Strongly Regular Graphs and Finite Ramsey Theory

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ABSTRACT

Some connections between strongly regular graphs and finite Ramsey theory are drawn. Let B_n denote the graph $K_2 + \bar{K}_n$. If there exists a strongly regular graph with parameters (v, k, λ, μ), then the Ramsey number $r(B_{\lambda+1}, B_{v-2k+\mu-1}) \geqslant v+1$. We consider the implications of this inequality for both Ramsey theory and the theory of strongly regular graphs.

INTRODUCTION

We discuss some connections between strongly regular graphs and finite Ramsey theory. The ideas involved in the theory of strongly regular graphs are linear algebraic, and their consequences are employed in this paper, even though linear algebraic techniques do not appear explicitly. Thus this paper can be regarded as an instance of the application of those techniques in another, related field. All graphs in this paper are both finite and simple. Let G_1 and G_2 be graphs. Then *the Ramsey number*, $r(G_1, G_2)$, *of* G_1 *and* G_2 is the smallest integer n such that in any 2-coloring (E_1, E_2) of the edges of K_n either $\langle E_1 \rangle \supseteq G_1$ or $\langle E_2 \rangle \supseteq G_2$. So, thinking of E_1 and E_2 as being "red" and "blue" edges respectively, if the edges of K_n are colored red and blue, then there exists either a red G_1 or a blue G_2. Furthermore, since n is minimal, there must exist a graph G on $n-1$ vertices such that $G \not\supseteq G_1$ and its complement $\bar{G} \not\supseteq G_2$. A *strongly regular graph with parameters* (v, k, λ, μ) [or more briefly we say a (v, k, λ, μ)-graph] is a graph which is regular of degree

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k on v vertices and is such that there exist exactly λ (μ) vertices mutually adjacent to any two distinct adjacent (nonadjacent) vertices. Excellent elementary introductory articles on strongly regular graphs by Cameron and Seidel appear in [1] and [3] respectively. Notice that if G is a (v, k, λ, μ)-graph, then $\bar{G}$ is a $(v, v-1-k, v-2k+\mu-2, v-2k+\lambda)$-graph. Now let B_n $(n \geq 1)$ denote the graph $K_2 + \bar{K}_n$ (see [9] for notation). Then the interaction between strongly regular graphs and Ramsey theory which we wish to discuss is made formally by the following observation.

OBSERVATION. *If there exists a (v, k, λ, μ)-graph G, then*

$$r(B_{\lambda+1}, B_{v-2k+\mu-1}) \geq v+1.$$

This follows because $G \not\supseteq B_{\lambda+1}$, $\bar{G} \not\supseteq B_{v-2k+\mu-1}$, and G has exactly v vertices. Now we can consider this inequality from two viewpoints. If a particular (v, k, λ, μ)-graph exists, then this determines a lower bound for the corresponding Ramsey number. We give an example of this approach in Section 1. On the other hand, if we can independently determine an upper bound for a particular Ramsey number, this gives some information on the nonexistence of strongly regular graphs with the appropriate parameters (and usually of course on the nonexistence of a much larger class of graphs). We take this viewpoint in Section 2. We mention [6], [7], [10], [11], [12], [13], and an article by T. D. Parsons in [1] for readers interested in this area.

This paper essentially contains just two original theorems, viz., Theorem 2 and Theorem 3. These theorems are discussed in Section 2 but not proved. We include their proofs as appendices. We do not recommend the reader to pursue all the details of these proofs, but simply to note their elementary nature and their dependence on the lemma stated at the beginning of Appendix 1.

1. A SPECIAL CASE

We prove in [12] the following theorem and corollary:

THEOREM 1. *If $2(m+n)+1 > (n-m)^2/3$, then $r(B_m, B_n) \leq 2(m+n+1)$. By refinement, $r(B_{n-1}, B_n) \leq 4n-1$ and, if $n \equiv 2$ (mod3), then $r(B_{n-2}, B_n) \leq 4n-3$.*

COROLLARY. *If $4n+1$ is a prime power, then $r(B_n, B_n) = 4n+2$. If $4n+1$ cannot be expressed as the sum of two integer squares, then $r(B_n, B_n) \leq 4n+1$. In the first example of the latter, $r(B_5, B_5) = 21$.*

The proof of Theorem 1 yields, as pointed out by T. D. Parsons [1], $r(B_m, B_n) \leqslant m + n + 2 + [\frac{2}{3}\sqrt{3(m^2 + mn + n^2)}\,]$. We can indicate the main idea of the proof of the theorem by directly proving the Corollary. This proof illustrates the first viewpoint on the observation made in the introduction.

Proof of the Corollary. Let $p, n \geqslant 1$. Suppose there exists a 2-coloring (E_1, E_2) of the edges of K_p such that $\langle E_i \rangle \not\supseteq B_n$ $(i = 1,2)$. Let M be the number of monochromatic triangles produced by this coloring. Then a classical result of Goodman [8] gives

$$M \geqslant \frac{p(p-1)(p-5)}{24}. \tag{1}$$

On the other hand, since on each red (blue) edge there exist at most $n-1$ red (blue) triangles,

$$M \leqslant \frac{|E_1|(n-1) + |E_2|(n-1)}{3} = \frac{p(p-1)(n-1)}{6}. \tag{2}$$

From (1) and (2), $p \leqslant 4n+1$. Hence $r(B_n, B_n) \leqslant 4n+2$. Suppose $p = 4n+1$. Then equality holds in (1) and (2). Write $G = \langle E_1 \rangle$. Goodman's result also tells us that since equality holds in (1), G is regular of degree $2n$. Equality in (2) implies that on each edge of G there are exactly $n-1$ triangles and on each edge of $\overline{G}$ there are exactly $n-1$ triangles. So G is a $(4n+1, 2n, n-1, n)$-graph. Hence, using our observation, $r(B_n, B_n) = 4n+2$ if and only if there exists a $(4n+1, 2n, n-1, n)$-graph. Such graphs [5] are called conference graphs and are well known to exist if $4n+1$ is a prime power. No such graph exists if $4n+1$ cannot be expressed as the sum of two integer squares. In the first example of the latter, $r(B_5, B_5) = 21$. This is proved by giving [12] a direct construction of a graph on 20 vertices with the required properties. ■

2. RESULTS

We prove in the appendices:

THEOREM 2. *Suppose* $1 \leqslant k < n$. *Then* $r(B_k, B_n) = 2n+3$ *for* $n \geqslant (k-1)(16k^3 + 16k^2 - 24k - 10) + 1$.

THEOREM 3.

(i) $r(B_1, B_n) = 2n+3$ $(n \geq 2)$.

(ii) $$2n+3 \leq r(B_2, B_n) \leq \begin{cases} 2n+6 & (2 \leq n \leq 11), \\ 2n+5 & (12 \leq n \leq 22), \\ 2n+4 & (23 \leq n \leq 37), \\ 2n+3 & (n \geq 38). \end{cases}$$

COROLLARY. $r(B_2, B_n) = 2n+6$, $n = 2, 5, 11$.

COMMENT. We now simply interpret these results (or at least some of them) in the context of our viewpoint that upper bounds for Ramsey numbers provide information on the existence of certain strongly regular graphs.

Let $t \geq 1$. Write $n = 3t-1$, and let $G(n)$ denote (if it exists) a $(6t+3, 2t+2, 1, t+1)$-graph. Then if $G(n)$ exists, $r(B_2, B_n) \geq 2n+6$, and so by Theorem 3(ii), $r(B_2, B_n) = 2n+6$. Now someone (see Cameron [1]) with knowledge of the theory of strongly regular graphs would proceed as follows. If $G(n)$ exists, then (using the so-called *integrality condition*)

$$\frac{(v-1)(\mu-\lambda)-2k}{\sqrt{(\mu-\lambda)^2+4(k-\mu)}}$$

is an integer, where $k = 2t+2$, $\lambda = 1$, $\mu = t+1$, and $v = 6t+3$. Hence $t = 1, 2, 4, 10$. The line graph $L(K_{3,3})$ of $K_{3,3}$, the complement of $L(K_6)$, and the line graph of the 27 lines on a cubic surface show respectively that $G(2)$, $G(5)$, and $G(11)$ exist. However, as yet we have not determined whether $G(29)$ exists. The only other general necessary conditions for the existence of a (v, k, λ, μ)-graph are the so-called *Krein conditions* (see Seidel [3]). This states that if

$$r+s = \lambda - \mu \text{ and } rs = \mu - k \qquad (r > s),$$

then

(i) $(r+1)(k+r+2rs) \leq (k+r)(s+1)^2$,
(ii) $(s+1)(k+s+2rs) \leq (k+s)(r+1)^2$.

In our case $r+s = -10$, $rs = -11$, and so $r = 1$, $s = -11$. We see that these values of r, s, and $k = 22$ do not satisfy the second Krein condition. Hence no $G(29)$ exists.

Now suppose we know nothing about the theory of the existence of strongly regular graphs. Then, by Theorem 3(ii), no $G(n)$ exists for $n \geqslant 12$. So in particular $G(29)$ does not exist. Hence we do not require the Krein conditions to prove the nonexistence of $G(29)$. However, against this the existence of $G(8)$ is undecided by our Ramsey theory, whereas the integrality test shows that no $G(8)$ exists.

Generally we may interpret Theorems 2 and 3 as follows: "if $r(B_m, B_n) \leqslant N$, then there exists no $(v, k, m-1, n+2k-v+1)$-graph with $v \geqslant N$."

Of course a much stronger statement is also true, viz., for $v \geqslant N$ there exists no graph G on v vertices such that: (i) on each edge of G there are at most $m-1$ triangles, and (ii) on each edge of $\overline{G}$ there are at most $n-1$ triangles.

3. CONJECTURES

We have conjectured in [12]:

CONJECTURE 1. There exists a constant $A > 0$ such that

$$r(B_m, B_n) \leqslant 2(m+n+1) + A.$$

Our theorems support this conjecture, although of course they are a very long way from giving the whole picture. A well-known $(253, 112, 36, 60)$-graph shows that $r(B_{37}, B_{88}) \geqslant 254$ and so $A \geqslant 2$. In this context, a $(275, 112, 30, 56)$-graph and a $(162, 56, 10, 24)$-graph show respectively that $r(B_{31}, B_{106}) \geqslant 276$ and $r(B_{11}, B_{73}) \geqslant 163$. Notice that these parameters are well away from the parabolic region of Theorem 1. Conjecture 1 would imply, if true, the truth of:

CONJECTURE 2. There exists a constant A $(A \geqslant 2)$ such that for every (v, k, λ, μ)-graph we have

$$2(\alpha + \beta) - v \leqslant A,$$

where $\alpha = k - \lambda - 1$ and $\beta = k - \mu$.

COMMENT. This conjecture is true for conference graphs. It is also true when $\lambda = \mu$. For readers not familiar with the parameters α and β, it is worthwhile recalling in this context that if we write $l = v - k - 1$, then since

$k(k-\lambda-1)=l\mu$,

$$\frac{\alpha}{l}+\frac{\beta}{k}=1.$$

Finally we would like to mention that if instead of discussing $r(B_m, B_n)$ we consider the Ramsey number $r(K_m+\bar{K}_n)$, then in [6] and [13] conference graphs are used to provide lower bounds. In [13] especially the asymptotic lower bounds are discussed in some depth.

APPENDIX 1. PROOF OF THEOREM 2

Almost all of our notation in this section will be standard [2, 4, 9]. There is one exception. Let G be a graph and $x \in V(G)$. Then $N(x)$ denotes the neighborhood of G, and for any subset $Y \subseteq V(G)$ we write

$$Y(x)=:N(X)\cap Y.$$

Furthermore if $x_1, x_2 \in V(G)$ we write

$$Y(x_1 \cap x_2)=:Y(x_1)\cap Y(x_2)$$

and

$$Y(x_1 \cup x_2)=:Y(x_1)\cup Y(x_2).$$

This is simply a notational device to restrict the number of symbols used.

In the proof of the theorem below we shall need to consider a 2-coloring (E_1, E_2) of the edges of K_{2n+3} $(n \geqslant 1)$. As above, we call the edges of E_1 red and those of E_2 blue. In general a suffix i $(i=1,2)$ will refer to the ith color. For example, if $v \in V(G)$, then $N_1(v)$ is the red neighborhood of v, and if $Y \subseteq V(K_{2n+3})$, then $Y_2(v)$ is the blue neighborhood of v contained in Y. Again if $Y \subseteq V(K_{2n+3})$, then $\langle Y \rangle_1$ is the subgraph of K_{2n+3} with vertex set Y and edge set consisting of all red edges with both end vertices in Y. We abuse this notation very slightly when we present and use the next lemma, but it is only in this context that we shall do this and no confusion should arise. This lemma plays a crucial role in the proofs of both Theorems 2 and 3.

LEMMA. *Let $A_1, A_2, \ldots, A_m$ $(m \geqslant 2)$ be subsets of a finite set A. Suppose δ and μ are integers such that for all $i, j \in \{1,2,\ldots,m\}$, $i \neq j$, we have $|A_i \cup A_j| \geqslant \delta$ and $|A_i \cap A_j| \leqslant \mu$. Then*

(1) $2(m-1)|A| \geqslant m(m-1)\delta - 2(m-2)(\Sigma |A_i \cap A_j|)$, *where the summation is over all unordered pairs $\{i,j\}$, $i, j \in \{1,2,\ldots,m\}$, $i \neq j$.*

(2) $2|A| \geqslant m\delta - m(m-2)\mu$.

Proof. Choose $i, j \in \{1,2,\ldots,m\}$, $i \neq j$. Then

$$|A_i| + |A_j| = |A_i \cup A_j| + |A_i \cap A_j|.$$

Summing over all possible such pairs i and j we obtain

$$(m-1)\left(\sum_{i=1}^{m} |A_i|\right) = \sum_{i \neq j} |A_i \cup A_j| + \sum_{i \neq j} |A_i \cap A_j|. \tag{3}$$

But

$$|A| \geqslant \left|\bigcup_{i=1}^{m} A_i\right| \geqslant \sum_{i=1}^{m} |A_i| - \sum_{i \neq j} |A_i \cap A_j|. \tag{4}$$

The lemma now follows from (3), (4) and the definitions of δ and μ. ■

THEOREM 2. *Suppose k and n are integers such that $1 \leqslant k < n$. Then*

$$r(B_k, B_n) = 2n+3$$

provided $n \geqslant (k-1)(16k^3 + 16k^2 - 24k - 10) + 1$.

Proof. We may in fact suppose $k > 1$, since the case $k = 1$ is proved in [12]. Since $K_{n+1,n+1}$ does not contain a B_k, and its complement does not contain a B_n we have

$$r(B_k, B_n) \geqslant 2n+3. \tag{5}$$

Unfortunately to prove equality is not so straightforward. Suppose that there is a 2-coloring (E_1, E_2) of the edges of K_{2n+3} such that $\langle E_1 \rangle \not\supseteq B_k$ and $\langle E_2 \rangle \not\supseteq B_n$. Suppose $n \geqslant (k-1)(16k^3 + 16k^2 - 24k - 10) + 1$. Choose (E_1, E_2)

to be a 2-coloring of the edges of K_{2n+3} such that $\langle E_1 \rangle \not\supseteq B_k$ and $\langle E_2 \rangle \not\supseteq B_n$. Choose $\alpha, \beta \in V(k_{2n+3})$ so that $\alpha\beta \in E_2$ and $|N_1(\alpha) \cap N_1(\beta)|$ is as large as possible. Write $D = N_1(\alpha) \cap N_1(\beta)$, $A = N_2(\alpha) \cap N_2(\beta)$, $B = N_2(\alpha) \setminus A$, $C = N_2(\beta) \setminus A$, and $H = B \cup C \cup D$ (see Figure 1; the broken lines indicate red edges). We assume $|B| \geqslant |C|$.

We emphasize the choice, in particular the maximality, of $|D|$. It will play a prominent role throughout the subsequent arguments. We proceed by a series of propositions.

PROPOSITION 1.

(1) $|A| \leqslant n-1$,
(2) $|H| = (2n+1) - |A| \geqslant n+2$,
(3) $|(B \cup D)_1(b)| \leqslant k-1$ $(b \in B)$, $|(C \cup D)_1(c)| \leqslant k-1$ $(c \in C)$, $|B_1(d)| \leqslant k-1$, $|C_1(d)| \leqslant k-1$, $|D_1(d)| \leqslant k-1$ $(d \in D)$,
(4) $|H_1(x)| \leqslant (k-1) + |D|$, $|D_1(x)| \leqslant k-1$ $(x \in B \cup C)$,
(5) $|H_1(d)| \leqslant 2(k-1)$ $(d \in D)$,
(6) $|H_2(x)| \geqslant (|H|-1) - \max\{2(k-1), (k-1)+|D|\}$ $(x \in H)$.

Proof. The proof follows directly from the various definitions. For example Proposition 1(4) (which we abbreviate to P.1.4) is proved by using P.1.3 and the maximality of $|D|$. ■

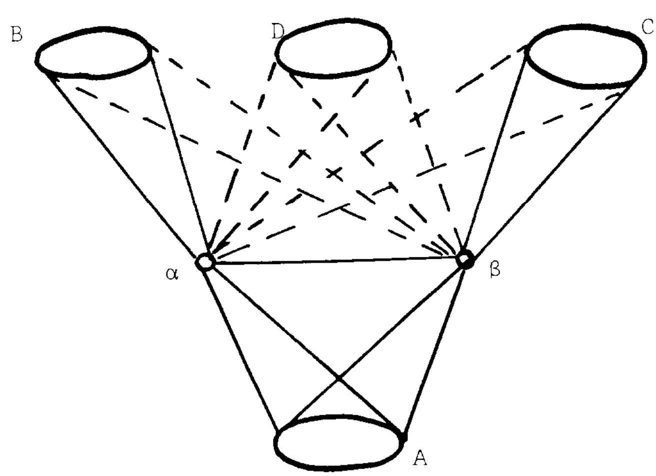

FIG. 1.

When the reader is in doubt, he should refer back to this proposition, which will not always be cited.

PROPOSITION 2. $\langle H\rangle_2 \supseteq K_3$.

Proof. Write $G=\langle B\rangle_2$. Then the minimal degree, $\delta(G)$, of G satisfies, using P.1.3,

$$\delta(G) \geqslant |B|-k. \tag{6}$$

If $G \not\supseteq K_3$, then, by Turan's theorem [4],

$$|E(G)| \leqslant |B|^2/4. \tag{7}$$

Hence, from (6) and (7), $|B| \leqslant 2k$. Therefore, using P.1.1 and $|B| \geqslant |C|$,

$$\begin{aligned} |D| &= (2n+1)-(|A|+|B|+|C|) \\ &\geqslant (n+2)-4k. \end{aligned} \tag{8}$$

We now use the same argument for $K=:\langle D\rangle_2$. Again if $K \not\supseteq K_3$, $|D| \leqslant 2k$. So from (8), $n \leqslant 6k-2$. This is a contradiction. ■

PROPOSITION 3. $|D| \geqslant 2k^2+1$.

Proof. Let v_1, v_2, v_3 be the vertices of a triangle in $\langle H\rangle_2$. Write $\theta = \max\{2(k-1),\ k-1+|D|\}$. Then, from P.1.6, for $i, j \in \{1,2,3\}$, $i \neq j$,

$$|H_2(v_i \cap v_j)| \geqslant |H|-2(\theta+1). \tag{9}$$

Since $\langle E_2\rangle \not\supseteq B_n$, from (9),

$$\begin{aligned} |A_2(v_i \cap v_j)| &\leqslant (n-1)-(|H|-2(\theta+1)) \\ &= |A|-n+2\theta. \end{aligned} \tag{10}$$

By the maximality of $|D|$, for all pairs i and j above, $|A_1(v_i \cap v_j)| \leqslant |D|$. Hence

$$|A_2(v_i \cup v_j)| \geqslant |A|-|D|. \tag{11}$$

Write $A_i = A_2(v_i)$ $(i=1,2,3)$, $\mu = |A|-n+2\theta$, $\delta = |A|-|D|$, and $m=3$.

Then from the lemma, (10), (11), and P.1.1,

$$3|D| \geqslant (n+2)-6\theta. \tag{12}$$

Now suppose $\theta = 2(k-1)$. Then $|D| \leqslant k-1$, and from (12), $n \leqslant 15k-17$, which is not true. On the other hand, if $\theta = (k-1)+|D|$, then

$$|D| \geqslant \frac{n-6k+8}{9}, \tag{13}$$

and so from (13) and the magnitude of n, $|D| \geqslant 2k^2+1$. ■

PROPOSITION 4. *$\langle D \rangle_1$ has at least $2k+1$ independent vertices.*

Proof. Let $G = \langle D \rangle_1$. Then, from P.1.3 and P.3, G is a graph with maximal degree at most $k-1$, and G has at least $2k^2+1$ vertices. The result now follows as an elementary exercise in graph theory. ■

PROPOSITION 5. $|D| \geqslant [(2k-1)|A|]/(2k+1)-(8k^2-14k+3)$.

Proof. Let $v_1, v_2, \ldots, v_{2k+1}$ be distinct independent vertices of $\langle D \rangle_1$. Let $m = 2k+1$, and write $A_i = A_2(v_i)$ $(i = 1,2,\ldots,m)$. Now with minor modifications (allowing for the fact that the v_i's belong not only to H but also to D) we repeat the argument of P.3. Let $i, j \in \{1,2,\ldots,m\}$, $i \neq j$. From P.1.5

$$|H_2(v_i \cap v_j)| \geqslant 2[|H|-2-2(k-1)]-(|H|-2).$$

Hence, since $\langle E_2 \rangle \not\supseteq B_n$ and using P.1.2,

$$\begin{aligned} |A_i \cap A_j| &\leqslant (n-1)-|H_2(v_i \cap v_j)| \\ &\leqslant (n-1)-|H|+4(k-1)+2 \\ &\leqslant 4k-5. \end{aligned} \tag{14}$$

Again, by the maximality of $|D|$, and since $\alpha, \beta \in N_1(v_i) \cap N_1(v_j)$, we have $|A_1(v_i \cap v_j)| \leqslant |D|-2$. Hence

$$|A_i \cup A_j| \geqslant |A|-|D|+2. \tag{15}$$

Write $\delta = |A|-|D|+2$, $\mu = 4k-5$. Then the result follows from the lemma, using $m = 2k+1$ and Equations (14) and (15). ■

PROPOSITION 6.

(1) *If* $x_1, x_2 \in D$ *and* $x_1x_2 \in E_2$, *then*

$$|H_2(x_1 \cap x_2)| \geqslant n-4(k-1).$$

(2) $|A| \geqslant n-4(k-1)$.

Proof. (1): From P.1.5

$$|H_2(x_1 \cap x_2)| \geqslant (|H|-2)-4(k-1)$$
$$=(2n-1)-|A|-4(k-1). \tag{16}$$

The result now follows from P.1.1

(2): Since $|D| \geqslant 2k^2+1$ and $\langle D \rangle_1 \not\supseteq B_k$, there exists at least one blue edge x_1x_2 with $x_i \in D$, $i=1,2$. The result now follows from (16) and the fact that $|H_2(x_1 \cap x_2)| \leqslant n-1$, since $\langle D \rangle_2 \not\supseteq B_n$. ■

PROPOSITION 7. $\langle D \rangle_1$ *does not contain 2 independent edges* x_1y_1 *and* x_2y_2 *such that all of* $x_1x_2, x_1y_2, y_1x_2, y_1y_2$ *are blue edges.*

Proof. Suppose $\langle D \rangle_1$ does contain 2 such independent edges. Then we may choose $x_i, y_i \in D$ $(i=1,2)$ so that x_1y_1 and x_2y_2 are red edges and such that $x_1x_2, x_1y_2, y_1x_2, y_1y_2$ are all blue edges. Then, for $i=1,2$, since $\alpha, \beta \in N_1(x_i \cap y_i)$ and since $\langle E_1 \rangle \not\supseteq B_k$,

$$|A_1(x_i \cap y_i)| \leqslant k-3.$$

Hence

$$|A_2(x_i \cup y_i)| \geqslant |A|-(k-3). \tag{17}$$

Therefore, with no loss of generality, we may suppose that

$$|A_2(x_1)| \geqslant \frac{|A|-(k-3)}{2}. \tag{18}$$

Now, from (17),

$$|A_2(x_1) \setminus A_2(x_2 \cup y_2)| \leqslant k-3. \tag{19}$$

Therefore from (19), again with no loss of generality, we may suppose that

$$|A_2(x_1 \cap x_2)| \geqslant \frac{|A_2(x_1)| - (k-3)}{2}. \tag{20}$$

Hence, from P.6.1, P.6.2, (18), and (20), and since $\langle E_2 \rangle \not\supseteq B_n$,

$$\begin{aligned} n-1 &\geqslant |A_2(x_1 \cap x_2)| + |H_2(x_1 \cap x_2)| \\ &\geqslant \frac{n-7k+13}{4} + [n-4(k-1)], \end{aligned}$$

i.e., $n \leqslant 23k - 33$, which is a contradiction. ■

PROPOSITION 8. *Write* $X = B \cup C$, *and let* $D^* = \{d \in D : |X_1(d)| \geqslant 1\}$. *Then*

$$|D^*| \geqslant |D| - (2k^2 - 6k + 7).$$

Proof. Firstly notice that at most one element of D is isolated in $\langle H \rangle_1$, i.e., $|H_1(d)| \geqslant 1$ for all but at most one element of D. Otherwise choose two such elements $d_1, d_2 \in D$, $d_1 \neq d_2$. Then, using P.1.2,

$$n-1 \geqslant |H_2(d_1 \cap d_2)| \geqslant |H| - 2 \geqslant n.$$

Now write $G = \langle D \rangle_1$. Then G is a graph with at least $2k^2+1$ vertices and maximal degree at most (using P.1.3) $k-1$. Since G has at most one independent edge xy, an elementary exercise in graph theory (see Figure 2) shows that G contains at least $|D| - [2+2(k-2)+2(k-2)^2]$ isolated vertices. This, together with the opening remark, proves the proposition. ■

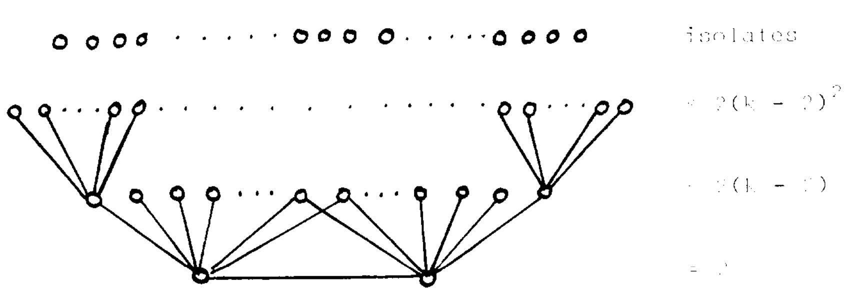

FIG. 2.

PROPOSITION 9. $|D| \leqslant [(2n+1-|A|)(k-1)+2k^2-6k+7)]/k$.

Proof. Let $E_1(B\cup C, D)$ denote the set of red edges with one end vertex in $B\cup C$ and the other end vertex in D. Then, from P.8,

$$|E_1(B\cup C, D)| \geqslant |D^*|$$

$$\geqslant |D| - (2k^2-6k+7). \tag{21}$$

Since $\langle E_1 \rangle \not\supseteq B_k$, if $x \in B\cup C$, then $|D_1(x)| \leqslant k-1$. Hence

$$|E_1(B\cup C, D)| \leqslant |B\cup C|(k-1)$$

$$= (2n+1-|A|-|D|)(k-1). \tag{22}$$

The proposition follows from (21) and (22). ■

Proof of Theorem 2. From Propositions 5 and 9

$$k(2k-1)|A| - k(2k+1)(8k^2-14k+3)$$

$$\leqslant (2k+1)(2n+1-|A|)(k-1)+(2k+1)(2k^2-6k+7).$$

Now use P.6.2 to obtain a contradiction to the magnitude of n. Hence $r(B_k, B_n) \leqslant 2n+3$ and so, from (5), $r(B_k, B_n) = 2n+3$. ■

A similar, but very much more delicate, analysis proves Theorem 3(ii). Theorem 3(i) is proved in [12].

APPENDIX 2. PROOF OF THEOREM 3(ii)

The corollary to Theorem 3 is proved in Section 2, and Theorem 3(i) is proved in [12].

THEOREM 3(ii).

$$2n+3 \leqslant r(B_2, B_n) \leqslant \begin{cases} 2n+6 & (2 \leqslant n \leqslant 11), \\ 2n+5 & (12 \leqslant n \leqslant 22), \\ 2n+4 & (23 \leqslant n \leqslant 37), \\ 2n+3 & (n \geqslant 38). \end{cases}$$

NOTATION AND ASSUMPTIONS. It follows from Theorem 1 that $r(B_2, B_n) \leqslant 2n+6$ for $2 \leqslant n \leqslant 10$. Therefore, we shall assume henceforth that $n \geqslant 11$. Suppose that (E_1, E_2) is a 2-coloring of the edges of K_{2n+t} such that $\langle E_1 \rangle \not\supseteq B_2$ and $\langle E_2 \rangle \not\supseteq B_n$, and where

$$t = \begin{cases} 6 & \text{if} \quad n = 11, \\ 5 & \text{if} \quad 12 \leqslant n \leqslant 22, \\ 4 & \text{if} \quad 23 \leqslant n \leqslant 37, \\ 3 & \text{if} \quad n \geqslant 38. \end{cases}$$

We shall show that the assumption that such a 2-coloring exists leads to a contradiction, and this will establish the theorem. We retain all the notation introduced in the proof of Theorem 2 (see Appendix 1), i.e. in the case when $t = 3$ and n is large. For example α, β, A, B, C, and D are defined as before, and we use the same notational tricks. In P.1 we are now dealing with $2n + t$ vertices rather than simply $2n+3$ vertices, so P.1.2 becomes $|H| \geqslant (2n + t - 2) - |A| \geqslant n+2$. We recall, since this was buried in the proof, that $X = B \cup C$. We shall in addition use the following notation. Let $h_1, h_2 \in H$ $(h_1 \neq h_2)$. Write $\theta(h_1, h_2) = |H_1(h_1 \cup h_2)|$, $\omega(h_1, h_2) = |N_1(h_1) \cap N_1(h_2) \cap \{\alpha, \beta\}|$. The proof now proceeds, as for Theorem 2, by a series of Propositions:

PROPOSITION 10. $r(B_2, B_n) \geqslant 2n+3$.

Proof. Put $k = 2$ in Equation (5). ■

PROPOSITION 11. *Let* $h_1, h_2 \in H$ *and* $h_1 h_2 \in E_2$. *Then*

(1) $|A_1(h_1 \cap h_2)| \leqslant |D| - \omega(h_1, h_2)$,
(2) $|A_2(h_1 \cup h_2)| \geqslant |A| - |D| + \omega(h_1, h_2)$,
(3) $|A_2(h_1 \cap h_2)| \leqslant |A| - n - t + 3 + \theta(h_1, h_2)$,
(4) $|A_1(h_1 \cup h_2)| \geqslant n + t - 3 - \theta(h_1, h_2)$.

Proof. (1): This is an immediate consequence of the definition of D.

(2): This follows from P.11.1.

(3): By P.1.1 and P.1.2

$$\begin{aligned} n-1 &\geqslant |A_2(h_1 \cap h_2)| + |H_2(h_1 \cap h_2)| \\ &= |A_2(h_1 \cap h_2)| + \big[\,|H| - 2 - |H_1(h_1 \cup h_2)|\,\big] \\ &= |A_2(h_1 \cap h_2)| + \big[2n + t - 4 - |A| - \theta(h_1, h_2)\big]. \end{aligned}$$

(4): This follows from P.11.3.

PROPOSITION 12. *Let $h_1, h_2 \in H$ $(h_1 \neq h_2)$.*

(1) $h_1h_2 \in E_2$ $(h_1, h_2 \in D)$.

(2) $0 \leqslant \theta(h_1, h_2) \leqslant 2(|D|+1)$, $\omega(h_1, h_2) \geqslant 0$.

(3) $4 \geqslant \theta(h_1, h_2) \geqslant t-2$; $\omega(h_1, h_2) = 2$ $(h_1, h_2 \in D)$.

(4) $\omega(h_1, h_2) = 1$ $(h_1 \in B,\ h_2 \in D$ *or* $h_1 \in C,\ h_2 \in D)$.

(5) $0 \leqslant \theta(h_1, h_2) \leqslant |D| + 3$, $\omega(h_1, h_2) = 1$ $(h_1 \in X,\ h_2 \in D)$.

Proof. (1): This follows because $\omega(h_1, h_2) = 2$.

(2): From P.1.4 and P.1.5, $|H_1(x)| \leqslant \max\{2, 1 + |D|\}$ $(x \in H)$. Hence $\theta(h_1, h_2) \leqslant 2(|D|+1)$.

(3): Let $d_1, d_2 \in D$. Then, from P.12.1, P.1.1, and P.1.2,

$$\begin{aligned} n-1 \geqslant |H_2(d_1 \cap d_2)| &= |H| - 2 - \theta(d_1, d_2) \\ &= (2n + t - 2 - |A|) - 2 - \theta(d_1, d_2) \\ &\geqslant n + t - 3 - \theta(d_1, d_2). \end{aligned}$$

Hence $\theta(d_1, d_2) \geqslant t-2$. Also, from P.1.5, $\theta(d_1, d_2) \leqslant 4$. By definition $\omega(d_1, d_2) = 2$.

(4): From the definition of ω.

(5): By P.1.4 and P.1.5. ∎

PROPOSITION 13.

(1) *Suppose $|D| \leqslant 2$. Then there exist $h_1, h_2, h_3 \in H$ such that $h_ih_j \in E_2$ $(i \neq j;\ i, j = 1, 2, 3)$.*

(2) *Suppose $|D| = 2$. Then there exist $h_1, h_2 \in D$, $h_3 \in H$ such that $h_3h_i \in E_2$, $i = 1, 2$.*

Proof. Now suppose $|D|=2$. Then $2|B| \geqslant n+t-3 \geqslant 7$. Hence $|B| \geqslant 3$. Let $D=\{h_1, h_2\}$. By P.12.1, $h_1h_2 \in E_2$. By P.1.3, $|B_1(h_i)| \leqslant 1$. So there exists $h_3 \in B$ with $h_3h_i \in E_2$ $(i=1,2)$.

Suppose $|D|=1$. Then $2|B| \geqslant n+t-2 \geqslant 8$. Hence $|B| \geqslant 4$. Let $h_1 \in D$. Since $|B_1(h_1)| \leqslant 1$, there exist $b_1, b_2, b_3 \in B_2(h_1)$. Now for all $b \in B$, by P.1.3, $|B_1(b)| \leqslant 1$. Hence for some $i, j=1,2,3$, $i \neq j$, we have $b_ib_j \in E_2$. Write $h_2=b_i$ and $h_3=b_j$.

Suppose $|D|=0$. Then $2|B| \geqslant n+t-1 \geqslant 9$. Hence $|B| \geqslant 5$. Since $|B_1(b)| \leqslant 1$ for all $b \in B$, it follows that there exist $h_1, h_2, h_3 \in B$ such that $h_ih_j \in E_2$ $(i \neq j;\ i, j=1,2,3)$. ■

PROPOSITION 14.

(1) *If* $t \in \{5,6\}$ *then* $|D| \geqslant 2$.
(2) *If* $t \in \{3,4\}$ *then* $|D| \geqslant 3$.

Proof. Suppose $|D| \leqslant 2$. Select (see P.13) $h_1, h_2, h_3 \in H$ so that $h_ih_j \in E_2$ $(i \neq j;\ i, j=1,2,3)$. Write $A_i=A_2(h_i)$, $i=1,2,3$; $\delta=|A|-|D|$; and $\mu=|A|-n-t+5+2|D|$. Then, by P.11.2 and P.12.2, $|A_i \cup A_j| \geqslant \delta$, and by P.11.3 and P.12.2, $|A_i \cap A_j| \leqslant \mu$. Then, by the second part of the lemma of Appendix 1 (which we shall denote by L.2, etc.) with $m=3$ and using P.1.1,

$$2(n-1) \geqslant 3(\delta-\mu)=3(n+t-5-3|D|). \tag{23}$$

If $|D| \leqslant 1$ and $t \in \{5,6\}$ or if $|D| \leqslant 2$ and $t \in \{3,4\}$, (23) yields a contradiction to the magnitude of n. ■

PROPOSITION 15. $|D| \geqslant 3$.

Proof. By P.14 we may suppose that $t \in \{5,6\}$ and $|D|=2$. Select (see P.13.2) $h_1, h_2 \in D$, $h_3 \in H$ so that $h_3h_i \in E_2$, $i=1,2$. Write $A_i=A_2(h_i)$, $i=1,2,3$, $\delta=|A|-|D|+1$, and $\mu=|A|-n+6-t+|D|$. Then, by P.11.2, P.12.3, and P.12.4, $|A_i \cup A_j| \geqslant \delta$, and by P.11.3, P.12.3, and P.12.5, $|A_i \cap A_j| \leqslant \mu$. Therefore, by L.2 with $m=3$ and using P.1.1,

$$2(n-1) \geqslant 3(\delta-\mu)=3(n+t-5-2|D|)=3(n+t-9).$$

This is a contradiction of the magnitude of n for $t \in \{5,6\}$. ■

PROPOSITION 16. $|D| \geqslant 10-t$, $t \in \{3,4,5,6\}$.

Proof. Suppose $|D| \leqslant 9-t$. Since, by P.15, $|D| \geqslant 3$, we may select $h_1, h_2, h_3 \in D$, $h_i \neq h_j$ $(i \neq j;\ i,j=1,2,3)$. By P.12.1, $h_i h_j \in E_2$. Write $A_i = A_1(h_i)$, $\delta = n-7+t$, and $\mu = |D|-2$. Then, by P.11.4 and P.12.3, $|A_i \cup A_j| \geqslant \delta$, and by P.11.1 and P.12.3, $|A_i \cap A_j| \leqslant \mu$ $(i \neq u;\ i,j=1,2,3)$. Therefore, by L.2 with $m=3$ and using P.1.1,

$$2(n-1) \geqslant 3(\delta-\mu) = 3(n+t-5-|D|) \geqslant 3(n+2t-14).$$

This is a contradiction of the magnitude of n. ■

Notation. Let x be any real number. Then $\lfloor x \rfloor$ and $\lceil x \rceil$ denote respectively the lower and upper integer part of x.

Proposition 17. *Let* $D^* = \left\{ d \in D \colon |X_1(d)| \geqslant \left\lceil \frac{t-2}{2} \right\rceil \right\}$, $t \in \{3,4,5,6\}$. *Then* $|D^*| \geqslant |D|-1$.

Proof. Suppose $|D^*| \leqslant |D|-2$. Choose $d_1, d_2 \in D \setminus D^*$. Then, by P.12.1 and P.12.3,

$$t-2 \leqslant \theta(d_1,d_2) \leqslant |X_1(d_1)| + |X_1(d_2)| \leqslant 2\left(\left\lceil \frac{t-2}{2} \right\rceil - 1 \right).$$

This leads to a contradiction. ■

Notation. Write $x = \left| \{ d \in D \colon |X_1(d)| = 1 \} \right|$ and $y = \left| \{ d \in D \colon |X_1(d)| = 0 \} \right|$. Then $0 \leqslant x \leqslant |D|$ and, from P.17, $0 \leqslant y \leqslant 1$. Recall that in general $|X_1(d)| \leqslant 2$.

Proposition 18. $|D| \leqslant \lfloor (2n+t-|A|+x+2y-2)/3 \rfloor$.

Proof. Let $E_1(D,X)$ denote the set of red edges with exactly one end vertex in D and exactly one end vertex in X. Then, since $|D_1(x)| \leqslant 1$ for all $x \in X$ (P.1.4),

$$|E_1(D,X)| \leqslant |X| = 2n+t-|A|-|D|-2. \tag{24}$$

However, since $|X_1(d)| \leq 2$,

$$|E_1(D, X)| = 2|D| - (x + 2y). \tag{25}$$

The result now follows from (24) and (25). ■

NOTATION. Let $m = 10 - t$ $(t \in \{3,4,5,6\})$. Then, by P.16, $3 \leq m \leq |D|$. Choose m distinct elements $d_1, d_2, \ldots, d_m \in D$ so that $\sum_{i=1}^{m} |X_1(d_i)|$ is as small as possible. Write $\mathcal{D}(m) = \{d_1, d_2, \ldots, d_m\}$. Suppose $\mathcal{D}(m)$ contains exactly a elements d with $|X_1(d)| = 1$ and exactly b elements d with $|X_1(d)| = 0$. Then, by definition, $0 \leq a \leq x \leq |D|$ and $0 \leq b \leq y \leq 1$, $a + b \leq m$. Let $\mathcal{P}(m)$ denote the set of $\binom{m}{2}$ unordered pairs $\{i, j\}$, $i \neq j$, chosen from the set $\{1, 2, \ldots, m\}$. Write

$$\Lambda(m) = \sum_{\mathcal{P}(m)} \theta(d_i, d_j).$$

PROPOSITION 19. $\Lambda(m) \leq 4\binom{m}{2} - (m-1)(a + 2b)$.

Proof. $|X_1(d)| \in \{0, 1, 2\}$ for all $d \in \mathcal{D}(m)$, and $\theta(d_i, d_j) \leq |X_1(d_i)| + |X_1(d_j)|$ for all $\{i, j\} \in \mathcal{P}(m)$. Hence

$$\Lambda(m) \leq 2\binom{a}{2} + 4\binom{m-(a+b)}{2} + 3a[m - (a+b)] + ab + 2b[m - (a+b)].$$

The result now follows. ■

PROPOSITION 20. $a + b < 10 - t$.

Proof. Write $A_i = A_2(d_i)$, $i = 1, 2, \ldots, m$ $(m = 10 - t)$. Let $\delta = |A| - |D| + 2$. Then, by P.11.2 and P.12.3, $|A_i \cup A_j| \geq \delta$, and by P.11.3, $|A_i \cap A_j| \leq |A| - n - t + 3 + \theta(d_i, d_j)$ for all $\{i, j\} \in \mathcal{P}(m)$. Then, by L.1,

$$2(9-t)|A| \geq (10-t)(9-t)(|A| - |D| + 2) - 2(8-t)\Big(\sum_{\mathcal{P}(m)} (|A| - n - t + 3) + \Lambda(m)\Big). \tag{26}$$

Using P.19 and (26) gives

$$2(9-t)|A| \geqslant (10-t)(9-t)(|A|-|D|+2) \\ -2(8-t)\left[\sum_{\mathcal{P}(m)} (|A|-n-t+3) \right. \\ \left. +4\binom{10-t}{2} - (9-t)(a+2b)\right]. \tag{27}$$

Now assume that $a+b=m=10-t$. From (27) and P.18 we obtain (after some simplification)

$$|A|[6+(10-t)(20-3t)] \\ \geqslant (10-t)[8-2n-t-(x+2y)-3(8-t)(5-n-t)]. \tag{28}$$

Now by P.12.1, $\langle D \rangle$ is a blue complete graph, so that $|D| \leqslant n+1$, since $\langle E_2 \rangle \not\supseteq B_n$. Then from P.17 we have $x+2y \leqslant |D|+1 \leqslant n+2$. Using this in (28) gives

$$|A|[6+(10-t)(20-3t)] \\ \geqslant (10-t)[n(21-3t)+(5-t)(3t-23)+1]. \tag{29}$$

Putting $t=4$ in (29) and using the fact that $n \geqslant 23$ gives the estimate $|A| > n-2$, so by P.1.1 we have $|A| = n-1$. Similarly putting $t=3$ in (29) and using $n \geqslant 38$ gives $|A| > n-2$, so that $|A| = n-1$ by P.1.1. Since now, from P.18, $|D| \leqslant (n+t)/2$ for $t=3,4$, we get that $x+2y \leqslant |D|+1 \leqslant (n+t+2)/2$; and now putting this in (28) for each of the cases $t=4$, $t=3$ gives $|A| > n-1$, which is a contradiction.

This leaves the cases $t=5,6$. These are easy because of P.17, which implies that for these values of t, we get $a+b \leqslant 1 < 4 \leqslant 10-t$. ■

Proof of Theorem 3(ii). By P.20 and the minimality condition imposed on $\mathcal{D}(m)$ $(m=10-t)$ it follows that $a=x$ and $y=b$. Write $\theta=a+2b$. From (27) (see P.20) and P.18, we have

$$2|A| \geqslant (10-t)\left(|A|+2-\left\lfloor \frac{2n+t-|A|+\theta-2}{3} \right\rfloor\right) \\ -(8-t)[(10-t)(|A|-n-t+3)+r(10-t)-2\theta]. \tag{30}$$

Collecting the terms involving $|A|$ on the left hand side of this inequality, we see that the coefficient of $|A|$ is at least $(9-t)(8-t)-\frac{1}{3}$, which, since $t \leqslant 6$, is positive. Therefore, using P.1.1 and (30),

$$2(n-1) \geqslant (10-t)\left(n+1-\left\lfloor \frac{n+t+\theta-1}{3} \right\rfloor\right) \\ -[8-t)((10-t)(6-t)-2\theta]. \tag{31}$$

Hence

$$6(n-1) \geqslant (10-t)(2n-t-\theta+4) \\ -3(8-t)[(10-t)(6-t)-2\theta]. \tag{32}$$

Hence

$$2(7-t)n \leqslant -6+(10-t)(t-7)(3t-20)+(5t-38)\theta. \tag{33}$$

When $t \in \{3,5,6\}$, (33) gives a contradiction of the magnitude of n. However, when $t=4$ we obtain only that $n \leqslant 23$. Now suppose $t=4$ and $n=23$. From (33) we have

$$132 \leqslant 6n \leqslant 138-18\theta.$$

Hence $\theta=0$. Put $n=23$, $t=4$, and $\theta=0$ in Equation (31) to obtain a contradiction. This is the final contradiction. ■

REMARKS.

(i) We do not know how sharp the bounds are in Theorem 3. It is true that $r(B_2, B_n)=2n+6$ when $n=11$, and $r(B_2, B_n)=2n+3$ $(n \geqslant 38)$.

Suppose $r(B_2, B_n)=2n+4$ when $n=37$. Then there exists a coloring (E_1, E_2) of K_{2n+3} such that $\langle E_1 \rangle \not\supseteq B_2$ and $\langle E_2 \rangle \not\subseteq B_n$. Write $G=\langle E_2 \rangle$. If G is strongly regular, we shall call the coloring a strongly regular coloring. In this case G must have parameters (77,50,36,26), which is impossible by the integrality condition. Of course this says very little about the existence of a general coloring (E_1, E_2) of K_{2n+3}. Again if $r(B_2, B_n)=2n+5$ when $n=21$ then any strongly regular coloring of K_{2n+4} would imply the existence of a strongly regular graph G with parameters (46,29,20,15). Once more, these parameters do not satisfy the integrality condition. ■

(ii) If $r(B_2, B_n)=2n+5$ when $n=22$, then there exists no strongly regular coloring of K_{48}. We can give some information about any coloring of

K_{48} such that $\langle E_1 \rangle \not\supseteq B_2$ and $\langle E_2 \rangle \not\supseteq B_{22}$. As in the proof of Theorem 3(ii), put $t=4$, $\theta=0$, $n=22$ in (30) to obtain $|A|=21$. From (27), $|D| \geq 8$, and from P.18, $|D| \leq 8$. Hence $|D|=8$ and $|B|+|C|=17$. From (26), $\Lambda(m) \geq 60$. But since $\theta(d_i, d_j) \leq 4$ for all $\{i,j\} \in \mathcal{P}(m)$, this means $\Lambda(m)=60$. In particular $\theta(d_i, d_j)=4$ for all such i, j. This implies $|C_1(d)|=1$ for all $d \in D$, and $|C_1(d_i \cap d_j)|=0$. This establishes a bijection between C and D, so that $|C|=8$ and so $|B|=9$. Furthermore $\langle D\rangle, \langle C\rangle, \langle B\rangle$ are blue complete graphs, and the only red edges from C to D, and from D to B, are matchings of size 8. ■

The authors are even more than usual indebted to the referee for his extremely helpful comments and for his correction of an error in the proof of Theorem 3. The first author would like to thank John Sheehan and the University of Aberdeen for their hospitality during the fall semester of 1980.

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Received 24 August 1981; revised 7 January 1982