaux (Hermann, Paris, 1958).

[6] K.R. Goodearl, von Neumann Regular Rings (Pitman, London, 1979).

[7] K.R. Goodearl and A.K. Boyle, Dimension theory for nonsingular injective modules, Memoirs Amer. Math. Soc. 177 (1976).

[8] I. Kaplansky, Rings of Operators (Benjamin, New York, 1968).

[9] B. Mitchell, Theory of Categories (Academic Press, New York, 1965).

[10] C. Năstăsescu and C. Nită, Sur le centre d'une catégorie de Grothendieck, C.R. Acad. Sci. Paris 265 (1967) 373–375.

[11] R.S. Pierce, Modules over commutative regular rings, Memoirs Amer. Math. Soc. 70 (1967).

[12] N. Popescu, Abelian Categories with Applications to Rings and Modules, (Academic Press, London, 1973).

[13] G. Renault, Anneaux réguliers auto-injectifs à droite, Bull. Soc. Math. France 101 (1973) 237–254.

[14] J.-E. Roos, Locally distributive spectral categories and strongly regular rings in: Reports of the Midwest Category Seminar, Lecture Notes in Mathematics No. 47 (Springer, Berlin-Heidelberg-New York, 1967) 156–181.

[15] J.-P. Serre, Faisceaux algébriques cohérents, Ann. of Math. 61 (1955) 197–278.