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Split dual Dyer–Lashof operations

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For each admissible monomial of Dyer–Lashof operations Q_I, we define a corresponding natural function $\hat{Q}_I : T\bar{H}_*(X) \to H^*(\Omega^n \Sigma^n X)$, called a *Dyer–Lashof splitting*. For every homogeneous class x in $H^*(X)$, a Dyer–Lashof splitting $\hat{Q}_I$ determines a canonical element y in $H^*(\Omega^n \Sigma^n X)$ so that y is connected to x by the dual homomorphism to the operation Q_I. The sum of the images of all the admissible Dyer–Lashof splittings contains a complete set of algebra generators for $H^*(\Omega^n \Sigma^n X)$.

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1. Introduction

Let X be a connected space and let $\Omega^n \Sigma^n X$ be denoted X_n. In this paper we provide a way to name specific cohomology classes in $H^*(X_n)$. We do this by defining functions $\hat{Q}_I$ from the tensor algebra $T\bar{H}^*(X)$ to $H^*(X_n)$, where the subscript I denotes an admissible sequence in a sense suitable for use with Dyer–Lashof operations. Naively, $\hat{Q}_I x$ may be regarded as "the" dual to $Q_I \bar{x}$, where $x \in H^*(X)$ is dual to $\bar{x} \in H_*(X)$. That is, we have the relation

$$\langle \hat{Q}_I x, Q_I \bar{x} \rangle = \langle x, \bar{x} \rangle.$$

(Here and elsewhere, we identify $H_*(X)$ with a submodule of $H_*(X_n)$ via the monomorphism induced by the standard map $\eta : X \to X_n$.)

The main theorem is as follows; some particulars of the notation will be explained below.

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Theorem 1.1. *For each admissible sequence* $I=(r_1,\dots,r_s)$ *there is a natural function* $\hat{Q}_I : T\bar{H}^*(X)\to H^*(X_n)$, *which satisfies the following properties*:

(1) *If* $r_1>0$ *then* $\hat{Q}_I$ *is a* $\mathbb{Z}/p$*-module homomorphism.*

(2) *We have the duality relation*

$$\begin{aligned}&\langle \hat{Q}_I(x_1,\dots,x_m),\ Q_J\lambda_{n-1}(\bar{x}_1,\dots,\lambda_{n-1}(\bar{x}_{m-1},\bar{x}_m)\dots)\rangle\\ &\quad=\delta_{IJ}\langle s^{n-1}x_1\otimes\cdots\otimes s^{n-1}x_m,\ [s^{n-1}\bar{x}_1,\dots,[s^{n-1}\bar{x}_{m-1},s^{n-1}\bar{x}_m]\dots]\rangle.\end{aligned}$$

(3) *Let* $\bar{Q}(X)\subset H^*(X_n)=\sum \operatorname{im}\hat{Q}_I$, *where the sum (not a direct sum) is taken over all admissible* I. *Then the projection* π *takes* $\bar{Q}(X)$ *onto* $H^*(X_n)$.

In (2), $[\,,]$ denotes the (graded) commutator in $T\bar{H}_*(\Sigma^{n-1}X)$, s^{n-1} is the isomorphism that increases degrees by $(n-1)$, and δ_{IJ} is the Kronecker delta on the sequences I and J. We allow I (or J) to be the empty sequence (denoted $\emptyset$), in which case Q_I is taken to be the identity.

The functions $\hat{Q}_I$ can be thought of as splittings of duals to Dyer–Lashof operations, and we will refer to them as *Dyer–Lashof splittings*. They generalize the "dual extended Dyer–Lashof operations" defined by Kuhn et al. in [5], and by Foskey and Slack in [4]. These latter operations were not shown to be natural transformations, and they did not generate the entire cohomology of X_n. They were, however, sufficient to allow Slack in [7] to show that an infinite loop space with trivial Dyer–Lashof action must be (p-locally) homotopy equivalent to a product of Eilenberg–Mac Lane spaces, and in [8] to provide a similar classification of spaces with p-torsion free homology for p odd.

All spaces in this paper will be connected, of the homotopy type of a CW-complex with finitely many cells in each dimension, and possessing a nondegenerate basepoint; and X will always denote an arbitrary space in this category. All coefficients for homology and cohomology will be in $\mathbb{Z}/p$ for p an odd prime, except in the final section where we will briefly discuss the case $p=2$. The notation ΣX represents the reduced suspension, and ΩX is the Moore loops on X. Finally, we will generally be working with functors from the category of spaces (as described above) to the category of $\mathbb{Z}/p$-modules. If we remark that a homomorphism is natural, we will mean that it is a natural transformation between two such functors. These transformations will not always be homomorphisms of graded modules, but they will preserve the property of being of finite type.

2. Homology operations

In this paper we will use the "lower notation" of Campbell et al. [2] for Dyer–Lashof operations. That is, if $\bar{x}\in H_q(\Omega^n X)$, $i+q$ is even, and $0\le i<n-1$, we define

$$Q_{i(p-1)}: H_q(\Omega^n X)\to H_{pq+i(p-1)}(\Omega^n X)$$

to be $Q^{(i+q)/2}\bar{x}$, and we define $Q_{i(p-1)-1}$ to be $\beta Q_{i(p-1)}$.

We use $Q_{(n-1)(p-1)}$ to represent the "top" operation, denoted ξ_{n-1} by Cohen in [1], which is special because it is not a homomorphism. Also, we use $Q_{(n-1)(p-1)-1}$ to represent the operation denoted ζ_{n-1} in [1]. This operation is not equal to $\beta Q_{(n-1)(p-1)}$, but rather differs from it by a correction term involving Browder operations. However, in most respects it resembles the other operations of the form $Q_{i(p-1)-1}$; in particular, it is a homomorphism.

Now let I represent the sequence

$$(i_1(p-1)-\varepsilon_1,\dots,i_s(p-1)-\varepsilon_s),$$

where ε_j is 0 or 1, and let Q_I represent the operation

$$Q_{i_1(p-1)-\varepsilon_1}\cdots Q_{i_s(p-1)-\varepsilon_s}.$$

The terms $i_1(p-1)-\varepsilon_1$ and $i_s(p-1)-\varepsilon_s$ will be called the leading and trailing terms of I, respectively, and we say that I is *admissible* if

(1) $0\le i_j\le i_{j+1}-\varepsilon_{j+1}$ for each $j\ge 1$, and

(2) $\varepsilon_{j+1}\equiv i_{j+1}-i_j \bmod 2$.

This is equivalent to the standard definition in [1] for admissible sequences in upper notation, with the second condition added to ensure that Q_I is defined. We note that our notation is slightly different from that given in [2].

We conclude this section with two theorems that render into lower notation some standard useful facts about Dyer–Lashof operations. Proofs, in upper notation, may be found in [1].

Theorem 2.1 (Suspension relations). *For any $\bar{x},\bar{y}\in H_*(\Omega^nX)$,*

$$\sigma_*(Q_{i(p-1)}\bar{x})=Q_{(i-1)(p-1)}(\sigma_*\bar{x})$$

and

$$\sigma_*\lambda_{n-1}(\bar{x},\bar{y})=\lambda_{n-2}(\sigma_*\bar{x},\sigma_*\bar{y}),$$

where σ_* denotes the suspension homomorphism $H_*(\Omega W)\to H_{*+1}(W)$.

Theorem 2.2 (External Cartan formula). *If $\bar{x}\otimes\bar{y}\in H_*(\Omega^nX\times\Omega^nY)$, then*

$$Q_{i(p-1)}(\bar{x}\otimes\bar{y})=\sum_{r+s=i}Q_{r(p-1)}\bar{x}\otimes Q_{s(p-1)}\bar{y}$$

where we ignore all terms for which $r+|\bar{x}|$ or $s+|\bar{y}|$ is odd.

The internal Cartan formula has essentially the same form, provided that $i<n-1$. For the $i=n-1$ case, see [1].

3. The homology of loop-suspension spaces

We will rely on Cohen's structure theorem for $H_*(X_n)$ [1, III]. In this section we restate that theorem in the notation of this paper.

Let TM denote the tensor algebra on a graded module M, and define the free Lie algebra LM to be the sub Lie algebra of TM generated by M. That is, we can inductively define a generating set A for LM by saying that $M \subset A$, and the commutator $[a,b] \in A$ whenever a and b are both in A.

If S is some arbitrary subset of $(TM)_{\text{even}}$, define ξS to be the submodule of TM generated by the set $\{\xi a \mid a \in S\}$, where ξ is the pth power map $\xi a = a^{\otimes p}$. We may then define the free graded restricted Lie algebra $L_R M$ to be the submodule

$$LM + \xi LM_{\text{even}} + \xi^2 LM_{\text{even}} + \cdots \subset TM.$$

It is an infinite sum rather than an infinite direct sum because the pth power map is not a homomorphism on a non-commutative ring. One may show that $L_R M$, defined this way, is still closed under the Lie bracket operation.

The notion of $L_R M$ is useful because $H_*(X_n)$ contains a degree-shifted copy of $L_R \bar{H}_*(\Sigma^{n-1} X)$, which we will call S_*. To see this, let $\bar{\sigma}$ denote the composition

$$H_*(X_n) = H_*(\Omega^n \Sigma^n X) \xrightarrow{(\sigma_*)^{n-1}} H_*(\Omega \Sigma^n X) \cong T\bar{H}_*(\Sigma^{n-1} X),$$

and let $\bar{\sigma}^{\text{split}} : L_R H_*(\Sigma^{n-1} X) \to H_*(\Omega^n \Sigma^n X)$ be determined by the following formal procedure: replace every $[\, ,]$ by $\lambda_{n-1}(\, ,)$, every ξ by $Q_{(n-1)(p-1)}$, and every $s^{n-1}\bar{x} \in H_*(\Sigma^{n-1} X)$ by $\eta_*(\bar{x}) \in H_*(X_n)$. For example,

$$\bar{\sigma}^{\text{split}}(\xi^2[s^{n-1}\bar{x}_1, s^{n-1}\bar{x}_2]) = Q_{(n-1)(p-1)} Q_{(n-1)(p-1)} \lambda_{n-1}(\eta_*(\bar{x}_1), \eta_*(\bar{x}_2)).$$

It follows from [1] that $\bar{\sigma}^{\text{split}}$ is a well-defined homomorphism, and the suspension relations, coupled with the fact that Q_0 is the pth power and λ_0 is the graded commutator, show that $\bar{\sigma}\bar{\sigma}^{\text{split}} = \text{id}$. Define $S_* \subset H_*(X_n)$ to be the image of $\bar{\sigma}^{\text{split}}$. We see that S_* is an isomorphic copy of $L_R \bar{H}_*(\Sigma^{n-1} X)$, except that degrees have been lowered by $n-1$.

If I is an admissible sequence with trailing term $i(p-1) - \varepsilon$, let $d(I)$ denote the set of nonnegative integers congruent to i mod 2. Then we may speak, for instance, of Q_I acting on $S_{d(I)}$. If I is the empty sequence, then let $S_{d(I)}$ be the set of all nonnegative integers. Using this notation, define $M_*(X_n)$, for $n > 1$, to be $\bigoplus Q_I S_{d(I)}$, with the direct sum taken over all admissible sequences I with leading term nonzero and trailing term not greater than $(n-1)(p-1) - 1$. Sequences meeting this criterion (including the empty sequence) will be referred to as *simple.*

We now state, in the notation of this section, Cohen's structure theorem:

Theorem 3.1. *Let $n > 1$. For any admissible sequence I with trailing term less than $(n-1)(p-1)$, the restriction of Q_I to $S_{d(I)}$ is a monomorphism. As a Hopf algebra, $H_*(X_n)$ is isomorphic to the free commutative algebra generated by*

$\bar{M}_*(X_n) = M_*(X_n) \cap \bar{H}_*(X_n)$, *with the coalgebra structure determined by the Cartan formulas for the Dyer–Lashof and Browder operations.*

In the definition of $M_*(X_n)$, the leading term must be nonzero because Q_0 is the pth power on homology, and so $Q_I\bar{x}$ is not a generator if I has leading term zero. On the other hand, the trailing term must be no more than $(n-1)(p-1)-1$ because $Q_{(n-1)(p-1)}\bar{x}$ is already accounted for as the class $\bar{\sigma}^{\mathrm{split}}\xi s^{n-1}\bar{x}$, and Q_r is undefined on $H_*(X_n)$ for $r > (n-1)(p-1)$.

In the case that $n=1$ we may define $M_*(X_n)$ as $\eta_* H_*(X)$, so that $M_*(X_n)$ will still be naturally isomorphic to $QH_*(X_n)$.

4. Defining the Dyer–Lashof splittings

In defining $\hat{Q}_I$, we will consider three cases of increasing generality: I simple, I with trailing term $(n-1)(p-1)$ (but leading term nonzero), and I with leading term zero. In all but the last case, n will be assumed greater than 1.

Case 1: *I simple.* In this case, $\hat{Q}_I$ will be the dual of a homomorphism $H_*(X_n) \to T\bar{H}_*(X)$, relying on the fact that, as a module, $(T\bar{H}_*(X))^*$ is naturally isomorphic to $T\bar{H}^*(X)$. Recall from Theorem 3.1 that $M_*(X_n)$ is defined to be the direct sum

$$\bigoplus_{I\ \text{simple}} Q_I S_{d(I)},$$

with each Q_I a monomorphism. Thus, for each simple I there is a splitting Q_I^{split}: $M_*(X_n) \to S_{d(I)}$ and we can construct the following composition:

$$\begin{aligned} H_*(X_n) &\twoheadrightarrow QH_*(X_n) \cong M_*(X_n) \xrightarrow{Q_I^{\mathrm{split}}} S_{d(I)} \\ &\xrightarrow{\bar{\sigma}} L_R\bar{H}_*(\Sigma^{n-1}X) \hookrightarrow T\bar{H}_*(\Sigma^{n-1}X) \xrightarrow{Ts^{1-n}} T\bar{H}_*(X). \end{aligned}$$

We define $\hat{Q}_I$ to be the dual homomorphism to this composition. Note that Ts^{1-n}, the result of applying the tensor algebra functor to the isomorphism $s^{1-n} : \bar{H}_*(\Sigma^{n-1}X) \to \bar{H}_{*-n+1}(X)$, is a ring isomorphism, but not a morphism of graded objects.

Case 2: *Trailing term* $(n-1)(p-1)$. For any k, let $I(k,t)$ equal k iterated t times. Let $k = (n-1)(p-1)$, let J be a simple sequence, and assume that the concatenation $JI(k,t)$ is admissible. Our goal is to define $\hat{Q}_{JI(k,t)}$. The difficulty in this case is that, as we noted earlier, the top homology operation $Q_{(n-1)(p-1)}$ is not a homomorphism and thus has no obvious splitting. We work around this problem by observing that the composition

$$\begin{aligned} (L_R\bar{H}_*(\Sigma^{n-1}X))_{\mathrm{even}} &\xrightarrow{\xi} L_R\bar{H}_*(\Sigma^{n-1}X) \\ &\twoheadrightarrow L_R\bar{H}_*(\Sigma^{n-1}X)/L\bar{H}_*(\Sigma^{n-1}X) \end{aligned}$$

is a homomorphism since the deviation from linearity of ξ is contained in $L\bar{H}_*(\Sigma^{n-1}X)$ (see, for instance, [1, III]). It follows from the Poincaré–Birkhoff–Witt theorem (see [6]) that

$$\xi^i : (L\bar{H}_*(\Sigma^{n-1}X))_{\text{even}} \to L_R\bar{H}_*(\Sigma^{n-1}X)/L\bar{H}_*(\Sigma^{n-1}X)$$

is a monomorphism for all $i \geq 1$, and so

$$L_R\bar{H}_*(\Sigma^{n-1}X)/L\bar{H}_*(\Sigma^{n-1}X) \cong \bigoplus_{i>0} \xi^i(L\bar{H}_*(\Sigma^{n-1}X))_{\text{even}}.$$

Thus for each i we have

$$(\xi^i)^{\text{split}} : L_R\bar{H}_*(\Sigma^{n-1}X)/L\bar{H}_*(\Sigma^{n-1}X) \to (L\bar{H}_*(\Sigma^{n-1}X))_{\text{even}}$$

which amounts to projection on the ith summand on the above direct sum splitting. Given J and $I(k,t)$, we define $\hat{Q}_{JI(k,t)}$ to be the dual of the following composition of homomorphisms:

$$\begin{aligned} H_*(X_n) \to M_*(X_n) &\xrightarrow{Q_J^{\text{split}}} S_{d(J)} \xrightarrow{\bar{\sigma}} L_R\bar{H}_*(\Sigma^{n-1}X) \\ &\twoheadrightarrow L_R\bar{H}_*(\Sigma^{n-1}X)/L\bar{H}_*(\Sigma^{n-1}X) \xrightarrow{(\xi^t)^{\text{split}}} (L\bar{H}_*(\Sigma^{n-1}X))_{\text{even}} \\ &\hookrightarrow T\bar{H}_*(\Sigma^{n-1}X) \xrightarrow{Ts^{1-n}} T\bar{H}_*(X). \end{aligned}$$

The reader should note that $\operatorname{im}\hat{Q}_{JI(k,t)} \subset \operatorname{im}\hat{Q}_J$. Thus, the definition of $\hat{Q}_I$ in the case of trailing term $(n-1)(p-1)$ is not necessary to define the set $\bar{Q}(X)$ of Theorem 1.1. However, as a way of labelling individual generators, this case is useful. In particular, the applications that have appeared [4, 7, 8] have used a variant of $Q_{(n-1)(p-1)}$.

Case 3: *Leading term zero.* The difficulty with defining $\hat{Q}_I$ in this case lies in the fact that Q_0 is the pth power on homology. Thus, for instance, $\hat{Q}_0x$ for x primitive should be a divided power $\gamma_p x$, characterized by the property that

$$\bar{\Delta}\gamma_p x = \sum_{\substack{i+j=p \\ i,j>0}} \frac{1}{i!\,j!} x^i \otimes x^j.$$

In general this property does not uniquely determine $\gamma_p x$, since the addition of a primitive does not change the reduced coproduct. However, in the special case of $H^*(X_n)$, we can make the following inductive definition:

Let x be an element of $PH^{\text{even}}(X_n)$ for $n > 1$. If $k < 2$, then let $\gamma_k x = x^k$. If $k \geq 2$ then let $\gamma_k x$ be an element y determined by the conditions

(1) $\bar{\Delta} y = \sum_{\substack{i+j=k \\ i,j>0}} \gamma_i x \otimes \gamma_j x,$

(2) $\langle y, \bar{a}\rangle = 0$ for any $\bar{a} \in M_*(X_n)$.

Proposition 4.1. *The class y, as defined above, exists and is unique.*

Proof. For both existence and uniqueness the proof is by induction, assuming that $\gamma_j x$ is already known to be well-defined for $j<k$. We note that there is no difficulty when $k=0$. We first prove existence. Let the subspace of $H^{|x|}(X_n)$ spanned by x be denoted $\langle x\rangle$, and write $PH^*(X_n)$ as $\langle x\rangle \oplus C^*$, where C^* is graded, with $C^{|x|}$ some complementary subspace to $\langle x\rangle$ in $PH^{|x|}(X_n)$. This splitting determines a dual splitting $M_*(X_n) \cong \langle \bar{x}\rangle \oplus C_*$, using the natural isomorphism $M_*(X_n) \cong QH_*(X_n)$. By Theorem 3.1, it follows that

$$H_{k|x|}(X_n) \cong \langle \bar{x}^k\rangle \oplus D,$$

with D spanned by products of the form $\prod_i \bar{w}_i$, where each $\bar{w}_i$ is in $M_*(X_n)$, and at least one factor $\bar{w}_i$ is in C_*. Then y is determined by

$$\langle y, \bar{x}^k\rangle = 1; \qquad \langle y, D\rangle = 0.$$

By construction, $\langle y, \bar{a}\rangle = 0$ for $\bar{a} \in M_*(X_n)$, and a simple calculation using the inductive hypothesis shows that y has the appropriate reduced coproduct.

To prove uniqueness, suppose y_1 and y_2 satisfy the definition of y. Then, since y_1 and y_2 have the same reduced coproduct, $y_1 - y_2$ must be primitive. Choose $\bar{y} \in H_*(X_n)$ such that $\langle y_1 - y_2, \bar{y}\rangle \neq 0$. Then, since $y_1 - y_2$ is primitive, the class $[\bar{y}] \in QH_*(X_n)$ must be nonzero. Hence we can write $\bar{y} = \bar{y}' + \bar{d}$ where $\bar{y}' \in M_*(X_n)$ and $\bar{d}$ is decomposable. But, since y_1 and y_2 were assumed to satisfy the definition of y, they must annihilate elements of $M_*(X_n)$. Hence $\langle y_1, \bar{y}'\rangle = \langle y_2, \bar{y}'\rangle = 0$. And, because $y_1 - y_2$ is primitive, $\langle y_1 - y_2, \bar{d}\rangle = 0$. Hence $\langle y_1 - y_2, \bar{y}\rangle = 0$, a contradiction. □

Now let J be a sequence with leading term nonzero such that the concatenation $I(0,t)J$ is admissible. For compactness of notation, let x denote an element of $T\bar{H}^*(X)$. If $n>1$ then we define $\hat{Q}_{I(0,t)J}x$ to be $\gamma_t \hat{Q}_J x$, bearing in mind that the definition of $\hat{Q}_J x$ ensures that it is primitive.

If $n=1$, then we take $\hat{Q}_{I(0,t)}x$ to be the image of $x^{\otimes p^t}$ under the isomorphism of coalgebras $T\bar{H}^*(X) \to H^*(X_1)$.

5. Properties of the Dyer–Lashof splittings

The following theorem and its corollary restate all but part 3 of the properties given in Theorem 1.1.

Theorem 5.1. *For I, J admissible, $x \in TH^*(X)$, and $\bar{x} \in S \subset H_*(X_n)$,*

$$\langle \hat{Q}_I x, Q_J \bar{x}\rangle = \delta_{IJ} \langle x, Ts^{1-n}\bar{\sigma}\bar{x}\rangle.$$

If I has leading term nonzero, then $\hat{Q}_I$ is a natural transformation of functors $T\bar{H}_*(\cdot)$ to $H_*(\Omega^n\Sigma^n\cdot)$. Otherwise, $\hat{Q}_I$ is a natural transformation of functors $\mathscr{S}T\bar{H}_*(\cdot)$ to $\mathscr{S}H_*(\Omega^n\Sigma^n\cdot)$, where $\mathscr{S}$ is the forgetful functor from the category of $\mathbb{Z}/p$-modules to the category of sets.

Corollary 5.2. *For I, J admissible, $x_i \in H^*(X)$, and $\bar{x}_i \in H_*(X) \hookrightarrow H_*(X_n)$,*

$$\begin{aligned}&\langle \hat{Q}_I(x_1,\ldots,x_m), Q_J\lambda_{n-1}(\bar{x}_1,\ldots,\lambda_{n-1}(\bar{x}_{m-1},\bar{x}_m)\ldots)\rangle\\ &\quad = \delta_{IJ}\langle s^{n-1}x_1\otimes\cdots\otimes s^{n-1}x_m, [s^{n-1}\bar{x}_1,\ldots,[s^{n-1}\bar{x}_{m-1},s^{n-1}\bar{x}_m]\ldots]\rangle.\end{aligned}$$

Proof of Theorem 5.1. The duality relation follows directly from the definition. If I has leading term nonzero, then $\hat{Q}_I$ is the dual of a composition of natural transformations, and hence natural. For the case of I with leading term zero, we will in fact show that the following diagram commutes for any k and any map $f: X \to Y$:

$$\begin{array}{ccc} PH^{\text{even}}(Y_n) & \xrightarrow{(\Omega^n\Sigma^n f)^*} & PH^{\text{even}}(X_n) \\ \big\downarrow{\scriptstyle \gamma_k} & & \big\downarrow{\scriptstyle \gamma_k} \\ H^*(Y_n) & \xrightarrow{(\Omega^n\Sigma^n f)^*} & H^*(X_n) \end{array}$$

The proof will be by induction. In the case $k=0$, commutativity is clear. For $k>0$, we will show that $(\Omega^n\Sigma^n f)^*\gamma_k y$ satisfies the two defining properties of $\gamma_k(\Omega^n\Sigma^n f)^* y$.

(1) By the naturality of the reduced coproduct and of γ_j for $j<k$,

$$\bar{\Delta}(\Omega^n\Sigma^n f)^*\gamma_k y = \sum_{\substack{i+j=k\\ i,j>0}} \gamma_i(\Omega^n\Sigma^n f)^* y\otimes\gamma_j(\Omega^n\Sigma^n f)^* y.$$

(2) For any $\bar{a}\in M_*(X_n)$

$$\langle(\Omega^n\Sigma^n f)^*\gamma_k y,\bar{a}\rangle = \langle\gamma_k y,(\Omega^n\Sigma^n f)_*\bar{a}\rangle = 0$$

because, by the naturality of the Dyer–Lashof operations with respect to n-fold loop maps, $(\Omega^n\Sigma^n f)_*\bar{a}$ must be an element of $M_*(Y_n)$. Thus γ_k is natural for all k, and hence $\hat{Q}_I$ is natural. □

Although $\hat{Q}_I$ is not a homomorphism if I has leading term zero, it behaves reasonably well with respect to the module structure of $H^*(X_n)$. It is easiest to state the results in terms of γ_k:

Proposition 5.3. *For any integer $k\geq 0$, and any $x_1,x_2\in PH^{\text{even}}(X_n)$,*

$$\begin{aligned}\gamma_k(x_1+x_2) &= \sum_{i+j=k}(\gamma_i x_1)(\gamma_j x_2)\\ &= \gamma_k x_1+\gamma_k x_2+\sum_{\substack{i+j=k\\ i,j\neq 0}}(\gamma_i x_1)(\gamma_j x_2),\end{aligned}$$

and, for any $c\in\mathbb{Z}/p$, $\gamma_k cx = c\gamma_k x$.

Proof. We prove the addition formula by induction. The result is trivial when $k=0$. Using the inductive hypothesis, a direct calculation shows that both sides of the equation have the same reduced coproduct. It remains to show that

$$\left\langle \sum_{i+j=k} (\gamma_i x_1)(\gamma_j x_2), \bar{a} \right\rangle = 0$$

for any $\bar{a} \in M_*(X_n)$. But it follows from the Cartan formula (Theorem 2.2) that the coproduct map takes $M_*(X_n)$ to $M_*(X_n) \otimes M_*(X_n)$. Thus

$$\langle (\gamma_i x_1)(\gamma_j x_2), \bar{a} \rangle = \langle \gamma_i x_1 \otimes \gamma_j x_2, \Delta \bar{a} \rangle = 0$$

since $(\gamma_i x_1)$ annihilates elements of $M_*(X_n)$.

The fact that $\gamma_k cx = c\gamma_k x$ follows easily by a similar method. □

Corollary 5.4. *If π denotes the projection $H^*(X_n) \to QH^*(X_n)$, then*

$$\pi\gamma_k : PH^{\text{even}}(X_n) \to QH^{\text{even}}(X_n)$$

is a homomorphism, nontrivial when $k = p^t$.

We now prove part (3) of Theorem 1.1.

Theorem 5.5. *The projection π maps $\bar{Q}(X)$ surjectively onto $QH^*(X_n)$.*

Proof. The theorem is true (but not helpful) when $n=1$ because $\bar{Q}(X) = H_*(\Omega\Sigma X)$. For the rest of the proof, let n be greater than 1, making $H_*(X_n)$ commutative as well as associative. Assume there exists a class $[a] \in QH^*(X_n)$ such that $[a] \notin \pi\bar{Q}(X)$. We can then choose $\bar{a} \in PH_*(X_n)$ such that $\langle [a], \bar{a} \rangle = 1$ but $\langle x, \bar{a} \rangle = 0$ for any $x \in \bar{Q}(X)$. Since $\bar{a}$ is primitive, it must be either indecomposable or a pth power.

If $\bar{a}$ is indecomposable, it can be written as

$$\sum_J Q_J \bar{a}_J + \bar{d}$$

for J simple, at least one $\bar{a}_J \in S_{d(J)}$ nonzero, and $\bar{d}$ decomposable. For some K such that $\bar{a}_K \neq 0$, choose $b \in TH^*(X)$ such that $\langle b, Ts^{1-n}\bar{\sigma}\bar{a}_K \rangle \neq 0$. By Theorem 5.1, it follows that $\langle \hat{Q}_K b, Q_K \bar{a}_K \rangle \neq 0$ and that, for $J \neq K$,

$$\langle \hat{Q}_K b, Q_J \bar{b}_i \rangle = \langle \hat{Q}_K b, \bar{d} \rangle = 0.$$

Hence $\langle \hat{Q}_K b, \bar{a} \rangle \neq 0$ even though $\hat{Q}_I b \in \bar{Q}(X)$, a contradiction.

If $\bar{a}$ is a pth power, it must be a pth power of a primitive. By induction, $\bar{a}$ must be of the form $(\bar{a}')^{p^t}$ for some t, where $\bar{a}'$ is indecomposable. Choosing b' with regard to $\bar{a}'$ just as b was chosen with regard to $\bar{a}$, we know that $\langle \hat{Q}_K b', \bar{a}' \rangle \neq 0$ for some appropriate sequence K. Then, since $\bar{a} = (\bar{a}')^{p^t}$,

$$\langle \gamma_{p^t} \hat{Q}_K b', \bar{a} \rangle \neq 0.$$

Again, this is a contradiction because $\gamma_{p^t} \hat{Q}_K b' = Q_{I(0,t)K} \in \bar{Q}(X)$. □

Unfortunately, the restriction of π to $\bar{Q}(X)$ is not injective. However, it is not hard to see that any class $\hat{Q}_I w \in \bar{Q}(X)$ is either indecomposable or a pth power. Thus the next theorem, which tells how the pth power map relates to the Dyer–Lashof splittings, makes it possible to determine in individual cases whether an element of $\bar{Q}(X)$ determines a generator.

Let the pth power map on cohomology be denoted ζ, to distinguish it from the restriction ξ on homology. Extend ζ on $\bar{H}^*(X)$ to $T\bar{H}^*(X)$ by defining it to be zero on $\bar{H}^*(X)^{\otimes m}$ for $m > 1$. If $I = (i_1(p-1), \dots, i_s(p-1))$ and $pi_s \leq n-1$ then let $pI = (pi_1(p-1), \dots, pi_s(p-1))$ and, conversely, define I/p to be J if I can be written as pJ. We adopt the convention that Q_{pI}, $Q_{I/p}$, $\hat{Q}_{pI}$, and $\hat{Q}_{I/p}$ are all the zero homomorphism if their respective indexing sequences are undefined. For instance, $\hat{Q}_{pI} = 0$ if Q_I contains a nontrivial Bockstein or $pi_s > n-1$.

Theorem 5.6. *For $w \in T\bar{H}^*(X)$, $\zeta \hat{Q}_I w = \hat{Q}_{pI} \zeta w$.*

Proof. First, we observe that ζ commutes with γ_k for any k. To see this, note that $\zeta \gamma_k x$ must have the correct coproduct since ζ is a morphism of Hopf algebras, and that $\zeta \gamma_k x$ annihilates $M_*(X_n)$ since ζ_* takes $M_*(X_n)$ to itself. Therefore our theorem will hold in general provided that it is true when I has leading term nonzero. We may thus assume that I has leading term nonzero for the rest of the proof.

With this assumption it will suffice to show that

$$(\zeta \hat{Q}_I)_* \bar{a} = (\hat{Q}_{pI} \zeta)_* \bar{a}$$

for all $\bar{a} \in H_*(X_n)$. If we allow I to represent either a sequence or a potentially undefined "sequence" of the form pJ for some J then, by the definition of $\hat{Q}_I$,

$$(\hat{Q}_I)_* = \begin{cases} Ts^{1-n} \bar{\sigma} Q_I^{\mathrm{split}} \pi & (I \text{ simple}) \\ Ts^{1-n} (\xi^t)^{\mathrm{split}} \phi \bar{\sigma} Q_K^{\mathrm{split}} \pi & (I \text{ has trailing term } (n-1)(p-1)) \\ 0 & (I \text{ undefined}) \end{cases}$$

where π is the projection $H_*(X_n) \twoheadrightarrow M_*(X_n)$, and ϕ is the projection

$$L_R \bar{H}_*(\Sigma^{n-1} X) \twoheadrightarrow L_R \bar{H}_*(\Sigma^{n-1} X) / L\bar{H}_*(\Sigma^{n-1} X).$$

Our strategy will be to independently evaluate both sides of the equation $(\zeta \hat{Q}_I)_* \bar{a} = (\hat{Q}_{pI} \zeta)_* \bar{a}$ using each of the three cases in the above formula for $(\hat{Q}_I)_*$. To do this we need to write an arbitrary $\bar{a}$ in an appropriate form. We begin by writing

$$\bar{a} = \sum_{J \text{ simple}} Q_J \bar{a}_J + \bar{d}$$

as in the proof of Theorem 5.5. Because

$$\begin{aligned} S_* / (\bar{\sigma}^{\mathrm{split}} L\bar{H}_*(\Sigma^{n-1} X)) &\cong L_R \bar{H}_*(\Sigma^{n-1} X) / L\bar{H}_*(\Sigma^{n-1} X) \\ &\cong \bigoplus_{t > 0} \xi^t (L\bar{H}_*(\Sigma^{n-1} X))_{\mathrm{even}}, \end{aligned}$$

the class of $\bar{a}_J$ in $S_*/(\bar{\sigma}^{\mathrm{split}} L\bar{H}_*(\Sigma^{n-1}X))$ may be written uniquely as

$$[\bar{a}_J] = [Q_{I(k,1)}\bar{a}_{J,1}] + [Q_{I(k,2)}\bar{a}_{J,2}] + \cdots,$$

where $k = (n-1)(p-1)$ and each $\bar{a}_{J,s} \in \bar{\sigma}^{\mathrm{split}} LH_*(\Sigma^{n-1}X)$. If

$$\bar{a}_{J,0} = \bar{a}_J - \sum_{s>0} Q_{I(k,s)}\bar{a}_{J,s}$$

then

$$\bar{a}_J = \sum_{s\geq 0} Q_{I(k,s)}\bar{a}_{J,s}$$

and each $\bar{a}_{J,s} \in \bar{\sigma}^{\mathrm{split}}(LH_*(\Sigma^{n-1}X))$. We thus have the (unique) decomposition

$$\bar{a} = \sum_{\substack{J \text{ simple} \\ s\geq 0}} Q_J Q_{I(k,s)}\bar{a}_{J,s} + \bar{d}.$$

It will be useful to further decompose each $\bar{a}_{J,t}$. Because

$$\bar{a}_{J,t} \in \bar{\sigma}^{\mathrm{split}} LH_*(\Sigma^{n-1}X),$$

we may write

$$\bar{a}_{J,t} = \eta_*\bar{b}_{J,t} + \sum \lambda_{n-1}(\bar{c}_{J,t,i}, \bar{c}'_{J,t,i})$$

for some $\bar{b}_{J,t} \in H_*(X)$ and $\bar{c}_{J,t,i}, \bar{c}'_{J,t,i} \in H_*(X_n)$. Here $\bar{b}_{J,t}$ is uniquely determined by $\bar{a}_{J,t}$, but $\bar{c}_{J,t,i}$ and $\bar{c}'_{J,t,i}$ are not. If I has the form $JI(k,t)$, then let $\bar{b}_I$ denote $\bar{b}_{J,t}$. If I is undefined, let $\bar{b}_I = 0$. We will show that

$$(\zeta\hat{Q}_I)_*\bar{a} = (\hat{Q}_{pI}\zeta)_*\bar{a} = \zeta_*\bar{b}_{pI}.$$

First we will show that $(\zeta\hat{Q}_I)_*\bar{a} = \zeta_*\bar{b}_{pI}$. If I is simple we have

$$Ts^{1-n}\bar{\sigma}Q_I^{\mathrm{split}}\pi\zeta_*\left(\sum_{\substack{J \text{ simple} \\ s\geq 0}} Q_J Q_{I(k,s)}\bar{a}_{J,s} + \bar{d}\right) = Ts^{1-n}\bar{\sigma}Q_I^{\mathrm{split}}\zeta_* \sum_{\substack{J \text{ simple} \\ s\geq 0}} Q_J Q_{I(k,s)}\bar{a}_{J,s}.$$

Wellington [9] has proven the general formula $\zeta_* Q_I\bar{x} = Q_{I/p}\zeta_*\bar{x}$. If we let $pI = KI(k,t)$, where t may be 0, then we can write $I = (K/p)I(k/p,t)$, where $I(k/p,0)$ is the empty sequence. Then

$$\begin{aligned} Ts^{1-n}\bar{\sigma}Q_I^{\mathrm{split}}\zeta_* \sum_{\substack{J \text{ simple} \\ s\geq 0}} Q_J Q_{I(k,s)}\bar{a}_{J,s} &= Ts^{1-n}\bar{\sigma}Q^{\mathrm{split}}_{(K/p)I(k/p,t)} \sum_{\substack{J \text{ simple} \\ s\geq 0}} Q_{J/p}Q_{I(k/p,s)}\zeta_*\bar{a}_{J,s} \\ &= Ts^{1-n}\bar{\sigma}\zeta_*\bar{a}_{K,t}. \end{aligned}$$

By [9], ζ_* annihilates classes in the image of the Browder operation λ_{n-1}, provided $n>0$. Thus

$$\zeta_*\bar{a}_{K,t}=\zeta_*\left(\eta_*\bar{b}_{K,t}+\sum\lambda_{n-1}(\bar{c}_{K,t,i},\bar{c}'_{K,t,i})\right)=\zeta_*\eta_*\bar{b}_{K,t}$$

and, since $\bar{b}_{K,t}=\bar{b}_{pI}$,

$$Ts^{1-n}\bar{\sigma}\zeta_*\bar{a}_{K,t}=Ts^{1-n}\bar{\sigma}\zeta_*\eta_*\bar{b}_{pI}=Ts^{1-n}\bar{\sigma}\eta_*\zeta_*\bar{b}_{pI}=Ts^{1-n}s^{n-1}\zeta_*\bar{b}_{pI}=\zeta_*\bar{b}_{pI}.$$

Thus, if I is simple, $(\zeta\hat{Q}_I)_*\bar{a}=\zeta_*\bar{b}_{pI}$.

If I has trailing term $(n-1)(p-1)$, then pI is undefined, so proving that $(\zeta\hat{Q}_I)_*\bar{a}=\zeta_*\bar{b}_{pI}$ amounts to proving that $(\zeta\hat{Q}_I)_*\bar{a}=0$. Let $I=KI(k,t)$. Then

$$\begin{aligned}Ts^{1-n}(\xi^t)^{\mathrm{split}}\phi\bar{\sigma}Q_K^{\mathrm{split}}\pi\zeta_*&\left(\sum_{\substack{J\text{ simple}\\ s\geq 0}}Q_JQ_{I(k,s)}\bar{a}_{J,s}+\bar{d}\right)\\ &=Ts^{1-n}(\xi^t)^{\mathrm{split}}\phi\bar{\sigma}Q_K^{\mathrm{split}}\left(\sum_{\substack{J\text{ simple}\\ s\geq 0}}Q_{J/p}Q_{I(k/p,s)}\zeta_*\bar{a}_{J,s}\right).\end{aligned}$$

But observe that, for any J and s, the sequence $(J/p)I(k/p,s)$ is, if defined, simple. If $K\neq(J/p)I(k/p,s)$ for any J or s, then

$$Q_K^{\mathrm{split}}\left(\sum_{\substack{J\text{ simple}\\ s\geq 0}}Q_{J/p}Q_{I(k/p,s)}\zeta_*\bar{a}_{J,s}\right)=0.$$

If $K=(L/p)I(k/p,t)$ for some L and t, then

$$Ts^{1-n}(\xi^t)^{\mathrm{split}}\phi\bar{\sigma}Q_K^{\mathrm{split}}\left(\sum_{\substack{J\text{ simple}\\ s\geq 0}}Q_{J/p}Q_{I(k/p,s)}\zeta_*\bar{a}_{J,s}\right)=Ts^{1-n}(\xi^t)^{\mathrm{split}}\phi\bar{\sigma}\zeta_*\bar{a}_{L,t}=0$$

because $\bar{\sigma}\zeta_*\bar{a}_{L,t}\in L\tilde{H}_*(\sum^{n-1}X)$.

If I is undefined, then $\hat{Q}_I$ is the zero homomorphism, so

$$(\zeta\hat{Q}_I)_*\bar{a}=0=\zeta_*\bar{b}_{pI}.$$

This completes our evaluation of the left-hand side of the equation $(\zeta\hat{Q}_I)_*\bar{a}=(\hat{Q}_{pI}\zeta)_*\bar{a}$; now we must show that $(\hat{Q}_{pI}\zeta)_*\bar{a}=\zeta_*\bar{b}_{pI}$. If pI is simple,

$$\zeta_*Ts^{1-n}\bar{\sigma}Q_{pI}^{\mathrm{split}}\pi\left(\sum_{\substack{J\text{ simple}\\ s\geq 0}}Q_JQ_{I(k,s)}\bar{a}_{J,s}+\bar{d}\right)=\zeta_*Ts^{1-n}\sum_{s\geq 0}\xi^s\bar{\sigma}\bar{a}_{pI,s}.$$

Now

$$\bar{\sigma}\bar{a}_{pI,s} = s^{n-1}b_{pI,s} + \sum_i [\bar{\sigma}\bar{c}_{pI,s,i}, \bar{\sigma}\bar{c}'_{pI,s,i}].$$

Since ζ_* annihilates tensor algebra decomposables in $T\bar{H}_*(\Sigma^{n-1}X)$,

$$\zeta_* Ts^{1-n} \sum_{s \ge 0} \xi^s \bar{\sigma}\bar{a}_{pI,s} = \zeta_* \bar{b}_{pI,0} = \zeta_* \bar{b}_{pI}$$

noting that the concatenation $(pI)I(k,0)$ is the same as the sequence pI.

Now assume that pI has trailing term $(n-1)(p-1)$. By the isomorphism

$$L_R\bar{H}_*(\Sigma^{n-1}X)/L\bar{H}_*(\Sigma^{n-1}X) \cong \bigoplus_{t>0} \xi^t (L\bar{H}_*(\Sigma^{n-1}X))_{\text{even}},$$

observe that $\phi \sum_{s \ge 0} \xi^s \bar{x}_s = \sum_{s>0} \xi^s \bar{x}_s$. Using this fact,

$$\begin{aligned}
&\zeta_* Ts^{1-n}(\xi^t)^{\text{split}} \phi \bar{\sigma} Q_K^{\text{split}} \pi \left(\sum_{\substack{J \text{ simple} \\ s \ge 0}} Q_J Q_{I(k,s)} \bar{a}_{J,s} + \bar{d} \right) \\
&\quad = \zeta_* Ts^{1-n}(\xi^t)^{\text{split}} \sum_{s>0} \xi^s \bar{\sigma}\bar{a}_{K,s} \\
&\quad = \zeta_* Ts^{1-n} \bar{\sigma}\bar{a}_{K,t}.
\end{aligned}$$

As with the case of pI simple,

$$\zeta_* Ts^{1-n} \bar{\sigma}\bar{a}_{K,t} = \zeta_* \bar{b}_{pI}.$$

Finally, if pI is undefined then $(\hat{Q}_{pI}\zeta)_* \bar{a} = 0 = \zeta_* \bar{b}_{pI}$. □

We use similar techniques to prove our concluding theorem, which relates the suspension homomorphism $\sigma^* : H^*(\Omega^n \Sigma^{n+1} X) \to H^{*-1}(\Omega^{n+1}\Sigma^{n+1}X)$ to the Dyer–Lashof splittings. We have been allowing s to represent both the isomorphisms $\bar{H}_*(X) \to \bar{H}_{*+1}(\Sigma X)$ and $\bar{H}^*(X) \to \bar{H}^{*+1}(\Sigma X)$. Under this convention, the dual homomorphism to s is s^{-1}.

Theorem 5.7. *If* $I = (i_1(p-1) - \varepsilon_1, \ldots, i_s(p-1) - \varepsilon_s)$, *then*

$$\sigma^* \hat{Q}_I Tsw = \hat{Q}_{I^{(+1)}} w,$$

where $I^{(+1)} = ((i_1 + 1)(p-1) - \varepsilon_1, \ldots, (i_s + 1)(p-1) - \varepsilon_s)$.

Proof. When I is simple or has trailing term $(n-1)(p-1)$, the proof is similar to that of Theorem 5.6. It is useful to write

$$(\sigma_*)^{n-1} : H_*(X_n) \to T\bar{H}_*(\Sigma^{n-1}X)$$

for $\bar{\sigma}$. Then, when I is simple, we must prove

$$(Ts^{-1})Ts^{1-n}(\sigma_*)^{n-1}Q_I^{\mathrm{split}}\pi\sigma_*\bar{a} = Ts^{-n}(\sigma_*)^n Q_{I^{(+1)}}^{\mathrm{split}}\pi\bar{a}.$$

Notice that on the right-hand side of the equation we are working with $\Omega^{n+1}\Sigma^{n+1}X$, so we must replace n by $n+1$ in the composition of functions we use to define $\hat{Q}_{I^{(+1)}}$. We will again write $\bar{a} = \sum_{J \text{ simple}} Q_J\bar{a}_J + \bar{d}$. Also, if $I = (i_1(p-1)-\varepsilon_1, \ldots, i_s(p-1)-\varepsilon_s)$, then we will write $I^{(-1)}$ for $((i_1-1)(p-1)-\varepsilon_1, \ldots, (i_s-1)(p-1)-\varepsilon_s)$. If I has leading term 0, then $Q_{I^{(-1)}} = 0$. Then

$$(Ts^{-1})Ts^{1-n}(\sigma_*)^{n-1}Q_I^{\mathrm{split}}\pi\sigma_*\left(\sum_{J \text{ simple}} Q_J\bar{a}_J + \bar{d}\right)$$

$$= Ts^{-n}(\sigma_*)^{n-1}Q_I^{\mathrm{split}}\sum_{J \text{ simple}} Q_{J^{(-1)}}\sigma_*\bar{a}_J$$

$$= Ts^{-n}(\sigma_*)^n\bar{a}_{I^{(+1)}},$$

while

$$Ts^{-n}(\sigma_*)^n Q_{I^{(+1)}}^{\mathrm{split}}\pi\left(\sum_{J \text{ simple}} Q_J\bar{a}_J + \bar{d}\right) = Ts^{-n}(\sigma_*)^n\bar{a}_{I^{(+1)}}.$$

This completes the proof in the case that I is simple.

If I has trailing term $(n-1)(p-1)$ then we write $\bar{a}$ as in the proof of Theorem 5.6 and I as $KI((n-1)(p-1),t)$, and we must show that

$$(Ts^{-1})Ts^{1-n}(\xi^t)^{\mathrm{split}}\phi(\sigma_*)^{n-1}Q_K^{\mathrm{split}}\pi\sigma_*\bar{a} = Ts^{-n}(\xi^t)^{\mathrm{split}}\phi(\sigma_*)^n Q_{K^{(+1)}}^{\mathrm{split}}\pi\bar{a}.$$

A calculation similar to the previous ones shows that both sides of the equation are equal to $Ts^{-n}(\sigma_*)^n\bar{a}_{K^{(+1)},t}$.

The leading-term-zero case is more complicated. If $I = I(0,t)J$, then $I^{(+1)} = I(p-1,t)J^{(+1)}$, and we are trying to show that

$$\sigma^*\gamma_{p'}\hat{Q}_J Tsw = \hat{Q}_{I^{(+1)}}w.$$

We will prove that

$$\langle\sigma^*\gamma_{p'}\hat{Q}_J Tsw, \bar{a}\rangle = \langle\hat{Q}_{I^{(+1)}}w, \bar{a}\rangle$$

for all $\bar{a} \in H_*(X_{n+1})$. Again we consider each side of the equation separately, proving that both sides are equal to $\langle\hat{Q}_{J^{(+1)}}w, \bar{a}_t\rangle$. With regard to the left-hand side,

$$\langle\sigma^*\gamma_{p'}\hat{Q}_J Tsw, \bar{a}\rangle = \langle\gamma_{p'}\hat{Q}_J Tsw, \sigma_*\bar{a}\rangle.$$

Since σ_* annihilates decomposables, we can assume that $\bar{a} \in M_*(X_{n+1})$. By the definitions of $M_*(\cdot)$ and of admissible sequences, we can write

$$\bar{a} = Q_{p-2}\bar{a}' + \sum_{i \geq 0} Q_{I(p-1,i)}\bar{a}_i,$$

where $\bar{a}', \bar{a}_i \in M_*(X_{n+1})$ have the property that $Q_{p-2}\bar{a}'$ and $Q_{I(p-1,i)}\bar{a}_i$ are also in $M_*(X_{n+1})$. Thus by Theorem 2.1,

$$\sigma_*\bar{a} = \sum_{i\geq 0} Q_{I(0,i)}\sigma_*\bar{a}_i = \sum_{i\geq 0} (\sigma_*\bar{a}_i)^{p^i},$$

noting that $\sigma_* Q_{p-2}\bar{a}' = \beta(\sigma_*\bar{a}')^p = 0$.

We will defer the proof of the following lemma to the end of this section:

Lemma 5.8. *If $z \in PH^*(\Omega^n\Sigma^{n+1}X)$, and $\bar{b}_i \in M_*(\Omega^n\Sigma^{n-1}X)$ for all i, then*

$$\left\langle \gamma_{p^t}z, \sum \bar{b}_i^{p^i} \right\rangle = \langle z, \bar{b}_t \rangle.$$

By the lemma, since $\hat{Q}_J Tsw$ is primitive and each $\sigma_*\bar{a}_i$ is in $M_*(\Omega^n\Sigma^{n+1}X)$, we have

$$\langle \gamma_{p^t}\hat{Q}_J Tsw, \sigma_*\bar{a} \rangle = \langle \hat{Q}_J Tsw, \sigma_*\bar{a}_t \rangle.$$

Using the theorem in the case of J simple, we know that

$$\langle \hat{Q}_J Tsw, \bar{\sigma}\bar{a}_t \rangle = \langle \hat{Q}_{J^{(+1)}} w, \bar{a}_t \rangle$$

and so $\langle \gamma_{p^t}\hat{Q}_J Tsw, \sigma_*\bar{a} \rangle = \langle \hat{Q}_{J^{(+1)}} w, \bar{a}_t \rangle$, as we wanted.

To finish the proof we must show that $\langle \hat{Q}_{I^{(+1)}} w, \bar{a} \rangle = \langle \hat{Q}_{J^{(+1)}} w, \bar{a}_t \rangle$. By definition, $\langle \hat{Q}_{I^{(+1)}} w, \bar{a} \rangle = \langle w, Ts^{-n}(\sigma_*)^n Q^{\text{split}}_{I^{(+1)}} \pi\bar{a} \rangle$. Because we are already assuming that $\bar{a} \in M_*(\Omega^{n+1}\Sigma^{n+1}X)$, we can disregard the homomorphism π, and, since $I^{(+1)} = I(p-1,t)J^{(+1)}$,

$$Q^{\text{split}}_{I^{(+1)}}\bar{a} = Q^{\text{split}}_{I^{(+1)}} \left(Q_{p-2}\bar{a}' + \sum_{i\geq 0} Q_{I(p-1,i)}\bar{a}_i \right) = Q^{\text{split}}_{I^{(+1)}} Q_{I(p-1,t)}\bar{a}_t = Q^{\text{split}}_{J^{(+1)}}\bar{a}_t.$$

Thus

$$\langle w, Ts^{-n}(\sigma_*)^n Q^{\text{split}}_{I^{(+1)}}\pi\bar{a} \rangle = \langle w, Ts^{-n}(\sigma_*)^n Q^{\text{split}}_{J^{(+1)}}\pi\bar{a}_t \rangle = \langle \hat{Q}_{J^{(+1)}} w, \bar{a}_t \rangle$$

and so, finally, $\langle \hat{Q}_{I^{(+1)}} w, \bar{a} \rangle = \langle \hat{Q}_{J^{(+1)}} w, \bar{a}_t \rangle$ as desired.

Proof of Lemma 5.8. We calculate:

$$\begin{aligned}\langle \gamma_{p^t}z, \bar{b}_i^{p^i} \rangle &= \langle \gamma_{p^t}z, \bar{\mu}_*(\bar{\mu}_* \otimes 1)\cdots(\bar{\mu}_* \otimes 1 \otimes \cdots \otimes 1)\bar{b}_i \otimes \cdots \otimes \bar{b}_i \rangle \\ &= \langle (\bar{\Delta} \otimes 1 \otimes \cdots \otimes 1)\cdots(\bar{\Delta} \otimes 1)\bar{\Delta}\gamma_{p^t}z, \bar{b}_i \otimes \cdots \otimes \bar{b}_i \rangle.\end{aligned}$$

Here we let $\bar{\mu}_*$ denote the Pontryagin product restricted to reduced homology, so that its dual homomorphism is the reduced coproduct. Using our definition of γ_t, one can compute that

$$\langle(\bar{\Delta}\otimes 1^{\otimes p^t-2})\cdots(\bar{\Delta}\otimes 1)\bar{\Delta}\gamma_{p^t}z,\bar{b}_i^{\otimes p^t}\rangle=\begin{cases}\langle(\gamma_{p^t-i}z)^{\otimes p^i},\bar{b}_i^{\otimes p^i}\rangle=0 & (i<t)\\ \langle z^{\otimes p^t},\bar{b}_t^{\otimes p^t}\rangle=\langle z,\bar{b}_t\rangle & (i=t)\\ \langle 0,\bar{b}_i^{\otimes p^i}\rangle=0 & (i>t)\end{cases}$$

The result follows. □

6. Variants

There are a number of other circumstances under which analogous results apply. In this section we briefly sketch the relevant differences.

First we consider the prime 2. When $p=2$, the Dyer–Lashof operations in lower notation take the form

$$Q_i: H_q(\Omega^n X)\to H_{2q+i}(\Omega^n X),$$

where q is arbitrary. A sequence $I=(i_1,\ldots,i_s)$ is admissible provided that $0\le i_j\le i_{j+1}$ for all j, and simple if $i_1>0$ and $i_s<n-1$. Cohen's structure theorem differs only in that $M_*(X_n)$ is defined to be $\bigoplus Q_I S_*$, where the sum ranges over all simple I. We need not use $S_{d(I)}$, since Q_I can act on elements in any degree. With these changes in the setup, the definitions of the $\hat{Q}_I$, and the proofs of their properties, go exactly as in the odd primary case.

At both odd primes and the prime 2, it is easy to see that the definition of $\hat{Q}_I$ can be carried over to the infinite loop space $QX=\varinjlim \Omega^n\Sigma^n X$. The structure theorem for $H_*(QX)$ again takes the same form as the theorem for $H_*(\Omega^n\Sigma^n X)$, except that $M_*(QX)$ is defined to be $\bigoplus Q_I\eta_*\bar{H}_{d(I)}(X)$ (or, if $p=2$, $\bigoplus Q_I\eta_*\bar{H}_*(X)$), where I ranges over all admissible sequences with leading term nonzero. Since there are no nontrivial Browder operations in $H_*(QX)$, the subspace S_* does not appear, and the Dyer–Lashof splittings $\hat{Q}_I$ are defined on $H^*(X)$ rather than on $TH^*(X)$.

Finally, we note that all of our results apply without change to the Milgram–May combinatorial models CX and C_nX, for QX and $\Sigma^n\Omega^n X$, respectively.

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