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STABLE TORSION RADICALS OVER NOETHERIAN RINGS

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Gabriel introduced the notion of a stable torsion radical in [6]. Recall that the torsion radical τ is stable if any essential extension of a τ-torsion module is τ-torsion. Equivalently, τ is stable if the topology it defines on any submodule $N \subseteq M$ coincides with the subspace topology induced from M. Damiano and Papp [5] have shown that τ is stable if and only if the quotient functor $Q_\tau : R\text{-Mod} \to R\text{-Mod}/\tau$ preserves essential monomorphisms. It is well known that the Goldie torsion radical defined in terms of the singular submodule is always stable. Gabriel showed that an ideal I of the ring R has the left Artin–Rees property if and only if the torsion radical defined by powers of I is stable. In addition, he showed that any torsion radical over a commutative Noetherian ring is stable.

One application of this theory is to the study of essential extensions of Artinian modules, and more generally, to the study of essential extensions of modules of Krull dimension α. The torsion theoretic methods provide a strong connection between the class of torsion modules (generated in the first case by all modules of finite length) and the associated filter of left ideals. Gabriel used the theory of stable torsion radicals to show that if R is left Noetherian and finitely generated (as a module) over its center, then any finitely generated essential extension of an Artinian module is again Artinian. Chamarie and Hudry [4] extended this result to left Noetherian rings integral over their centers. They also showed more generally that for this class of rings a finitely generated essential extension of a module with Krull dimension α again has Krull dimension α. This result holds for any left and right fully bounded Noetherian ring, as shown previously in the fundamental paper of Jategaonkar [11], using different techniques. These results were extended in [1] by giving necessary and sufficient conditions under which a torsion radical of a left fully bounded left Noetherian ring is stable.

Brown [2] studied the following weaker condition: if ${}_RN$ is an essential submodule of the finitely generated module ${}_RM$ and $R/\mathrm{Ann}(N)$ has Krull dimension α, then so does $R/\mathrm{Ann}(M)$. His work motivates the introduction in this paper of the notion of a stably bounded torsion radical. This notion can be used to charac-

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terize a certain class of rings which has been studied by Jategaonkar [13]. A ring will be called left fully stably bounded if it satisfies Jategaonkar's condition (‡). For this class of rings the characterization of stability given in [1] can be extended to characterize stably bounded torsion radicals. (For left FBN rings a torsion radical is stable if and only if it is stably bounded.)

After establishing some basic torsion theoretic results, a number of applications will be given. For Noetherian fully stably bounded rings, a semiprime ideal is localizable if and only if the associated torsion radicals are stably bounded (both on the left and the right). This and other conditions equivalent to localizability are given in Theorem 8. Specializing to the torsion radicals defined by Krull dimension, Theorem 9 characterizes left Noetherian rings which are smooth in the sense of Brown [2] (i.e. if ${}_RN$ is an essential submodule of the finitely generated module ${}_RM$ and $R/\mathrm{Ann}(N)$ has Krull dimension α, then so does $R/\mathrm{Ann}(M)$.) In particular, any left smooth left Noetherian ring is left fully stably bounded. The theorem can be applied easily to give Brown's sufficient condition for smoothness [2, Theorem 3.2]. It is shown in Theorem 10 that a Noetherian, weakly K-symmetric ring is smooth if and only if it is K-symmetric and fully stably bounded. It should be noted that there are still no examples of Noetherian rings which fail to be K-symmetric, so it may be reasonable to conjecture that the hypotheses involving symmetry of Krull dimension can be dropped.

Throughout the paper, R will be assumed to be an associative ring with identity element. Furthermore, it will be assumed to be left Noetherian. The categories of unital left R-modules and unital right R-modules will be denoted by R-Mod and Mod-R, respectively. The injective envelope of a module ${}_RM$ will be denoted by $E(M)$.

The book by Stenström [15] will be used as a basic reference for facts regarding torsion radicals. If τ is a torsion radical (equivalently, τ is the torsion functor defined by an hereditary torsion theory), then a module ${}_RM$ is said to be τ-torsion if $\tau(M)=M$ and τ-torsionfree if $\tau(M)=(0)$; a submodule $N\subseteq M$ is said to be τ-dense if M/N is τ-torsion, and τ-closed if M/N is τ-torsionfree. The associated filter of τ-dense left ideals determines τ, since $\tau(M)=\{m\in M \mid Dm=(0)$ for some τ-dense left ideal $D\}$. Using this characterization of τ, it is easy to see tha a prime ideal of R must be either τ-closed or τ-dense.

Stenström [15, p. 150] calls a torsion radical τ bounded if the associated topology has a basis of two-sided ideals. This is equivalent to the condition that if ${}_RM$ is finitely generated and τ-torsion, then $R/\mathrm{Ann}(M)$ is τ-torsion, since if $M=\sum_{i=1}^n Rm_i$, then $\bigcap_{i=1}^n \mathrm{Ann}(m_i)$ must contain a τ-dense ideal, which implies that $\mathrm{Ann}(M)$ is τ-dense. It can be shown that a left Noetherian ring is left fully bounded if and only if every torsion radical of R is bounded. Recall that R is said to be left fully bounded (abbreviated FBN when R is left Noetherian) if for each prime ideal P of R, each essential left ideal of R/P contains a nonzero ideal. It is important to note [15, Chapter VII, Proposition 3.5] that a bounded torsion radical τ of a left Noetherian ring is completely determined by the set of τ-dense prime ideals, since a left ideal is τ-dense if and only if it contains a (finite) product of τ-dense prime ideals.

Given a torsion radical τ, there is an associated bounded torsion radical $\hat{\tau}$ with its filter defined as follows: a left ideal $D \subseteq R$ is $\hat{\tau}$-dense if D contains a τ-dense ideal. Thus every $\hat{\tau}$-dense left ideal is τ-dense, and so $\hat{\tau} \leq \tau$ since $\hat{\tau}(M) \subseteq \tau(M)$ for any module ${}_RM$. Note that a finitely generated module ${}_RM$ is $\hat{\tau}$-torsion if and only if $\text{Ann}(M)$ is τ-dense. Thus, in the following definition, τ is stably bounded if and only if $\hat{\tau}$ is a stable torsion radical.

Definition. Let τ be a torsion radical of R-mod. Then τ is said to be stably bounded if $R/\text{Ann}(M)$ is τ-torsion for any finitely generated module ${}_RM$ which has an essential submodule $N \subseteq M$ such that $R/\text{Ann}(N)$ is τ-torsion.

An ideal $I \subseteq R$ is said to have the left Artin–Rees property if for each left ideal A of R there exists a positive integer n such that $I^n \cap A \subseteq IA$. Since R is left Noetherian, the powers of I generate a torsion radical τ by defining $\tau(M) = \{m \in M \mid I^n M = (0) \text{ for some } n > 0\}$ for any module ${}_RM$. As observed by Gabriel [6], the ideal I has the left AR-property if and only if the associated torsion radical τ, which coincides with $\hat{\tau}$ since it is bounded, is stable. It follows immediately that every torsion radical of R-Mod is stably bounded if and only if every ideal of R satisfies the left AR-property, since [15, Chapter VII, Theorem 4.4] shows that every bounded torsion radical of R-Mod is stable if and only if R has the left AR-property.

The following theorem characterizes stably bounded torsion radicals via conditions on the prime ideals of R. If τ is a torsion radical of R-Mod, then for any ideal I of R the 'restriction' τ^* of τ to R/I-Mod is defined in the obvious way by letting $\tau^*(M) = \tau(M)$ for any R/I-module M.

Theorem 1. *Let R be a left Noetherian ring, and let τ be a torsion radical of R-*Mod*. Then τ is stably bounded if and only if for each τ-closed prime ideal P the restriction of τ to R/P-*Mod *is stably bounded and for each τ-dense prime ideal Q there exists a τ-dense ideal D such that $PD \subseteq QP$.*

Proof. If τ is stably bounded, then it is clear that the restriction of τ to R/I-Mod is stably bounded for any ideal I of R. Given any ideal T and any τ-dense ideal I, let A be a left ideal maximal in the set $\{X \subseteq R \mid X \cap T = IT\}$. Then $\text{Ann}((T+A)/A)$ is τ-dense since it contains I, and so by assumption $D = \text{Ann}(R/A)$ is τ-dense since R/A is an essential extension of $(T+A)/A$. Thus $D \cap T \subseteq A \cap T = IT$, and so $TD \subseteq IT$.

Conversely, assume that ${}_RM$ is a finitely generated essential extension of N and $\text{Ann}(N)$ is a τ-dense ideal. If $\text{Ann}(M)$ is not τ-dense, then since R is left Noetherian there exists an ideal I maximal in the set

$$\{\text{Ann}(Y) \mid Y \text{ is a submodule of } M \text{ and } \text{Ann}(Y) \text{ is not } \tau\text{-dense}\},$$

with $I = \mathrm{Ann}(X)$ for some submodule X of M. The crux of the proof is to show that I is a prime ideal, and this follows exactly as in the proof of [1, Theorem 1.1]. Thus $X \supseteq X \cap N$ is an essential extension of R/I-modules such that $\mathrm{Ann}(X \cap N)$ is τ-dense and $I = \mathrm{Ann}(X)$ is τ-closed. This contradicts the assumption that τ is stably bounded on R/I-Mod, and so $\mathrm{Ann}(M)$ must be τ-dense. □

It is desirable to find a general class of rings in which it is not necessary to verify the condition on relative stability over certain prime factor rings. This is provided by a class of rings introduced by Jategaonker [13]. In that paper a left Noetherian ring R is said to satisfy condition (‡) on the left if for any prime ideal P and any finitely generated module ${}_RM$ with $P \subseteq \mathrm{Ann}(M)$, $P \neq \mathrm{Ann}(M)$ if there exists an essential submodule $N \subseteq M$ with $P \neq \mathrm{Ann}(N)$.

If R is a prime left Goldie ring, let γ denote the Goldie torsion radical, defined for any module ${}_RM$ by $\gamma(M) = \{m \in M \mid cm = 0$ for some regular element $c \in R\} = \{m \in M \mid \mathrm{Ann}(m)$ is essential in $R\}$. Since a two-sided ideal is essential as a left ideal if and only if it is nonzero, γ is stably bounded if and only if $\mathrm{Ann}(M) \neq 0$ for any finitely generated module ${}_RM$ which has an essential submodule $N \subseteq M$ such that $\mathrm{Ann}(N) \neq (0)$. It seems reasonable to give the following definition, which is equivalent to Jategaonkar's condition (‡).

Definition. The left Noetherian ring R is said to be left fully stably bounded if for each prime factor ring R/P the Goldie torsion radical of R/P-Mod is stably bounded.

Since a left Noetherian ring is left fully bounded if and only if for each prime factor ring R/P the Goldie torsion radical is bounded, it follows that any left FBN is left fully stably bounded, since the Goldie torsion radical is always stable. Jategaonkar [13] has shown that HNP rings with enough invertible ideals, enveloping algebras of solvable Lie algebras, and group rings of polycyclic-by-finite groups over commutative rings are fully stably bounded.

If R is left Noetherian, then the torsion radical τ is stably bounded if for each τ-dense prime ideal P there exists a stably bounded torsion radical σ such that $\sigma \leq \tau$ and P is σ-dense. To show this it is sufficient to consider a finitely generated uniform module ${}_RU$ which contains a submodule N such that $\mathrm{Ann}(N)$ is τ-dense. If P is an associated prime ideal of N, then P is τ-dense since $\mathrm{Ann}(N) \subseteq P$. By assumption, P is σ-dense for a stably bounded torsion radical σ such that $\sigma \leq \tau$. Since σ is stably bounded, $\mathrm{Ann}(U)$ must be σ-dense, and this implies that $\mathrm{Ann}(U)$ is τ-dense since $\sigma \leq \tau$.

The left Noetherian ring R is said to be a left poly-AR ring if for any pair of prime ideals $Q \subsetneq P$ there exists an ideal I, with $Q \subsetneq I \subseteq P$, such that I/Q has the left AR-property in R/Q. If γ is the Goldie torsion radical of R/Q-Mod, then for any γ-dense prime ideal P the ideal I given by the definition defines a stably bounded torsion radical of R/Q-Mod for which P is dense. It then follows from the general remark in the previous paragraph that γ is stably bounded. This provides another proof of [13, Proposition 4.1], which shows that a left Noetherian, left poly-AR ring is left fully stably bounded.

Theorem 2. *If R is left Noetherian and left fully stably bounded, then the following conditions are equivalent for any torsion radical τ of R*-Mod:

(1) *The torsion radical τ is stably bounded.*

(2) *For any ideal T and any τ-dense ideal I of R, there exists a τ-dense ideal D such that $TD \subseteq IT$.*

(3) *For any τ-closed prime ideal P and any τ-dense prime ideal Q, there exists a τ-dense ideal D such that $PD \subseteq QP$.*

Proof. The proof differs from that of Theorem 1 only in the last step, in obtaining a contradiction from the following: ${}_RX$ is a finitely generated faithful module over the prime ring R/I, the submodule $X \cap N$ is essential, $\mathrm{Ann}(X \cap N)$ is τ-dense and I is τ-closed. By assumption R is left fully stably bounded, which forces $\mathrm{Ann}(X \cap N) = I$, a contradiction since no proper ideal can be both τ-closed and τ-dense. □

In practice condition (3) of Theorem 2 is often easy to verify. The following corollary is an extension of a result of Brown and Lenagan [3], who obtained the result for left fully bounded rings. Note that a more general result holds. Assume that R is left fully stably bounded and that every left primitive factor ring is Artinian. Then every finitely generated essential extension of a simple left R-module has finite length if and only if for each maximal ideal Q and each non-maximal prime ideal P there exists an ideal D such that R/D is left Artinian and $PD \subseteq QP$. The Jacobson radical of R will be denoted by $J(R)$.

Corollary 3. *If R is left fully stably bounded and each simple left R-module is finitely generated over a central subring of R, then $\bigcap_{n=1}^{\infty} J(R)^n = (0)$.*

Proof. It is well known that $\bigcap_{n=1}^{\infty} J(R)^n = (0)$ if every finitely generated essential extension of a simple left R-module has finite length. This is equivalent to the condition that τ is stable, where τ is the torsion radical generated by all modules of finite length. (By definition a left ideal D is τ-dense if and only if R/D has finite length.) If ${}_RS$ is a simple module generated by $x_1, \ldots, x_n$ over a central subring of R, then $\mathrm{Ann}(S) = \bigcap_{i=1}^{n} \mathrm{Ann}(x_i)$, and from this it follows that if ${}_RM$ has finite length then so does $R/\mathrm{Ann}(M)$. This observation shows that τ is bounded, and so verifying condition (3) of Theorem 2 will imply that τ is stable.

Let P and Q be prime ideals such that R/Q is Artinian, and consider the ideal P/QP. This is Artinian as a left ideal of R/QP, and so it follows from Lemma 3 of [3] that R/D is Artinian for $D = \mathrm{r}(P/QP) = \{a \in R \mid Pa \subseteq QP\}$. Then $PD \subseteq QP$ and the required condition holds. □

Corollary 4. *Let R be left Noetherian and left fully stably bounded, and let I be an ideal of R. Then I has the left AR-property if and only if for each prime ideal P with $P \not\supseteq I$ there exists an integer $n>0$ such that $PI^n \subseteq IP$.*

Proof. If I has the left AR-property, then for some n we have $PI^n \subseteq I^n \cap P \subseteq IP$. Conversely, let τ denote the torsion radical defined by powers of I. Then a prime ideal P is τ-closed if and only if $P \not\supseteq I$ and τ-dense if and only if $I \subseteq P$. If the stated condition holds, then Theorem 2 shows that τ is stably bounded, and so I has the left AR-property. □

The torsion radical τ of R-Mod will be called symmetric if there exists a torsion radical σ of Mod-R such that $\tau({}_RB/A) = \sigma(B/A_R)$ for any ideals $A \subseteq B \subseteq R$. In this case the pair (τ, σ) will be called a symmetric pair. This notion was introduced by Jategaonkar in [9], where the term 'biradical' was used. The term 'symmetric' is consistent with the terminology which has since been adopted for Krull dimension. Jategaonkar observed that over any Noetherian ring, a left and right Ore set defines a symmetric pair of torsion radicals. It should be noted that over a left Noetherian left fully stably bounded ring any symmetric torsion radical is stably bounded, since [1, Proposition 1.4] shows that the conditions of Theorem 2 are satisfied.

The pair of torsion radicals (τ, σ) of R-Mod and Mod-R, respectively, will be called weakly symmetric if the set of τ-dense prime ideals coincides with the set of σ-dense prime ideals. For a left Noetherian ring this occurs if and only if the set of all τ-dense ideals coincides with the set of all σ-dense ideals. For any ideal I the torsion radicals on the left and right defined by powers of I form a weakly symmetric pair, which need not be symmetric. In addition, if S is a semiprime ideal of R and $\mathscr{C}(S)$ denotes the set of regular elements of R, then the torsion radicals defined by $\mathscr{C}(S)$ are weakly symmetric since an ideal is $\mathscr{C}(S)$-dense if and only if it contains an element of $\mathscr{C}(S)$.

Finally, let T be an ideal of R. The torsion radical τ of R-Mod is said to be invariant under T if for each τ-dense left ideal D, the left ideal TD is τ-dense in T. If τ is invariant under every ideal of R, then it is said to be ideal invariant. This definition, together with some comments, can be found in [1].

Lemma 5. *Let R be a Noetherian ring, and let τ and σ be torsion radicals of R-Mod and Mod-R, respectively. Then (τ, σ) is a symmetric pair if and only if (τ, σ) is a weakly symmetric pair and both $\hat{\tau}$ and $\hat{\sigma}$ are ideal invariant.*

Proof. First, assume that (τ, σ) is a weakly symmetric pair such that $\hat{\tau}$ and $\hat{\sigma}$ are ideal invariant. Let $A \subseteq B$ be ideals of R such that ${}_RB/A$ is τ-torsion, and let $D = \mathrm{Ann}({}_RB/A)$. Since B/A is finitely generated as a right R-module, the generators can be used to embed ${}_RR/D$ in a finite direct sum of copies of ${}_RB/A$, which shows that D is τ-dense. By assumption, D is σ-dense, but then B/DB_R must be $\hat{\sigma}$-torsion since $\hat{\sigma}$ is ideal invariant. It follows that B/A_R must be σ-torsion since it is a homomorphic image of B/DB_R and $\hat{\sigma} \leq \sigma$. Similarly, it can be shown that if B/A is σ-torsion, then ${}_RB/A$ is τ-torsion.

Conversely, assume that (τ, σ) is a symmetric pair. Let D be a $\hat{\tau}$-dense left ideal, and let T be any ideal of R. Then D contains a τ-dense ideal D', and to show that $\hat{\tau}$ is ideal invariant it suffices to show that ${}_RT/TD'$ is $\hat{\tau}$-torsion. By assumption D' is σ-dense, and this implies that T/TD'_R is σ-torsion, so by assumption ${}_RT/TD'$ is τ-torsion. As in the first part of the proof, T/TD' is $\hat{\tau}$-torsion since T/TD' is finitely generated on the right. Similarly, it can be shown that $\hat{\sigma}$ is ideal invariant. □

Proposition 6. *Let R be a Noetherian, fully stably bounded ring, and let* (τ, σ) *be a weakly symmetric pair of torsion radicals. Then the following conditions are equivalent:*

(1) *Both* τ *and* σ *are stably bounded.*

(2) *Both* $\hat{\tau}$ *and* $\hat{\sigma}$ *are ideal invariant.*

(3) *The pair* (τ, σ) *is symmetric.*

Proof. Let T be any ideal of R and let I be a τ-dense ideal. Since T/TI is finitely generated on the right, it follows as in the previous lemma that TI is τ-dense in T if and only if $\mathrm{Ann}({}_RT/TI)$ is τ-dense. Using the fact that (τ, σ) is a weakly symetric pair, this shows that $\hat{\tau}$ is ideal invariant if and only if σ is stably bounded. It follows easily that $(1) \Leftrightarrow (2)$, and then $(2) \Leftrightarrow (3)$ by the preceding lemma. □

If S is a semiprime ideal of the left Noetherian ring R, then the set $\mathscr{C}(S)$ of elements regular modulo S defines a torsion radical τ of R-Mod as follows: $\tau(M) = \{m \in M \mid \text{for each } r \in R \text{ there exists } c \in \mathscr{C}(S) \text{ such that } crm = 0\}$, for any module ${}_RM$. It is well-known that τ is the largest torsion radical for which R/S is torsionfree, and so it coincides with the torsion radical cogenerated by $E(R/S)$. A τ-torsion module will be said to be $\mathscr{C}(S)$-torsion, etc. The ideal S is said to be left localizable if $\mathscr{C}(S)$ is a left Ore set. The following lemma holds in a more general setting than stated. The proof requires only that S is finitely generated as a right ideal and that the second layer of $E(R/S)$ is tame (see [13] for the necessary definitions).

Lemma 7. *Let R be a Noetherian, fully stably bounded ring, and let S be a semiprime ideal of R. Then S is left localizable if and only if for each* $\mathscr{C}(S)$*-dense prime ideal P there exists a* $\mathscr{C}(S)$*-dense ideal D such that* $DS \subseteq SP$.

Proof. Assume that S is left localizable. As noted in [1] this is equivalent to the condition that S is invariant under the torsion radical defined by $\mathscr{C}(S)$. Thus if I is a $\mathscr{C}(S)$-dense ideal, then SI is $\mathscr{C}(S)$-dense in S. Since R is Noetherian, S is finitely generated as a right ideal, and it follows that $D = \mathrm{Ann}(S/SI)$ must be $\mathscr{C}(S)$-dense.

Conversely, assume that the given condition holds. Using the localization criterion in [10], to show that S is left localizable it suffices to show that $Q =$

$\{x \in E(R/S) \mid Sx = (0)\}$ is a $\mathscr{C}(S)$-closed submodule of $E = E(R/S)$. Suppose not, and assume that U is a finitely generated submodule of E such that U/Q is $\mathscr{C}(S)$-torsion. By choosing a maximal annihilator ideal of U/Q it is possible to assume without loss of generality that U/Q is a uniform module and $P = \mathrm{Ann}(U/Q)$ is a prime ideal. It follows from [13, Lemma 2.4] that U/Q is isomorphic to a submodule of $E(R/P)$, and therefore P must be $\mathscr{C}(S)$-dense. By assumption there exists a $\mathscr{C}(S)$-dense ideal D such that $DS \subseteq SP$. Thus $DSU \subseteq SPU \subseteq SQ \subseteq (0)$, and so $SU = (0)$ since E is $\mathscr{C}(S)$-torsionfree. This shows that $U \subseteq Q$, a contradiction, so E/Q is $\mathscr{C}(S)$-torsionfree. □

Theorem 8. *Let R be a Noetherian, fully stably bounded ring, and let S be a semiprime ideal of R. Then the following conditions are equivalent for the torsion radicals τ in R-*Mod *and τ' in* Mod-R *defined by $\mathscr{C}(S)$:*

(1) *The ideal S is localizable.*

(2) *The torsion radicals τ and τ' are stably bounded.*

(3) *The bounded torsion radicals $\hat{\tau}$ and $\hat{\tau}'$ are ideal invariant.*

(4) *The pair (τ, τ') is symmetric.*

Proof. It follows from Proposition 6 that conditions (2), (3) and (4) are equivalent. Since the conditions hold on both sides, condition (3) is equivalent to the conditions necessary in Lemma 7 to show that S is left and right localizable. Thus condition (1) is equivalent to condition (3). □

The Krull dimension of a module $_RM$ will be denoted by $|M|$. It is defined recursively, as follows: if M is Artinian, then $|M| = 0$; if α is an ordinal and $|M| \not< \alpha$, then $|M| = \alpha$ if there is no infinite descending chain $M = M_0 \supsetneq M_1 \supsetneq \cdots$ of submodules M_i such that $|M_{i-1}/M_i| \not< \alpha$ for $i = 1, 2, \ldots$. Further details can be found in [8]. It can be shown that any Noetherian module has Krull dimension. For a given ordinal α, the set of left ideals $D \subseteq R$ such that $|R/D| < \alpha$ defines a topology, and the associated torsion radical of R-Mod will be denoted by τ_α.

Using standard terminology (see [7], for example), a Noetherian ring R is called K-symmetric if $|_RB/A| = |B/A_R|$ for any ideals $A \subseteq B$ of R. Equivalently, for each ordinal α, the corresponding torsion radicals τ_α of R-Mod and τ'_α of Mod-R form a symmetric pair. Similarly, R is called weakly K-symmetric if $|_RR/P| = |R/P_R|$ for all prime ideals P of R, that is, if for each α the pair $(\tau_\alpha, \tau'_\alpha)$ is weakly symmetric. The ring R is said to be ideal invariant on the left if for any ideals T and I of R, $|_RR/I| < \alpha$ implies $|_RT/TI| < \alpha$. This is just the condition that $\hat{\tau}_\alpha$ is ideal invariant for each α. Finally, Brown [2] calls the ring R left smooth if τ_α is stably bounded, for each α.

Theorem 9. *Let R be a left Noetherian ring. Then R is left smooth if and only if R is left fully stably bounded and for each pair of ideals I, T such that $|_RR/I| < \alpha$ there exists an ideal D such that $|_RR/D| < \alpha$ and $TD \subseteq IT$.*

Proof. Since R is left smooth if and only if τ_α is stably bounded for each ordinal α, the theorem will follow from Theorem 2 after it is shown that a left smooth ring is left fully stably bounded. Given a prime ideal P and a finitely generated module $_RM$ with $P = \mathrm{Ann}(M)$, let $|_RR/P| = \alpha$. If N is an essential submodule of M, let Q be an associated prime ideal of N. If $P \subsetneq Q$, then $|_RR/Q| < \alpha$, and so Q is τ_α-dense. By assumption τ_α is stably bounded, and so this gives a contradiction. It follows that $P = \mathrm{Ann}(N)$ for any essential submodule of M, and so R is left fully stably bounded. □

Theorem 10. *Let R be a Noetherian ring. Then R is smooth and weakly K-symmetric if and only if R is fully stably bounded and K-symmetric.*

Proof. If R is smooth, then it is fully stably bounded. It then follows from Proposition 6 that R *is* K-symmetric.

Conversely, Proposition 6 shows that if R is fully stably bounded and K-symmetric, then R is smooth. □

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References

[1] J.A. Beachy, Stable torsion radicals over FBN rings, J. Pure Appl. Algebra 24 (1982) 235-244.
[2] K.A. Brown, Module extensions over Noetherian rings, J. Algebra 69 (1981) 247-260.
[3] K.A. Brown and T.H. Lenagan, A note on Jacobson's conjecture for right Noetherian rings, Glasgow Math. J. 23 (1982) 7-8.
[4] M. Chamarie and A. Hudry, Anneaux Noethériens à droit entiers sur un sous-anneau de leur centre, Comm. Algebra 6 (1978) 203-222.
[5] R.F. Damiano and Z. Papp, On consequences of stability, Commun. Algebra 9 (1981) 747-764.
[6] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90 (1962) 323-448.
[7] A. Goldie and G. Krause, Artinian quotient rings of ideal invariant Noetherian rings, J. Algebra 63 (1980) 374-388.
[8] R. Gordon and J.C. Robson, Krull dimension, Mem. Amer. Math. Soc. 133 (1973).
[9] A.V. Jategaonkar, Injective modules and classical localization in Noetherian rings, Bull. Amer. Math. Soc. 79 (1973) 152-157.
[10] A.V. Jategaonkar, Injective modules and localization in non-commutative Noetherian rings, Trans. Amer. Math. Soc. 190 (1974) 109-123.
[11] A.V. Jategaonkar, Jacobson's conjecture and modules over fully bounded Noetherian rins, J. Algebra 30 (1974) 103-121.
[12] A.V. Jategaonkar, Noetherian bimodules, primary decomposition, and Jacobson's conjecture, J. Algebra 71 (1982) 379-400.

[13] A.V. Jategaonkar, Solvable Lie algebras, polycyclic-by-finite groups, and bimodule Krull dimension, Comm. Algebra 10 (1982) 19–69.
[14] T.H. Lenagan, Artinian ideals in Noetherian rings, Proc. Amer. Math. Soc. 51 (1975) 499–500.
[15] B. Stenström, Rings of Quotients (Springer, Berlin, 1975).