ELSEVIER

Journal of Pure and Applied Algebra 130 (1998) 49–84

JOURNAL OF
PURE AND
APPLIED ALGEBRA

Spreads and the symmetric topos II

M. Bunge*, J. Funk

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke St. W., Montréal, QC, Canada H3A 2K6

Communicated by P.J. Freyd; received 7 October 1996

Dedicated to Bill Lawrence on his 60th birthday

Abstract

We continue our investigation of distributions on a topos, the symmetric monad, and complete spreads. Our tool is the pure dense/complete spread, or comprehensive, factorization of a geometric morphism. We obtain, among other results, a characterization of the algebras for the symmetric monad that translates into a "Waelbroeck" theorem for toposes. We also investigate the connection between complete spreads and the fundamental group of a topos. © 1998 Elsevier Science B.V. All rights reserved.

AMS Classification: 18B25; 18C15; 54B30; 18A32

1. Introduction

The purpose of this paper is to pursue our previous investigations [6] of distributions on a topos [14, 15], of the symmetric monad [4, 5], and of complete spreads [8, 10].

The pure dense/complete spread factorization of a geometric morphism [6] is important to our discussion. (For the purposes of this paper, we need only consider the case when the domain topos of the geometric morphism is locally connected [2, 16].) We begin in Section 1 with a review of pure dense geometric morphisms and complete spreads, and we record some facts about them which will be used throughout the paper. Then in Section 2 the pure dense/complete spread factorization is interepreted as a "comprehensive factorization", and compared with the work of Street and Walters [18]. Closely related to comprehension is the notion of the density of a distribution (see [15]). In Section 3, we analyze density in terms of complete spreads. We use

* Corresponding author. E-mail: bunge@triples.math.mcgill.ca.

PII: S0022-4049(97)00086-8

this analysis to show in Section 7 that a locally connected topos whose connected components functor preserves products is simply connected.

A probability distribution is a distribution which preserves the terminal object. In Section 4, we give an explicit construction of the probability distribution classifier in terms of the finite non-empty connected limit completion of a small category, thus answering a question left open in [6].

A theorem of Waelbroeck [19] says that the space of distributions (with compact support) on a smooth manifold is the free vector space on the manifold, relative to a certain class of test spaces (the "b-spaces"). Lawvere [14] has suggested that this result could be interpreted in other suitable contexts. Kock [12] has considered the question for an object in a ringed topos. The question for toposes has already been partly answered with the result that the symmetric topos is part of a monad, denoted by M, on the 2-category of toposes [6]. Further, in Sections 5 and 6, we show that for a topos $\mathscr{E}$, the unit $\delta_{\mathscr{E}} : \mathscr{E} \to M\mathscr{E}$ (the "Dirac delta") has a universal property which can be interpreted as a "Waelbroeck" theorem for toposes (in two forms). This amounts to characterizing M-algebras as "linear" toposes, so that for a topos $\mathscr{E}$, $M\mathscr{E}$ becomes the free linear topos on $\mathscr{E}$. Similar characterizations are obtained for the algebras of the lower bagdomain monad [11], and of the probability distribution classifier. We also show that if the carrier topos of an M-algebra is locally connected, then the topos has the stronger property of being totally connected.

One should expect that an arbitrary locally constant object in a topos be a complete spread [8, 10, Proposition 5.18]. We begin Section 7 by showing that this is indeed the case. It is then natural to consider the question of whether every local homeomorphism which is a complete spread is locally constant. We present some preliminary investigations of this question. In particular, we show that in a connected presheaf topos, the locally constant objects coincide with the complete spread objects. Thus, the fundamental group [1, 7] of a connected presheaf topos coincides with its full subcategory of "clopens". The general case remains an open question.

We will often use the terms "pullback" and "comma object" in a 2-categorical context. We always mean bi-pullback and bi-comma object.

1. Pure dense geometric morphisms and complete spreads

The notions of a pure dense geometric morphism and of a complete spread are used throughout this paper. These notions are defined relative to a base topos, for which we reserve the symbol $\mathscr{S}$. For a topos $\mathscr{G}$ over $\mathscr{S}$, its structure geometric morphism will be denoted by g, as part of an obvious general rule. We occasionally omit these labels altogether. If $\mathscr{G}$ is locally connected, we denote the left adjoint of g^* by $g_!$.

Let $\Omega_{\mathscr{S}}$ denote the subobject classifier in the base topos $\mathscr{S}$. A geometric morphism $\mathscr{E} \xrightarrow{\varphi} \mathscr{F}$ over $\mathscr{S}$ is said to be *pure* (respectively, *dense*) [6] if the unit $\Omega_{\mathscr{S}} \to \varphi_* \varphi^* \Omega_{\mathscr{S}}$ is an epimorphism (respectively, a monomorphism). We say that φ_* *preserves* $\mathscr{S}$*-coproducts* if for every $I \in \mathscr{S}$, the unit $I \to \varphi_* \varphi^* I$ is an isomorphism. We have the following.

Proposition 1.1. *Let $\mathscr{E}$ and $\mathscr{F}$ be locally connected toposes. Then the following are equivalent for a geometric morphism $\mathscr{E} \xrightarrow{\varphi} \mathscr{F}$ over $\mathscr{S}$.*

1. *φ_* preserves $\mathscr{S}$-coproducts.*
2. *φ is pure and dense.*
3. *The canonical morphism $e_! \cdot \varphi^* \to f_!$ is an isomorphism.*

Proof. This follows from [6], Proposition 2.7. □

Proposition 1.2. *Consider a commutative triangle*

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "p"] \arrow[dr, "q"'] & & \mathscr{F} \\ & \mathscr{G} \arrow[ur, "r"'] & \end{tikzcd}$$

of toposes and geometric morphisms (over a base topos $\mathscr{S}$).

1. *If r_* is faithful and p is pure, then q is pure.*
2. *If r_* is full and p is dense, then q is dense.*
3. *If r is an inclusion and p_* preserves $\mathscr{S}$-coproducts, then so does q_*.*

Proof. Let $I \in \mathscr{S}$. We have a commutative square

$$\begin{tikzcd} r^*f^*I \arrow[r, "r^*\eta"] \arrow[d] & r^*r_*q_*q^*g^*I \arrow[d, "\varepsilon"] \\ g^*I \arrow[r, "\kappa"'] & q_*q^*g^*I \end{tikzcd}$$

where $\eta : f^*I \to p_*p^*f^*I \cong r_*q_*q^*g^*I$ is the unit of $p^* \dashv p_*$, and κ is the unit of $q^* \dashv q_*$. The morphism ε is the counit of $r^* \dashv r_*$, and the left-hand morphism is an isomorphism. For 1, we take I to be $\Omega_{\mathscr{S}}$. We are assuming that η is an epimorphism. If r_* is faithful, ε is an epimorphism, so that κ is also. For 2, again with I equal to $\Omega_{\mathscr{S}}$, ε is a (split) monomorphism, so that if η is a monomorphism, κ is as well. The preservation of $\mathscr{S}$-coproducts by the direct image functors means that the counits η and κ are isomorphisms, so 3 is clear. □

It is well-known [2, 16] that locally connected geometric morphisms are stable under pullback. We use this fact in the proofs of the next two results.

Proposition 1.3. *Pure dense geometric morphisms and geometric morphisms whose direct image functors preserve $\mathscr{S}$-coproducts are stable under pullback along locally connected geometric morphisms.*

Proof. Let

$$\begin{tikzcd} \mathscr{D} \arrow[r, "q"] \arrow[d, "\beta"'] & \mathscr{G} \arrow[d, "\gamma"] \\ \mathscr{E} \arrow[r, "p"'] & \mathscr{F} \end{tikzcd}$$

be a pullback with γ locally connected and p pure dense. Then β is locally connected and we have $\beta_! q^* \cong p^* \gamma_!$, or equivalently $\gamma^* p_* \cong q_* \beta^*$. By assumption, the unit $f^* \Omega \to p_* p^* f^* \Omega$ is an isomorphism, where Ω is the subobject classifier of $\mathscr{S}$. Apply γ^* to this isomorphism to yield

$$g^* \Omega \to \gamma^* p_* p^* f^* \Omega \cong q_* \beta^* e^* \Omega \cong q_* d^* \Omega \cong q_* q^* g^* \Omega.$$

This isomorphism is equal to the unit $g^* \Omega \to q_* q^* g^* \Omega$. Thus, q is pure dense.

The above argument works also for geometric morphisms whose direct image functors preserve $\mathscr{S}$-coproducts. □

We recall [6] that a locally connected topos is said to be *totally connected* (respectively, to have *totally connected components*) if its connected components functor is left exact (respectively, preserves pullbacks). A locally connected topos is totally connected if and only if it has totally connected components and is also connected.

Proposition 1.4. *The class of geometric morphisms having totally connected components is closed under composition and under arbitrary pullback.*

Proof. Closure under composition is clear. For pullback stability, let $\mathscr{D} \xrightarrow{d} \mathscr{S}$ have totally connected components. Assume we are pulling back along a geometric morphism γ. We can factor d as $\mathscr{D} \xrightarrow{\hat{d}} \mathscr{S}/I \xrightarrow{I} \mathscr{S}$ such that $\hat{d}$ is locally connected and $\hat{d}_!$ is left exact. The object I is $d_! 1$. There is $\mathscr{S}/I \xrightarrow{p} \mathscr{D}$ over $\mathscr{S}$ which is right adjoint to $\hat{d}$ ($p^* = \hat{d}_!$). We know that the pullback geometric morphism $\gamma^\# d$ is locally connected, but we must show that $(\gamma^\# d)_!$ preserves pullbacks. We have $\gamma^\# \hat{d} \cong \widehat{\gamma^\# d}$, and $\gamma^* I \cong (\gamma^\# d)_!(1)$. A pullback of an adjoint pair is again an adjoint pair. This applies to $\hat{d} \dashv p$, so that $(\gamma^\# \hat{d})_!$ is isomorphic to $(\gamma^\# p)^*$, whence left exact. Thus, $(\gamma^\# d)_!$ preserves pullbacks. □

The notion of a *complete spread* is the complement of pure dense; on the one hand, complete spreads and pure dense geometric morphisms are a factorization system on $\mathbf{Top}_{\mathscr{S}}$ [6, Theorem 2.15], and on the other, the notion of complete spread is to pure dense as closed is to dense in topology. For this paper, the reader may take the following as the definition of a complete spread. Let $\mathscr{F}$ be a topos presented over $\mathscr{S}$ by a site with underlying category $\mathbf{C}$. Let $\mathscr{E} \xrightarrow{\varphi} \mathscr{F}$ denote an arbitrary geometric morphism over $\mathscr{S}$, where $\mathscr{E}$ is locally connected (we consider only complete spreads with locally connected domains). Consider the discrete opfibration $\mathbf{D} \xrightarrow{D} \mathbf{C}$ corresponding to the functor $\mathbf{C} \to \mathscr{F} \xrightarrow{e_!\varphi^*} \mathscr{S}$. A typical object of $\mathbf{D}$ is a pair (c,x), $x \in e_!(\varphi^* c)$. We say φ is a complete spread if the following square is a pullback in $\mathbf{Top}_{\mathscr{S}}$.

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "v"] \arrow[d, "\varphi"'] & \mathscr{S}^{\mathbf{D}^{op}} \arrow[d, "u"] \\
\mathscr{F} \arrow[r] & \mathscr{S}^{\mathbf{C}^{op}}
\end{tikzcd}$$

In this diagram, u denotes the geometric morphism induced by D, and v is that induced by the flat functor $V : \mathbf{D} \to \mathscr{E}$ such that $V(c,x)$ is the pullback

$$\begin{tikzcd}
V(C,x) \arrow[r] \arrow[d] & \varphi^* c \arrow[d] \\
1 \arrow[r, "e^* x"'] & e^* e_!(\varphi^* e).
\end{tikzcd}$$

The right-hand vertical arrow is the unit of $e_! \dashv e^*$. This definition of complete spread does not depend on the choice of site for the codomain topos [6, Proposition 2.11].

Proposition 1.5. *Complete spreads are stable under pullback along local homeomorphisms.*

Proof. Let $\mathscr{D} \xrightarrow{\varphi} \mathscr{F}$ denote a complete spread (over $\mathscr{S}$) with $\mathscr{D}$ locally connected. Fix an object $X \in \mathscr{F}$. We wish to show that in the pullback

$$\begin{tikzcd}
\mathscr{D}/\varphi^* X \arrow[r] \arrow[d, "\varphi_X"'] & \mathscr{D} \arrow[d, "\varphi"] \\
\mathscr{F}/X \arrow[r] & \mathscr{F}
\end{tikzcd}$$

the geometric morphism φ_X is a complete spread. Let $(\mathbf{C},J)$ be an $\mathscr{S}$-site for $\mathscr{F}$, and regard X as a discrete fibration $\mathbf{X} \xrightarrow{X} \mathbf{C}$ so that $\mathbf{X}$ can be taken as (the underlying category of) a site for $\mathscr{F}/X$. Let $\mathbf{D} \xrightarrow{D} \mathbf{C}$ denote the discrete opfibration corresponding to the cosheaf $d_! \cdot \varphi^* \cdot \varepsilon$ in the following commutative diagram:

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\varepsilon'"] \arrow[d, "X"'] & \mathscr{F}/X \arrow[r, "\varphi_X{}^*"] \arrow[d, "\Sigma"] & \mathscr{D}/\varphi^* X \arrow[d, "\Sigma"] \arrow[dr] & \\
\mathbf{C} \arrow[r, "\varepsilon"'] & \mathscr{F} \arrow[r, "\varphi^*"'] & \mathscr{D} \arrow[r, "d_!"'] & \mathscr{S}
\end{tikzcd}$$

The functor ε' sends an object $(c,s) \in \mathbf{X}$ to the object $\varepsilon c \xrightarrow{s} X$. Let $P : \mathbf{X} \to \mathscr{S}$ denote the cosheaf corresponding to φ_X. The above diagram shows that P is isomorphic to $d_! \cdot \phi^* \cdot \varepsilon$ composed with $\mathbf{X} \xrightarrow{X} \mathbf{C}$, so that the discrete opfibration corresponding to P is the pullback

$$\begin{tikzcd}
\mathbf{P} \arrow[r] \arrow[d, "P"'] & \mathbf{D} \arrow[d, "D"] \\
\mathbf{X} \arrow[r, "X"'] & \mathbf{C}
\end{tikzcd}$$

of functors and small categories. Our aim is to show that

$$\begin{tikzcd}
\mathscr{D}/\varphi^* X \arrow[r] \arrow[d, "\varphi_X"'] & \mathscr{S}^{\mathbf{P}^{\mathrm{op}}} \arrow[d, "v"] \\
\mathscr{F}/X \arrow[r] & \mathscr{S}^{\mathbf{X}^{\mathrm{op}}}
\end{tikzcd}$$

is a pullback of toposes, where v is the geometric morphism induced by P. This square appears as the back face in the following commutative cube. The geometric morphism

u is that induced by D.

$$\begin{tikzcd}
& \mathscr{D}/\varphi^*X \arrow[rr] \arrow[dl] \arrow[dd] & & \mathscr{S}^{\mathbf{D}^{op}}/u^*X \arrow[dl] \arrow[dd] \\
\mathscr{D} \arrow[rr, crossing over] \arrow[dd, "\varphi"'] & & \mathscr{S}^{\mathbf{D}^{op}} & \\
& \mathscr{F}/X \arrow[rr] \arrow[dl] & & \mathscr{S}^{\mathbf{C}^{op}}/X \arrow[dl] \\
\mathscr{F} \arrow[rr] & & \mathscr{S}^{\mathbf{C}^{op}} \arrow[from=uu, crossing over, "u"] &
\end{tikzcd}$$

The left, right, bottom and front faces are pullbacks, and therefore, so is the back face. This concludes the proof. □

2. Comprehension for distributions on toposes

The *comprehensive factorization* considered by Street and Walters [18] arises from the comprehension schema [13] for a certain fibration, which we will denote by P, on the category of small categories **Cat**. For a small category **B**, the fiber $P(\mathbf{B})$ is defined to be the category of $\mathscr{S}$-valued functors on **B**. Transition along a functor F between small categories is given by composition with F. This fibration has a terminal $\top$ (for each **B**, the terminal functor $\top_{\mathbf{B}}$), and also Σ (left Kan extension). That it has *comprehension* is a consequence of the fact that to an $\mathscr{S}$-valued functor there corresponds a discrete opfibration, as this constitutes the right adjoint to the functor

$$\mathbf{Cat}/\mathbf{B} \to P(\mathbf{B})$$

assigning to a functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$, the functor $\Sigma_F(\top_{\mathbf{A}})$. Let k denote the functor $\Sigma_F(\top_{\mathbf{A}})$, and $\{k\} : \mathbf{D} \to \mathbf{B}$ the discrete opfibration associated with k. Then the unit $F \to \{k\}$ is an *initial* functor $\mathbf{A} \xrightarrow{Q} \mathbf{D}$, unique with the property that $\{k\} \cdot Q \cong F$ and that $\Sigma_Q(\top_{\mathbf{A}}) \cong \top_{\mathbf{D}}$. Thus, the comprehensive factorization of F is its initial/discrete opfibration factorization, hence the name.

The pure dense/complete spread factorization [6] of a geometric morphism may also be regarded as "comprehensive". Consider the fibration of distributions on the category $\mathbf{lcTop}_{\mathscr{S}}$ of locally connected toposes over $\mathscr{S}$. We denote this fibration by D. For a topos $\mathscr{E}$, the fiber $D\mathscr{E}$ is defined to be the category of ($\mathscr{S}$-valued) distributions on $\mathscr{E}$. For a geometric morphism $\mathscr{E} \xrightarrow{\phi} \mathscr{F}$, composition with φ^* gives Σ_ϕ. The transition functor $D\varphi$ is by definition the right adjoint of Σ_φ (coherence for the $D\phi$ is satisfied since it holds for the Σ_φ). The functors $D\varphi$ exist since the fiber categories are locally presentable (e.g., they appear as a suitable coinverter [5]), and since the Σ_φ are cocontinuous).

The fibration D has a terminal $\top$. For each locally connected topos $\mathscr{E}$, the terminal distribution $\top_{\mathscr{E}}$ is the connected components functor $e_!$. The comprehension schema requires for each locally connected topos $\mathscr{F}$, the existence of a right adjoint of the functor

$$\mathbf{lcTop}_{\mathscr{S}}/\mathscr{F} \to D\mathscr{F} \tag{1}$$

assigning to a geometric morphism $\mathscr{E} \xrightarrow{\phi} \mathscr{F}$ over $\mathscr{S}$, the distribution $\Sigma_{\phi}(\top_{\mathscr{E}})$ (this distribution is equal to $e_! \cdot \varphi^*$). The functor (1) was considered in [6], and it was there observed that its right adjoint $\{\cdot\}$ exists. This right adjoint associates with a distribution μ on $\mathscr{F}$, a complete spread $\{\mu\} : \mathscr{D} \to \mathscr{F}$. In this way, we obtain the pure dense/complete spread factorization of a geometric morphism between locally connected toposes as a comprehensive factorization; a geometric morphism $\mathscr{E} \xrightarrow{\varphi} \mathscr{F}$ is factored as $\{\mu\} \cdot \eta$, where μ denotes $\Sigma_{\varphi}(\top_{\mathscr{E}})$, and $\mathscr{E} \xrightarrow{\eta} \mathscr{D}$ is the pure dense unit. We summarize these remarks in the following:

Theorem 2.1. *The fibration of distributions on locally connected toposes over $\mathscr{S}$ satisfies the comprehension schema. The corresponding comprehensive factorization of a geometric morphism is its pure dense/complete spread factorization.*

Of course, the concepts of complete spread and pure dense apply to presheaf toposes [6, Example 2.16, 2]. We have the following analysis of this case. We recall some facts, which are well-known, about the *Karoubi envelope* of a small category. If $\mathbf{C}$ denotes a small category, we shall denote its Karoubi envelope by $K\mathbf{C}$. Its objects are the idempotents $c \xrightarrow{f} c$ of $\mathbf{C}$, and its morphisms $f \xrightarrow{\varphi} g$ are commutative diagrams as follows (which compose in an obvious manner).

$$\begin{tikzcd} c \arrow[r, "\varphi"] \arrow[d, "f"'] \arrow[dr, "\varphi"'] & d \arrow[d, "g"] \\ c \arrow[r, "\varphi"'] & d \end{tikzcd}$$

There is a fully faithful functor $\mathbf{C} \to K\mathbf{C}$ which sends an object c to the idempotent 1_c. This functor induces by pullback an equivalence

$$\mathscr{S}^{K\mathbf{C}} \simeq \mathscr{S}^{\mathbf{C}}. \tag{2}$$

The pseudo-inverse of this equivalence associates with a discrete opfibration $\mathbf{P} \xrightarrow{P} \mathbf{C}$ the functor KP. KP is a discrete opfibration on $K\mathbf{C}$, and in fact, for an idempotent e in C,

$$KP(e) = \{x \mid Pe(x) = x\} = \{x \mid \exists y \; Pe(y) = x\}.$$

The equivalence (2) says in particular that for a discrete opfibration $P:\mathbf{P}\to\mathbf{C}$, the commutative square

$$\begin{tikzcd} \mathbf{P} \arrow[r] \arrow[d, "P"'] & K\mathbf{P} \arrow[d, "KP"] \\ \mathbf{C} \arrow[r] & K\mathbf{C} \end{tikzcd}$$

of functors and small categories is a pullback.

Since $K(\mathbf{C}^{\mathrm{op}})\simeq(K\mathbf{C})^{\mathrm{op}}$, the statements of the previous paragraph can be similarly expressed for discrete fibrations.

The Karoubi envelope is in general not enough to recover the small category from its topos of presheaves (it is if $\mathcal{S}$ has 'choice'). The general case requires the *stack completion* of a category [3]. Consider the canonical embedding of a small category into its stack completion (which is not always small).

$$\mathbf{D}\to\widetilde{\mathbf{D}} \tag{3}$$

This functor is a weak equivalence, and since $\mathcal{S}$ is an $\mathcal{S}$-stack, composition with it induces an equivalence

$$\mathcal{S}^{\widetilde{\mathbf{D}}}\simeq\mathcal{S}^{\mathbf{D}}$$

of $\mathcal{S}$-indexed categories. For a discrete opfibration $\mathbf{P}\xrightarrow{P}\mathbf{D}$, the functor $\widetilde{P}$ is a discrete opfibration, and we have a pullback

$$\begin{tikzcd} \mathbf{P} \arrow[r] \arrow[d, "P"'] & \widetilde{K\mathbf{P}} \arrow[d, "\widetilde{P}"] \\ \mathbf{D} \arrow[r] & \widetilde{D} \end{tikzcd}$$

of functors and categories. This pullback combines with the previous one to give the following pullback, for a discrete opfibration P.

$$\begin{tikzcd} \mathbf{P} \arrow[r] \arrow[d, "P"'] & \widetilde{K\mathbf{P}} \arrow[d, "\widetilde{KP}"] \\ \mathbf{C} \arrow[r] & \widetilde{K\mathbf{C}} \end{tikzcd}$$

Since $\mathscr{S}^{\mathrm{op}}$ is an $\mathscr{S}$-stack, composition with the opposite of (3) gives an equivalence

$$\mathscr{S}^{(\widetilde{\mathbf{D}})^{\mathrm{op}}} \simeq \mathscr{S}^{\mathbf{D}^{\mathrm{op}}},$$

and it follows that the previous square is a pullback for discrete fibrations as well.

Finally, we recall [3] that the geometric morphism $\mathscr{S}^{\mathbf{A}^{\mathrm{op}}} \to \mathscr{S}^{\mathbf{B}^{\mathrm{op}}}$ induced by a functor $F : \mathbf{A} \to \mathbf{B}$ is an equivalence if and only if $\widetilde{KF}$ is an equivalence.

We are now ready to examine the pure dense/complete spread factorization of a geometric morphism between presheaf toposes induced by a small functor F in terms of the comprehensive factorization of F. We include the case when F is a discrete fibration, as it is of interest to us in Section 7.

Proposition 2.2. *Consider the geometric morphism $f : \mathscr{S}^{\mathbf{A}^{\mathrm{op}}} \to \mathscr{S}^{\mathbf{B}^{\mathrm{op}}}$ induced by a functor $F : \mathbf{A} \to \mathbf{B}$. Then we have the following.*

1. *If F is a discrete opfibration, then f is a complete spread.*
2. *If f is a complete spread, then $\widetilde{KF}$ is a discrete opfibration.*
3. *If f is a complete spread and if F is a discrete fibration, then F is a discrete opfibration.*
4. *f is pure dense if and only if F is initial.*

Proof. 1. This follows immediately from the definition of a complete spread given in Section 1.

2. The assumption that f is a complete spread amounts to the assumption that the geometric morphism q induced by the initial factor Q of the comprehensive factorization of F is an equivalence, as depicted in the following diagrams.

$$\begin{tikzcd} \mathbf{A} \arrow[r, "Q"] \arrow[dr, "F"'] & \mathbf{D} \arrow[d, "D"] \\ & \mathbf{B} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{S}^{\mathbf{A}^{\mathrm{op}}} \arrow[r, "q"] \arrow[dr, "f"'] & \mathscr{S}^{\mathbf{D}^{\mathrm{op}}} \arrow[d, "d"] \\ & \mathscr{S}^{\mathbf{B}^{\mathrm{op}}} \end{tikzcd}$$

The functor D is a discrete opfibration. Then $\widetilde{KD}$ is a discrete opfibration, and $\widetilde{KQ}$ is an equivalence. Thus, $\widetilde{KF}$ must also be a discrete opfibration.

3. By *2*, $\widetilde{KF}$ is a discrete opfibration. Under the assumption that F is a discrete fibration we have the pullback square

$$\begin{tikzcd} \mathbf{A} \arrow[r] \arrow[d, "F"'] & \widetilde{K\mathbf{A}} \arrow[d, "\widetilde{KF}"] \\ \mathbf{B} \arrow[r] & \widetilde{K\mathbf{B}} \end{tikzcd}$$

so that in this case F must be a discrete opfibration.

4. If f is pure dense, then the geometric morphism d is an equivalence, whence $\widetilde{KD}$ is also an equivalence. Then D is an equivalence as it is the pullback of $\widetilde{KD}$ along $\mathbf{B} \to \widetilde{K\mathbf{B}}$, and so F is initial. Conversely, if F is initial, then F^{op} is final, so that for a presheaf P on $\mathbf{B}$,

$$\varinjlim (f^*P) \cong \varinjlim (P \cdot F^{\mathrm{op}}) \cong \varinjlim P.$$

This says that f is pure dense (Proposition 1.1). $\square$

Remark 2.3. A discrete fibration F on a small category $\mathbf{B}$ is at the same time a discrete opfibration if and only if for each morphism $c \xrightarrow{x} d$ in $\mathbf{B}$, Fx is an isomorphism. We leave it to the reader to verify that if $\mathbf{B}$ is connected, such presheaves are exactly the locally constant presheaves on $\mathbf{B}$ (see 7.9).

3. The density of a distribution

In the classical theory of distributions, functions act on measures producing new measures. The Radon–Nikodym derivative "inverts" this action; it can be thought of as the density of a measure relative to another. Lawvere [15] explains:

> It is in terms of such "action" (or "multiplication") of intensive quantities on extensive quantities that the role of the former as "ratios" of the latter must be understood.

In this section we describe the density of a distribution on a topos. We will do this by considering a natural enrichment over sheaves that distributions possess. Then we pass to an action of sheaves on distributions by copowers which the density inverts.

We denote the category of distributions on a topos $\mathscr{E}$ by $D\mathscr{E}$, as in Section 2. However, here we consider $D\mathscr{E}$ as having the structure of an $\mathscr{E}$-indexed category, i.e., of a fibration over $\mathscr{E}$. For X an object of $\mathscr{E}$, we take for the fibre $(D\mathscr{E})^X$ the category $D(\mathscr{E}/X)$. The transition functor $(D\mathscr{E})^Y \to (D\mathscr{E})^X$ along a morphism $X \to Y$ of $\mathscr{E}$ for this fibration is given by composition with the coproduct functor $\mathscr{E}/X \xrightarrow{\Sigma} \mathscr{E}/Y$. A detailed explanation of this structure on $D\mathscr{E}$, and a proof of the following result are given in [9].

Theorem 3.1. *$D\mathscr{E}$ is cocomplete and locally small as an $\mathscr{E}$-indexed category.*

In particular, $D\mathscr{E}$ has $\mathscr{E}$-copowers. The copower of a distribution μ by an object X of $\mathscr{E}$ is a distribution which we denote by $X.\mu$. The following natural bijection expresses the universal property of the copower $X.\mu$.

$$\frac{\text{morphisms } \mu \cdot \Sigma_X \to \lambda \cdot \Sigma_X \text{ in } (D\mathscr{E})^X}{\text{morphisms } X.\mu \to \lambda \text{ in } (D\mathscr{E})^1}.$$

We have

$$X.\mu(F) = \mu(X \times F), \quad F \in \mathscr{E}.$$

The local smallness of $D\mathscr{E}$ means that for distributions μ and λ, there is an object $D\mathscr{E}(\mu,\lambda)$ in $\mathscr{E}$ which represents morphisms in $D\mathscr{E}$, i.e., for which there is a natural bijection

$$\frac{\text{morphisms } X \to D\mathscr{E}(\mu,\lambda) \text{ in } \mathscr{E}}{\text{morphisms } \mu\cdot\Sigma_X \to \lambda\cdot\Sigma_X \text{ in } (D\mathscr{E})^X}.$$

If a site is given for $\mathscr{E}$, say with underlying category $\mathbf{C}$, then the sheaf $D\mathscr{E}(\mu,\lambda)$ is given by

$$D\mathscr{E}(\mu,\lambda)(c) = \{\text{natural transformations } c.\mu \to \lambda\}, \quad c \in \mathbf{C}.$$

For a fixed distribution μ, we have adjoint functors

$$\mathscr{E} \to D\mathscr{E};\ X \mapsto X.\mu, \quad D\mathscr{E} \to \mathscr{E};\ \lambda \mapsto D\mathscr{E}(\mu,\lambda).$$

Definition 3.2. Let $\mathscr{E}$ denote a locally connected topos with components functor $e_!$. For a distribution λ on $\mathscr{E}$, we will refer to the object $D\mathscr{E}(e_!,\lambda)$ of $\mathscr{E}$ as the *density of* λ. We denote this object by $\mathbf{d}\lambda$. The functor $\mathbf{d}$ is the right adjoint of $X \mapsto X.e_!$.

Example 3.3. The density of $e_!$ is the terminal sheaf; $\mathbf{d}e_! = 1$. This is because $e_!$ is the terminal distribution.

We next relate the density of a distribution to the comprehensive, or pure dense/complete spread, factorization of a geometric morphism. We will call the complete spread of this factorization of a geometric morphism (with locally connected domain) its *associated complete spread*. The domain of the associated complete spread is locally connected.

Proposition 3.4. *For $X \in \mathscr{E}$, the complete spread corresponding to the copower $X.e_!$ is the associated complete spread of $\mathscr{E}/X \to \mathscr{E}$, as in the following diagram:*

$$\begin{tikzcd} \mathscr{E}/X \arrow[r, "p"] \arrow[dr] & \mathscr{D} \arrow[d, "X.e_!"] \\ & \mathscr{E} \end{tikzcd}$$

The pure dense p can be factored by the unit $X \to \mathbf{d}(X.e_!)$ as indicated below.

$$\begin{tikzcd} \mathscr{E}/X \arrow[r] \arrow[dr] & \mathscr{E}/\mathbf{d}(X.e_!) \arrow[r] \arrow[d] & \mathscr{D} \arrow[dl, "X.e_!"] \\ & \mathscr{E} & \end{tikzcd}$$

Proof. Let $\mathscr{D} \xrightarrow{\varphi} \mathscr{E}$ denote the associated complete spread of $\mathscr{E}/X \to \mathscr{E}$. Let λ denote an arbitrary distribution on $\mathscr{E}$ with corresponding complete spread $\mathscr{T} \xrightarrow{\lambda} \mathscr{E}$. We have the natural bijection

$$\frac{\text{morphisms } e_! \cdot \Sigma_X \to \lambda \cdot \Sigma_X \text{ in } (D\mathscr{E})^X}{\text{morphisms } X.e_! \to \lambda \text{ in } (D\mathscr{E})^1.}$$

The complete spread over $\mathscr{E}/X$ corresponding to the distribution $\lambda \cdot \Sigma_X$ on $\mathscr{E}/X$ is the pullback $\mathscr{T}/\lambda^*X \to \mathscr{E}/X$ (Proposition 1.5). The one corresponding to $e_! \cdot \Sigma_X$ is the identity geometric morphism on $\mathscr{E}/X$. Geometric morphisms $\mathscr{D} \to \mathscr{T}$ over $\mathscr{E}$ are in bijection with geometric morphisms $\mathscr{E}/X \to \mathscr{T}$ over $\mathscr{E}$, whence geometric morphisms $\mathscr{E}/X \to \mathscr{T}/\lambda^*X$ over $\mathscr{E}/X$. Thus, φ has the universal property required of the complete spread corresponding to the copower $X.e_!$. $\square$

One way to say that a geometric morphism $\mathscr{Y} \to \mathscr{E}$ is *localic* is to say that there is a locale Y in $\mathscr{E}$ whose topos of sheaves $Sh_{\mathscr{E}}(Y)$ is equivalent to $\mathscr{Y}$ over $\mathscr{E}$. We may consider the *associated local homeomorphism*

$$\begin{tikzcd} \mathscr{E}/E \arrow[r] \arrow[dr] & \mathscr{Y} \arrow[d, "\gamma"] \\ & \mathscr{E} \end{tikzcd}$$

of a localic geometric morphism γ over $\mathscr{E}$. The object E is the object of points of the locale Y, i.e.,

$$E = \{\mathscr{E}\text{-internal frame morphisms } \mathscr{O}(Y) \to \Omega_{\mathscr{E}}\},$$

where $\Omega_{\mathscr{E}}$ is the subobject classifier of $\mathscr{E}$, and where $\mathscr{O}(Y)$ denotes the frame in $\mathscr{E}$ which corresponds to Y. We recall that a spread is a localic geometric morphism [6, Proposition 1.3].

Proposition 3.5. *Let $\mu \in D\mathscr{E}$. Then $\mathbf{d}\mu$ is isomorphic to the associated local homeomorphism of the complete spread corresponding to μ:*

$$\begin{tikzcd} \mathscr{E}/\mathbf{d}\mu \arrow[r] \arrow[dr] & \mathscr{T} \arrow[d, "\mu"] \\ & \mathscr{E} \end{tikzcd}$$

In terms of complete spreads over $\mathscr{E}$, the counit $(\mathbf{d}\mu).e_! \to \mu$ is the second horizontal morphism in the following diagram:

$$\begin{tikzcd} \mathscr{E}/\mathbf{d}\mu \arrow[r] \arrow[dr] & \mathscr{D} \arrow[r] \arrow[d, "\mathbf{d}\mu.e_!"] & \mathscr{T} \arrow[dl, "\mu"] \\ & \mathscr{E} & \end{tikzcd}$$

Proof. Let $F \in \mathscr{E}$. To be shown is that morphisms $F \to \mathbf{d}\mu$ in $\mathscr{E}$ correspond to geometric morphisms $\mathscr{E}/F \to \mathscr{T}$ over $\mathscr{E}$. By definition, morphisms $F \to \mathbf{d}\mu$ correspond to morphisms $F.e_! \to \mu$ in $D\mathscr{E}$. By Proposition 3.4, these correspond to geometric morphisms from the associated complete spread of F to $\mathscr{T}$ over $\mathscr{E}$, whence to geometric morphisms $\mathscr{E}/F \to \mathscr{T}$ over $\mathscr{E}$. □

Corollary 3.6. *Let $X \in \mathscr{E}$. If $\mathscr{E}/X \to \mathscr{E}$ is a complete spread, then the unit $X \to \mathbf{d}(X.e_!)$ is an isomorphism.*

Proof. This follows from Propositions 3.4 and 3.5. □

4. Probability distributions and connected limits

A finite connected limit is one whose diagram is finite, *non-empty* and connected. Finite connected limits can be freely adjoined to an arbitrary small category. Let us denote by $\mathbf{C}^{\oplus}$ the finite connected limit completion of a small category C. We have the unit $\kappa : \mathbf{C} \to \mathbf{C}^{\oplus}$.

Lemma 4.1. *κ is a final functor.*

Proof. Let $\mathbf{D}_{\oplus}$ denote the finite connected *colimit* completion of a small category $\mathbf{D}$. $\mathbf{D}_{\oplus}$ can be constructed as the full subcategory of $\mathscr{S}^{\mathbf{D}^{\mathrm{op}}}$ determined by those presheaves which are finite connected colimits of representables. Then the canonical functor $\mathbf{D} \to \mathbf{D}_{\oplus}$ is an initial functor, so that, since $\mathbf{C}^{\oplus} = (\mathbf{C}^{\mathrm{op}}_{\oplus})^{\mathrm{op}}$, the functor κ is final. □

A *probability distribution* on a topos is a distribution which preserves the terminal object. The topos classifier of probability distributions on a topos $\mathscr{E}$ is denoted by $C\mathscr{E}$; the category of geometric morphisms $\mathscr{G} \to C\mathscr{E}$ is naturally equivalence to the category of probability distributions $\mathscr{E} \to \mathscr{G}$. The existence of $C\mathscr{E}$ can be established by constructing it as a subtopos of $M\mathscr{E}$ using a forcing topology [6, Theorem 3.15]. The following result gives a direct construction of $C\mathscr{E}$.

Theorem 4.2. *Let $\mathscr{E}$ denote an arbitrary topos presented by a site $(\mathbf{C},J)$, so that $M\mathscr{E}$ is presented by $(\mathbf{C}^\star,J^\star)$, where $\mathbf{C}^\star$ is the finite limit free completion of $\mathbf{C}$. Then $C\mathscr{E}$ can be constructed as the pullback*

$$\begin{tikzcd} C\mathscr{E} \arrow[r] \arrow[d] & M\mathscr{E} \arrow[d] \\ \mathscr{S}^{\mathbf{C}^{\oplus\,\mathrm{op}}} \arrow[r] & \mathscr{S}^{\mathbf{C}^{\star\,\mathrm{op}}} \end{tikzcd}$$

in $\mathbf{Top}_{\mathscr{S}}$, where the bottom geometric morphism is that induced by the unique factorization of δ through κ:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\mathscr{K}"] \arrow[dr, "\delta"'] & \mathbf{C}^{\oplus} \arrow[d] \\ & \mathbf{C}^{\star} \end{tikzcd}$$

Proof. We must show that for an arbitrary topos $\mathscr{G}$, the category of $\mathscr{G}$-valued probability distributions on $\mathscr{E}$ is equivalent to the category of cones

$$\begin{tikzcd} \mathscr{G} \arrow[r, "x"] \arrow[d, "k"'] & M\mathscr{E} \arrow[d] \\ \mathscr{S}^{\mathbf{C}^{\oplus\,\mathrm{op}}} \arrow[r] & \mathscr{S}^{\mathbf{C}^{\star\,\mathrm{op}}} \end{tikzcd}$$

by an equivalence which is natural in $\mathscr{G}$. Suppose we are given a $\mathscr{G}$-valued probability distribution. We have its corresponding geometric morphism $\mathscr{G} \xrightarrow{x} M\mathscr{E}$, and also its corresponding cosheaf $K : \mathbf{C} \to \mathscr{G}$. K has the property that $\varinjlim K \cong 1$. Then K lifts to a functor $K^{\oplus} : \mathbf{C}^{\oplus} \to \mathscr{G}$ which preserves finite connected limits, and which furthermore, since κ is final (Lemma 4.1), satisfies $\varinjlim K^{\oplus} \cong 1$. It follows that the left extension $k^* : \mathscr{S}^{\mathbf{C}^{\oplus\,\mathrm{op}}} \to \mathscr{G}$ of $K^{\oplus}$ preserves finite connected limits and also 1. Therefore, k^* is left exact, so that we have a geometric morphism k and a cone as above.

Conversely, a cone such as above gives a cosheaf $X : \mathbf{C} \to \mathscr{G}$ corresponding to x, and at the same time a flat functor $K : \mathbf{C}^{\oplus} \to \mathscr{G}$ corresponding to k. K satisfies $\varinjlim K \cong 1$, so that, again since κ is final, we have $\varinjlim (K \cdot \kappa) \cong 1$ also. The commutativity of the cone gives that X and $K \cdot \kappa$ coincide, so that the corresponding distribution is a probability distribution. $\square$

The following diagram (left) depicts the canonical factorizations of $\mathscr{E} \xrightarrow{\delta} M\mathscr{E}$ through the bag-domain $\mathscr{B}_L\mathscr{E}$, and through the probability distribution classifier $C\mathscr{E}$. In terms of freely adjoining finite limits to a site $\mathbf{C}$ for $\mathscr{E}$, it corresponds to the center diagram. The remaining diagram describes the distributions classified.

$$\begin{tikzcd}[column sep=small]
 & B_L\mathscr{E} \arrow[dr] & \\
\mathscr{E} \arrow[ur] \arrow[dr] & & M\mathscr{E} \\
 & C\mathscr{E} \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathbf{C}^{\times} \arrow[dr, "\text{equalizers}"] & \\
\mathbf{C} \arrow[ur, "\text{finite products}"] \arrow[dr, "\text{finite connected}"'] & & \mathbf{C}^{\star} \\
 & \mathbf{C}^{\oplus} \arrow[ur, "\text{finite products}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \text{partial points} \arrow[dr] & \\
\text{points} \arrow[ur] \arrow[dr] & & \text{distributions} \\
 & \text{probability distributions} \arrow[ur] &
\end{tikzcd}$$

5. The Waelbroeck theorem for toposes (first version)

Our goal in this section is to obtain a characterization of the algebras for the symmetric monad M on $\mathbf{Top}_{\mathscr{S}}$. We will here characterize M-algebras in terms of a certain cocompleteness property (Theorem 5.4 below), and then in Section 6 in terms of a "linear" structure (Theorem 6.4). We call these results Waelbroeck theorems for toposes, by analogy with [19].

One way to express the cocompleteness of an object in a bicategory is to ask for the existence of certain left Kan extensions. The following diagram depicts the left extension of a 1-cell p along another one φ:

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[dr, "p"'] & B \arrow[d, "\Sigma_\varphi p"] \arrow[dl, phantom, "\Rightarrow" near start] \\
 & X
\end{tikzcd}$$

We should not expect the existence of such extensions along every 1-cell φ; it is natural to require restrictions on φ. For instance, consider when A,B,X are categories and we wish to express the *small* cocompleteness of X. In this case we would restrict φ by requiring that for every $b \in B$, the comma category $\varphi \downarrow b$ be small. In addition to requiring the existence of left extensions, we should expect that the Beck–Chevalley condition (BCC) for comma objects hold. In the example of categories and small cocompleteness, it holds for the comma object

$$\begin{tikzcd}
\varphi \downarrow b \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & 1 \arrow[d, "b"] \\
A \arrow[r, "\varphi"'] & B\,.
\end{tikzcd}$$

The following definition expresses in this manner a cocompleteness property pertaining to an object in the bicategory $\mathbf{Top}_{\mathscr{S}}$. We take the class of morphisms φ to be the class of $\mathscr{S}$-essential geometric morphisms. We will use the fact [6, Theorem 3.6], first discovered by Pitts, that in a comma object

$$\begin{tikzcd} \mathscr{I} \arrow[r, "\psi"] \arrow[d] & \mathscr{K} \arrow[d] \\ \mathscr{G} \arrow[r, "\varphi"'] \arrow[ur, phantom, "\Rightarrow"] & \mathscr{H} \end{tikzcd}$$

in $\mathbf{Top}_{\mathscr{S}}$ in which φ is $\mathscr{S}$-essential, ψ is locally connected.

Definition 5.1. A topos $\mathscr{F}$ (over $\mathscr{S}$) will be said to be *cocomplete* (*for $\mathscr{S}$-essential geometric morphisms*) if it satisfies the following two conditions:

1. $\mathscr{F}$ admits left Kan extensions along essential geometric morphisms.

$$\begin{tikzcd} \mathscr{G} \arrow[r, "\varphi"] \arrow[dr, "p"'] & \mathscr{H} \arrow[d, "\Sigma_\varphi p"] \arrow[dl, phantom, "\Rightarrow"] \\ & \mathscr{F} \end{tikzcd}$$

   In other words, for every essential geometric morphism $\mathscr{G} \xrightarrow{\varphi} \mathscr{H}$, precomposition with φ has a left adjoint.

$$\varphi^{\#} : \mathbf{Top}_{\mathscr{S}}(\mathscr{H}, \mathscr{F}) \to \mathbf{Top}_{\mathscr{S}}(\mathscr{G}, \mathscr{F}), \qquad \Sigma_\varphi : \mathbf{Top}_{\mathscr{S}}(\mathscr{G}, \mathscr{F}) \to \mathbf{Top}_{\mathscr{S}}(\mathscr{H}, \mathscr{F})$$

2. The Beck–Chevalley condition for comma objects holds. This means that for every comma object

$$\begin{tikzcd} \mathscr{I} \arrow[r, "\psi"] \arrow[d, "\kappa"'] & \mathscr{K} \arrow[d, "\rho"] \\ \mathscr{G} \arrow[r, "\varphi"'] \arrow[ur, phantom, "\Rightarrow"] & \mathscr{H} \end{tikzcd}$$

   in $\mathbf{Top}_{\mathscr{S}}$ in which φ is essential, the canonical natural transformation

$$\Sigma_\psi \cdot \kappa^{\#} \Rightarrow \rho^{\#} \cdot \Sigma_\varphi$$

is an isomorphism. In other words, for any geometric morphism $\mathscr{G} \xrightarrow{p} \mathscr{F}$, the extension along ψ of $p \cdot \kappa$ is canonically isomorphic to ρ composed with the extension along φ of p.

A geometric morphism $\mathcal{F} \xrightarrow{\gamma} \mathcal{F}'$ between toposes which are cocomplete in the above sense will be said to be *cocontinuous* (*for essential geometric morphisms*) if for any essential geometric morphism $\mathcal{G} \xrightarrow{\varphi} \mathcal{H}$, and any "diagram" $\mathcal{G} \xrightarrow{p} \mathcal{F}$, the canonical morphism $\Sigma_\varphi(\gamma \cdot p) \Rightarrow \gamma \cdot \Sigma_\varphi p$ is an isomorphism.

$$
\begin{tikzcd}
\mathcal{G} \arrow[rr, "\varphi"] \arrow[dr, "p"'] & \arrow[d, phantom, "\Rightarrow"] & \mathcal{H} \arrow[dl, "\Sigma_\varphi p"] \arrow[dr, "\Sigma_\varphi(\gamma\cdot p)"] \arrow[d, phantom, "\Rightarrow"] & \\
& \mathcal{F} \arrow[rr, "\gamma"'] & {} & \mathcal{F}'
\end{tikzcd}
$$

We encourage the reader to review the fact [6, Theorem 3.1] that a topos $\mathcal{F}$ is an M-algebra if and only if $\delta : \mathcal{F} \to M\mathcal{F}$ has a left adjoint in $\mathbf{Top}_{\mathcal{S}}$.

Proposition 5.2. *M-algebras are cocomplete and M-homomorphisms are cocontinuous. For an M-algebra $M\mathcal{F} \xrightarrow{\sigma} \mathcal{F}$ and a geometric morphism $\mathcal{G} \xrightarrow{p} \mathcal{F}$, we have $\Sigma_\varphi p \cong \sigma \cdot \Sigma_\varphi(\delta \cdot p)$.*

$$
\begin{tikzcd}
\mathcal{G} \arrow[r, "\varphi"] \arrow[d, "p"'] & \mathcal{H} \arrow[d, "\Sigma_\varphi(\delta\cdot p)"] \\
\mathcal{F} \arrow[r, "\delta"'] \arrow[ur, phantom, "\Rightarrow"] \arrow[dr, "1_{\mathcal{F}}"'] & M\mathcal{F} \arrow[d, "\sigma"] \\
& \mathcal{F} \arrow[u, phantom, shift right=4, "\Rightarrow"]
\end{tikzcd}
$$

Furthermore, we have $\Sigma_\delta \delta \cong 1_{M\mathcal{F}}$, $\Sigma_\delta 1_{\mathcal{F}} \cong \sigma$, and the canonical morphism $1_{\mathcal{F}} \Rightarrow \Sigma_\delta 1_{\mathcal{F}} \cdot \delta$ is an isomorphism.

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, "\delta"] \arrow[dr, "\delta"'] & M\mathcal{F} \arrow[d, "\Sigma_\delta \delta"] \\
& M\mathcal{F} \arrow[u, phantom, shift right=4, "\Rightarrow"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F} \arrow[r, "\delta"] \arrow[dr, "1_{\mathcal{F}}"'] & M\mathcal{F} \arrow[d, "\Sigma_\delta 1_{\mathcal{F}}"] \\
& \mathcal{F} \arrow[u, phantom, shift right=4, "\Rightarrow"]
\end{tikzcd}
$$

Proof. Fix an essential geometric morphism $\mathcal{G} \xrightarrow{\varphi} \mathcal{H}$. We first work with free M-algebras. Under the fundamental adjointness

$$\mathbf{Top}_{\mathcal{S}}(\mathcal{X}, M\mathcal{F}) \simeq \mathbf{Cocts}_{\mathcal{S}}(\mathcal{F}, \mathcal{X}),$$

$\varphi^\#$ for $M\mathcal{F}$ is identified with composition with the inverse image functor φ^*. Then Σ_φ for $M\mathcal{F}$ can be identified with composition with the left adjoint $\varphi_!$. Given this, one verifies easily that $\Sigma_\delta \delta \cong 1_{M\mathscr{F}}$. The BCC (for comma objects) for the free algebra $M\mathscr{F}$ follows because in a comma object:

$$\begin{tikzcd} \mathscr{I} \arrow[r, "\psi"] \arrow[d, "\kappa"'] & \mathscr{K} \arrow[d, "\rho"] \\ \mathscr{G} \arrow[r, "\varphi"'] \arrow[ur, phantom, "\Rightarrow"] & \mathscr{H} \end{tikzcd}$$

we have $\psi_! \cdot \kappa^* \cong \rho^* \cdot \varphi_!$. Thus, free algebras are cocomplete. Now let $M\mathscr{F} \xrightarrow{\sigma} \mathscr{F}$ denote an arbitrary M-algebra. We have $\sigma \dashv \delta$. For any $\mathscr{G} \xrightarrow{p} \mathscr{F}$, define

$$\Sigma_\varphi p = \sigma \cdot \Sigma_\varphi(\delta \cdot p), \tag{1}$$

where on the right side, Σ_φ refers to the free algebra $M\mathscr{F}$. The following series of bijections shows that this definition gives the left adjoint to $\varphi^\#$. Let $\mathscr{H} \xrightarrow{q} \mathscr{F}$ be arbitrary:

$$\frac{\dfrac{\dfrac{\sigma \cdot \Sigma_\varphi(\delta \cdot p) \Rightarrow q}{\Sigma_\varphi(\delta \cdot p) \Rightarrow \delta \cdot q}}{\delta \cdot p \Rightarrow \delta \cdot q \cdot \varphi}}{p \Rightarrow q \cdot \varphi = \varphi^\# q}$$

The last bijection holds because δ is an inclusion. A consequence of (1) is

$$\Sigma_\delta 1_{\mathscr{F}} = \sigma \cdot \Sigma_\delta(\delta \cdot 1_{\mathscr{F}}) = \sigma \cdot \Sigma_\delta \delta \cong \sigma \cdot 1_{M\mathscr{F}} = \sigma.$$

We now verify the BCC for the algebra $\mathscr{F}$. Referring to the above comma object, by (1) we have

$$\rho^\# \Sigma_\varphi p = \rho^\#(\sigma \cdot \Sigma_\varphi(\delta \cdot p)) = \sigma \cdot \rho^\# \Sigma_\varphi(\delta \cdot p).$$

By the BCC for free algebras, this is isomorphic to

$$\sigma \cdot \Sigma_\psi \kappa^\#(\delta \cdot p) = \sigma \cdot \Sigma_\psi(\delta \cdot \kappa^\# p) = \Sigma_\psi \kappa^\# p.$$

The last equality is (1) again.

Finally, we show that M-homomorphisms are cocontinuous. It is not difficult to verify that free homomorphisms, i.e., geometric morphisms of the form $M\gamma$, are cocontinuous. Now let $\mathscr{F} \xrightarrow{\gamma} \mathscr{F}'$ be an arbitrary M-homomorphism. Then for φ essential, and $\mathscr{G} \xrightarrow{p} \mathscr{F}$, by (1) for $\mathscr{F}'$ we have

$$\Sigma_\varphi(\gamma \cdot p) = \sigma' \cdot \Sigma_\varphi(\delta' \cdot \gamma \cdot p) \cong \sigma' \cdot \Sigma_\varphi(M\gamma \cdot \delta \cdot p).$$

Since $M\gamma$ is cocontinuous, this is isomorphic to

$$\sigma' \cdot M\gamma \cdot \Sigma_\varphi(\delta \cdot p) \cong \gamma \cdot \sigma \cdot \Sigma_\varphi(\delta \cdot p) = \gamma \cdot \Sigma_\varphi p,$$

where the last equality is by (1). $\square$

Corollary 5.3. *Let $M\mathscr{F} \xrightarrow{\sigma} \mathscr{F}$ denote an M-algebra. Let $\mathscr{E} \xrightarrow{e} \mathscr{S}$ be locally connected* (i.e., let e be $\mathscr{S}$-essential). *The left extension of a geometric morphism $\mathscr{E} \xrightarrow{\psi} \mathscr{F}$ along e is isomorphic to $\sigma \cdot q$, where the point $\mathscr{S} \xrightarrow{q} M\mathscr{F}$ corresponds to the distribution $e_! \cdot \psi^* : \mathscr{F} \to \mathscr{S}$.*

$$\begin{tikzcd} \mathscr{E} \arrow[r, "e"] \arrow[dr, "\psi"'] & \mathscr{S} \arrow[d, "\Sigma_e\psi = \sigma \cdot q"] \arrow[dl, phantom, "\Rightarrow"] \\ & \mathscr{F} \end{tikzcd}$$

If $\mathscr{E} \xrightarrow{p} \mathscr{D} \xrightarrow{\varphi} \mathscr{F}$ denotes the pure dense/complete spread factorization of ψ, then $\Sigma_d \varphi \cong \Sigma_e \psi$.

Proof. The first statement is seen to be true by examining the constructions in the Proof of Proposition 5.2. The second statement follows from the first because the distributions $d_! \cdot \varphi^*$ and $e_! \cdot \psi^*$ are isomorphic. $\square$

Theorem 5.4 (Waelbroeck Theorem, first version). *A topos is an M-algebra if and only if it is cocomplete* (Definition 5.1). *A geometric morphism is an M-homomorphism if and only if it is cocontinuous. For any topos $\mathscr{F}$, the topos $M\mathscr{F}$ is its free cocompletion. That is, if $\mathscr{F} \xrightarrow{\xi} \mathscr{G}$ is a geometric morphism into a cocomplete topos $\mathscr{G}$, then there is an essentially unique cocontinuous geometric morphism $\tilde{\xi}$ such that*

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\delta"] \arrow[dr, "\xi"'] & M\mathscr{F} \arrow[d, "\tilde{\xi}"] \\ & \mathscr{G} \end{tikzcd}$$

commutes.

Proof. Assume that a given topos $\mathscr{F}$ is cocomplete. Let $\sigma : M\mathscr{F} \to \mathscr{F}$ denote the left extension $\Sigma_\delta(1_\mathscr{F})$. We will show that $\sigma \dashv \delta$, so that $(\mathscr{F}, \sigma)$ is an M-algebra. Let $\mathscr{H} \xrightarrow{q} M\mathscr{F}$ and $\mathscr{H} \xrightarrow{m} \mathscr{F}$ be arbitrary geometric morphisms, and consider the following comma object:

$$\begin{tikzcd} \mathscr{G} \arrow[r, "\varphi"] \arrow[d, "p"'] & \mathscr{H} \arrow[d, "q"] \arrow[dl, phantom, "\Rightarrow"] \\ \mathscr{F} \arrow[r, "\delta"'] & M\mathscr{F} \end{tikzcd}$$

We have the following natural bijections:

$$\frac{\sigma \cdot q \Rightarrow m}{\dfrac{q^\# \Sigma_\delta 1_\mathscr{F} \Rightarrow m}{\dfrac{\Sigma_\varphi p^\# 1_\mathscr{F} \Rightarrow m \text{ by the BCC for the above comma object}}{\dfrac{p \Rightarrow \varphi^\# m}{\dfrac{\delta \cdot p \Rightarrow \delta \cdot \varphi^\# m \text{ since } \delta \text{ is an inclusion}}{\dfrac{p^\# \delta \Rightarrow \varphi^\# (\delta \cdot m)}{\dfrac{\Sigma_\varphi p^\# \delta \Rightarrow \delta \cdot m}{\dfrac{q^\# \Sigma_\delta \delta \Rightarrow \delta \cdot m}{\dfrac{q^\# 1_{M\mathscr{F}} \Rightarrow \delta \cdot m \text{ by Prop. 5.2}}{q \Rightarrow \delta \cdot m.}}}}}}}}}$$

It remains to show that a cocontinuous geometric morphism between cocomplete toposes (equivalently, M-algebras) is a homomorphism. Let $\mathscr{F} \xrightarrow{\gamma} \mathscr{F}'$ be such a geometric morphism. We must show that the canonical natural transformation $\sigma' \cdot M\gamma \overset{\zeta}{\Rightarrow} \gamma \cdot \sigma$ is an isomorphism. There are canonical isomorphisms

$$\sigma' \cdot M\gamma \cdot \delta \Rightarrow \sigma' \cdot \delta' \cdot \gamma \Rightarrow \gamma$$

to which we apply Σ_δ giving the top row of the following diagram:

$$\begin{tikzcd}
\Sigma_\delta(\sigma' \cdot M\gamma \cdot \delta) \arrow[r] \arrow[d] & \Sigma_\delta(\sigma' \cdot \delta' \cdot \gamma) \arrow[r] & \Sigma_\delta \gamma \arrow[d] \\
\sigma' \cdot M\gamma \arrow[r, "\zeta"'] & \gamma \cdot \sigma \arrow[r] & \gamma \cdot \Sigma_\delta 1_{\mathscr{F}}
\end{tikzcd}$$

The left vertical arrow is the counit of $\Sigma_\delta \dashv \delta^\#$. The right vertical arrow is an isomorphism because γ is assumed to be cocontinuous. The bottom right morphism is the isomorphism obtained by applying γ to $\sigma \cong \Sigma_\delta 1_{\mathscr{F}}$. This diagram can be seen to commute by unraveling the adjoints. Next observe that the left vertical arrow is an isomorphism. In fact, both σ' and $M\gamma$ are M-homomorphisms, whence cocontinuous (Proposition 5.2), so that

$$\Sigma_\delta(\sigma' \cdot M\gamma \cdot \delta) \cong \sigma' \cdot M\gamma \cdot \Sigma_\delta \delta \cong \sigma' \cdot M\gamma \cdot 1_{M\mathscr{F}} = \sigma' \cdot M\gamma.$$

This shows that ζ is an isomorphism. $\square$

6. The Waelbroeck theorem for toposes (second version)

For the second Waelbroeck theorem (Theorem 6.4 below) we shift our attention from essential geometric morphisms and comma objects to locally connected geometric morphisms and pullback squares.

Definition 6.1. A topos $\mathscr{F}$ (over $\mathscr{S}$) will be said to be a *linear topos* if it admits left Kan extensions along locally connected geometric morphisms (see Definition 5.1). Furthermore, we require that $\mathscr{F}$ satisfy the BCC for pullback squares. Explicitly, this means that for every pullback square

$$\begin{tikzcd}
\mathscr{J} \arrow[r, "\psi"] \arrow[d, "\kappa"'] \arrow[dr, phantom, "\cong"] & \mathscr{K} \arrow[d, "\rho"] \\
\mathscr{G} \arrow[r, "\varphi"'] & \mathscr{H}
\end{tikzcd}$$

in $\mathbf{Top}_{\mathscr{S}}$ in which φ is locally connected, the canonical natural transformation

$$\Sigma_\psi \cdot \kappa^\# \Rightarrow \rho^\# \cdot \Sigma_\varphi$$

is an isomorphism. A geometric morphism $\mathscr{F} \xrightarrow{\gamma} \mathscr{F}'$ between linear toposes will be said to be a *linear geometric morphism* if for any locally connected geometric morphism $\mathscr{G} \xrightarrow{\varphi} \mathscr{H}$, and any $\mathscr{G} \xrightarrow{p} \mathscr{F}$, the canonical morphism $\Sigma_\varphi(\gamma \cdot p) \Rightarrow \gamma \cdot \Sigma_\varphi p$ is an isomorphism (see Definition 5.1).

The following fact is also shown in [17].

Lemma 6.2. *Assume there is given a comma object*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "g"'] & B \arrow[d, "k"] \arrow[dl, phantom, "\overset{t}{\Rightarrow}"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

in a 2-category. Also assume that k has a left adjoint l. Then g has a left adjoint $C \xrightarrow{m} A$ such that the unit $1_C \Rightarrow g \cdot m$ is an isomorphism. We also have $h \cdot m \cong l \cdot f$.

Proof. Let η denote the unit of $l \dashv k$. The cone

$$\begin{tikzcd} C \arrow[r, "l \cdot f"] \arrow[d, "1_C"'] & B \arrow[d, "k"] \arrow[dl, phantom, "\overset{\eta f}{\Rightarrow}"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

induces a 1-cell $C \xrightarrow{m} A$ and isomorphisms $i : 1_C \cong g \cdot m$ and $j : h \cdot m \cong l \cdot f$ such that $tm \cdot fi = \eta f \cdot kj$. Let $X \xrightarrow{c} C$ and $X \xrightarrow{a} A$ be arbitrary 1-cells. By the universal property of the given comma object, there is a natural bijection between 2-cells $m \cdot c \Rightarrow a$ and cone 2-cells

$$\begin{tikzcd} X \arrow[r, "l \cdot f \cdot c"] \arrow[d, "c"'] & B \arrow[d, "k"] \arrow[dl, phantom, "\overset{(\eta f)c}{\Rightarrow}"] & \overset{(\beta,\gamma)}{\Rightarrow} & X \arrow[r, "h \cdot a"] \arrow[d, "g \cdot a"'] & B \arrow[d, "k"] \arrow[dl, phantom, "\overset{ta}{\Rightarrow}"] \\ C \arrow[r, "f"'] & D & & C \arrow[r, "f"'] & D . \end{tikzcd}$$

The 2-cells $l \cdot f \cdot c \overset{\beta}{\Rightarrow} h \cdot a$ and $c \overset{\gamma}{\Rightarrow} g \cdot a$ satisfy $ta \cdot f\gamma = k\beta \cdot (\eta f)c$, and this equation transposes under $l \dashv k$ to $\widetilde{ta} \cdot l(f\gamma) = \beta$. We conclude, since β is determined by γ, that such pairs (β, γ) are in bijection with 2-cells $c \overset{\gamma}{\Rightarrow} g \cdot a$. Note that the unit of $m \dashv g$ is the isomorphism i. $\square$

Remark 6.3. When Lemma 6.2 is interpreted in toposes and geometric morphisms, the adjointness $m \dashv g$ reads $g^* \dashv g_* = m^* \dashv m_*$, so that g is a connected local geometric morphism and m is an essential inclusion for which $m_!$ is left exact.

Theorem 6.4 (Waelbroeck Theorem, second version). *A topos is an M-algebra if and only if it is a linear topos. A geometric morphism is an M-homomorphism if and only if it is linear. For a topos $\mathscr{F}$, $M\mathscr{F}$ is the free linear topos, i.e., an arbitrary geometric morphism $\mathscr{F} \to \mathscr{G}$ into a linear topos lifts uniquely to a linear geometric morphism $M\mathscr{F} \to \mathscr{G}$.*

Proof. Let $\mathscr{F}$ denote an arbitrary topos over $\mathscr{S}$. If $\mathscr{F}$ is an M-algebra, then it can be shown that $\mathscr{F}$ is a linear topos as in the proof of Proposition 5.2, proceeding first with free M-algebras. Note that in a pullback

$$\begin{tikzcd} \mathscr{J} \arrow[r, "\psi"] \arrow[d, "\kappa"'] & \mathscr{K} \arrow[d, "\rho"] \arrow[dl, phantom, "\cong"] \\ \mathscr{G} \arrow[r, "\varphi"'] & \mathscr{H} \end{tikzcd}$$

in which φ is locally connected, ψ is locally connected and we have $\psi_! \cdot \kappa^* \cong \rho^* \cdot \varphi_!$.

Now assume that $\mathscr{F}$ is a linear topos. We will show that $\mathscr{F}$ is cocomplete (Definition 5.1), so that by Theorem 5.4, $\mathscr{F}$ is an M-algebra. Let $\mathscr{G} \xrightarrow{\varphi} \mathscr{H}$ be an arbitrary essential geometric morphism and consider the comma object

$$\begin{tikzcd} \mathscr{J} \arrow[r, "\psi"] \arrow[d, "\kappa"'] & \mathscr{H} \arrow[d, "1_{\mathscr{H}}"] \arrow[dl, phantom, "\Rightarrow"] \\ \mathscr{G} \arrow[r, "\varphi"'] & \mathscr{H} \end{tikzcd}$$

in $\mathbf{Top}_{\mathscr{S}}$. We know that ψ is locally connected. By Lemma 6.2, κ has a left adjoint $\mathscr{G} \xrightarrow{\sigma} \mathscr{J}$ for which the unit is an isomorphism, and such that $\varphi \cong \psi \cdot \sigma$. For an arbitrary $\mathscr{G} \xrightarrow{x} \mathscr{F}$, define

$$\Sigma_\varphi x = \Sigma_\psi(x \cdot \kappa).$$

The following series of natural bijections shows that $\Sigma_\varphi \dashv \varphi^\#$:

$$\frac{\dfrac{\dfrac{\Sigma_\psi(x \cdot \kappa) \Rightarrow y}{x \cdot \kappa \Rightarrow y \cdot \psi}}{x \Rightarrow y \cdot \psi \cdot \sigma}}{x \Rightarrow y \cdot \varphi}$$

It remains to show that the BCC for comma objects holds. A comma object

$$\begin{tikzcd} \mathscr{J} \arrow[r, "\psi"] \arrow[d, "\kappa"'] & \mathscr{K} \arrow[d, "\rho"] \arrow[dl, phantom, "\Rightarrow"] \\ \mathscr{G} \arrow[r, "\varphi"'] & \mathscr{H} \end{tikzcd}$$

in which φ is essential, and therefore ψ is locally connected, can be rewritten as the following composite of a pullback and a comma object:

$$\begin{tikzcd}
\mathscr{I} \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathscr{K} \arrow[d, "\rho"] \\
\mathscr{D} \arrow[r, "\psi'"'] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathscr{H} \arrow[d, "1_{\mathscr{H}}"] \\
\mathscr{G} \arrow[r, "\varphi"'] & \mathscr{H}
\end{tikzcd}$$

Note that also ψ' is locally connected. The BCC for pullbacks now gives the BCC for comma objects.

The statement concerning M-homomorphisms can be proved in a manner similar to the above. $\square$

The algebras for the bagdomain monad $\mathscr{B}_L$ have a characterization comparable to the one for M-algebras in Theorem 6.4. The class of geometric morphisms having totally connected components is closed under arbitrary pullback (Proposition 1.4), so the BCC makes sense for this class.

Theorem 6.5. *A topos is a $\mathscr{B}_L$-algebra if and only if it admits left extensions along geometric morphisms with totally connected components, so that the BCC for pullbacks holds.*

Proof. (Johnstone [11, Theorem 5.1]). Johnstone has characterized $\mathscr{B}_L$-algebras as those toposes $\mathscr{F}$ having the property that for every topos $\mathscr{G}$, $\mathbf{Top}_{\mathscr{S}}(\mathscr{G},\mathscr{F})$ has $\mathscr{G}$-indexed coproducts, which are natural in $\mathscr{G}$. This property is easily seen to be equivalent to the property that the topos admit left extensions along local homeomorphisms, such that the BCC for pullbacks holds. We shall show that this is equivalent to the property stated in the theorem. Let $\mathscr{F}$ denote an arbitrary topos over $\mathscr{S}$ which admits left extensions along local homeomorphisms. Let $\mathscr{D} \xrightarrow{d} \mathscr{G}$ and $\mathscr{D} \xrightarrow{x} \mathscr{F}$ be arbitrary geometric morphisms and assume that d has totally connected components. We want to show that $\Sigma_d x$ exists. As in the proof of Proposition 1.4, we factor d as $\mathscr{D} \xrightarrow{\hat{d}} \mathscr{G}/I \xrightarrow{I} \mathscr{G}$ for which there is $\mathscr{G}/I \xrightarrow{p} \mathscr{D}$ over $\mathscr{G}$ satisfying $\hat{d} \dashv p$. Define $\Sigma_d x$ to be $\Sigma_I(x \cdot p)$. For any $\mathscr{G} \xrightarrow{y} \mathscr{F}$, we have natural bijections

$$\frac{\dfrac{\dfrac{\Sigma_I(x \cdot p) \Rightarrow y}{x \cdot p \Rightarrow y \cdot I}}{x \Rightarrow y \cdot I \cdot \hat{d}}}{x \Rightarrow y \cdot d,}$$

which shows that $\Sigma_d \dashv d^{\#}$. That the BCC can be lifted from local homeomorphisms to geometric morphisms with totally connected components follows easily. □

Remark 6.6. The algebras for the probability distribution classifier (see Section 4) can be characterized as those toposes admitting left extensions along *connected* locally connected geometric morphisms, such that the BCC holds. We omit the proof, as it is similar to the proof of Theorem 6.4.

We will refer to an M-algebra for which the carrier topos is locally connected as a *locally connected M-algebra.* Recall that a topos $\mathscr{E}$ is locally connected if and only if $M\mathscr{E}$ has a terminal point [6, Theorem 3.10]. We conclude this section by showing that the action of a locally connected M-algebra preserves this terminal point.

Proposition 6.7. *Let $M\mathscr{E} \xrightarrow{\sigma} \mathscr{E}$ be a locally connected M-algebra, and let $\mathscr{S} \xrightarrow{\top} M\mathscr{E}$ denote the terminal point of $M\mathscr{E}$. Then $\sigma\top$ is terminal. In particular, $\mathscr{E}$ is totally connected. (A topos is totally connected if and only if it has a terminal point.)*

Proof. We will show that $\sigma\top$ is pure dense, hence terminal by [6], Proposition 3.12. We have $\mathscr{E} \xrightarrow{\delta} M\mathscr{E}$ satisfying $\sigma \dashv \delta$ and $\sigma\delta \cong 1_{\mathscr{E}}$. We consider the unit $\top \xrightarrow{\eta} \delta\sigma\top$ in terms of complete spreads over $\mathscr{E}$, as in the following diagram:

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "\eta"] \arrow[dr, "1"'] & & \mathscr{F} \arrow[dl, "\varphi"] \\ & \mathscr{E} & \end{tikzcd}$$

The geometric morphism φ is the complete spread of the pure dense/complete spread factorization of $\sigma\top$:

$$\begin{tikzcd} \mathscr{S} \arrow[rr, "p"] \arrow[dr, "\sigma\top"'] & & \mathscr{F} \arrow[dl, "\varphi"] \\ & \mathscr{E} & \end{tikzcd}$$

The complete spread φ also appears in the following comma object:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "f"] \arrow[d, "\varphi"'] & \mathscr{S} \arrow[d, "\delta\sigma\top"] \\ \mathscr{E} \arrow[r, "\delta"'] \arrow[ur, phantom, "\Rightarrow"] & M\mathscr{E} \end{tikzcd}$$

The uniqueness of the universal property of this comma object shows that

$$\begin{tikzcd} \mathscr{S} \arrow[rr, "\sigma\top"] \arrow[dr, "p"'] & & \mathscr{E} \arrow[dl, "\eta"] \\ & \mathscr{F} & \end{tikzcd}$$

commutes. Observe that η is an inclusion since spreads are localic [6, Proposition 1.3], and since a point of a locale is an inclusion. By Proposition 1.2, $\sigma\top$ is pure dense. □

Corollary 6.8. *Let* $M\mathscr{E} \xrightarrow{\sigma} \mathscr{E}$ *be a locally connected algebra* (with terminal point $\mathscr{S} \xrightarrow{\top} \mathscr{E}$, by 6.7). *Let* $\mathscr{F} \xrightarrow{f} \mathscr{S}$ *be locally connected, and let* $\mathscr{F} \xrightarrow{p} \mathscr{E}$ *be an arbitrary pure dense geometric morphism. Then the left extension of* p *along* f *is isomorphic to* $\top$.

$$\begin{tikzcd} \mathscr{F} \arrow[r, "f"] \arrow[dr, "p"'] & \mathscr{S} \arrow[d, "\Sigma_f p = \top"] \arrow[dl, phantom, "\Rightarrow"] \\ & \mathscr{E} \end{tikzcd}$$

Proof. By Corollary 5.3, $\Sigma_f p \cong \sigma \cdot q$, where the point $\mathscr{S} \xrightarrow{q} M\mathscr{E}$ corresponds to the distribution $f_! \cdot p^* : \mathscr{E} \to \mathscr{S}$. If p is pure dense, this distribution is isomorphic to $e_!$, so that q must be isomorphic to the terminal point of $M\mathscr{E}$. By Proposition 6.7, we are done. □

7. Complete spreads and the fundamental group of a topos

The subject of the fundamental group of a topos has a long and ongoing "trajectory" (see [7] for a brief review and references). The *Grothendieck fundamental group* G of a locally connected topos $\mathscr{E}$ is a (localic) group which represents first-degree cohomology of $\mathscr{E}$ with coefficients in (discrete) groups. Its classifying topos $\mathscr{B}G$, the category of continuous G-sets, may be identified with the full subcategory of $\mathscr{E}$ determined by those objects which are *sums of locally constant objects.* If in addition, $\mathscr{E}$ is locally simply connected [1], the latter reduces to the full subcategory of *locally constant objects* in $\mathscr{E}$. Moreover, in the stronger case where $\mathscr{E}$ is locally paths simply connected, the Grothendieck fundamental group can be constructed "by paths" [7], i.e., it is equivalent to the *(Moerdijk-Wraith) paths fundamental group* of $\mathscr{E}$.

There is a connection between complete spreads and the fundamental group of a topos. That locally constant objects are complete spreads (7.3), and that in a connected presheaf topos the converse holds as well (7.9), provide evidence for this. The general question of the converse is still open, as are most questions about the category of objects in a topos which are complete spreads (Definition 7.2).

We have seen that complete spreads are stable under pullback along a local homeomorphism (1.5). We begin this section by showing that they are reflected under pullback along a surjective local homeomorphism.

Proposition 7.1. *"(Complete) spread" is a local property. Let $\mathscr{E} \xrightarrow{\varphi} \mathscr{F}$ be an arbitrary geometric morphism* (over a base topos $\mathscr{S}$), *and assume that $\mathscr{E}$ is locally connected. Let U be a cover of 1 in $\mathscr{F}$. If the pullback $\varphi_U : \mathscr{E}/\varphi^* U \to \mathscr{F}/U$ is a (complete) spread, then so is φ.*

Proof. We first show that if φ_U is a spread, then so is φ. To do this recall that the notion of a spread in the theory of geometric morphisms [6, Definition. 1.1] is defined in terms of *definable* morphisms [2, p. 139]. Fix $E \in \mathscr{E}$. If φ_U is a spread, there is a commutative square

$$\begin{tikzcd}
S \arrow[r, two heads] \arrow[d, "m"'] & E \times \varphi^* U \arrow[d] \\
\varphi^* F \arrow[r] & \varphi^* U
\end{tikzcd}$$

for some $F \to U$, where m is definable in $\mathscr{E}/\varphi^* U$. The coproduct functors $\mathscr{E}/X \xrightarrow{\Sigma} \mathscr{E}$ preserve definable morphisms, so m is definable in $\mathscr{E}$, and we have the diagram

$$\begin{tikzcd}
S \arrow[r, two heads] \arrow[d, "m"'] & E \times \varphi^* U \arrow[r, two heads] & E \\
\varphi^* F & &
\end{tikzcd}$$

in $\mathscr{E}$. The second morphism above is an epimorphism because $U \to 1$ is.

Now assume that φ_U is a *complete* spread. Factor φ as a pure dense geometric morphism followed by a complete spread (where the middle topos is locally connected), and form their respective pullbacks along $\mathscr{F}/U \to \mathscr{F}$.

$$\begin{tikzcd}
\mathscr{E}/\varphi^* U \arrow[r, "p_U"] \arrow[d] & \mathscr{D}/\psi^* U \arrow[r, "\psi_U"] \arrow[d] & \mathscr{F}/U \arrow[d] \\
\mathscr{E} \arrow[r, "p"'] & \mathscr{D} \arrow[r, "\psi"'] & \mathscr{F}
\end{tikzcd}$$

We will show that p is both an inclusion and a surjection. By the first paragraph, φ is a spread, so p is also [6, Proposition 1.2], and pure dense spreads are inclusions [6, Proposition 2.4]. By Proposition 1.3, p_U is pure dense, and by 1.5, ψ_U is a complete spread. By the uniqueness of the pure dense/complete spread factorization, p_U must be an equivalence, and therefore p is a surjection. $\square$

Definition 7.2. An object X in a locally connected topos $\mathscr{E}$ over $\mathscr{S}$ will be said to be a *complete spread* if the geometric morphism $\mathscr{E}/X \to \mathscr{E}$ is a complete spread (over $\mathscr{S}$).

Recall that an object X in a topos $\mathscr{E}$ is said to be *locally constant* if there is $U \in \mathscr{E}$ with global support and $I \in \mathscr{S}$ such that $U \times X \cong U \times e^*I$ over U. The object U is said to split X.

Theorem 7.3. *Locally constant objects in a locally connected topos are complete spreads.*

Proof. We first show that constant objects in a locally connected topos are complete spreads. Let $\mathscr{E}$ denote a locally connected topos with $e_! \dashv e^*$. Let I be an arbitrary object of $\mathscr{S}$. We want to show that $\mathscr{E}/I \to \mathscr{E}$ is a complete spread. If $(\mathbf{C}, J)$ is a site for $\mathscr{E}$, then the square

$$\begin{tikzcd} \mathscr{E}/I \arrow[r] \arrow[d] & \mathscr{S}^{\mathbf{C}^{op}}/I \arrow[d] \\ \mathscr{E} \arrow[r] & \mathscr{S}^{\mathbf{C}^{op}} \end{tikzcd}$$

is a pullback of toposes. Since $\mathscr{E}$ is locally connected, we may assume that the chosen site $(\mathbf{C}, J)$ has the property that for all objects $c \in \mathbf{C}$, $e_! c = 1$. Then the cosheaf $\mathbf{C} \to \mathscr{S}$ corresponding to the cocontinuous functor $\mathscr{E} \xrightarrow{I^*} \mathscr{E}/I \xrightarrow{e^I_!} \mathscr{S}$ is the constant cosheaf $c \mapsto I$. The discrete opfibration corresponding to this is $\mathbf{C} \times I \to \mathbf{C}$, and the complete spread it defines is the following pullback:

$$\begin{tikzcd} \mathscr{D} \arrow[r] \arrow[d] & \mathscr{S}^{(\mathbf{C} \times I)^{op}} \arrow[d] \\ \mathscr{E} \arrow[r] & \mathscr{S}^{\mathbf{C}^{op}} \end{tikzcd}$$

This coincides with the first pullback, so $\mathscr{E}/I \to \mathscr{E}$ must be a complete spread.

The result for locally constant objects now follows from Proposition 7.1. □

Is every complete spread object of a topos locally constant? We take the opportunity to report some preliminary findings.

Recall that a topos is said to be *simply connected* if every locally constant object is constant.

Proposition 7.4. *If the connected components functor of a locally connected topos preserves products, then every complete spread object of the topos is constant. In particular, such toposes are simply connected* (*by* Theorem 7.3).

Proof. Let $\mathcal{E}$ denote a locally connected topos. If $e_!$ preserves products, and if $X \in \mathcal{E}$, then $X.e_! \cong I.e_!$, where I is $e_!X$. Let X be a complete spread. Then by Corollary 3.6, $X \cong \mathbf{d}(X.e_!)$, where $\mathbf{d}$ is the density functor (Definition 3.2). We have

$$X \cong \mathbf{d}(X.e_!) \cong \mathbf{d}(I.e_!) \cong I,$$

where I is $e_!X$. This says that X is constant. □

We now take our investigation in a different direction. We begin with a basic characterization of objects in a topos which are complete spreads. Recall [2] that a topos is locally connected if and only if a site for the topos can be chosen such that the constant presheaves are sheaves. It is also true that an inclusion of a topos into presheaves is pure dense if and only if the constant presheaves are sheaves. Thus, every locally connected topos is a pure dense subtopos of a presheaf topos.

Proposition 7.5. *Let $\mathcal{E}$ be locally connected with pure dense inclusion $\mathcal{E} \to \mathcal{S}^{\mathbf{C}^{op}}$. Let X be an arbitrary object of $\mathcal{E}$, and let Q denote the initial factor of X* (when X is considered as a discrete fibration over $\mathbf{C}$). *Let q denote the geometric morphism induced by Q, and p the pullback of q along $\mathcal{E} \to \mathcal{S}^{\mathbf{C}^{op}}$. Then X is a complete spread if and only if p is an equivalence.*

$$\begin{tikzcd} \mathbf{X} \arrow[r, "Q"] \arrow[dr, "X"'] & \mathbf{D} \arrow[d, "D"] \\ & \mathbf{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}^{\mathbf{X}^{op}} \arrow[r, "q"] \arrow[dr] & \mathcal{S}^{\mathbf{D}^{op}} \arrow[d] \\ & \mathcal{S}^{\mathbf{C}^{op}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}/X \arrow[r, "p"] \arrow[dr] & \mathcal{D} \arrow[d] \\ & \mathcal{E} \end{tikzcd}$$

Proof. The discrete opfibration over $\mathbf{C}$ associated with X as an object of $\mathcal{E}$ is the functor $c \mapsto e_!(X \times c)$. The one associated with X as an object of $\mathcal{S}^{\mathbf{C}^{op}}$ is the functor $c \mapsto Dc = \varinjlim (X \times c)$. These two opfibrations are isomorphic because $\mathcal{E} \to \mathcal{S}^{\mathbf{C}^{op}}$ was chosen to be pure dense. Thus, the geometric morphism $\mathcal{D} \to \mathcal{E}$ in the above diagram (right) is the associated complete spread of X. □

We next recall a fact about locally constant objects first reported by Barr and Diaconescu [1] (herein, Proposition 7.6). Let $\mathscr{E}$ denote an arbitrary locally connected topos. For objects $A, E \in \mathscr{E}$, the pairing of the unit of $e_! \dashv e^*$ with the projection $E \times A \to A$ is a morphism

$$\tau_E^A : E \times A \to e^* e_!(E \times A) \times A$$

in $\mathscr{E}$, which is natural in A and E. We also have the projection

$$\pi_E^A : e^* e_!(E \times A) \times A \to A.$$

Now fix an object $U \in \mathscr{E}$. We can regard a component $c \in e_! U$ as a subobject of U in $\mathscr{E}$. Consider the coproduct of the morphisms τ_E^c as c ranges over the components of U:

$$\tau_E : E \times U \to \coprod_{c \in e_! U} e^* e_!(E \times c) \times c.$$

The object on the right we will denote by TE. The coproduct over $e_! U$ of the projections π_E^c gives a morphism

$$\pi_E : TE \to \coprod_{c \in e_! U} c = U,$$

such that the composite $\pi_E \cdot \tau_E$ is equal to the projection $E \times U \to U$.

Proposition 7.6 (Barr and Diaconescu [1]). *Let $\mathscr{E}$ be connected and locally connected. Let X be an object of $\mathscr{E}$, and let U be a cover of* 1. *Then U splits X if and only if τ_X is an isomorphism.*

We identify the following condition on an object X of a locally connected topos $\mathscr{E}$.

∇ There is a site $\mathbf{C}$ for $\mathscr{E}$ so that $\mathscr{E} \to \mathscr{S}^{\mathbf{C}^{\mathrm{op}}}$ is pure dense (see Proposition 7.5), and such that for every morphism $c \xrightarrow{m} d$ in $\mathbf{C}$, the map

$$e_!(X \times m) : e_!(X \times c) \to e_!(X \times d)$$

(equivalently, by Proposition. 7.5, the map $\overset{lim}{\to}(X \times m) : \overset{lim}{\to}(X \times c) \to \overset{lim}{\to}(X \times d)$) is an isomorphism.

Proposition 7.7. *Let X be an object of a connected locally connected topos $\mathscr{E}$. If X is a complete spread, and satisfies condition* ∇ (for site $\mathbf{C}$), *then X is locally constant, split by the coproduct $\coprod\{c \mid c \in \mathbf{C}\}$.*

Proof. We will show that τ_X, for $U = \coprod\{c \mid c \in \mathbf{C}\}$, is an isomorphism (Proposition 7.6). The components of this U are the objects c. First, we have some notation and terminology to define. Consider the category $\mathbf{C}^{\rightarrow}$ whose objects are the morphisms of $\mathbf{C}$, and whose morphisms are commutative squares in $\mathbf{C}$. There are the domain and codomain functors $\partial_0, \partial_1 : \mathbf{C}^{\rightarrow} \to \mathbf{C}$. The domain functor ∂_0 is a split fibration (∂_1 is a split opfibration), and so can be thought of as a category object in $\mathscr{S}^{\mathbf{C}^{\mathrm{op}}}$. We denote this category object by $\mathbf{U}$. The object U is (the associated sheaf of) the "objects" presheaf of $\mathbf{U}$. Let $\mathbf{D} \xrightarrow{D} \mathbf{C}$ denote an arbitrary discrete opfibration, and consider the pullbacks

$$\begin{tikzcd} \mathbf{D}^{\rightarrow} \arrow[r, "\partial_0"] \arrow[d, "\partial_0^* D"'] & \mathbf{D} \arrow[d, "D"] \\ \mathbf{C}^{\rightarrow} \arrow[r, "\partial_0"'] & \mathbf{C} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{T} \arrow[r] \arrow[d, "\partial_1^* D"'] & \mathbf{D} \arrow[d, "D"] \\ \mathbf{C}^{\rightarrow} \arrow[r, "\partial_1"'] & \mathbf{C} \end{tikzcd}$$

of functors and small categories. Observe that by composing with ∂_0, we can regard $\partial_1^* D$ as a discrete opfibration on $\mathbf{U}$ in the topos $\mathscr{S}^{\mathbf{C}^{\mathrm{op}}}$. We denote this discrete opfibration by $\mathbf{T} \xrightarrow{\pi} \mathbf{U}$. By sheafifying, we regard π as a discrete opfibration in $\mathscr{E}$, i.e., as an object of $\mathscr{E}^{\mathbf{U}}$. There is a functor

$$\theta : \partial_0^* D \to \partial_1^* D$$

over $\mathbf{C}^{\rightarrow}$ which sends an object $(c \xrightarrow{f} d, x \in Dc)$ to the object $(c \xrightarrow{f} d, Df(x))$. If every Df is an isomorphism, then θ is an isomorphism.

We return to the task at hand. Consider the comprehensive factorization

$$\mathbf{X} \xrightarrow{Q} \mathbf{D} \xrightarrow{D} \mathbf{C}$$

of X. For this D, we have the functor θ and the discrete opfibration $\mathbf{T} \xrightarrow{\pi} \mathbf{U}$ in $\mathscr{E}$, as explained in the previous paragraph. The sheaf of objects for this $\mathbf{T}$ is the sheaf TX, defined in the paragraph preceding Proposition 7.6. There is the pullback $\partial_0^* Q$,

$$\begin{tikzcd} \mathbf{X} \times \mathbf{U} \arrow[r, "\partial_0^* Q"] \arrow[d] & \mathbf{D}^{\rightarrow} \arrow[d, "\partial_0"] \\ \mathbf{X} \arrow[r, "Q"'] & \mathbf{D} \end{tikzcd}$$

which when composed with θ gives the following commutative diagram of functors and categories in $\mathscr{E}$:

$$\begin{tikzcd}
X \times \mathbf{U} \arrow[rr, "\theta \cdot \partial_0^* Q"] \arrow[dr] & & \mathbf{T} \arrow[dl, "\pi"] \\
& \mathbf{U} &
\end{tikzcd}$$

The functor $\theta \cdot \partial_0^* Q$ is a morphism of $\mathscr{E}^{\mathbf{U}}$, and its "objects" part is the morphism τ_X of Proposition 7.6. We have the following pullbacks of toposes:

$$\begin{tikzcd}
\mathscr{E}^{X \times \mathbf{U}} \arrow[r, "r"] \arrow[d] & \mathscr{D}^{\mathbf{U}} \arrow[r] \arrow[d] & \mathscr{E}^{\mathbf{U}} \arrow[d] \\
\mathscr{E}/X \arrow[r, "q"'] & \mathscr{D} \arrow[r] & \mathscr{E}
\end{tikzcd}$$

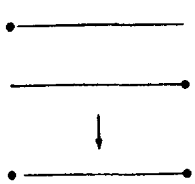
In the above diagram, $\mathscr{D}$ is the associated complete spread of X, and q and r denote the geometric morphisms induced by Q and $\partial_0^* Q$, respectively. Since X is assumed to be a complete spread, q is an equivalence, so that r is also. Condition ∇ says that θ is an isomorphism, so that the geometric morphism induced by θ, denoted Θ in following diagram, is an equivalence:

$$\begin{tikzcd}
\mathscr{E}^{X \times \mathbf{U}} \arrow[r, "\gamma"] \arrow[dr] & \mathscr{D}^{\mathbf{U}} \arrow[r, "\Theta"] \arrow[d] & \mathscr{E}^{\mathbf{T}} \arrow[dl, "\mathscr{E}^{\pi}"] \\
& \mathscr{E}^{\mathbf{U}} &
\end{tikzcd}$$

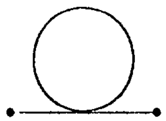
We conclude that τ_X is an isomorphism. $\square$

Example 7.8. Proposition 7.7 is false without the hypothesis that X is a complete spread, as the following simple example shows:

X:



A solid dot indicates that the endpoint is included. Here X is a surjective local homeomorphism and the connected open sets in the bottom space are a base $\{U\}$ such that every $\pi_0(X^{-1}U) \to \pi_0(X^{-1}U')$ is an isomorphism (i.e., ∇ is satisfied), but X is not a covering space. This example can be modified so that the top space is connected. Consider as a bottom space a circle and a tangent closed line segment.



Take as the top space the *connected* 2-to-1 cover of the circle with the same two line segments as above tangent to each of the points of the fiber of the point p. This provides a surjective sheaf space for which the top space is connected, and such that condition ∇ is satisfied, but the sheaf space is not locally constant.

Corollary 7.9. *In a connected presheaf topos, the locally constant objects coincide with the complete spread objects. A connected presheaf topos is locally simply connected, and its fundamental group is equivalent to its full subcategory of complete spread objects.*

Proof. Let $\mathbf{D}$ denote a small connected category. If a presheaf X on $\mathbf{D}$ is a complete spread, then it is a discrete opfibration (Proposition 2.2(3)) so that for every morphism $c \xrightarrow{m} d$ in $\mathbf{D}$, the transition map Xm is an isomorphism (Remark 2.3). Then for every $d \in \mathbf{D}$, $Xd \cong \varinjlim(X \times d)$, and every map

$$\varinjlim(X \times m) : \varinjlim(X \times c) \to \varinjlim(X \times d)$$

is an isomorphism. By Proposition 7.7, X is locally constant.

The second statement holds because, as we have just shown, the presheaf $\coprod\{d \mid d \in \mathbf{D}\}$ splits every complete spread object. But every locally constant object is a complete spread. $\square$

Example 7.8 shows that condition ∇ is not equivalent to X being locally constant; however, we can say the following.

Proposition 7.10. *Let X be an object of a connected locally connected topos $\mathscr{E}$. Then X satisfies ∇ if and only if the associated complete spread of X is a locally constant object of $\mathscr{E}$.*

Proof. Assume that X satisfies ∇. By Corollary 7.9, the discrete opfibration D in the comprehensive factorization of X must be a locally constant presheaf. The initial factor of this factorization can then be regarded as a natural transformation $X \to D$ in $\mathscr{S}^{\mathbf{C}^{op}}$.

The associated sheaf of D is again locally constant and it is the associated complete spread of X.

We only sketch a proof of the converse. Let $\bar{X}$ denote the associated complete spread of X, a locally constant object of $\mathscr{E}$. Then there is a site $\mathbf{C}$ for $\mathscr{E}$ (such that $\mathscr{E} \to \mathscr{S}^{\mathbf{C}^{\mathrm{op}}}$ is pure dense and) such that every restriction map $\bar{X}m : \bar{X}d \to \bar{X}c$ is an isomorphism. For this site we have $\bar{X}c \cong e_!(\bar{X} \times c)$ for every c, so every $e_!(\bar{X} \times m)$ is an isomorphism. Then every $e_!(X \times m)$ is an isomorphism because $X \to \bar{X}$ is pure dense. $\square$

We close with a question posed by Lawvere. Is there a suitable single universe in which complete spreads and local homeomorphisms over a topos $\mathscr{E}$ coexist and interact? Gluing along the density functor $\mathbf{d} : M\mathscr{E} \to \mathscr{E}$ produces such a universe, but the question requires further study. Other (still open) questions about distributions on toposes can be found in the writings and lectures of Lawvere (e.g., [14, 15]).

Acknowledgements

We thank Bill Lawvere for his constant inspiration and encouragement. We also thank Claudio Hermida for his expert advice on the subject of fibrations. Our research has been partially supported by grants from NSERC and FCAR.

References

[1] M. Barr, R. Diaconescu, On locally simply connected toposes and their fundamental groups, Cahiers de Top. et Géom. Diff. Catégoriques, 22 (3) (1981) 301–314.
[2] M. Barr, R. Paré, Molecular toposes, J. Pure Appl. Alg. 17 (1980) 127–152.
[3] M. Bunge, Stack completions and Morita equivalence for categories in a topos, Cahiers de Top. et Géom. Diff. Catégoriques 20 (4) (1979) 401–436.
[4] M. Bunge, Cosheaves and distributions on toposes, Algebra Universalis, 34 (1995) 469–484.
[5] M. Bunge, A. Carboni, The symmetric topos, J. Pure Appl. Alg. 105 (1995) 233–249.
[6] M. Bunge, J. Funk, Spreads and the symmetric topos, J. Pure Appl. Alg. 113 (1996) 1–38.
[7] M. Bunge, I. Moerdijk, On the construction of the Grothendieck fundamental group of a topos by paths, J. Pure Appl. Alg. 116 (1997) 99–113.
[8] R.H. Fox, Covering spaces with singularities, in: R.H. Fox et al. (Eds.), Algebraic Geometry and Topology: A Symposium in Honor of S. Lefschetz, Princeton University Press, Princeton, 1957, pp. 243–257.
[9] J. Funk, Descent for cocomplete categories, Ph.D. Thesis, McGill University, 1990.
[10] J. Funk, The display locale of a cosheaf, Cahiers de Top. et Géom. Diff. Catégoriques 36 (1) (1995) 53–93.
[11] P.T. Johnstone, Partial products, bagdomains and hyperlocal toposes, in: M.P. Fourman, P.T. Johnstone, A.M. Pitts (Eds.), Applications of Categories in Computer Science, London Mathematical Society Lecture Notes Series 177, Cambridge University Press, Cambridge, 1992, pp. 315–339.
[12] A. Kock, Some problems and results in synthetic functional analysis, in: A. Kock (Ed.), Category Theoretic Methods in Geometry, Aarhus, Matematisk Institut, Aarhus Universitet, Various Publications Series 35, 1983, pp. 168–191.
[13] F.W. Lawvere, Equality in hyperdoctrines and the comprehension schema as an adjoint functor, in: Proc. Symp. Pure Math. 17, Amer. Math. Soc., 1968, pp. 1–14.
[14] F.W. Lawvere, Entensive and extensive quantities, Notes for the lectures given at the workshop on Categorical Methods in Geometry, Aarhus, 1983.

[15] F.W. Lawvere, Categories of space and of quantity, in: J. Echeverria et al. (Eds.), The Space of Mathematics, W. de Gruyter, Berlin, pp. 14–30.
[16] A.M. Pitts, On product and change of base for toposes, Cahiers de Top. et Géom. Diff. Catégoriques, 26 (1) (1985) 43–61.
[17] R. Street, Fibrations and Yoneda's lemma in a 2-category, in: Category Seminar, Sydney 1972/73, LNM 420, Springer, Berlin, 1974, pp. 104–133.
[18] R. Street, R.F.C. Walters, The comprehensive factorization of a functor, Bull. Amer. Math. Soc. 79 (1973) 936–941.
[19] L. Waelbroeck, Differentiable mappings into *b*-spaces, J. Funct. Anal. 1 (1967) 409–418.