Simultaneous Diagonalization of Rectangular Matrices*

Sujit Kumar Mitra
Purdue University
West Lafayette, Indiana 47907
and
Indian Statistical Institute
New Delhi 110016, India

Submitted by David H. Carlson

ABSTRACT

A matrix D is said to be diagonal if its (i,j)th element is null whenever i and j are unequal. For a set $\{A_\theta\}$ of matrices A_θ of the same order, the paper gives necessary and sufficient conditions for nonsingular matrices S and T to exist, such that $SA_\theta T = D_\theta$ is diagonal for each matrix A_θ in the set.

1. INTRODUCTION

Let A, B be matrices of order $m \times n$ with elements from a field $\mathfrak{F}$. The vector space spanned by such matrices is denoted by $\mathfrak{F}^{m\times n}$. A matrix $D \in \mathfrak{F}^{m\times n}$ is said to be diagonal if $(D)_{ij}$, the element in the (i,j)th position of D, is 0 whenever $i \neq j$. We ask ourselves the following question: Given a pair of matrices $A, B \in \mathfrak{F}^{m\times n}$, do there exist nonsingular matrices $S \in \mathfrak{F}^{m\times m}$ and $T \in \mathfrak{F}^{n\times n}$ such that

$$SAT = D_a, \qquad SBT = D_b, \tag{1.1}$$

where D_a and D_b are diagonal matrices in $\mathfrak{F}^{m\times n}$?

If A and B represent linear transformations from an n-dimensional vector space $V_n(\mathfrak{F})$ to an m-dimensional vector space $V_m(\mathfrak{F})$ with reference to chosen bases in $V_m(\mathfrak{F})$ and $V_n(\mathfrak{F})$, we are thus essentially seeking changes in bases so that the transformations can be described in simpler terms through diagonal matrices.

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52 Vanderbilt Ave., New York, NY 10017 0024-3795/82/060139-12$02.75

Theorem 3.1 provides necessary and sufficient conditions for (1.1) to hold. Simultaneous diagonability of a set $\{A_\theta\}$ of matrices in $\mathcal{F}^{m\times n}$ is studied in Theorem 4.1 We note here that since the vector space $\mathcal{F}^{m\times n}$ is finite-dimensional, one may without any loss of generality assume that set $\{A_\theta\}$ so studied consists of only a finite number of such matrices.

Williamson [12] showed that complex matrices A and B can be simultaneously diagonalized as in (1.1) through unitary matrices S and T iff AB^* and A^*B are normal, where * on a matrix indicates its complex-conjugate transpose. Necessary and sufficient conditions for the existence of unitary matrices S and T such that

$$SA_\theta T = D_\theta$$

is diagonal for each A_θ in a set $\{A_\theta\}$ of complex matrices are given by Gibson [3]. The reader is referred to Gibson [3] for a bibliography on other related work in this area.

2. SOME OTHER NOTATION AND PRELIMINARY RESULTS

$\mathcal{F}^m$ denotes the vector space of m-tuples with elements in $\mathcal{F}$. Lowercase letters a, b indicate column-vector representations of such m-tuples. For a matrix A; $\mathcal{M}(A)$ denotes its column span and $\mathcal{N}(A)$ its null space. A' denotes the transpose of A. A^-, a generalized inverse (g-inverse) of A, is a matrix A^- satisfying the equation $AA^-A = A$ [11]. The class of all possible g-inverses of A is denoted by $\{A^-\}$. Two subspaces of a vector space are said to be virtually disjoint if they have only the null vector in common.

Definition 2.1. Given a matrix $A \in \mathcal{F}^{m\times n}$ and subspaces $\mathcal{S} \subset \mathcal{F}^m$, $\mathcal{T} \subset \mathcal{F}^n$, the shorted matrix $S(A|\mathcal{S},\mathcal{T})$ is a matrix $C \in \mathcal{F}^{m\times n}$ such that

$$\mathcal{M}(C) \subset \mathcal{S}, \qquad \mathcal{M}(C') \subset \mathcal{T}, \tag{2.1}$$

and if E is any matrix $\in \mathcal{F}^{m\times n}$ that satisfies (2.1), then

$$\operatorname{Rank}(A - E) \geqslant \operatorname{Rank}(A - C). \tag{2.2}$$

This definition extends the notion of a shorted positive operator studied by Krein [6], Anderson and Trapp [1], and Mitra and Puri [8]. Shorted matrices are studied in greater detail elsewhere [9].

Let $X \in \mathfrak{F}^{m \times p}$, $Y \in \mathfrak{F}^{q \times n}$ be such that

$$\mathcal{S} = \mathfrak{M}(X), \qquad \mathcal{T} = \mathfrak{M}(Y')$$

and 0 be the null matrix in $\mathfrak{F}^{q \times p}$. We consider the bordered matrix

$$F = \begin{pmatrix} A & X \\ Y & 0 \end{pmatrix} \tag{2.3}$$

and let

$$G = \begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix} \in \{F^-\}, \tag{2.4}$$

where $C_1 \in \mathfrak{F}^{n \times m}$, $C_2 \in \mathfrak{F}^{n \times q}$, $C_3 \in \mathfrak{F}^{p \times m}$, and $C_4 \in \mathfrak{F}^{p \times q}$.

Theorem 2.1 gives a set of necessary and sufficient conditions for the existence of a unique shorted matrix $S(A|\mathcal{S},\mathcal{T})$ and provides an explicit expression for it.

THEOREM 2.1.

(a) *The shorted matrix* $S(A|\mathcal{S},\mathcal{T})$ *exists and is unique iff the matrix* F *satisfies the rank addivity conditions*

$$\operatorname{Rank} F = \operatorname{Rank}(A \vdots X) + \operatorname{Rank} Y = \operatorname{Rank}\begin{pmatrix} A \\ Y \end{pmatrix} + \operatorname{Rank} X. \tag{2.5}$$

Further, (2.5) *is also necessary for the existence of an unique shorted matrix, unless precisely one of* $\mathcal{S}$ *or* $\mathcal{T}$ *is zero-dimensional.*

(b) *When* (2.5) *is satisfied,*

(i) $C_2 \in \{Y^-\}$, $C_3 \in \{X^-\}$;

(ii) AC_2Y, XC_3A, *and* XC_4Y *are invariant under the choice of* G *in* (2.4), *and further*

$$AC_2Y = XC_3A = XC_4Y = A - AC_1A = C \quad (\text{say}); \tag{2.6}$$

(iii) *The matrix* C *in* (2.6) *is the unique shorted matrix* $S(A|\mathcal{S},\mathcal{T})$.

Proof. The "if" part of (a) and the whole of (b) are proved for complex matrices in [7]: see Theorems 1 and 2 and Remark 1 following Theorem 2.

Theorem 1 in [7] is a generalization of similar theorems due to Khatri [5] and Rao [10]. The transition from the complex field to an arbitrary field $\mathcal{F}$ presents no special difficulties. As in [7], it can be shown for example that $A = AC_1A + XC_3A$ and that $\mathcal{M}(AC_1A)$ and $\mathcal{M}(X) = \mathcal{S}$ are virtually disjoint, as are $\mathcal{M}(AC_1A)'$ and $\mathcal{M}(Y') = \mathcal{T}$; thus for any E satisfying (2.1),

$$\begin{aligned}\text{Rank}(A-E) &= \text{Rank}(A - XC_3A + XC_3A - E)\\ &= \text{Rank}(A - XC_3A) + \text{Rank}(XC_3A - E)\\ &\geqslant \text{Rank}(A - XC_3A),\end{aligned}$$

with equality iff $E = XC_3A$. To prove the "only if" part of (a) observe that (2.5) is trivially true, when both $\mathcal{S}$ and $\mathcal{T}$ are zero-dimensional and the null matrix is the unique matrix satisfying the condition (2.1). In the general case when both $\mathcal{S}$ and $\mathcal{T}$ have positive dimensions, assume now that $A_0 = S(A|\mathcal{S},\mathcal{T})$ is the unique shorted matrix. Write $A = A_0 + A_1$, and observe that the uniqueness of the shorted matrix $S(A|\mathcal{S},\mathcal{T})$ implies that $\mathcal{M}(A_1)$ is virtually disjoint with $\mathcal{S}$, and $\mathcal{M}(A_1')$ with $\mathcal{T}$. If $\mathcal{M}(A_1)$ is not virtually disjoint with $\mathcal{S}$, let l_1 be a nonnull m-tuple in $\mathcal{M}(A_1)\cap\mathcal{S}$. Let A_1 be of rank s. Consider a rank factorization of A_1:

$$A_1 = LR,$$

where $L = (l_1 \vdots l_2 \vdots \cdots \vdots l_s)$, $R' = (r_1 \vdots r_2 \vdots \cdots \vdots r_s)$. For any nonnull n-tuple t_1 in $\mathcal{T}$, the matrix $E = A_0 + l_1t_1'$ satisfies the condition (2.1), and further $\text{Rank}(A - E) \leqslant \text{Rank}(A - A_0) = \text{Rank}(A_1)$. This contradicts the uniqueness of the shorted matrix $S(A|\mathcal{S},\mathcal{T})$. A similar argument shows that $\mathcal{M}(A_1')$ is virtually disjoint with $\mathcal{T}$. If $\mathcal{M}\begin{pmatrix}A\\Y\end{pmatrix}$ is not virtually disjoint with $\mathcal{M}\begin{pmatrix}X\\0\end{pmatrix}$, let vectors $a\in\mathcal{F}^n$, $b\in\mathcal{F}^m$ be such that

$$\begin{aligned}Aa &= Xb \neq 0,\\ Ya &= 0.\end{aligned} \tag{2.7}$$

Then $A_1a = Xb \neq 0$, which contradicts the assumption that $\mathcal{M}(A_1)$ is virtually disjoint with $\mathcal{S}$. The other part of (2.5) is similarly established. ∎

We also need an explicit representation of a g-inverse of F, given in Theorem 2.2. The proof is by direct computation. The complex version of Theorem 2.2 appears as Theorem 3 in [7]. This generalizes a theorem of Hall and Meyer [4].

THEOREM 2.2. *For any choice of the g-inverses of* X, Y, *and* $E_X AF_Y$,

$$\begin{pmatrix} 0 & Y^- \\ X^- & -X^-AY^- \end{pmatrix} + \begin{pmatrix} I \\ -X^-A \end{pmatrix} Q \begin{pmatrix} I & -AY^- \end{pmatrix} \tag{2.8}$$

is a g-inverse of F, *where* $Q = F_Y(E_X AF_Y)^- E_X$, $E_X = I - XX^-$, *and* $F_Y = I - Y^- Y$.

3. SIMULTANEOUS DIAGONALIZATION OF A PAIR OF MATRICES

THEOREM 3.1. *Let* $A, B \in \mathfrak{F}^{m\times n}$. *There exists a pair of nonsingular matrices satisfying* (1.1) *iff*:

(a) *we have*

$$\operatorname{Rank}\begin{pmatrix} A & B \\ B & 0 \end{pmatrix} = \operatorname{Rank}\begin{pmatrix} A \vdots B \end{pmatrix} + \operatorname{Rank} B = \operatorname{Rank}\begin{pmatrix} A \\ B \end{pmatrix} + \operatorname{Rank} B \tag{3.1}$$

and

(b) *in addition*

$$AC_2BC_2 \text{ is semisimple} \tag{3.2}$$

(*or equivalently* C_3BC_3A *is semisimple*), *where*

$$\begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix} \text{ is any g-inverse of } F = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}.$$

Proof. "Only if" part: We assume here that nonsingular S and T exist such that

$$SAT = D_a, \qquad SBT = D_b$$

where D_a and D_b are diagonal matrices. It is easily seen that

$$\operatorname{Rank}\begin{pmatrix} D_a & D_b \\ D_b & 0 \end{pmatrix} = \operatorname{Rank}\begin{pmatrix} D_a \vdots D_b \end{pmatrix} + \operatorname{Rank} D_b = \operatorname{Rank}\begin{pmatrix} D_a \\ D_b \end{pmatrix} + \operatorname{Rank} D_b.$$

Hence (3.1) follows.

Further, the matrix

$$\begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix}$$

is a g-inverse of F iff $C_1 = T\bar{C}_1 S$, $C_2 = T\bar{C}_2 S$, $C_3 = T\bar{C}_3 S$, and $C_4 = T\bar{C}_4 S$, where

$$\begin{pmatrix} \bar{C}_1 & \bar{C}_2 \\ \bar{C}_3 & -\bar{C}_4 \end{pmatrix} \text{ is a g-inverse of } \begin{pmatrix} D_a & D_b \\ D_b & 0 \end{pmatrix}.$$

We now show that there exists a choice of a g-inverse of

$$\begin{pmatrix} D_a & D_b \\ D_b & 0 \end{pmatrix}$$

such that $\bar{C}_2$ and $\bar{C}_3$ are both diagonal. For this we use (2.8) and substitute for D_b^- and Q the matrices defined as follows:

$$\begin{aligned} (D_b^-)_{ii} &= 1/(D_b)_{ii} \quad \text{if} \quad (D_b)_{ii} \neq 0, \\ (D_b^-)_{ij} &= 0 \qquad\qquad \text{otherwise,} \end{aligned} \tag{3.3}$$

$$\begin{aligned} (Q)_{ii} &= 1/(D_a)_{ii} \quad \text{if} \quad (D_a)_{ii} \neq 0 \text{ and } (D_b)_{ii} = 0, \\ (Q)_{ij} &= 0 \qquad\qquad \text{otherwise.} \end{aligned} \tag{3.4}$$

Since D_b^- and Q are diagonal matrices,

$$\bar{C}_2 = D_b^- - Q D_a D_b^-$$

is diagonal and

$$AC_2BC_2 = S^{-1}D_aT^{-1}T\bar{C}_2SS^{-1}D_bT^{-1}T\bar{C}_2S = S^{-1}D_1S,$$

where $D_1 = D_a\bar{C}_2D_b\bar{C}_2 \in \mathfrak{F}^{m\times m}$ and is diagonal. This establishes the fact that AC_2BC_2 is semisimple. We now show that if (3.1) holds, the semisimplicity of AC_2BC_2 is equivalent to the semisimplicity of AC_2BB^- for any choice of B^-. This follows from the fact that if x is an eigenvector of AC_2BC_2 for a nonnull eigenvalue λ,

$$AC_2BC_2x = \lambda x \quad \Rightarrow \quad AC_2x = \lambda x, \tag{3.5}$$

since $x \in \mathfrak{M}(AC_2B) = \mathfrak{M}(BC_3A) \subset \mathfrak{M}(B)$ and $C_2 \in \{B^-\}$. For the same reason,

$$AC_2BB^-x = AC_2x = \lambda x. \tag{3.6}$$

This shows that x is an eigenvector of AC_2BB^- for the same eigenvalue λ and vice versa. Since $\text{Rank}(AC_2BC_2) = \text{Rank}(AC_2BB^-) = \text{Rank}(AC_2B)$, the equivalence of the two statements follows.

Since AC_2B is invariant under choice of a g-inverse of F, if AC_2BC_2 is semisimple for one choice of this g-inverse, it is so for every other choice.

"If" part: Let B be of rank r. Consider a rank factorization of B,

$$B = UV,$$

where $U \in \mathfrak{F}^{m\times r}$, $V \in \mathfrak{F}^{r\times n}$.

Since $\mathfrak{M}(AC_2B) \subset \mathfrak{M}(B)$, $\mathfrak{M}(B'C_2'A') \subset \mathfrak{M}(B')$, we have

$$AC_2B = UKV$$

for some $K \in \mathfrak{F}^{r\times r}$. Choose and fix a g-inverse of B, $B^- = V_R^{-1}U_L^{-1}$, where U_L^{-1} and V_R^{-1} are respectively left and right inverses of U and V. Semisimplicity of AC_2BC_2 implies semisimplicity of $AC_2BB^- = UKU_L^{-1}$, which in turn implies semisimplicity of K. Put $K = WDW^{-1}$, where $W, D \in \mathfrak{F}^{r\times r}$ and D is diagonal. Then

$$AC_2B = UKV = UWDW^{-1}V = S_1DT_1,$$

where $S_1 = UW$, $T_1 = W^{-1}V$. Check that $B = S_1T_1$. Also, let S_2T_2 be a rank factorization of $A - AC_2B$. Then $\mathfrak{M}(A - AC_2B) \cap \mathfrak{M}(B) = \{0\}$ and $\mathfrak{M}(A' - B'C_2'A') \cap \mathfrak{M}(B') = \{0\}$ follows from (3.1) and the proof of Theorem 2 of [7]. Hence $\mathfrak{M}(S_2)$ is virtually disjoint with $\mathfrak{M}(S_1)$, and $\mathfrak{M}(T_2')$ with $\mathfrak{M}(T_1')$. Let S_3 and T_3 be so chosen that $(S_1 \vdots S_2 \vdots S_3)$ and $(T_1' \vdots T_2' \vdots T_3')$ are nonsingular. Put $S^{-1} = (S_1 \vdots S_2 \vdots S_3)$, $(T')^{-1} = (T_1' \vdots T_2' \vdots T_3')$, and check that

$$SAT = D_a \quad \text{and} \quad SBT = D_b$$

where $D_a = \text{diag}(D, I, 0)$, $D_b = \text{diag}(I, 0, 0)$ are clearly diagonal matrices. This completes the proof of the "if" part and of Theorem 3.1. ■

4. SIMULTANEOUS DIAGONALIZATION OF SEVERAL MATRICES

Without any loss of generality let us assume here that $m \leqslant n$. We shall further assume here that the field $\mathcal{F}$ contains more than m distinct nonnull elements.

We need the following result.

Lemma 4.1.[1] *If matrices A and B satisfy the condition* (3.1), *there exists a nonnull scalar k such that*

$$\mathfrak{M}(A) \subset \mathfrak{M}(A+kB), \qquad \mathfrak{M}(A') \subset \mathfrak{M}(A'+kB'), \tag{4.1a}$$

or equivalently

$$\mathfrak{M}(B) \subset \mathfrak{M}(A+kB), \qquad \mathfrak{M}(B') \subset \mathfrak{M}(A'+kB'), \tag{4.1b}$$

and

$$\operatorname{Rank}\{B(A+kB)^{-}B\} = \operatorname{Rank} B. \tag{4.1c}$$

Conversely, (4.1a) *or* (4.1b) *and* (4.1c) *imply* (3.1).

Proof. Assume now that (3.1) holds, and let

$$\begin{pmatrix} C_1 & C_3 \\ C_2 & -C_4 \end{pmatrix} \in \left\{ \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}^{-} \right\}.$$

Let k be so chosen that $k \neq 0$ and

$$\det(BC_4 + kI) \neq 0.$$

Clearly

$$\mathfrak{M}(BC_4B) \subset \mathfrak{M}(B) = \mathfrak{M}(BC_4B + kB),$$

$$\mathfrak{M}(B'C_4'B') \subset \mathfrak{M}(B') = \mathfrak{M}(B'C_4'B' + kB'). \tag{4.2}$$

Since $\mathfrak{M}(A - BC_4B) \cap \mathfrak{M}(B) = \{0\}$ and $\mathfrak{M}(A' - B'C_4'B') \cap \mathfrak{M}(B') = \{0\}$

[1]Lemma 4.1 is false if the field contains only m distinct nonnull elements or less.

follows from (3.1) as in the proof of Theorem 2 of [7], we have

$$\mathfrak{M}(A) = \mathfrak{M}(A - BC_4B + BC_4B) = \mathfrak{M}(A - BC_4B) + \mathfrak{M}(BC_4B)$$
$$\subset \mathfrak{M}(A - BC_4B) + \mathfrak{M}(BC_4B + kB)$$
$$= \mathfrak{M}(A - BC_4B + BC_4B + kB) = \mathfrak{M}(A + kB),$$

and similarly $\mathfrak{M}(A') \subset \mathfrak{M}(A' + kB')$. This establishes (4.1a). Equation (4.1b) is trivial.

If (4.1b) holds, the matrix

$$\begin{pmatrix} A+kB & B \\ B & 0 \end{pmatrix}$$

can be reduced to

$$\begin{pmatrix} A+kB & 0 \\ B & B(A+kB)^{-}B \end{pmatrix}$$

through sweepout operations on its rows and columns. Hence

$$\operatorname{Rank}\begin{pmatrix} A & B \\ B & 0 \end{pmatrix} = \operatorname{Rank}\begin{pmatrix} A+kB & B \\ B & 0 \end{pmatrix}$$
$$= \operatorname{Rank}(A+kB) + \operatorname{Rank} B(A+kB)^{-}B$$
$$= \operatorname{Rank}\begin{pmatrix} A \\ B \end{pmatrix} + \operatorname{Rank} B(A+kB)^{-}B$$
$$= \operatorname{Rank}(A \vdots B) + \operatorname{Rank} B(A+kB)^{-}B,$$

and (3.1) implies (4.1c). Conversely the same argument shows that (4.1c) implies (3.1). ■

THEOREM 4.1. *Let $A_1, A_2, \dots, A_p \in \mathfrak{F}^{m\times n}$. The following two statements are equivalent:*

(a) *There exist nonsingular matrices $S \in \mathfrak{F}^{m\times m}$, $T \in \mathfrak{F}^{n\times n}$ such that*

$$SA_iT = D_i, \qquad i = 1, 2, \dots, p, \tag{4.3}$$

where each D_i is a diagonal matrix in $\mathfrak{F}^{m\times n}$.

(b) *There exist nonnull scalars $k_2,\ldots,k_p$ in $\mathcal{F}$ such that if*

$$A_0 = A_1 + k_2A_2 + \cdots + k_pA_p, \tag{4.4}$$

then for $i=1,2,\ldots,p$, $j=1,2,\ldots,p$,

$$\mathcal{M}(A_i)\subset\mathcal{M}(A_0), \qquad \mathcal{M}(A_i')\subset\mathcal{M}(A_0'), \tag{4.5}$$

$$A_iA_0^- \text{ is semisimple}, \tag{4.6}$$

$$A_iA_0^-A_j = A_jA_0^-A_i. \tag{4.7}$$

Proof. (a) ⇒ (b): Since A_1 and A_2 are simultaneously reducible to diagonal matrices using Theorem 3.1 and then Lemma 4.1, a nonnull scalar k_2 can be determined so that if

$$A_{(2)} = A_1 + k_2A_2,$$

then

$$\mathcal{M}(A_1)\subset\mathcal{M}(A_{(2)}), \qquad \mathcal{M}(A_1')\subset\mathcal{M}(A_{(2)}'),$$

$$\mathcal{M}(A_2)\subset\mathcal{M}(A_{(2)}), \qquad \mathcal{M}(A_2')\subset\mathcal{M}(A_{(2)}').$$

Since $A_{(2)}$ and A_3 are simultaneously reducible to diagonal matrices, the same argument can be repeated and the nonnull scalars $k_2, k_3,\ldots,k_p$ can be recursively determined so as to satisfy (4.5).

Let $D_0 = D_1 + k_2D_2 + \cdots + k_pD_p$. Then D_0 is diagonal and

$$SA_0T = D_0.$$

As in the proof of Theorem 3.1 it is seen that if $A_iA_0^-$ is semisimple for some choice of A_0^- it is so for every other choice. Choose for D_0^- the following diagonal matrix in $\mathcal{F}^{n\times m}$:

$$\begin{aligned} (D_0^-)_{ii} &= 1/(D_0)_{ii} && \text{if } (D_0)_{ii}\neq 0,\\ (D_0^-)_{ij} &= 0 && \text{otherwise.} \end{aligned}$$

It is seen that $TD_0^-S\in\{A_0^-\}$, and with this choice of A_0^- the truth of (4.6) and (4.7) is easily verified. We note that on account of (4.5), $A_i A_0^- A_j$ is independent of the choice of A_0^-.

(b) $\Rightarrow$ (a): Consider a rank factorization of A_0,

$$A_0 = UV,$$

where $U \in \mathfrak{F}^{m \times r}$, $V \in \mathfrak{F}^{r \times n}$, and $r = \text{Rank } A_0$. Choose and fix a g-inverse A_0^- where

$$A_0^- = V_R^{-1} U_L^{-1}$$

and U_L^{-1} and V_R^{-1} are respectively left and right inverses of U and V. Then (4.5) implies

$$A_i = UB_iV$$

for some matrix $B_i \in \mathfrak{F}^{r \times r}$ and

$$A_i A_0^- = UB_i U_L^{-1}.$$

Since on account of (4.6) and (4.7) the matrices $A_i A_0^-$ commute and are semisimple, it follows that the matrices B_i commute and are semisimple. Hence there exists a nonsingular matrix $W \in \mathfrak{F}^{r \times r}$ such that

$$W^{-1} B W = D_i, \qquad i = 1, 2, \ldots, p,$$

where $D_1, D_2, \ldots, D_p$ are diagonal matrices. The rest of the proof of Theorem 4.1 can be completed on the same lines as in the proof of the "if" part of Theorem 3.1. ∎

Theorem 4.2 is an extension of Theorem 6 of Bhimasankaram [2].

THEOREM 4.2. *Let $A_1, A_2, \ldots, A_p$ be complex hermitian matrices of order $n \times n$. Then there exists a nonsingular matrix T such that $T^* A_i T$ is diagonal for each i iff there exist nonnull real scalars $k_2, k_3, \ldots, k_p$ such that if*

$$A_0 = A_1 + k_2 A_2 + \cdots + k_p A_p$$

then for $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, p$,

(a) $\mathfrak{M}(A_i) \subset \mathfrak{M}(A_0)$,

(b) *$A_i A_0^-$ is semisimple with real eigenvalues for some g-inverse A_0^- of A_0,*

(c) $A_i A_0^- A_j = A_j A_0^- A_i$.

Proof. The "only if" part follows from the corresponding part of Theorem 4.1, since here without any loss of generality one can restrict the scalar k_i to be real. The "if" part follows from Theorem 6 of Bhimasankaram [2]. ■

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