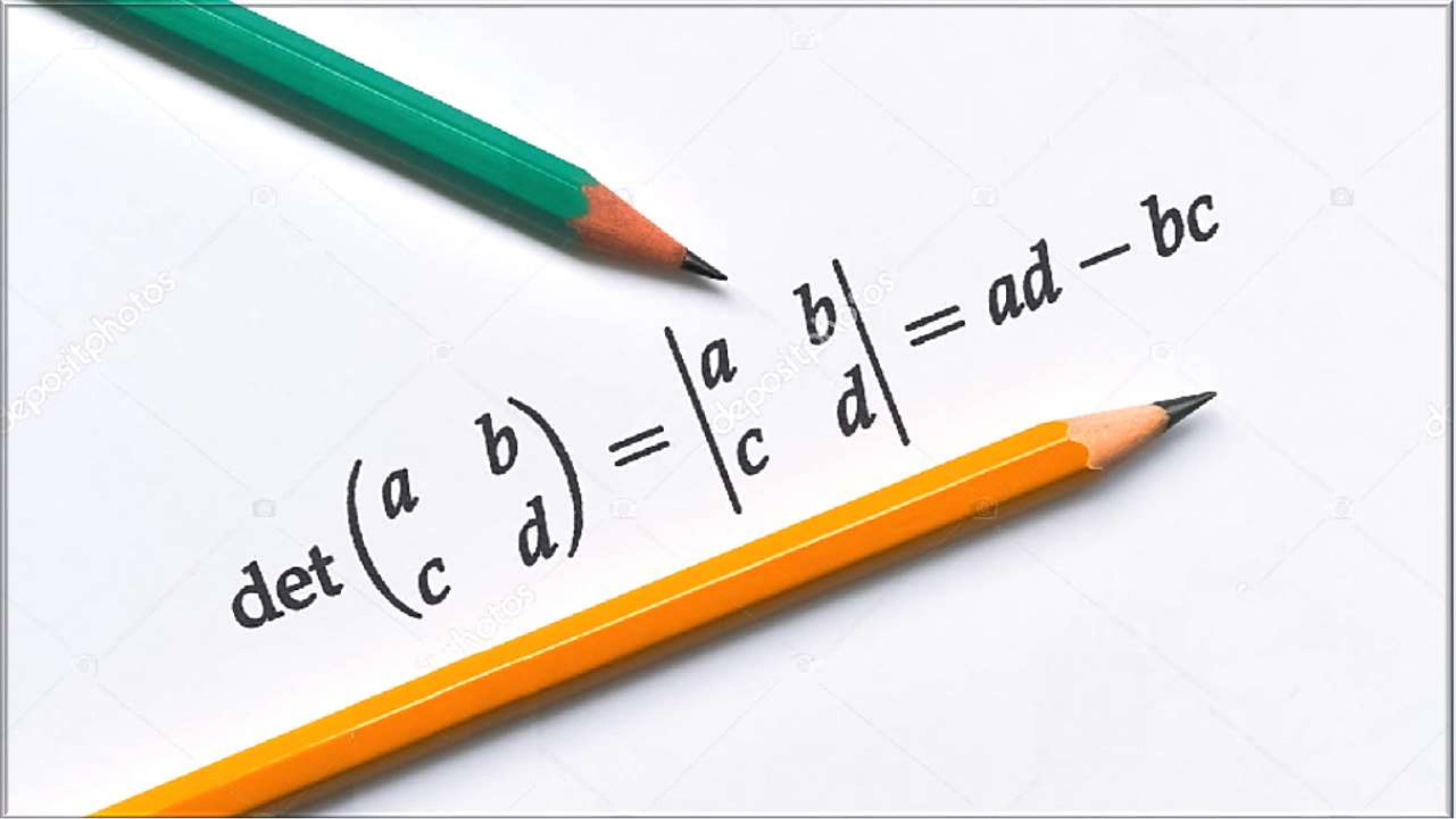
Determinantlar. Determinantlarning asosiy xossalari. Teskari matritsa. Matritsaning rangi.


$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
depositphotos

REJA:

1. Birinchi va ikkinchi tartibli determinantlar.
2. Uchinchi tartibli determinantlar. Ta’rif va hisoblash qoidalari.
3. Determinantlarning xossalari.
4. *n*-tartibli determinant haqida tushuncha.

Birinchi va ikkinchi tartibli determinantlar

$$|a|$$

$$|a_{11}|$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

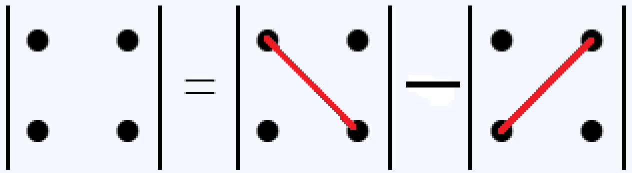
Ta’rif bo‘yicha *birinchi tartibli determinant* quyidagi $|a| = a$ songa aytiladi.

Ikkinchi tartibli determinant quyidagi $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ formadagi songa aytiladi va quyidagicha hisoblanadi:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

a_{11}, a_{12}, a_{21}, a_{22} sonlar *determinant elementlari* deb ataladi.

Hisoblash sxemasi:



Misollar:

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2. \qquad \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 2 - 2 = 0. \qquad \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2. \qquad \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2.$$

$$\begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = 6 - 5 = 1. \qquad \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1.$$

$$\begin{vmatrix} 0 & 3 \\ 0 & 2 \end{vmatrix} = 0 - 0 = 0. \qquad \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} = 6 - 0 = 6. \qquad \begin{vmatrix} 3 & 0 \\ 17 & 2 \end{vmatrix} = 6 - 0 = 6.$$

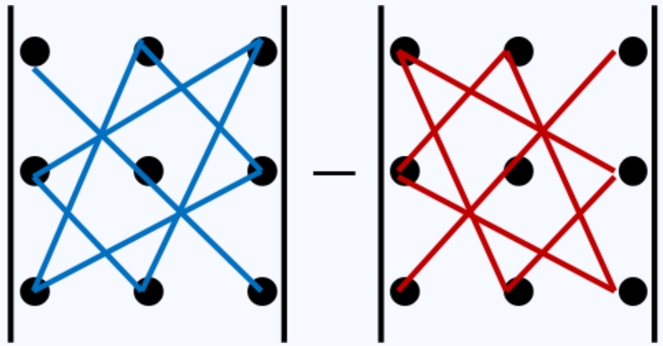
$$\begin{vmatrix} 3 & 1 \\ 15 & 2 \end{vmatrix} = 6 - 15 = -9. \qquad \begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} = 2 - 5 = -3. \qquad \begin{vmatrix} -1 & 1 \\ -5 & 2 \end{vmatrix} = -2 + 5 = 3.$$

$$\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix} = -3 - 1 = -4. \qquad \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} = -2 - 2 = -4.$$

$$\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ 5 & 2 \end{vmatrix} = -3 - 9 = -12. \qquad \begin{vmatrix} -1 & 1 \\ 10 & 2 \end{vmatrix} = -2 - 10 = -12.$$

Uchinchi tartibli determinantlar. Ta'rif va hisoblash qoidalari

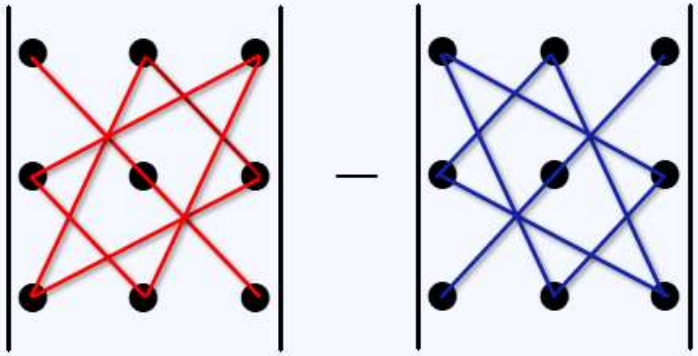
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$



Uchinchi tartibli determinantlarni hisoblash usullari:

Uchburchak usuli:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

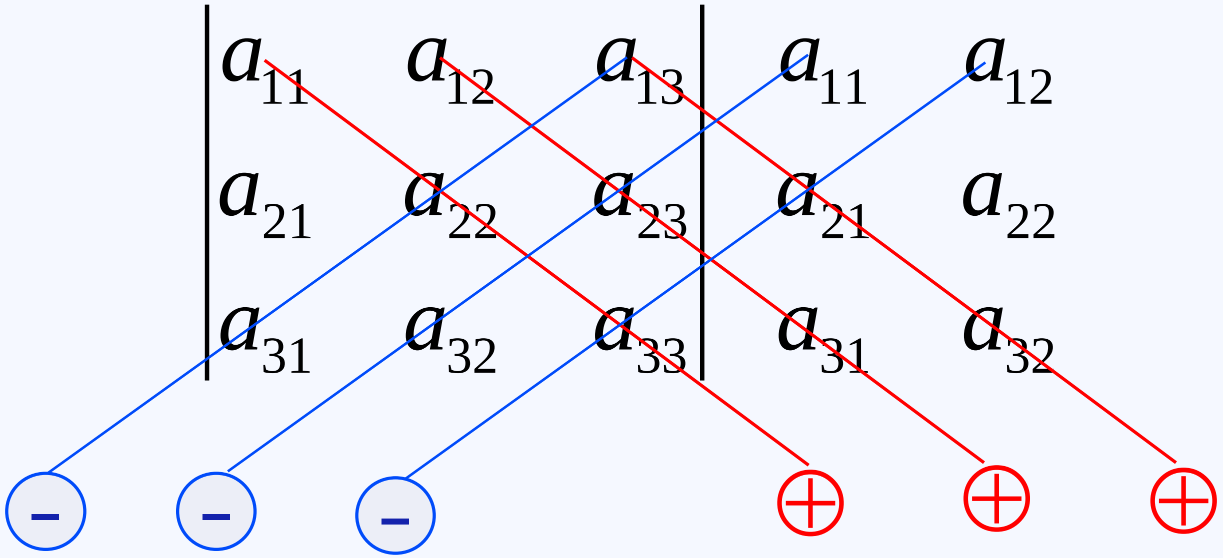


Misol:

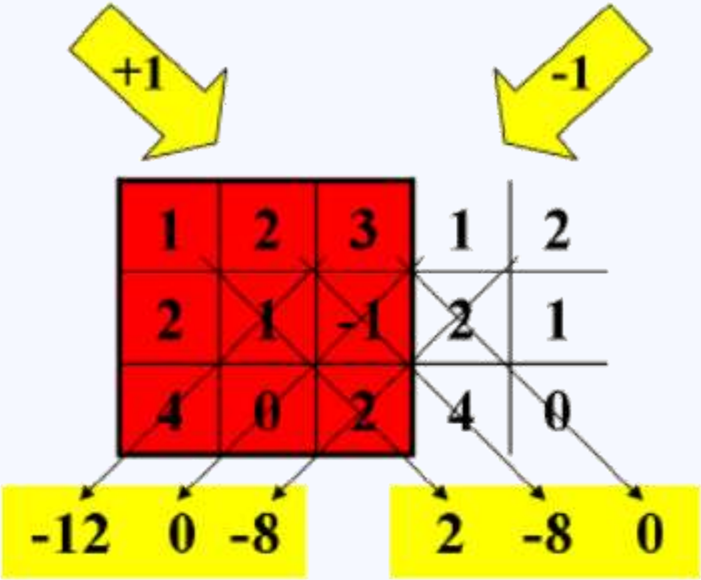
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\cdot5\cdot9 + 2\cdot6\cdot7 + 3\cdot4\cdot8 - 3\cdot5\cdot7 - 2\cdot4\cdot9 - 1\cdot6\cdot8 = 45 + 84 + 96 - 105 - 72 - 48 = 0.$$

Sarrius usuli:

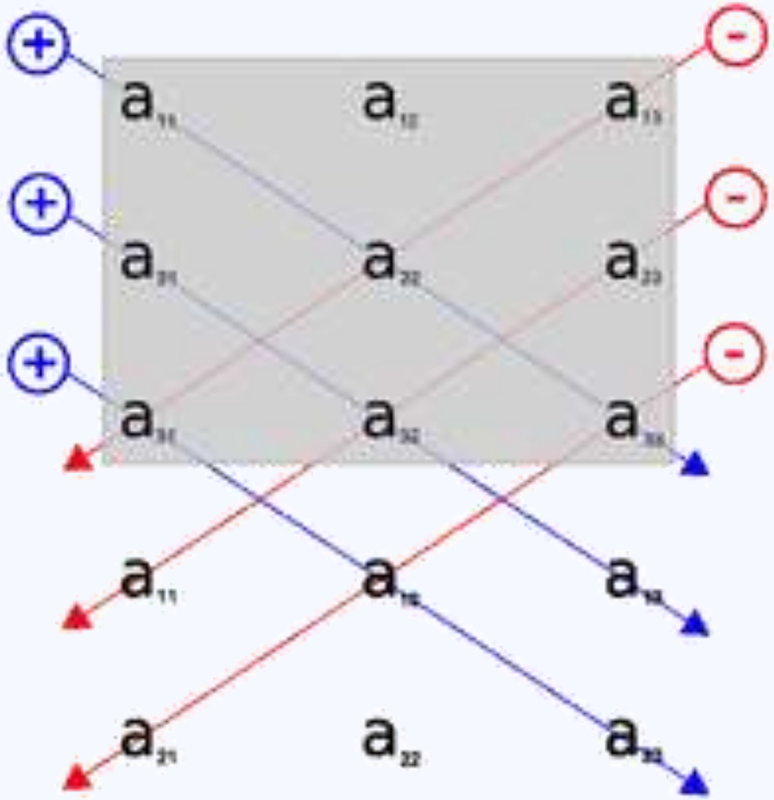
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$



$$\begin{vmatrix} 1 & -2 & -3 \\ -4 & 5 & -6 \\ 7 & -8 & 9 \end{vmatrix} \begin{matrix} 1 & -2 \\ -4 & 5 \\ 7 & -8 \end{matrix} = 45 + 84 - 96 - 72 - 48 + 105 = 18.$$



$$\Delta = -6 - \left(-20\right) = 14.$$



n-tartibli determinantdagi a_{ij} elementnig ***minori*** quyidagi, determinant o‘zida i-satr va j-ustun o‘chirilgandan keyin hosil bo‘lgan (n–1)-determinantga aytiladi.

a_{ij} elementning minori M_{ij} kabi belgilanadi.

Masalan, 3-tartibli $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ determinant uchun

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \qquad M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \qquad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

Quyidagi determinant uchun $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$

$$M_{11} = M_{22} = \cos\alpha,$$
$$M_{12} = M_{21} = \sin\alpha.$$

Algebraik to'ldiruvchisi: a_{ij} elementning algebraik to'ldiruvchisi deb, $A_{ij} = (-1)^{i+j} \cdot M_{ij}$ songa aytiladi.

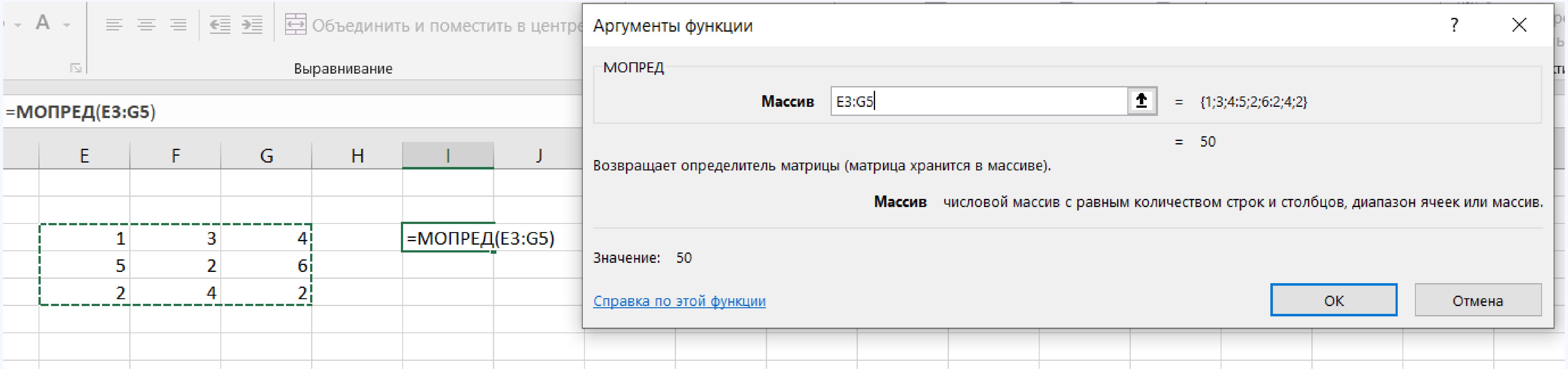
Masalan, 3-tartibli determinant uchun

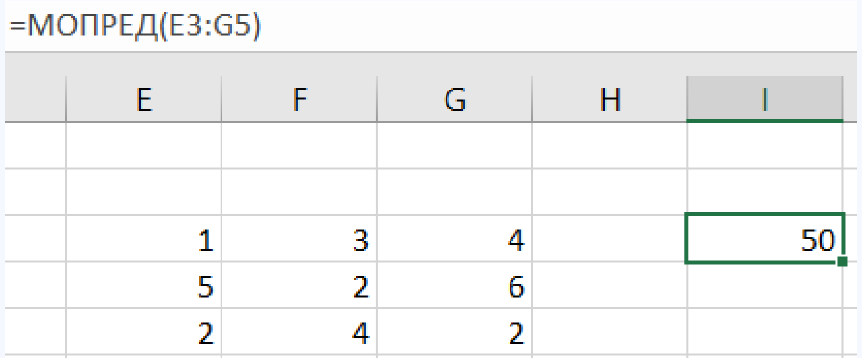
$$A_{11} = (-1)^2 \cdot M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \qquad A_{12} = (-1)^3 M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix}.$$

Keltirilgan $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$ determinant uchun:

$$A_{11} = A_{22} = \cos\alpha, \qquad A_{12} = A_{21} = -\sin\alpha.$$

Excelda hisoblash:





Determinantlarning xossalari

1. Determinantning satrlarini unga mos ustunlar bilan almashtirish natijasida determinantning qiymati o‘zgarmaydi:

$$\Delta=\begin{vmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} & \boldsymbol{a}_{13} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} & \boldsymbol{a}_{23} \\ \boldsymbol{a}_{31} & \boldsymbol{a}_{32} & \boldsymbol{a}_{33} \end{vmatrix}=\begin{vmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{21} & \boldsymbol{a}_{31} \\ \boldsymbol{a}_{12} & \boldsymbol{a}_{22} & \boldsymbol{a}_{32} \\ \boldsymbol{a}_{13} & \boldsymbol{a}_{23} & \boldsymbol{a}_{33} \end{vmatrix}.$$

2. Determinantning ikkita satr (yoki utsun) larini o‘rinlarini almashtirish natijasida determinantning ishorasi o‘zgaradi:

$$\begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix}=-\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

3 (Laplas teoremasi). Determinantning biror satr (yoki ustun) elementlarini ularning algebraik to‘ldiruvchilariga ko‘paytirib qo‘shsak yig‘indi determinantning o‘ziga teng bo’ladi:

$$\Delta = \boldsymbol{a}_{11}\boldsymbol{A}_{11} + \boldsymbol{a}_{12}\boldsymbol{A}_{12} + \boldsymbol{a}_{13}\boldsymbol{A}_{13}; \qquad \Delta = \boldsymbol{a}_{21}\boldsymbol{A}_{21} + \boldsymbol{a}_{22}\boldsymbol{A}_{22} + \boldsymbol{a}_{23}\boldsymbol{A}_{23}.$$

4. Agar determinantda ikkita bir xil ustun (yoki satr) bo‘lsa, u holda usha determinant 0 ga teng bo ‘ladi.

$$\begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} & \boldsymbol{a}_{23} \\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{vmatrix} = -\begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} & \boldsymbol{a}_{23} \\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{vmatrix} = 0.$$

5. Determinantning biror satr (yoki ustun) elementlarini biror songa ko‘paytirish determenantni shu songa ko‘paytirishga teng kuchlidir:

$$\begin{vmatrix} a_{11} & a_{12} & ka_{13} \\ a_{21} & a_{22} & ka_{23} \\ a_{31} & a_{32} & ka_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

6. Ikkita proporsional satr (yoki ustun) larga ega bo‘lgan determinant nolga teng:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{11} & ka_{12} & ka_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

7. Agar determinantning biror satr (yoki ustun) elementlari ikkita qo‘shiluvchining yig‘indisidan iborat bo‘lsa, u holda berilgan determinant ikkita determinant yig‘indisiga teng bo‘ladi, ulardan birining tegishli satri (ustuni) elementlari birinchi qo‘shiluvchilardan, ikkinchisining tegishli satri (ustuni) esa ikkinchi qo‘shiluvchilardan iborat bo‘lib, qolgan elementlari berilgan determinant elementlaridan iborat.

Masalan,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_1 + a_2 & b_1 + b_2 & c_1 + c_2 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_1 & b_1 & c_1 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_2 & b_2 & c_2 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Misol:

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & b \\ 1 & b & c \\ 1 & c & a \end{vmatrix} + \begin{vmatrix} 1 & a & c \\ 1 & b & a \\ 1 & c & b \end{vmatrix} = ab+ac+bc-b^2-a^2-c^2+b^2+a^2+c^2-bc-ab-ac=0.$$

8. Determinantning biror satr (yoki ustun) elementlarini biror songa ko‘paytirib boshqa bir satr (yoki ustun) ning mos elementlariga qo‘shish natijasida determinantning qiymati o‘zgarmaydi:

$$\begin{vmatrix} a_{11}+ka_{31} & a_{12}+ka_{32} & a_{13}+ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} ka_{31} & ka_{32} & ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

9. Determinantning ixtiyoriy sonlarga ko‘paytirilgan satri (yoki ustuni) elementlarining algebraik to‘ldiruvchilari yig‘indisi determinantga teng bo‘lib, bu determinantda ushbu satr (ustun) elementlari berilgan sonlar bilan almashtiriladi:

$$q_1A_{12}+q_2A_{22}+q_3A_{32}=\begin{vmatrix} a_{11} & q_1 & a_{13} \\ a_{21} & q_2 & a_{23} \\ a_{31} & q_3 & a_{33} \end{vmatrix}; \qquad q_1A_{31}+q_2A_{32}+q_3A_{33}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ q_1 & q_2 & q_3 \end{vmatrix}.$$

Misol:

$$5\cdot(-1)^{1+2}\cdot\begin{vmatrix} -2 & 4 \\ 7 & -1 \end{vmatrix}-6\cdot(-1)^{2+2}\cdot\begin{vmatrix} 1 & 3 \\ 7 & -1 \end{vmatrix}+2\cdot(-1)^{3+2}\cdot\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix}=\begin{vmatrix} 1 & 5 & 3 \\ -2 & -6 & 4 \\ 7 & 2 & -1 \end{vmatrix}.$$

10. Agar determinantning biror satri (ustuni) elementlari uning boshqa satri (ustuni) elementlarining mos algebraik to‘ldiruvchilarga ko‘paytirib qo‘shilsa, ushbu yig ‘indisi nolga teng bo‘ladi:

$$a_{13}A_{12} + a_{23}A_{22} + a_{33}A_{32} = \begin{vmatrix} a_{11} & a_{13} & a_{13} \\ a_{21} & a_{23} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{vmatrix} = 0.$$

11. Uchburchakli matritsalar (quyi uchburchakli va yuqori uchburchakli) ning determinant diagonal elementlari ko’paytmasiga teng bo‘ladi:

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}.$$

Misol:

$$\begin{vmatrix} 2 & 3 & -1 \\ 4 & 1 & 5 \\ -2 & -3 & 3 \end{vmatrix}.$$

$$\begin{vmatrix} 2 & 3 & -1 \\ 4 & 1 & 5 \\ -2 & -3 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -1 \\ 4+10 & 1+15 & 5-5 \\ -2+6 & -3+9 & 3-3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -1 \\ 14 & 16 & 0 \\ 4 & 6 & 0 \end{vmatrix} = 2 \cdot 2 \cdot \begin{vmatrix} 1 & 3 & -1 \\ 7 & 8 & 0 \\ 2 & 3 & 0 \end{vmatrix}$$

$$= 4 \cdot (-1) \cdot (-1)^{1+3} \cdot \begin{vmatrix} 7 & 8 \\ 2 & 3 \end{vmatrix} = -4 \cdot (21-16) = -20.$$

Laplas teoremasi yordamida to‘rtinchi tartibli determinantni ta’riflash mumkin:

$$\Delta_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} + a_{42}A_{42}.$$

va, umumiy holda, *n*-tartibli determinantni:

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n} .$$

Misol

$$\Delta = \begin{vmatrix} -2 & 7 & 0 & 6 & -2 \\ 1 & -1 & 3 & 2 & 2 \\ 3 & 4 & 0 & 5 & 3 \\ 2 & 5 & -4 & -2 & 2 \\ 0 & 3 & -1 & 1 & -4 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} -2 & 7 & 0 & 6 & -2 \\ 1+0 & -1+9 & 3-3 & 2+3 & 2-12 \\ 3 & 4 & 0 & 5 & 3 \\ 2 & 5 & -4 & -2 & 2 \\ 0 & 3 & -1 & 1 & -4 \end{vmatrix} = \begin{vmatrix} -2 & 7 & 0 & 6 & -2 \\ 1 & 8 & 0 & 5 & -10 \\ 3 & 4 & 0 & 5 & 3 \\ 2 & 5 & -4 & -2 & 2 \\ 0 & 3 & -1 & 1 & -4 \end{vmatrix} =$$

$$= \begin{vmatrix} -2 & 7 & 0 & 6 & -2 \\ 1 & 8 & 0 & 5 & -10 \\ 3 & 4 & 5 & 0 & 3 \\ 2 & 5-12 & -4+4 & -2-4 & 2+16 \\ 0 & 3 & -1 & 1 & -4 \end{vmatrix} = \begin{vmatrix} -2 & 7 & 0 & 6 & -2 \\ 1 & 8 & 0 & 5 & -10 \\ 3 & 4 & 0 & 5 & 3 \\ 2 & -7 & 0 & -6 & 18 \\ 0 & 3 & -1 & 1 & -4 \end{vmatrix} =$$

$$=(-1)\cdot(-1)^{5+3}\begin{vmatrix}-2 & 7 & 6 & -2\\ 1 & 8 & 5 & -10\\ 3 & 4 & 5 & 3\\ 2 & -7 & -6 & 18\end{vmatrix}=-\begin{vmatrix}-2+2 & 7+16 & 6+10 & -2-20\\ 1 & 8 & 5 & -10\\ 3-3 & 4-24 & 5-15 & 3+30\\ 2-2 & -7-16 & -6-10 & 18+20\end{vmatrix}=$$

$$=-\begin{vmatrix}0 & 23 & 16 & -22\\ 1 & 8 & 5 & -10\\ 0 & -20 & -10 & 33\\ 0 & -23 & -16 & 38\end{vmatrix}=-1\cdot(-1)^{2+1}\cdot\begin{vmatrix}23 & 16 & -22\\ -20 & -10 & 33\\ -23 & -16 & 38\end{vmatrix}=$$

$$=\begin{vmatrix}23-23 & 16-16 & -22+38\\ -20 & -10 & 33\\ -23 & -16 & 38\end{vmatrix}=\begin{vmatrix}0 & 0 & 16\\ -20 & -10 & 33\\ -23 & -16 & 38\end{vmatrix}=16\cdot\begin{vmatrix}-20 & -10\\ -23 & -16\end{vmatrix}=$$

$$=16\cdot(-10)\cdot\begin{vmatrix} 2 & 1 \\ -23 & -16 \end{vmatrix}=-160\cdot(-32+23)=1440.$$

Teskari matritsa.

Ushbu A kvadrat matritsani qaraymiz:

$$A=\begin{pmatrix} a_{11} \ a_{12}\ldots a_{1i}\ldots a_{1n} \\ a_{21} \ a_{22}\ldots a_{2i}\ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1} \ a_{n2}\ldots a_{ni}\ldots a_{nn} \end{pmatrix}$$

Agar $A\cdot B=B\cdot A=E$ bo'lsa, B matritsa A matritsaga teskari matritsa deb ataladi va u A^{-1} kabi belgilanadi.

1-Teorema: Agar A matritsa xosmas, ya'ni $\det A \neq 0$ bo'lsa, u holda uning uchun A^{-1} teskari matritsa mavjud.

2-Teorema: Agar A matritsa xos, ya'ni $\det A = 0$ bo'lsa, u holda uning uchun A^{-1} teskari matritsa mavjud emas.

3-Teorema: Agar A matritsa xosmas, ya'ni $\det A \neq 0$ bo'lsa, u holda A^{-1} teskari matritsa yagonadir.

1) A kvadrat matritsalar uchun $\det A \neq 0$ bo'lsa, teskari A^{-1} matritsa mavjud.

2)

$$A^* = \begin{pmatrix} A_{11} & A_{21} & A_{31} \dots A_{k1} \\ A_{12} & A_{22} & A_{32} \dots A_{k2} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ A_{1n} & A_{2n} & A_{3n} \dots A_{nn} \end{pmatrix}$$

matritsa A matritsaga biriktirilgan matritsa deyiladi.

Bu yerda A_{ij} lar a_{ij} elementlarning algebraik to'ldi-ruvchilari. Agar $\det A \neq 0$ bo'lsa

$$B = A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} A_{11} & A_{21} & A_{31} & \dots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & A_{3n} & \dots & A_{nn} \end{pmatrix}$$

matritsaga A matritsaga teskari matritsa deyiladi.

Misol. 1) $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$, $\det A = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8-3 = 5 \neq 0$

$$A_{11} = (-1)^{1+1} \cdot 4 = 4, \quad A_{12} = (-1)^{3} \cdot 3 = -3, \quad A_{21} = (-1)^{3} \cdot = -1, \quad A_{22} = (-1)^{4} \cdot 2 = 2$$

$$A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$$

$$A^{-1} \cdot A = \frac{1}{5} \cdot \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} 8-3 & 4-4 \\ -6+6 & -3+9 \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E$$

Misol 1. Ushbu

$$A=\begin{pmatrix}1 & 2 & 0\\ 3 & 2 & 1\\ 0 & 1 & 2\end{pmatrix}$$

kvadrat matritsaga *teskari matritsa* A^{-1} ni toping.

Yechish. 1) Kvadrat matritsaning determinantini hisoblaymiz. Buning uchun uni birinchi satr elementlari bo'yicha yoyamiz:

$$|A|=\begin{vmatrix}1 & 2 & 0\\ 3 & 2 & 1\\ 0 & 1 & 2\end{vmatrix}=\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}-2\begin{vmatrix}3 & 1\\ 0 & 2\end{vmatrix}=3-12=-9\,.$$

$|A|=-9\neq 0$ bo'lganligidan, berilgan matritsaning xosmas matritsa ekanligi va uning uchun A^{-1} teskari matritsa mavjudligi kelib chiqadi.

2) Determinantning a_{ij} elementlarni A_{ij} algebraik to'ldiruvchilarini hisoblaymiz:

$$A_{11}=(-1)^{1+1}\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}=3,\quad A_{12}=(-1)^{1+2}\begin{vmatrix}3 & 1\\ 0 & 2\end{vmatrix}=6,\quad A_{13}=(-1)^{1+3}\begin{vmatrix}3 & 2\\ 0 & 1\end{vmatrix}=3,$$

$$A_{21}=(-1)^{2+1}\begin{vmatrix}2 & 0\\ 1 & 2\end{vmatrix}=-4,\quad A_{22}=(-1)^{2+2}\begin{vmatrix}1 & 0\\ 0 & 2\end{vmatrix}=2,\quad A_{23}=(-1)^{2+3}\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix}=-1,$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix}2 & 0\\ 2 & 1\end{vmatrix}=2,\quad A_{32}=(-1)^{3+2}\begin{vmatrix}1 & 0\\ 3 & 1\end{vmatrix}=-1,\quad A_{33}=(-1)^{3+3}\begin{vmatrix}1 & 2\\ 3 & 2\end{vmatrix}=4.$$

3) Teskari matritsa formulasidan topamiz:

$$A^{-1} = -\frac{1}{9}\begin{pmatrix} 3 & -4 & 2 \\ -6 & 2 & -1 \\ 3 & -1 & -4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{4}{9} & -\frac{2}{9} \\ \frac{2}{3} & -\frac{2}{9} & \frac{1}{9} \\ -\frac{1}{3} & \frac{1}{9} & \frac{4}{9} \end{pmatrix}.$$

4) Teskari matritsani to'g'ri topganligimizni tekshiramiz. Buning uchun $A \cdot A^{-1}$ ko'paytmani hisoblaymiz:

$$A \cdot A^{-1} = -\frac{1}{9}\begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & -4 & 2 \\ -6 & 2 & -1 \\ 3 & -1 & -4 \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}.$$

Matritsalarni ko'paytirish formulasidan foydalanib, topamiz:

$$e_{11} = -\frac{1}{9}(1\cdot 3 - 2\cdot 6 + 0\cdot 3) = 1,\ e_{12} = -\frac{1}{9}(-1\cdot 4 + 2\cdot 2 - 0\cdot 1) = 0,$$

$$e_{13} = -\frac{1}{9}(1\cdot 2 - 1\cdot 2 + 0\cdot 4) = 0,\ e_{21} = -\frac{1}{9}(3\cdot 3 - 2\cdot 6 + 3\cdot 1) = 0,$$

$$e_{22} = -\frac{1}{9}(-3\cdot 4 + 2\cdot 2 - 1\cdot 1) = 1,\ e_{23} = -\frac{1}{9}(3\cdot 2 - 2\cdot 4 - 1\cdot 4) = 0,$$

$$e_{31} = -\frac{1}{9}(0\cdot 3 - 1\cdot 6 + 2\cdot 3) = 0,\ e_{32} = -\frac{1}{9}(0\cdot 4 + 1\cdot 2 - 1\cdot 2) = 0,$$

$$e_{33} = -\frac{1}{9}(0\cdot 2 - 1\cdot 1 - 4\cdot 2) = 1.$$

Shunday qilib

$$A\cdot A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

tenglikni hosil qilamiz. Bu tenglik teskari matritsani to'g'ri topganligimizni bildiradi.

Matritsa rangi. Matritsaning rangi haqidagi teorema.

O'lchami $m \times n$ bo'lgan A matritsaning biror satr va ustunlarini o'chirish natijasida k-tartibli kvadrat matritsalarni hosil qilish mumkin, bu yerda $k \le \min\{m, n\}$. Bunday kvadrat matritsalarning determinantlari berilgan A *matritsaning minorlari* deb ataladi.

A *matritsaning rangi* deb uning noldan farqli minorlarining eng yuqori tartibiga aytiladi va $rang\, A$ yoki $r(A)$ ko'rinishda belgilanadi.

Matritsa rangining xossalari:

- agar A matritsaning o'lchami $m \times n$ bo'lsa, u holda $r(A) \le \min\{m, n\}$ bo'ladi;
- $rang\, A = 0$ bo'lishi uchun A matritsaning barcha elementlari 0 ga teng bo'lishi zarur va yetarli;
- tartibi n bo'lgan A kvadrat matritsaning rangi n ga teng $(rang\, A = n)$ bo'lishi uchun $|A| \ne 0$ bo'lishi zarur va yetarli.

Matritsa rangini o'zgartirmaydigan *elementar almashtirishlar*:

- barcha elementlari noldan iborat satrni (ustunni) tashlab yuborish;

- matritsa satr (ustun)idagi barcha elementlarini noldan farqli songa ko'paytirish;

- matritsa satr (ustun) o'rinlarini almashtirish;

- matritsaning berilgan satr (ustun) elementlariga boshqa bir satr (ustun) elementlarini biror songa ko'paytirib qo'shish;

- matritsani transponirlash.

Elementar almashtirishlar yordamida matritsani quyidagi zinapoyasimon ko'rinishga keltirish mumkin .

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2k} & \dots & a_{2n} \\ - & - & - & - & - & - \\ 0 & 0 & \dots & a_{kk} & \dots & a_{kn} \end{pmatrix},$$

bu yerda $a_{ii} \neq 0, i=1,...,k; \quad k \leq n.$

Zinapoyasimon matritsaning rangi k ga teng.

A matritsaning satrlarini $e_1=(a_{11}a_{12}...a_{1n})$, $e_2=(a_{21}a_{22}...a_{2n}),...,$ $e_m=(a_{m1}a_{m2}...a_{mn})$ ko'rinishda belgilaymiz va ular ustidagi chiziqli amallarni quyidagicha kiritamiz:

$$\lambda e_k=(\lambda a_{k1}\lambda a_{k2}...\lambda a_{kn});\; e_k+e_s=\left[(a_{k1}+a_{s1})(a_{k2}+a_{s2})...(a_{kn}+a_{sn})\right].$$

Xuddi shu usul bilan matritsaning ustunlari ustida chiziqli amallarni kiritish mumkin.

Agar matritsaning $e_1,e_2,...,e_s$ satrlari uchun, bir vaqtda nolga teng bo'lmagan shunday $\lambda_1,\lambda_2,...,\lambda_s$ sonlar topilsaki, ushbu satrlarining chiziqli kombinasiyasi nol satrga teng bo'lsa, ya'ni $\lambda_1e_1+\lambda_2e_2+...+\lambda_se_s=\bar{0}$ tenglik o'rinli bo'lsa, bu yerda $\bar{0}=(0,0,...,0)$, u holda $e_1,e_2,...,e_s$ satrlar *chiziqli bog'liq satrlar* deb ataladi. Aks holda matritsaning satrlari *chiziqli bog'liq bo'lmagan satrlar* deb ataladi. Demak, chiziqli bog'liq bo'lmagan satrlar uchun $\lambda_1e_1+\lambda_2e_2+...+\lambda_se_s=\bar{0}$ tenglik faqat $\lambda_1=\lambda_2=...=\lambda_s=0$ bo'lgandagina o'rinli bo'ladi.

Matritsaning chiziqli bog’liq bo’lmagan satrlarning maksimal soni chiziqli bog’liq bo’lmagan ustunlarning maksimal soniga teng bo'ladi.

Matritsaning rangi haqidagi teorema: Matritsa rangi undagi chiziqli bog’liq bo’lmagan satrlarning (ustunlarning) maksimal soniga teng bo'ladi.

Matritsa rangini o'zgartirmaydigan elementar almashtirishlar yordamida Matritsa rangini topish jarayononi ancha soddalashtirish mumkin.

Misol 3. Matritsa rangini toping.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 5 \\ 2 & 0 & 0 & 0 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 \\ 2 & 11 \end{pmatrix}, \qquad \begin{vmatrix} 1 & 5 \\ 2 & 11 \end{vmatrix} = 11 - 10 = 1 \neq 0 \Rightarrow \ rang\, A = 2.$$

Misol 4. Matritsa rangini toping.

$$\begin{pmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 4 & 8 & 12 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}, \quad \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = 1 \neq 0 \Rightarrow \ rang\,A = 2\,.$$

Misol 5. Matritsa rangini toping.

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 6 & 8 \\ 1 & 2 & 1 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 6 & 8 \end{pmatrix}, \ \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0. \Rightarrow \ rang\,A = 2\,.$$

Matritsalar ustida elementar almashtirishlar:

1) matritsa biror satri (ustuni) har bir elementini biror noldan farqli songa ko‘paytirish;
2) matritsa biror satri (ustuni) elementlariga uning boshqa parallel satri (ustuni) mos elementlarini biror noldan farqli songa ko‘paytirib, so‘ngra qo‘sh;
3) matritsa satrlari (ustunlari) o‘rinlari almashtirilish.

Agar matritsalardan biri ikkinchisidan elementar almashtirishlar yordamida olinsa, u holda bunday matritsalarga *ekvivalent matritsalar* deb ataladi va $A \sim B$ kabi yoziladi.

Elementar almashtirishlar yordamida ixtiyoriy matritsani shunday ko‘rinishga keltirish mumkinki, xosil bo‘lgan matritsaning bosh diagonalning boshida ketma-ket bir nechta birliklar mavjud va boshqa barcha elementlar nolga teng bo‘ladi. Ushbu matritsa *kanonik* deb ataladi.


E'tiborlaringiz uchun rahmat!