Spectral Radius, Norms of Iterates, and the Critical Exponent

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1. INTRODUCTION

Let B be a Banach space and denote by $L(B)$ the Banach algebra of all bounded linear operators on B. If $A \in L(B)$, then the connection between the spectral radius $|A|_\sigma$ of A and the norms of the successive powers of A is given by the well-known formula

$$|A|_\sigma = \lim \sqrt[r]{|A^r|}.$$

This formula is, in fact, nothing more than the statement that the radius of convergence of the power series $E + \lambda A + \lambda^2 A^2 + \cdots$ coincides with the reciprocal of the spectral radius of A.

In particular, the series $E + A + A^2 + \cdots$ will be convergent if and only if $|A|_\sigma < 1$ and this is equivalent to the requirement that $|A^r| < 1$ for some r. Hence if $|A| = 1$ and $|A|_\sigma < 1$ there is some power of A which will be < 1. It is thus natural to ask how far one has to go in order to find a power $|A^r| < 1$ and, furthermore, if these exponents have a common bound. More precisely, let us denote by $\mathscr{C}$ the set of all operators A with $|A| \leqq 1$ and $|A|_\sigma < 1$. For each $A \in \mathscr{C}$, let us denote by $e(A)$ the smallest exponent r for which $|A^r| < 1$. Is there a common bound for the function $e(A)$ on $\mathscr{C}$? This leads to the following definition.

1. *Let B be a finite dimensional Banach space. The number q is said to be the critical exponent of the space B if the following two conditions are satisfied:*

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(1) *if* $A \in L(B)$ *and* $|A| = |A^q| = 1$, *then* $|A|_\sigma = 1$;

(2) *there exists a* $T \in L(B)$ *such that*

$$|T| = |T^{q-1}| = 1 \quad \textit{and} \quad |T|_\sigma < 1.$$

The problem of the existence of the critical exponent was first introduced and solved by J. Mařík and the present author [1] for the n-dimensional space with norm $|x| = \max|x_i|$. The critical exponent turns out to be $n^2 - n + 1$. Later, the present author [3] showed that the critical exponent of the n-dimensional Euclidean space is equal to n. Since the critical exponent of a space B and of its adjoint B' are clearly equal, the critical exponent of the n-dimensional space with norm $|x| = \sum |x_i|$ is the same as that of the first space. All these spaces belong to the class of Hölder spaces of type l_p, which may be described as follows.

Given any natural number n and any number p such that $1 \leqq p \leqq \infty$, we shall denote by $B_{n,p}$ the (real or complex) n-dimensional vector space, the norm of the vector $x = (x_1, \ldots, x_n)$ being defined by the formula

$$|x| = (\sum |x_i|^p)^{1/p}.$$

Of course, this reduces to $|x| = \max|x_i|$ if $p = \infty$.

If we agree to write $q(B)$ for the critical exponent of B, provided it is finite, the results mentioned above may be reformulated as follows:

$$q(B_{n,\infty}) = q(B_{n,1}) = n^2 - n + 1;$$

$$q(B_{n,2}) = n.$$

The existence of the critical exponent for finite dimensional l_p spaces, p different from 1, 2, and ∞, is still an open problem. For certain particular values of p, its existence has been announced by M. Perles [2]; however, the bounds that he has been able to give are very large.

The failure of the attempts to compute the critical exponent of l_p spaces is largely due to the fact that, in a certain sense, the definition of the critical exponent is based on a qualitative statement: if $|A| = 1$ and $|A^q| = 1$ then the spectral radius $|A|_\sigma = 1$. It is the purpose of the present note to point out that the negative restatement of the definition of the critical exponent can very easily be given a quantitative character; this leads to many interesting problems, some of which might be of interest for immediate applications in numerical analysis.

We begin by defining, for each finite dimensional Banach space B, a series of constants which describes the behavior of the norms of the successive powers of linear operators in B.

2. *Given a Banach space B, a number $0 \leqq \rho < 1$, and a natural number r we shall denote by $C(B, \rho, r)$ the number*

$$C(B, \rho, r) = \sup\{|A^r|; A \in L(B), |A| \leqq 1, |A|_\sigma \leqq \rho\}.$$

Clearly $0 \leqq C(B, \rho, r) \leqq 1$ for any Banach space B, any $0 \leqq \rho < 1$, and any r. Furthermore, $C(B, \rho, r + 1) \leqq C(B, \rho, r)$.

Let us first clear up the connection of these constants with the critical exponent.

The following lemma is based on the continuity of the spectrum as a function of the operator A.

3. *Let B be a finite dimensional Banach space and let q be a natural number. Then the following two statements are equivalent:*

(1) *$q \geqq q(B)$, the critical exponent of B;*

(2) *$C(B, \rho, q) < 1$ for each $0 \leqq \rho < 1$.*

Proof. Suppose first that (2) is satisfied and that A is a linear operator on B such that $|A| = 1$ and $|A^q| = 1$. Suppose that $|A|_\sigma < 1$. It follows from the definition of our constants $C(B, \rho, q)$ that

$$1 = |A^q| \leqq C(B, |A|_\sigma, q) < 1,$$

which is a contradiction.

On the other hand, assume (1) and suppose that $C(B, \rho, q) = 1$ for some $\rho < 1$. It follows that there exists a sequence $A_n \in L(B)$ such that $|A_n| \leqq 1$, $|A_n|_\sigma \leqq \rho$ and $\lim|A_n{}^q| = 1$. The unit sphere in $L(B)$ being compact, there exists an infinite set R of real numbers such that the subsequence A_n, $n \in R$, converges to some operator A_0. Since $|A_n| \leqq 1$ and $|A_n|_\sigma \leqq \rho$ for each n, it follows that $|A_0| \leqq 1$ and $|A_0|_\sigma \leqq \rho$, the second inequality being a consequence of the continuity of the spectrum as a function of the operator. At the same time $|A_0{}^q| = \lim_{n \in R}|A_n{}^q| = 1$. Hence $|A_0| = |A_0{}^q| = 1$ and $|A_0|_\sigma \leqq \rho < 1$, so that $q \geqq q(B)$ is impossible.

It is the purpose of the present note to compute the constants $C(B_{n,2}, \rho, n)$ for n-dimensional Hilbert space. We propose to do so by constructing, for each $\rho < 1$, a certain operator $A(\rho)$ with $|A(\rho)| = 1$, $|A(\rho)|_\sigma = \rho$, and

$$|A(\rho)^n| = \max\{|A^n; |A| \leqq 1, |A|_\sigma \leqq \rho\}.$$

There is little doubt that, once the result is known, shorter ways of obtaining $C(B_{n,2}, \rho, n)$ will be devised; nevertheless we feel that the present approach is of interest inasmuch as it provides additional information about the behavior of iterates of operators.

2. NOTATION AND PRELIMINARIES

The algebra of all complex-valued matrices of type (n, n) will be denoted by $\mathcal{M}_n$.

Let E be an n-dimensional Hilbert space with scalar product (x, y) and norm $|x|$.

If B is a sequence of n vectors $b_1, \ldots, b_n$ in E, we shall denote by $G(B)$ or $G(b_1, \ldots, b_n)$ the Gram matrix of B. The elements g_{ik} of $G(B)$ are defined as $g_{ik} = (g_i, g_k)$ for $1 \leqq i, k \leqq n$.

If W is a matrix of type (n, n) with elements w_{ik}, we can form another sequence of vectors $c_i = \sum_k w_{ik} b_k$. It is easy to verify that

$$G(c_1, \ldots, c_n) = WG(b_1, \ldots, b_n)W^*.$$

The matrix $G(B)$ is always positive semidefinite; further, $G(B)$ is positive definite if and only if B is a basis; in other words, if and only if the vectors $b_1, \ldots, b_n$ are linearly independent.

If B is a basis of E and if $x \in E$ is given, we shall denote by $M(x; B)$ the (row) vector of the coordinates of x with respect to the basis B so that $M(x; B) = (\xi_1, \ldots, \xi_n)$ is equivalent to $x = \xi_1 b_1 + \cdots + \xi_n b_n$.

The algebra of all linear operators on E will be denoted by $L(E)$. Now let U and V be two bases of E consisting respectively of the vectors $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$. If $A \in L(E)$, the matrix of A in the bases U and V will be denoted by $M(A; U, V)$. Its ith row is taken to be $M(Au_i; V)$ so that

$$Au_i = \sum_k m_{ik} v_k.$$

Using this notation, we obtain, for each $x \in E$,

$$M(Ax; V) = M(x; U)M(A; U, V).$$

If U, V, W are three bases of E and $A, B \in L(E)$, then

$$M(AB; U, W) = M(B; U, V)M(A; V, W).$$

We shall frequently be using the following lemma:

4. *Let $A \in L(E)$ and let U, V be two bases of E. Denote by M the matrix $M(A; U, V)$. Then $|A| \leqq \lambda$ is equivalent to*

$$MG(V)M^* \leqq \lambda^2 G(U).$$

Proof. Let $x \in E$ be given, let $y = Ax$, and put $p = M(x; U)$, $q = M(y; V)$. Clearly

$$|x^2| = |\sum p_i u_i|^2 = pG(U)p^*.$$

Since $q = pM$, we have

$$|y|^2 = |\sum q_i v_i|^2 = qG(V)q^* = pMG(V)M^*p^*.$$

The inequality $|y|^2 \leqq \lambda^2|x|^2$ for each x is thus equivalent to the inequality

$$pMG(V)M^*p^* \leqq pG(U)p^*$$

for each p.

It is not difficult to see that $L(E)$ itself is a Hilbert space under the scalar product $(A, T) = \operatorname{tr} T^*A$. Hence every linear functional f on $L(E)$ may be obtained in the form

$$f(A) = \operatorname{tr}(W^*A)$$

for a suitable $W \in L(E)$. In particular, for fixed x and y, the expression (Ax, y) is a linear functional on $L(E)$. It is not difficult to see that

$$(Ax, y) = (A, T),$$

where T is the one-dimensional operator defined by $Tu = (u, x)y$.

3. THE MAXIMUM PROBLEM FOR OPERATORS SATISFYING A GIVEN CAYLEY-HAMILTON EQUATION

In the present section we intend to solve the maximum problem for the class of all operators which satisfy a given equation of the Cayley-Hamilton type.

Suppose we are given n complex numbers $\alpha_1, \ldots, \alpha_n$ such that all roots of the equation $x^n = \alpha_1 + \alpha_2 x + \cdots + \alpha_n x^{n-1}$ are < 1 in absolute value. To simplify the notation, we shall write simply a for the vector $a = (\alpha_1, \ldots, \alpha_n)$. We intend to investigate the class $\mathscr{A}$ of all operators $A \in L(E)$ such that $|A| \leqq 1$ and

$$A^n = \alpha_1 + \alpha_2 A + \cdots + \alpha_n A^{n-1}.$$

Clearly this polynomial identity is satisfied if and only if the minimal polynomial of A is a divisor of $x^n - (\alpha_1 + \alpha_2 x + \cdots + \alpha_n x^{n-1})$.

It will be useful to establish a connection of our class $\mathscr{A}$ with a class of matrices $\mathscr{Z}$, defined as follows. We denote by T the matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{pmatrix}$$

and observe that the characteristic polynomial of T is $\lambda^n - (\alpha_1 + \alpha_2\lambda + \cdots + \alpha_n\lambda^{n-1})$. We take $\mathscr{Z}$ to be the class of all (hermitian) symmetric matrices $Z \in \mathscr{M}_n$ which satisfy $TZT^* \leqq Z$ and $z_{11} = 1$.

In the following proposition we shall learn how to associate, with each vector $z \in E$ with $|z| = 1$, a certain mapping g which establishes a connection between $\mathscr{A}$ and $\mathscr{Z}$:

5. *Let $z \in E$ be a given vector with $|z| = 1$. Let g be the mapping of $L(E)$ into $\mathscr{M}_n$ which assigns to every $S \in L(E)$ the matrix*

$$g(S) = G(z, Sz, S^2z, \ldots, S^{n-1}z).$$

Then $g(\mathscr{A}) = \mathscr{Z}$.

Proof. Let $A \in \mathscr{A}$. For $i = 1, 2, \ldots, n$ define z_i as $A^{i-1}z$ so that $g(A) = G(z_1, \ldots, z_n)$. Consider now the vectors w_i defined by the relation $w_i = \sum_k t_{ik} z_k$ so that

$$Tg(A)T^* = TG(z_1, \ldots, z_n)T^* = G(w_1, \ldots, w_n).$$

At the same time, A being a contraction, we have $G(Az_1, \ldots, Az_n) \leqq G(z_1, \ldots, z_n)$ by Lemma 4. If we show that $Az_i = w_i$, we shall have, combining this with the above equation, $Tg(A)T^* \leqq g(A)$. Since $(z_1, z_1) =$

$(z, z) = 1$, this will show that $g(A) \in \mathscr{Z}$. To show that $Az_i = w_i$ take first the case $i < n$. Clearly $w_i = z_{i+1}$; at the same time $Az_i = z_{i+1}$ as well, so that $Az_i = w_i$. If $i = n$, we have $w_n = \alpha_1 z_1 + \cdots + \alpha_n z_n$. Now $A^n = \alpha_1 E + \alpha_2 A + \cdots + \alpha_n A^{n-1}$, whence $A^n z = \alpha_1 z + \alpha_2 Az + \cdots + \alpha_n A^{n-1} z = \alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_n z_n = w_n$. It follows that $w_n = A^n z = A(A^{n-1}z) = Az_n$ and the proof is complete.

On the other hand let $Z \in \mathscr{Z}$. Since $TZT^* \leqq Z$ it follows by induction that $T^r Z T^{*r} \leqq Z$. We note first that the characteristic polynomial of the matrix T is $x^n = \alpha_1 + \alpha_2 x + \cdots + \alpha_n x^{n-1}$ so that the spectral radius of T is less than one. It follows that $\lim T^r = 0$ so that, passing to the limit, we obtain $0 \leqq Z$. It follows that there exist vectors $z_1, \ldots, z_n \in E$ such that $Z = G(z_1, z_2, \ldots, z_n)$. Since $z_{11} = 1$, the first vector z_1 has norm 1 so that, taking a suitable unitary transformation, we may assume $z_1 = z$. Define now vectors $w_1, \ldots, w_n$ as follows:

$$w_i = z_{i+1} \quad \text{for} \quad 1 \leqq i < n;$$

$$w_n = \alpha_1 z_1 + \cdots + \alpha_n z_n.$$

Let us show now that, for each $\xi_1, \ldots, \xi_n$, the inequality

$$|\xi_1 w_1 + \cdots + \xi_n w_n|^2 \leqq |\xi_1 z_1 + \cdots + \xi_n z_n|^2$$

is satisfied. To see that denote by u the row vector $(\xi_1, \ldots, \xi_n)$. Since $w_i = \sum_k t_{ik} z_k$, we have, since $TZT^* \leqq Z$,

$$|\sum \xi_i w_i|^2 = \left(\sum \xi_i w_i, \sum \xi_j w_j\right) = uG(w_1, \ldots, w_n)u^*$$

$$= uTG(z_1, \ldots, z_n)T^*u^* = uTZT^*u^* \leqq uZu^* = |\sum \xi_i z_i|^2$$

so that the inequality is established. In particular, it follows from this inequality that a relation of the form $\sum \xi_i z_i = 0$ implies $\sum \xi_i w_i = 0$. Accordingly there exists on the subspace E_0 generated by $z_1, \ldots, z_n$ a linear operator A_0 which takes z_i into w_i. Let us extend A_0 to an operator A on the whole of E by putting $A = 0$ on $E_0^\perp$. Let us show first that A is a contraction. If $x \in E$ is given, it may be expressed in the form

$$x = \xi_1 z_1 + \cdots + \xi_n z_n + y$$

with $y \in E_0^\perp$ so that $Ax = \xi_1 w_1 + \cdots + \xi_n w_n$. By the inequality above, we have

$$|Ax|^2 = |\sum \xi_i w_i|^2 \leqq |\sum \xi_i z_i|^2 \leqq |\sum \xi_i z_i|^2 + |y|^2 = |x|^2.$$

The next step consists in showing that $z_i = A^{i-1}z$ for $i = 1, 2, \ldots, n$. The operator A has been defined so as to have $Az_i = w_i$ for $i = 1, 2, \ldots, n$. If $i < n$, we have $w_i = z_{i+1}$ so that $Az_i = z_{i+1}$, from which, together with $z_1 = z$, the equations $z_i = A^{i-1}z$ follow easily. This shows already that $g(A) = Z$. Since we know already that A is a contraction we shall have $A \in \mathscr{A}$ if we show that $A^n - (\alpha_1 + \alpha_2 A + \cdots + \alpha_n A^{n-1}) = 0$. First of all,

$$A^n z = A(A^{n-1}z) = Az_n = w_n = \alpha_1 z_1 + \cdots + \alpha_n z_n$$

$$= \sum_1^n \alpha_i A^{i-1} z,$$

whence $(A^n - \sum_1^n \alpha_i A^{i-1})z = 0$. If r, $1 \leqq r \leqq n$, is given, we have

$$\left(A^n - \sum_1^n \alpha_i A^{i-1}\right) z_r = \left(A^n - \sum_1^n \alpha_i A^{i-1}\right) A^{r-1} z$$

$$= A^{r-1}\left(A^n - \sum_1^n \alpha_i A^{i-1}\right) z = 0;$$

since $A = 0$ on $E_0{}^\perp$, we conclude that $A^n - \sum_1^n \alpha_i A^{i-1} = 0$. The proof is complete.

Now we are ready to attack the maximum problem.

Let $A \in \mathscr{A}$ and let $z \in E$. Define again $z_i = A^{i-1}z$ for $1 \leqq i \leqq n$. Let $m \geqq n$. Since $A \in \mathscr{A}$, we have $A^n = \alpha_1 + \alpha_2 A + \cdots + \alpha_n A^{n-1}$ so that $A^n z = \sum \alpha_i z_i$. Since $Az_i = \sum t_{ik} z_k$ we have $A^{m-n} z_i = \sum_k t_{ik}^{(m-n)} z_k$, where $t_{ik}^{(p)}$ are the elements of the matrix T^p. Hence $A^m z = A^{m-n}A^n z = A^{m-n} \sum_i \alpha_i z_i = \sum_{i,k} \alpha_i t_{ik}^{(m-n)} z_k = \sum_{i,k} t_{ni} t_{ik}^{(m-n)} z_k = \sum_j t_{nj}^{(m-n+1)} z_j$ so that

$$|A^m z|^2 = \left(\sum \beta_j z_j, \sum \beta_k z_k\right) = \sum \beta_j \bar{\beta}_k (z_j, z_k),$$

where we have put $\beta_j = t_{nj}^{(m-n+1)}$. If we denote by f_m the linear functional on $\mathscr{M}_n$ defined by $f_m(W) = \sum_{i,k} w_{ik} \beta_i \bar{\beta}_k$, we may write

$$|A^m z|^2 = f_m(g(A)),$$

g being the transformation defined in the preceding section. It follows that $\max |A^m z|^2$ for $A \in \mathscr{A}$ equals the maximum of f_m on the set $\mathscr{Z}$. The last set being compact and convex, the maximum of f_m will be attained at an extreme point of $\mathscr{Z}$.

Consider now the cone $\mathcal{T}$ of all symmetric matrices Z such that $TZT^* \leqq Z$. We have seen already that $TZT^* \leqq Z$ implies $Z \geqq 0$ so that $\mathcal{T}$ is a subcone of the cone $\mathcal{P}$ of all symmetric positive semidefinite matrices. In order to obtain the extreme rays of $\mathcal{T}$ let us establish a linear isomorphism between $\mathcal{T}$ and $\mathcal{P}$. If $Z \in \mathcal{T}$, denote by $p(Z)$ the matrix $p(Z) = Z - TZT^*$. Clearly p is a linear mapping of $\mathcal{T}$ into $\mathcal{P}$. Now $p(Z) = 0$ means $Z = TZT^*$ and, by iteration, $Z = T^rZT^{*r}$; the right-hand side, however, tends to zero so that $p(Z) = 0$ implies $Z = 0$. Given $P \in \mathcal{P}$, define Z as

$$Z = P + TPT^* + T^2PT^{*2} + \cdots,$$

so that $Z \in \mathcal{P}$ and $Z = P + TZT^*$. Hence $p(Z) = P$ and the mapping p is thus seen to be one-to-one and onto. It follows that the extreme rays of $\mathcal{T}$ are generated by matrices of the form $p^{-1}(P)$ where P are generators of extreme rays of $\mathcal{P}$. It follows that

$$\max |A^m z|^2 = \max f_m(p^{-1}(P)),$$

where P runs over the set of all matrices of the form $p_{ik} = p_i\bar{p}_k$ such that the matrix $Z = p^{-1}(P)$ has $z_n = 1$.

Let us now introduce the following notation: We denote by q_{ik} the linear functional on $\mathcal{M}$ defined as follows. Given a matrix M the value $q_{ik}(M) = m_{ik}$. Now our functional $f_m(W)$ may also be expressed as

$$q_{nn}(T^{m-n+1}WT^{*m-n+1}).$$

We shall show later that $t_{ni}^{(r)} = t_{1i}^{(r+n-1)}$ for each r and i. From these equations it follows that

$$q_{nn}(T^{m-n+1}WT^{*m-n+1}) = q_{11}(T^mWT^{*m}).$$

In particular, if W is of the form $w_{ik} = p_i\bar{p}_k$ the functional $q_{11}(T^rWT^{*r})$ assumes an especially simple form. In fact,

$$q_{11}(T^rWT^{*r}) = \sum_{i,k} t_{1i}^{(r)}p_i\bar{p}_k t_{k1}^{*(r)}$$

$$= \sum_{i,k} t_{1i}^{(r)}p_i\overline{t_{1k}^{(r)}}\bar{p}_k = \left|\sum_k t_{1k}^{(r)}p_k\right|^2.$$

Take now $Z = p^{-1}(P)$, where P is of the form $p_{ik} = p_i\bar{p}_k$. It follows that

$$q_{11}(Z) = q_{11}(P) + q_{11}(TPT^*) + q_{11}(T^2PT^{*2}) + \cdots,$$

while

$$q_{11}(T^m Z T^{*m}) = q_{11}(T^m P T^{*m}) + q_{11}(T^{m+1} P T^{*m+1}) + \cdots .$$

We introduce now an abbreviation. If $p_1, \ldots, p_n$ is a given vector and P the corresponding matrix, $p_{ik} = p_i \bar{p}_k$, we shall denote by $\xi_r(P)$ the functional

$$\xi_r(P) = \sum_k t_{1k}^{(r)} p_k$$

for $r = 0, 1, 2, \ldots$. It follows that

$$q_{11}(Z) = |\xi_0(P)|^2 + |\xi_1(P)|^2 + \cdots ,$$

while

$$q_{11}(T^m Z T^{*m}) = |\xi_m(P)|^2 + |\xi_{m+1}(P)|^2 + \cdots .$$

Consider now the Hilbert space H of all sequences $x = (x_0, x_1, x_2, \ldots)$ such that $\sum_{i \geqq 0} |x_i|^2$ is convergent. In H, let us consider the following n vectors b_i defined as follows:

$$b_i = (b_{i0}, b_{i1}, \ldots),$$

where $b_{ir} = t_{1i}^{(r)}$. It is not difficult to see that these n vectors belong to H. Indeed, since $|T|_\sigma < 1$, we have the estimate $|T^r| \leqq \lambda^r$ for large r and a suitable $0 < \lambda < 1$. It follows that

$$|(T^r u, v)| \leqq \lambda^r |u|\, |v| \qquad \text{for any} \quad u, v$$

so that any sequence of the type

$$s_r = (T^r u, v), \qquad r = 0, 1, 2, \ldots,$$

belongs to H. If we denote by $x(P)$ the sequence

$$x(P) = (\xi_0(P), \xi_1(P), \xi_2(P), \ldots),$$

clearly $x(P) = p_1 b_1 + p_2 b_2 + \cdots + p_n b_n$ (since $\xi_r(P) = \sum_k t_{1k}^{(r)} p_k = \sum_k p_k b_{kr}$). Let us denote by S the shift operator in H, so that, for $x = (x_0, x_1, x_2, \ldots)$, we have $Sx = (x_1, x_2, \ldots)$. Our task consists in finding the maximum of $|S^m x(P)|^2$ under the condition that $|x(P)|^2 = 1$. However, if we allow P to vary over all symmetric matrices of rank one, the vectors $x(P)$ will actually sweep out the subspace of H generated by $b_1, \ldots, b_n$.

Let us denote this subspace by B. It is not difficult to show that B is invariant with respect to S. Indeed, it is not difficult to see that $Sb_i = b_{i-1} + \alpha_i b_n$, where we put $b_0 = 0$. First of all, $t_{1i}^{(r+1)} = \sum_k t_{1k}^{(r)} t_{ki}$, whence

$$t_{11}^{(r+1)} = \alpha_1 t_{1n}^{(r)} \qquad \text{for} \quad i = 1,$$

$$t_{1i}^{r+1} = t_{1,i-1}^{(r)} + \alpha_i t_{1n}^{(r)} \qquad \text{for} \quad i > 1,$$

so that

$$b_{1,r+1} = \alpha_1 b_{n,r},$$

$$b_{i,r+1} = b_{i-1,r} + \alpha_i b_{n,r}.$$

We have thus shown that $\sup_{A \in \mathscr{A}} |A^m| = |S^m|_B$, where the norm is to be computed on the subspace B. Let us show now that the space B coincides with the space of all solutions of the recursive formula

$$x_{r+n} = \alpha_1 x_r + \alpha_2 x_{r+1} + \cdots + \alpha_n x_{r+n-1}.$$

Since the first coordinates of the vectors b_i are just

$$\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 1 \end{matrix}$$

it suffices to show that each of the sequences b_i satisfies this recursive formula. To see that, let us consider the matrix T^∞,

$$\begin{matrix} t_{11}^{(0)} & t_{12}^{(0)} & \cdots & t_{1n}^{(0)} \\ t_{11}^{(1)} & t_{12}^{(1)} & \cdots & t_{1n}^{(1)} \\ t_{11}^{(2)} & t_{12}^{(2)} & \cdots & t_{1n}^{(2)} \\ & & \vdots & \end{matrix}$$

(infinite number of rows and n columns). We observe that the ith column of this matrix is identical with the vector b_i. Let us show now that this matrix has the following simple property. Given any $r = 0, 1, \ldots,$ consider the n by n matrix consisting of the n consecutive rows of T^∞ starting with $t_{11}^{(r)}, t_{12}^{(r)}, \ldots, t_{1n}^{(r)}$. Then this section of T^∞ is identical with T^r. To prove this, let us show first that, for any $r > 0$ and any $q < n$

$$t_{qi}^{(r)} = t_{q+1,i}^{(r-1)}.$$

Since $t_{qi}^{(r)} = \sum_j t_{qj} t_{ji}^{(r-1)}$ and $q < n$, the only nonzero term of this sum is the one with $j = q + 1$ and $t_{q,q+1} = 1$. Using this reduction formula several times we obtain, for $p < n$,

$$t_{1i}^{(r+p)} = t_{1+p,i}^{(r)},$$

and this proves our statement.

Now let an i be given, $1 \leqq i \leqq n$, and let us prove that

$$t_{1i}^{(r+n)} = \alpha_1 t_{1i}^{(r)} + \alpha_2 t_{1i}^{(r+1)} + \cdots + \alpha_n t_{1i}^{(r+n-1)}.$$

Using the preceding formula the sum on the right-hand side reduces to

$$\begin{aligned} \alpha_1 t_{1i}^{(r)} &+ \alpha_2 t_{1i}^{(r+1)} + \alpha_3 t_{1i}^{(r+2)} + \cdots + \alpha_n t_{1i}^{(r+n-1)} \\ &= \alpha_1 t_{1i}^{(r)} + \alpha_2 t_{2i}^{(r)} + \alpha_3 t_{3i}^{(r)} + \cdots + \alpha_n t_{ni}^{(r)} \\ &= \sum_j t_{nj} t_{ji}^{(r)} = t_{ni}^{(r+1)}. \end{aligned}$$

Again by the formula above, $t_{ni}^{(r+1)} = t_{1i}^{(r+n)}$.

Summing up, we can now state the main theorem of this section:

6. *Let $\alpha_1, \ldots, \alpha_n$ be complex numbers such that all roots of the polynomial $x^n - (\alpha_1 + \alpha_2 x + \cdots + \alpha_n x^{n-1})$ are less than one in absolute value. Denote by $\mathscr{A}$ the set of all contractions on a fixed n-dimensional Hilbert space E which satisfy the equation*

$$A^n = \alpha_1 + \alpha_2 A + \cdots + \alpha_n A^{n-1}.$$

For any $m \geqq n$,

$$\max_{A \in \mathscr{A}} |A^m| = |S^m|_{H(\alpha_1, \ldots, \alpha_n)},$$

where S is the shift operator on l_2 and $H(\alpha_1, \ldots, \alpha_n)$ is the n-dimensional subspace of l_2 consisting of all solutions of the recurrent relation

$$x_{r+n} = \alpha_1 x_r + \alpha_2 x_{r+1} + \cdots + \alpha_n x_{r+n-1}.$$

The space $H(\alpha_1, \ldots, \alpha_n)$ is invariant with respect to S.

4. AN ESTIMATE

At this point, it is already possible to give a very simple estimate which constitutes a considerable strengthening of the original theorem on the critical exponent of the n-dimensional Hilbert space.

Denote by H the Hilbert space of all sequences of the form $x = \{x_0, x_1, \ldots\}$ with $|x| = (\sum |x_i|^2)^{1/2}$. In this space, consider the orthogonal projections P and Q such that $P + Q = I$ and

$$Px = \{x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots\}.$$

Further, let F be the n-dimensional Hilbert space of all vectors $y = \{y_0, \ldots, y_{n-1}\}$ with $|y| = (\sum |y_i|^2)^{1/2}$. If $y \in F$ is given, we shall denote by $T(y)$ the sequence $z_0, z_1, \ldots$ such that

$$y_0, y_1, \ldots, y_{n-1}, z_0, z_1, \ldots$$

satisfies the recurrence relation with coefficients $\alpha_1, \ldots, \alpha_n$. Clearly $T(y)$ is an element of H; we shall denote by $|T|$ the norm of T as an operator from F into H.

We intend to show now that

$$|S^n|_{H(\alpha_1, \ldots, \alpha_n)} \leqq \left(\frac{|T|^2}{1 + |T|^2}\right)^{1/2}.$$

The number on the right-hand side being less than one, this estimate clearly contains the earlier result that the critical exponent of E is n.

To prove the estimate above, take an arbitrary $x \in H(\alpha_1, \ldots, \alpha_n)$ and denote by y the vector $x_0, \ldots, x_{n-1}$ in F.

Clearly $|Qx| = |Ty|$ and $|y| = |Px|$ so that

$$|Qx|^2 = |Ty|^2 \leqq |T|^2|y|^2 = |T|^2|Px|^2.$$

Adding $|T|^2|Qx|^2$ to both sides of this inequality, we obtain

$$(1 + |T|^2)|Qx|^2 \leqq |T|^2(|Px|^2 + |Qx|^2) = |T|^2|x|^2.$$

Now it suffices to observe that $|S^n x| = |Qx|$ and our inequality is established.

5. THE GENERAL MAXIMUM PROBLEM

Having obtained theorem 6 it is now comparatively easy to compute $C(E, \rho, n)$. It suffices to consider all recurrent relations $\alpha_1, \ldots, \alpha_n$ for which the polynomial $x^n - (\alpha_1 + \alpha_2 x + \cdots + \alpha_n x^{n-1})$ has all roots $\leqq \rho$ in absolute value, take the corresponding space $H(\alpha_1, \ldots, \alpha_n)$, and find the maximum of $|S^n|_{H(\alpha_1, \ldots, \alpha_n)}$ for such $\alpha_1, \ldots, \alpha_n$. The main idea used for the solution of this maximum problem consists in the following: Denote again by $b_1, \ldots, b_n$ the solutions of the recurrent relations with unit initial values and express their coordinates in terms of $\rho_1, \ldots, \rho_n$, the roots of $x^n - (\alpha_1 + \alpha_2 x + \cdots + \alpha_n x^{n-1})$; the coordinates b_{ir} for $r \geqq n$ are obtained (if the ρ_i are considered as indeterminates) in the form of a quotient of two determinants of Vandermonde type; these quotients, in their turn, may be expressed as polynomials in $\rho_1, \ldots, \rho_n$. A closer inspection of the form of these polynomials suggests the conjecture that all the coefficients of all polynomials b_{ir}, $r \geqq n$, are of the same sign (which depends on i only). In fact, for $i = 1$, it is not difficult to verify directly that these coefficients are all equal to one. I am indebted to Professor V. Knichal, who, at my request, supplied a proof of this conjecture. This result is stated as Lemma 7 below. With the aid of this lemma, it is not difficult to show that the maximum of the norm of S^n is attained on the space corresponding to the case where all ρ_i are equal to ρ.

We shall denote, for $1 \leqq i \leqq n$, by E_i the polynomial

$$E_i(x_1, \ldots, x_n) = \sum_{\substack{0 \leqq e_t \leqq 1 \\ e_1 + e_2 + \cdots + e_n = i}} x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}.$$

Now let $\rho_1, \ldots, \rho_n$ be given complex numbers. For $r = 1, 2, \ldots, n$, put $\alpha_r = (-1)^{n-r} E_{n-r+1}(\rho_1, \ldots, \rho_n)$ so that the roots of the equation

$$x^n = \alpha_1 + \alpha_2 x + \cdots + \alpha_n x^{n-1}$$

are exactly $\rho_1, \ldots, \rho_n$. Consider the recursive relation

$$x_{r+n} = \alpha_1 x_r + \alpha_2 x_{r+1} + \cdots + \alpha_n x_{r+n-1}.$$

For each i, $1 \leqq i \leqq n$, we denote by $w_i(\rho_1, \ldots, \rho_n)$ the solution of this relation with initial conditions

$$w_{ik}(\rho_1, \ldots, \rho_n) = \delta_{i,k+1}, \qquad 0 \leqq k \leqq n - 1.$$

For the explicit expression of the w_{ik} as polynomials in $\rho_1, \ldots, \rho_n$ the following result may be proved:

7. *For each* $i = 1, 2, \ldots, n$ *and each* $r \geqq n$,

$$w_{ir}(\rho_1, \ldots, \rho_n) = \varepsilon_i Q_{ir}(\rho_1, \ldots, \rho_n),$$

where $\varepsilon_i = (-1)^{n-i}$ *and*

$$Q_{i,r}(\rho_1, \ldots, \rho_n) = \sum_{\substack{e_j \geqq 0 \\ e_1 + \cdots + e_n = r - i + 1}} c_{ir}(e_1, \ldots, e_n)\rho_1^{e_1} \cdots \rho_n^{e_n},$$

where all $c_{ir}(e_1, \ldots, e_n) \geqq 0$.

For any $\rho_1, \ldots, \rho_n$ we shall denote by $P(\rho_1, \ldots, \rho_n)$ the linear space consisting of all solutions of the recursive relation

$$x_{r+n} = \alpha_1 x_r + \cdots + \alpha_n x_{r+n-1}$$

or, in other words, the linear space spanned by the n vectors $w_1(\rho_1, \ldots, \rho_n), \ldots, w_n(\rho_1, \ldots, \rho_n)$. Now let $0 < \rho < 1$ be given and suppose that all $|\rho_i| \leqq \rho$. We have seen already that, in this case, $P(\rho_1, \ldots, \rho_n)$ is a subspace of H. We intend to show that

$$|S^n|_{P(\rho_1, \ldots, \rho_n)} \leqq |S^n|_{P(\rho, \ldots, \rho)}.$$

To prove this, we intend to show that, for each $x \in P(\rho_1, \ldots, \rho_n)$, there exists a $y \in P(\rho, \ldots, \rho)$ such that

$$\frac{|S^n x|}{|x|} \leqq \frac{|S^n y|}{|y|}.$$

We note first that, all coefficients of the forms Q_{ir} being nonnegative,

$$|Q_{ir}(\rho_1, \ldots, \rho_r)| \leqq Q_{ir}(\rho, \ldots, \rho).$$

Now put $y = \sum_{i=1}^n |x_{i-1}|\varepsilon_i w_i(\rho, \ldots, \rho)$. It follows that, for $0 \leqq r \leqq n-1$, we have $|y_r| = |x_r|$. If $r \geqq n$,

$$\begin{aligned}|x_r| &= \left|\sum_{i=1}^n x_{i-1} w_{ir}(\rho_1, \ldots, \rho_r)\right| \leqq \sum_{i=1}^n |x_{i-1}|\,|Q_{ir}(\rho_1, \ldots, \rho_n)| \\ &\leqq \sum_{i=1}^n |x_{i-1}| Q_{ir}(\rho, \ldots, \rho) = \sum_{i=1}^n y_{i-1}\varepsilon_i Q_{ir}(\rho, \ldots, \rho) \\ &= \sum_{i=1}^n y_{i-1} w_{ir}(\rho, \ldots, \rho) = y_r.\end{aligned}$$

We have thus $|y_r| = |x_r|$ for $0 \leqq r \leqq n-1$ and $y_r \geqq |x_r|$ for $r \geqq n$ and this implies the desired inequality. We have thus proved the following theorem:

8. *Let* $\rho < 1$. *The maximum of* $|A^n|$ *where* A *is a linear operator on an n-dimensional Hilbert space subject to the conditions* $|A| \leqq 1$ *and* $|A|_\sigma \leqq \rho$

is attained for the nth *power of the shift operator* S *on the space of all sequences* x_0, x_1, x_2 *which satisfy*

$$\sum_{j=0}^{n} \binom{n}{j} \rho^j x_{r+n-j} = 0$$

for each $r \geqq 0$.

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