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Spectra in model categories and applications to the algebraic cotangent complex

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Abstract

Consider a commutative simplicial ring B which is an algebra over the rational numbers. We show that the homotopy theory of simplicial B-modules and the stable homotopy theory of augmented commutative B-algebras are equivalent. In terms of this equivalence, we can identify André–Quillen homology as a stabilization process (suspension spectrum). © 1997 Elsevier Science B.V.

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0. Introduction

This paper is concerned with aspects of the stable homotopy theory of commutative simplicial rings. We will use the framework of Quillen's closed simplicial model categories [4]. So in the first sections, we review parts of homotopical algebra. To be able to talk about the stable homotopy theory of a model category, we introduce spectra in pointed closed simplicial model categories and use the small object argument to show that the category of spectra inherits a closed model category structure. We make precise the idea that for a linear model category, the passage to spectra gives the same homotopy theory.

This applies in particular to the linear model category of simplicial modules over a fixed simplicial ring B. Furthermore, in the case of a commutative simplicial $\mathbb{Q}$-algebra

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the stable homotopy theory of simplicial B-modules will be shown to be equivalent to the stable homotopy theory of augmented commutative B-algebras. Hence up to weak equivalence, a spectrum of augmented commutative B-algebras is the same as a simplicial B-module.

In [6], the cotangent complex of a morphism $A \to B$ of commutative rings is introduced, which gives rise to André-Quillen homology. It is a simplicial B-module well defined up to weak equivalence. Hence in the rational case we can ask what spectrum of algebras the cotangent complex corresponds to and we will see that the corresponding spectrum is the suspension spectrum of the unreduced suspension of A over B. We have thus reinterpreted the cotangent complex in terms of a stabilization process.

1. Review of homotopical algebra

1.1. Closed model categories

If $i : A \to B$ and $p : X \to Y$ are morphisms in a category, we will say that i has the *left lifting property* with respect to p and p has the *right lifting property* with respect to i, if given any commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

with unbroken arrows, there exists the dotted morphism $B \to X$ such that the resulting diagram is commutative.

Definition 1.1.1 (*Quillen* [5,II-1]). A *closed model category* is a category $\mathscr{C}$ equipped with three classes of morphisms called *cofibrations*, *fibrations* and *weak equivalences* respectively. A morphism is called an *acyclic cofibration* if it is both a cofibration and a weak equivalence and an *acyclic fibration* if it is both a fibration and a weak equivalence. There are five axioms to be satisfied:

(CM1) The category $\mathscr{C}$ is closed under finite limits and colimits; in particular it has an initial and a terminal object.

(CM2) Let f and g be composable morphisms in $\mathscr{C}$. Then if two of f, g and gf are weak equivalences, so is the third.

(CM3) If f is a retract of g and g is a fibration, cofibration or weak equivalence, so is f.

(CM4) Cofibrations have the left lifting property with respect to acyclic fibrations and fibrations have the right lifting property with respect to acyclic cofibrations.

(CM5) Any morphism in $\mathscr{C}$ can be factored as an acyclic cofibration followed by a fibration and it can also be factored as a cofibration followed by an acyclic fibration.

Acyclic (co-)fibrations are called trivial (co-)fibrations in [1, 4, 5]. However, Quillen changes to 'acyclic' in [6], so we also use this terminology. Consequences of the axioms are

Proposition 1.1.2 (Quillen [5.II-1]). *The cofibrations (resp. acyclic cofibrations) are precisely those maps having the left lifting property with respect to all acyclic fibrations (resp. fibrations). The fibrations (resp. acyclic fibrations) are precisely those maps having the right lifting property with respect to all acyclic cofibrations (resp. cofibrations).*

The following statements are all proved using these characterizations by lifting properties and they will be used frequently in the sequel.

Corollary 1.1.3. *The class of fibrations (resp. acyclic fibrations) is closed under composition and base change and contains all isomorphisms. The class of cofibrations (resp. acyclic cofibrations) is closed under composition and cobase change and contains all isomorphisms.*

Corollary 1.1.4. *Suppose that $A_0 \to A_1 \to \cdots \to A_i \to \cdots$ is a sequence of cofibrations (resp. acyclic cofibrations) and that the colimit of the A_i exists. Then the canonical map $A_0 \to \operatorname{colim}_{i\geq 0} A_i$ is also a cofibration (resp. acyclic cofibration).*

Corollary 1.1.5. *Suppose $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are functors between closed model categories such that F is left adjoint to G. Then F preserves cofibrations (resp. acyclic cofibrations) if and only if G preserves acyclic fibrations (resp. fibrations).*

In a closed model category we will usually speak of maps instead of morphisms. An object is *cofibrant* if the map from the initial object to it is a cofibration and it is *fibrant* if the map to the terminal object is a fibration. We will use a feathered arrow $\rightarrowtail$ to indicate that a morphism is a cofibration, double headed arrows $\twoheadrightarrow$ for fibrations and we will put a tilde $\xrightarrow{\sim}$ above an arrow to denote a weak equivalence.

For any closed model category $\mathscr{C}$, the *homotopy category* $Ho\,\mathscr{C}$ is the category obtained by formally inverting the weak equivalences. Quillen shows that it is equivalent to the concrete category with objects those objects of $\mathscr{C}$ which are both fibrant and cofibrant and where morphisms are morphisms in $\mathscr{C}$ modulo the homotopy equivalence relation (cf. [4, I.1] for the details). Furthermore, $Ho\,\mathscr{C}$ is not just a category but has some extra structure such as fibration sequences, cofibration sequences, and loop and suspension functors if $\mathscr{C}$ has a zero object. The localization functor $\mathscr{C} \to Ho\,\mathscr{C}$ has the particular property that a map in $\mathscr{C}$ is a weak equivalence if and only if it becomes an isomorphism in $Ho\,\mathscr{C}$ [4, I.5].

Definition 1.1.6 (*Bousfield and Friedlander* [1,1.2]). A model category is called *proper* if the following conditions hold: let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

be a pushout diagram such that $A \rightarrowtail B$ is a cofibration and $A \xrightarrow{\sim} C$ is a weak equivalence. Then $B \xrightarrow{\sim} D$ is a weak equivalence. Furthermore, if the diagram is a pullback square, $B \twoheadrightarrow D$ is a fibration and $C \xrightarrow{\sim} D$ is a weak equivalence, then $A \xrightarrow{\sim} B$ is also a weak equivalence.

Though it is seemingly weaker, properness is actually equivalent to the gluing lemma (Lemma 1.1.9) and its dual. Most examples of model categories, such as, e.g., topological spaces, simplicial sets or simplicial groups, are proper. We will see in Section 3.1 that simplicial modules and commutative simplicial rings also form proper model categories. For an example of a model category which is not proper, see [5, II, Remark 2.9].

We introduce homotopy cocartesian and homotopy cartesian squares. These notions only behave well in proper closed model categories.

Definition 1.1.7 (*Bousfield and Friedlander* [1,A.2]). A commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in a proper closed model category will be called *homotopy cocartesian* if for some factorization $A \rightarrowtail Z \xrightarrow{\sim} B$ of $A \to B$ as a cofibration followed by a weak equivalence, the induced morphism from the pushout $C \cup_A Z \xrightarrow{\sim} D$ is a weak equivalence.

Now suppose $A \rightarrowtail W \xrightarrow{\sim} C$ is a factorization of $A \to C$ as a cofibration followed by a weak equivalence. Then since $A \rightarrowtail Z$ is a cofibration, so is $W \rightarrowtail W \cup_A Z$. If we apply the properness assumption to the pushout diagram

$$\begin{array}{ccc} W & \rightarrowtail & W \cup_A Z \\ \sim\downarrow & & \downarrow \\ C & \rightarrowtail & C \cup_A Z \end{array}$$

we see that $W \cup_A Z \xrightarrow{\sim} C \cup_A Z$ is a weak equivalence. After another such step we obtain a commutative diagram

$$\begin{array}{ccccc} W \cup_A B & \xleftarrow{\sim} & W \cup_A Z & \xrightarrow{\sim} & C \cup_A Z \\ & \searrow & \downarrow & \swarrow & \\ & & D & & \end{array}$$

that tells us that there are several equivalent ways of defining homotopy cocartesian squares. We could have used a factorization of $A \to C$ instead of $A \to B$, we could have used factorizations of both maps or we could have required $C \cup_A Z \xrightarrow{\sim} D$ to be a weak equivalence for any factorization of $A \to B$ instead of just for one.

With this knowledge, the proof of the following lemma is straightforward.

Lemma 1.1.8. *Consider a commutative diagram in a proper closed model category*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

Then if the two inner squares are homotopy cocartesian, so is the outer square. Also if the left square and the outer square are homotopy cocartesian, so is the right square.

We have promised to show the gluing lemma:

Lemma 1.1.9. *In a proper closed model category consider a commutative diagram*

$$\begin{array}{ccccc} Y & \longleftarrow & A & \rightarrowtail & X \\ \sim\downarrow & & \sim\downarrow & & \downarrow\sim \\ Y' & \longleftarrow & A' & \rightarrowtail & X' \end{array}$$

such that the vertical maps are weak equivalences and two of the horizontal maps are cofibrations as indicated. Then the induced map on pushouts

$$Y \cup_A X \xrightarrow{\sim} Y' \cup_{A'} X'$$

is a weak equivalence.

Proof. Properness implies that $A' \rightarrowtail A' \cup_A X \xrightarrow{\sim} X'$ is a factorization of $A' \to X'$ as a cofibration followed by a weak equivalence. Since the square

$$\begin{array}{ccc} A' & \rightarrowtail & X' \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y' \cup_{A'} X' \end{array}$$

is homotopy cocartesian, the map $Y' \cup_{A'} (A' \cup_A X) \cong Y' \cup_A X \xrightarrow{\sim} Y' \cup_{A'} X'$ is thus a weak equivalence. But the map $Y \cup_A X \xrightarrow{\sim} Y' \cup_A X$ is also a weak equivalence because it is the pushout of $Y \xrightarrow{\sim} Y'$ along the cofibration $Y \rightarrowtail Y \cup_A X$. □

The gluing lemma can be rephrased: A map between homotopy cocartesian squares which is a weak equivalence on all corners except possibly the terminal ones is also a weak equivalence on the terminal corners.

We also need the dual concept. A commutative square as in Definition 1.1.7 will be called *homotopy cartesian* if for some factorization $B \xrightarrow{\sim} Y \twoheadrightarrow D$ of $B \to D$ as a weak equivalence followed by a fibration, the induced map into the pullback $A \xrightarrow{\sim} C \times_D Y$ is a weak equivalence. All we have said about homotopy cocartesian squares can be dualized.

1.2. Simplicial model categories

One fundamental example of a model category is the category of simplicial sets. A map of simplicial sets is a weak equivalence if it induces a homotopy equivalence on geometric realizations, it is a cofibration if it is dimensionwise injective, and the fibrations are the fibrations in the sense of Kan; this defines a structure of a closed model category (cf. [4, II.3]).

In many cases we have an additional structure on closed model categories, such that we have 'function complexes' and we can 'take products with simplicial sets'. This idea is made precise in the following definition. Here $\mathcal{S}$ denotes the category of simplicial sets and $\mathcal{S}_f$ the category of finite simplicial sets.

Definition 1.2.1 (*Quillen* [4, II.2]). A closed model category $\mathcal{C}$ becomes a *closed simplicial model category* if it is endowed with the following structure:

- Functors

$$\begin{array}{lll} \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}, & (X,Y) \mapsto \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) & \text{(function complex)} \\ \mathcal{C} \times \mathcal{S}_f \to \mathcal{C}, & (X,K) \mapsto X \otimes K & \\ \mathcal{C} \times \mathcal{S}_f{}^{op} \to \mathcal{C}, & (X,K) \mapsto X^K & \end{array}$$

- A natural composition map

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z)\,.$$

- A natural isomorphism

$$\underline{\mathrm{Hom}}_{\mathscr{C}}(X,Y)_0 \cong \mathrm{Hom}_{\mathscr{C}}(X,Y) \tag{1}$$

- Natural isomorphisms

$$\underline{\mathrm{Hom}}_{\mathscr{S}}(K,\ \underline{\mathrm{Hom}}_{\mathscr{C}}(X,Y)) \cong \underline{\mathrm{Hom}}_{\mathscr{C}}(X\otimes K,Y) \cong \underline{\mathrm{Hom}}_{\mathscr{C}}(X,Y^K) \tag{2}$$

This data is subject to some compatibility conditions (among them associativity of composition) which can be found in [4, II.2] and to one more axion which relates the simplicial structure to the model category structure:

(SM7) If $i : A \rightarrowtail B$ is a cofibration and $p: X \twoheadrightarrow Y$ a fibration in $\mathscr{C}$, then

$$\underline{\mathrm{Hom}}_{\mathscr{C}}(B,X) \to \underline{\mathrm{Hom}}_{\mathscr{C}}(A,X) \times_{\underline{\mathrm{Hom}}_{\mathscr{C}}(A,Y)} \underline{\mathrm{Hom}}_{\mathscr{C}}(B,Y)$$

is a fibration of simplicial sets which is a weak equivalence if i or p is.

A functor $F : \mathscr{C} \to \mathscr{D}$ between closed simplicial model categories is a *simplicial functor* if it is equipped with natural maps

$$\underline{\mathrm{Hom}}_{\mathscr{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathscr{D}}(F(X),F(Y))$$

compatible with composition and the isomorphism (1).

Quillen only requires the existence of the objects $X\otimes K$ and X^K, whereas we assume for simplicity that they are given by functors and that the appropriate isomorphisms are natural. The isomorphisms (1) and (2) show in particular that the functors $X \mapsto X\otimes K$ and $X \mapsto X^K$ are adjoint. Furthermore, one can deduce natural isomorphisms

$$X\otimes\Delta^0 \cong X \quad \text{and} \quad X\otimes(K\times L) \cong (X\otimes K)\otimes L$$

and similarly for X^K as in Proposition 1 of [4, II.1] Via the isomorphisms of (2) lifting diagrams can be translated to show (cf. Proposition 3 of [4, II.2]) that (SM7) is equivalent to either

(SM7a′) If $p: X \twoheadrightarrow Y$ is a fibration in $\mathscr{C}$ and $j: K \rightarrowtail L$ is a cofibration of finite simplicial sets, then

$$X^L \to X^K \times_{Y^K} Y^L$$

is a fibration which is a weak equivalence if p or j is.

or

(SM7b′) If $i : A \rightarrowtail B$ is a cofibration in $\mathscr{C}$ and $j: K \rightarrowtail L$ is a cofibration of finite simplicial sets, then

$$A\otimes L \cup_{A\otimes K} B\otimes K \to B\otimes L$$

is a cofibration which is a weak equivalence if i or j is.

A simplicial functor F gives rise to a natural transformation

$$F(X)\otimes K \to F(X\otimes K)$$

in the following way. We obtain a natural map

$$F(X) \otimes \Delta^n \to F(X \otimes \Delta^n)$$

as the image of the identity of $X \otimes \Delta^n$ under the map

$$\begin{aligned} &\mathrm{Hom}_{\mathscr{C}}(X \otimes \Delta^n, X \otimes \Delta^n) \cong \underline{\mathrm{Hom}}_{\mathscr{C}}(X, X \otimes \Delta^n)_n \\ &\to \underline{\mathrm{Hom}}_{\mathscr{D}}(F(X), F(X \otimes \Delta^n))_n \cong \mathrm{Hom}_{\mathscr{D}}(F(X) \otimes \Delta^n, F(X \otimes \Delta^n)). \end{aligned}$$

Since $X \otimes -$ commutes with colimits, this transformation extends to a natural map $F(X) \otimes K \to F(X \otimes K)$. This argument can be reversed to show that a transformation $F(X) \otimes K \to F(X \otimes K)$ with suitable compatibility assumptions makes F into a simplicial functor.

Here is the standard example of how the functor $\otimes$ can arise. In any category $\mathscr{C}$ with finite coproducts, we can define the product of objects with finite sets by taking the coproduct of copies of that object indexed by the sets in question. This extends to a functor $\mathscr{C} \times \mathscr{S} \to s\mathscr{C}$, where $s\mathscr{C}$ denotes the category of simplicial objects in $\mathscr{C}$. More precisely, if X is an object of $\mathscr{C}$ and K is a finite simplicial set, we define $X \times K$, a simplicial object in $\mathscr{C}$, to be $\coprod_{\sigma \in K_n} X_\sigma$ in dimension n, where $X_\sigma = X$ for all $\sigma \in K_n$. If $\alpha : [m] \to [n]$ is a morphism in the category Δ, the structural morphism $\alpha^* : (X \times K)_n \to (X \times K)_m$ maps X_σ to $X_{\alpha^*(\sigma)}$ by the identity of X. If the category $\mathscr{C}$ is already a category of simplicial objects, we thus obtain bisimplicial objects which we can diagonalize to obtain objects of $\mathscr{C}$ again, i.e., $X \otimes K = \mathrm{diag}(X \times K)$. This procedure applies for example in the case of simplicial sets, simplicial rings and simplicial modules.

If $\mathscr{C}$ is pointed by a zero object $*$, we define ΣX, the *suspension* of an object X, as the pushout of the diagram

$$\begin{array}{ccc} X \otimes \partial\Delta^1 & \longrightarrow & X \otimes \Delta^1 \\ \downarrow & & \\ * & & \end{array}$$

and we define ΩX, the *loop object* of X as the pullback of the diagram

$$\begin{array}{ccc} & & X^{\Delta^1} \\ & & \downarrow \\ * & \longrightarrow & X^{\partial\Delta^1} \end{array}$$

Suspension and loop define adjoint endofunctors of $\mathscr{C}$, suspension being the left adjoint.

Using axiom (SM7b′) one shows that suspension preserves cofibrations and acyclic cofibrations. So Σ preserves weak equivalences between cofibrant objects by Lemma 9.9 of [2]. Dually, the loop functor preserves fibrations, acyclic fibrations and weak equivalences between fibrant objects. Note also that $\Sigma(X) \otimes K \cong \Sigma(X \otimes K)$ and $\Omega(X^K) \cong (\Omega X)^K$ for any finite simplicial set K.

Recall that on the homotopy category of a pointed closed model category, not necessarily simplicial, there is a pair of adjoint functors, also called loop and suspension [4, I.2]. If $\mathscr{C}$ is closed simplicial, the relation between these functors on the homotopy category and the functors with the same names on $\mathscr{C}$ is as follows. If an object A is cofibrant, ΣA is a model for the suspension of A in the homotopy category. This amounts to saying that the suspension functor on $Ho\,\mathscr{C}$ is the total left derived functor [4, I.4] of the suspension on $\mathscr{C}$. Similarly, the loop functor on $Ho\,\mathscr{C}$ is the total right derived functor of $\Omega : \mathscr{C} \to \mathscr{C}$, which means that for a fibrant object X, ΩX is isomorphic to the loop object of X in the homotopy category. However, if objects are not cofibrant (resp. fibrant), the suspension or loop formed in $\mathscr{C}$ will in general have the 'wrong' homotopy type. So one important feature of a closed simplicial model category is that the loop and suspension functors can be lifted from the homotopy category to $\mathscr{C}$; this makes it possible to define spectra and to study the stable homotopy theory of $\mathscr{C}$.

If $\mathscr{C}$ is any category and B an object of $\mathscr{C}$, we can form the category $\mathscr{C}/\!/B$ of objects containing B as retract. The objects of this category are triples $(X, r : X \to B, s : B \to X)$, where X is an object of $\mathscr{C}$ and r and s satisfy $rs = 1_B$. Morphisms are the maps in $\mathscr{C}$ that respect the retractions and the sections.

The category $\mathscr{C}/\!/B$ inherits a structure of a closed simplicial model category from $\mathscr{C}$ in the following way. We define cofibrations, fibrations and weak equivalences via the forgetful functor, i.e., a morphism $(X, r, s) \to (X', r', s')$ is a cofibration (resp. fibration or weak equivalence) in $\mathscr{C}/\!/B$ if and only if $X \to X'$ is a cofibration (resp. fibration or weak equivalence) in $\mathscr{C}$.

If K is a finite simplicial set, we define $(X, r, s) \otimes K$ to be the pushout in $\mathscr{C}$ of the diagram

$$\begin{array}{ccc} B\otimes K & \longrightarrow & X\otimes K \\ \downarrow & & \\ B \cong B\otimes\Delta^0 & & \end{array}$$

with the induced structural maps. $(X, r, s)^K$ is defined as the pullback in $\mathscr{C}$ of the diagram

$$\begin{array}{ccc} & & X^K \\ & & \downarrow \\ B \cong B^{\Delta^0} & \longrightarrow & B^K \end{array}$$

with the induced structural map. $\underline{\mathrm{Hom}}_{\mathscr{C}/\!/B}((X, r, s), (X', r', s'))$ is the subcomplex of $\underline{\mathrm{Hom}}_{\mathscr{C}}(X, X')$ of 'maps which respect the retraction and the section' (cf. Proposition 6 of [4, II.2]).

Proposition 1.2.2. *With these definitions, $\mathscr{C}//B$ becomes a closed simplicial model category which is proper if $\mathscr{C}$ is.*

Proof. In [4, II.2, Proposition 6], the analogous statement for the category of objects over B is proved, and a similar proof works in our case. To see that properness is inherited, we note that the forgetful functor $\mathscr{C}//B \to \mathscr{C}$ commutes with pushouts and pullbacks (though not with limits or colimits in general). $\square$

1.3. The small object argument

The small object argument is used in [4, II.3] to construct factorizations of maps of spaces. We will use it systematically to insure good behaviour of colimits over sequences of cofibrations.

Definition 1.3.1. Let $\mathscr{C}$ be a closed simplicial model category which has sequential colimits. An object M of $\mathscr{C}$ is *small* if for every sequence $X_0 \rightarrowtail \cdots \rightarrowtail X_k \rightarrowtail \cdots$ of cofibrations in $\mathscr{C}$ and for all finite simplicial sets K the map $\operatorname{colim} \operatorname{Hom}_{\mathscr{C}}(M \otimes K, X_i) \to \operatorname{Hom}_{\mathscr{C}}(M \otimes K, \operatorname{colim} X_i)$ is a bijection. We say that $\mathscr{C}$ *admits the small object argument* if there is a set $\{L_j \rightarrowtail M_j\}_{j \in J}$ of cofibrations, called test maps, with small sources and targets, such that a map in $\mathscr{C}$ is an acyclic fibration (resp. fibration) if and only if it has the right lifting property with respect to the test maps (resp. with respect to all acyclic cofibrations among the test maps).

Examples of model categories that admit the small object argument are categories where the objects have underlying simplicial sets such as simplicial (abelian) groups, simplicial modules, simplicial rings. In these cases the objects freely generated by finite simplicial sets are small and the test maps are the ones induced by the inclusions $\partial\triangle^n \to \triangle^n$ of the boundaries and $\Lambda^n_k \to \triangle^n$ of the horns into the n-simplicies. (Λ^n_k is the union of those faces of $\triangle^n$ which contain the kth vertex.) Similarly, the category of topological spaces with its usual model category structure admits the small object argument (cf. [4, II.3]).

Let $\mathscr{C}^{\mathbb{N}}$ denote the category of infinite sequences $A: A_0 \to \cdots \to A_k \to \cdots$ of composable maps in $\mathscr{C}$, with morphisms the natural transformations. If $\mathscr{C}$ is a closed model category, $\mathscr{C}^{\mathbb{N}}$ also becomes one as follows. Fibrations and weak equivalences are the maps of sequences which are termwise fibrations or weak equivalences respectively. A map of sequences $A \to B$ is a cofibration if the maps $A_0 \to B_0$ and $A_n \cup_{A_{n-1}} B_{n-1} \to B_n$ are cofibrations in $\mathscr{C}$. Checking the axioms is straightforward. Note that every cofibrant sequence is a sequence of cofibrations, but not vice versa, unless the first object in the sequence is cofibrant. Using the characterization by lifting properties one sees that the colimit, as a functor $\mathscr{C}^{\mathbb{N}} \to \mathscr{C}$, preserves cofibrations and acyclic cofibrations.

Lemma 1.3.2. *Let $\mathscr{C}$ be a closed simplicial model category which admits the small object argument. Let X and Y be sequences of cofibrations in $\mathscr{C}$.*

(a) *A fibration (resp. acyclic fibration) of sequences $X \to Y$ induces a fibration (resp. acyclic fibration) on the colimits of the sequences. In particular, the colimit preserves weak equivalences between sequences of cofibrations.*

(b) *Suppose $\mathscr{C}$ is pointed and denote by ΩY the sequence obtained from Y by taking the loop termwise. Then a fibration (resp. acyclic fibration) of sequences $X \to \Omega Y$ induces a fibration (resp. acyclic fibration)* $\operatorname{colim} X \to \Omega \operatorname{colim} Y$.

(c) *(Homotopy colimits commute with finite inverse limits) Let $D : I \to \mathscr{C}^{\mathbb{N}}$ be a finite diagram of sequences in $\mathscr{C}$ (i.e., a functor from a finite category) such that for all objects i of I, the sequence $D(i)$ consists of cofibrations, and let X be another sequence of cofibrations. Then any weak equivalence $X \xrightarrow{\sim} \lim_I D$ of sequences gives a weak equivalence* $\operatorname{colim} X \xrightarrow{\sim} \lim_I(\operatorname{colim} \circ D)$.

Proof. (a) We prove the statement for acyclic fibrations; in the case of fibrations, the argument is similar. We check that the induced map on colimits has the right lifting property with respect to every test map $L_j \rightarrowtail M_j$. Since source and target are small we know that a lifting diagram factors

$$\begin{array}{ccccc} L_j & \longrightarrow & X_k & \longrightarrow & \operatorname{colim} X_i \\ \downarrow & \nearrow & \downarrow \sim & & \downarrow \\ M_j & \longrightarrow & Y_k & \longrightarrow & \operatorname{colim} Y_i \end{array}$$

for suitable k such that the left square commutes if k is chosen big enough. Hence a lifting exists.

(b) If an object M is small, so is its suspension ΣM. Since Σ and Ω are adjoint, we know that for such an M the map $\operatorname{colim} \operatorname{Hom}_{\mathscr{C}}(M, \Omega Y_i) \to \operatorname{Hom}_{\mathscr{C}}(M, \Omega \operatorname{colim} Y_i)$ is a bijection. The rest of the argument is the same as in part (a).

(c) Since colim preserves acyclic cofibrations, we can assume that $X \xrightarrow{\sim} \lim_I D$ is an acyclic fibration; we claim that in this case the map $\operatorname{colim} X \xrightarrow{\sim} \lim_I(\operatorname{colim} \circ D)$ is also an acyclic fibration. Since filtered colimits and finite limits of functors to the category of sets commute, for small L the map $\operatorname{colim} \operatorname{Hom}_{\mathscr{C}}(L, \lim_I D) \to \operatorname{Hom}_{\mathscr{C}}(L, \lim_I(\operatorname{colim} \circ D))$ is a bijection. The rest is again as in part (a) or (b). □

Lemma 1.3.3. *Suppose $\mathscr{C}$ is a closed simplicial model category with arbitrary coproducts which admits the small object argument. Then morphisms can functorially be factored as cofibrations followed by acyclic fibrations. Also morphisms can functorially be factored as acyclic cofibrations followed by fibrations.*

Proof. The argument is completely analogous to [4, II.3, Lemma 3] and we will not repeat it. We just use the test maps where Quillen uses the inclusions of the boundaries of simplices. Since the construction is using all appropriate diagrams in the gluing process, the factorization produced is in fact functorial. □

Lemma 1.3.4. *If $\mathscr{C}$ admits the small object argument, then so does $\mathscr{C}/\!/B$ for any object B of $\mathscr{C}$.*

Proof. Let $\{\tau_j : L_j \rightarrowtail M_j\}_{j\in J}$ be a set of test maps for $\mathscr{C}$. From a map $f : X \to B$ in $\mathscr{C}$ we can construct an object $(B\amalg X, 1_B \amalg f : B\amalg X \to B, B \to B\amalg X)$ of $\mathscr{C}/\!/B$ which we denote by $B\amalg_f X$. Using that $(B\amalg_f X)\otimes K \cong B\amalg_{\bar{f}}(X\otimes K)$ with respect to the map $\bar{f} : X\otimes K \to X \xrightarrow{f} B$ and the fact that the forgetful functor $\mathscr{C}/\!/B \to \mathscr{C}$ commutes with sequential colimits, one sees that if X is small in $\mathscr{C}$, then $B\amalg_f X$ is small in $\mathscr{C}/\!/B$. We obtain a set of test maps for $\mathscr{C}/\!/B$ if we take all the maps $B\amalg_{f\tau_j} L_j \rightarrowtail B\amalg_f M_j$ for $j\in J$ and $f\in \mathrm{Hom}_{\mathscr{C}}(M_j, B)$. □

2. Spectra

2.1. How spectra form a model category

In this section we introduce spectra in pointed closed simplicial model categories and show how the category of spectra forms a model category. We will use ideas of [1] where spectra of simplicial sets are treated. The fact that our definitions work relies on a theorem of [1].

Definition 2.1.1. A *spectrum* X in a pointed closed simplicial model category $\mathscr{C}$ is a collection of objects X_n of $\mathscr{C}$ and maps $\Sigma X_n \to X_{n+1}$ $(n = 0, 1, 2, \ldots)$. A morphism of spectra $f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ such that all the diagrams

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\Sigma f_n} & \Sigma Y_n \\ \downarrow & & \downarrow \\ X_{n+1} & \xrightarrow[f_{n+1}]{} & Y_{n+1} \end{array}$$

commute.

We denote by $\mathscr{C}^{\infty}$ the category of spectra in $\mathscr{C}$.

A spectrum in our sense is sometimes called a prespectrum by other authors. With the familiar example of spaces in mind one might think that our definition does not give enough maps and homotopies between spectra. But we have to keep in mind that when passing to the homotopy category in homotopical algebra, we have to replace the sources of maps by cofibrant objects and the targets by fibrant objects. We will see that cofibrant spectra are the ones where all objects are cofibrant and the structural maps are cofibrations; furthermore, the fibrant spectra turn out to be the degreewise fibrant Ω-spectra. As in the case of spaces, there are enough maps of the kind we allow between such spectra, so we do get the stable homotopy category we want.

In $\mathscr{C}^\infty$ there exist all the types of limits and colimits that exist in $\mathscr{C}$ and the (co-) limits are formed degreewise.

An important example is the *suspension spectrum* of an object X which we denote by $\Sigma^\infty X$. It is defined as $(\Sigma^\infty X)_n = \Sigma^n X$, the structure maps being identity maps. This extends to a functor $\Sigma^\infty : \mathscr{C} \to \mathscr{C}^\infty$ which is left adjoint to the functor which maps a spectrum to its degree zero term.

If X is a spectrum and K a finite simplicial set, define the spectrum $X \otimes K$ by $(X \otimes K)_n = X_n \otimes K$ with structure maps $\Sigma(X_n \otimes K) \cong \Sigma X_n \otimes K \to X_{n+1} \otimes K$. Similarly, define X^K by $(X^K)_n = X_n^K$ with structure maps adjoint to $X_n^K \to (\Omega X_{n+1})^K \cong \Omega X_{n+1}^K$.

The definition of $X \otimes K$ is a special case of the fact that simplicial functors induce functors of spectra via degreewise application. More precisely, suppose $F : \mathscr{C} \to \mathscr{D}$ is a simplicial functor between pointed closed simplicial model categories preserving the zero object. The maps $F(X) \otimes K \to F(X \otimes K)$ induce a natural transformation $\Sigma F(X) \to F(\Sigma X)$. Hence we get a functor $F^\infty : \mathscr{C}^\infty \to \mathscr{D}^\infty$ which is given on objects by $F^\infty(X)_n = F(X_n)$ with structural maps $\Sigma F(X_n) \to F(\Sigma X_n) \to F(X_{n+1})$. The diagram of functors

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\Sigma^\infty} & \mathscr{C}^\infty \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F^\infty} \\ \mathscr{D} & \xrightarrow[\Sigma^\infty]{} & \mathscr{D}^\infty \end{array}$$

does *not* in general commute, but we obtain a natural transformation $\Sigma^\infty F \to F^\infty \Sigma^\infty$ which is an isomorphism if $\Sigma F \to F\Sigma$ is one.

Next consider adjoint functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ between pointed closed simplicial model categories, F being the left adjoint. If F is simplicial and the maps $\Sigma F(X) \xrightarrow{\cong} F(\Sigma X)$ are all isomorphisms, we can define a transformation $\Sigma G \to G\Sigma$ by

$$\Sigma G(Y) \to GF\Sigma G(Y) \xleftarrow{\cong} G\Sigma FG(Y) \to G\Sigma(Y)$$

using the adjunction morphisms. Hence G also induces a functor $G^\infty : \mathscr{D}^\infty \to \mathscr{C}^\infty$ and F^∞ and G^∞ are again adjoint.

Using the model category structure on $\mathscr{C}$ we want to define a closed simplicial model category structure on $\mathscr{C}^\infty$ such that the weak equivalences are the maps inducing weak equivalences on the homotopy colimits of the sequences $X_n \to \Omega X_{n+1} \to \cdots \to \Omega^k X_{n+k} \to \cdots$.

Definition 2.1.2. A spectrum X is an *Ω-spectrum* if there are weak equivalences $X_n \xrightarrow{\sim} X_n^{\mathrm{f}}$ from the X_n to fibrant objects such that the maps $X_n \xrightarrow{\sim} \Omega X_{n+1}^{\mathrm{f}}$ are weak equivalences in $\mathscr{C}$.

Since we want this to be an honest definition, it should not depend on choices. In fact, if $X_{n+1} \xrightarrow{\sim} \bar{X}_{n+1}^{\mathrm{f}}$ is a different choice of weakly equivalent fibrant object, then $X_n \to \Omega X_{n+1}^{\mathrm{f}}$ is a weak equivalence if and only if $X_n \to \Omega \bar{X}_{n+1}^{\mathrm{f}}$ is one; one proves this by considering first an acyclic cofibration to a fibrant object. The definition takes into account that ΩX_{n+1} may have the wrong homotopy type if X_{n+1} is not fibrant.

We will say that a map of spectra is a *strict weak equivalence* if it is degreewise a weak equivalence. If $X \to Y$ is a strict weak equivalence of spectra, then X is an Ω-spectrum if and only if Y is one. A map $X \to Y$ of spectra is a *strict cofibration* if the maps

$$X_0 \to Y_0 \quad \text{and} \quad X_n \cup_{\Sigma X_{n-1}} \Sigma Y_{n-1} \to Y_n$$

are cofibrations in $\mathscr{C}$. The *strict fibrations* are the maps which are degreewise fibrations in $\mathscr{C}$. It is straightforward to check that the strict notions make $\mathscr{C}^\infty$ into a closed simplicial model category which is proper if $\mathscr{C}$ is.

We assume for the rest of this section that $\mathscr{C}$ admits the small object argument. Lemma 1.3.3 shows that in this case, the factorizations of axiom (CM5) can be chosen functorially. For a spectrum X we can then functorially construct a new spectrum $\bar{Q}X$ and a natural map $\eta_X : X \to \bar{Q}X$. $\bar{Q}X$ is the candidate for a weakly equivalent Ω-spectrum associated to X. We define objects Z_{ik}, $i,k \geq 0$ and a lot of maps connecting these. Set $Z_{0k} = X_k$. For $i > 0$ let Z_{ik} be the object in the functorial factorization

$$\begin{array}{ccc} & & Z_{i,k} \\ & \nearrow & \big\downarrow \sim \\ Z_{i-1,k} & \longrightarrow & \Omega Z_{i-1,k+1} \end{array}$$

and define $Z_{i,k} \to \Omega Z_{i,k+1}$ as the composition $Z_{i,k} \to \Omega Z_{i-1,k+1} \xrightarrow{\Omega(-)} \Omega Z_{i,k+1}$.

We end up with commutative diagrams

$$\begin{array}{ccccccccc} X_k = Z_{0k} & \rightarrowtail & Z_{1k} & \rightarrowtail & \cdots & \rightarrowtail & Z_{ik} & \rightarrowtail & \cdots \\ \big\downarrow & & \big\downarrow & & & & \big\downarrow & & \\ \Omega X_{k+1} = \Omega Z_{0,k+1} & \longrightarrow & \Omega Z_{1,k+1} & \longrightarrow & \cdots & \longrightarrow & \Omega Z_{i,k+1} & \longrightarrow & \cdots \end{array}$$

and we define $(\bar{Q}X)_k = \operatorname{colim}_{i \geq 0} Z_{ik}$ with structure maps adjoint to

$$(\bar{Q}X)_k = \operatorname{colim}_{i \geq 0} Z_{ik} \to \Omega \operatorname{colim}_{i \geq 0} Z_{i,k+1} = \Omega(\bar{Q}X)_{k+1}.$$

Then the canonical maps $X_k \to \operatorname{colim}_{i \geq 0} Z_{i,k} = (\bar{Q}X)_k$ assemble to a map of spectra $\eta_X : X \to \bar{Q}X$. We collect some properties of this construction:

Lemma 2.1.3. *Let $\mathscr{C}$ be a proper pointed closed simplicial model category which admits the small object argument.*

(a) *If X is degreewise fibrant, then $\bar{Q}X$ is an Ω-spectrum.*

(b) *If X is a degreewise fibrant Ω-spectrum, then $\eta_X : X \to \bar{Q}X$ is a strict weak equivalence.*

(c) *If $X \to Y$ is a strict weak equivalence between degreewise fibrant spectra, then $\bar{Q}X \to \bar{Q}Y$ is a strict weak equivalence.*

(d) *The two maps $\bar{Q}\eta_X, \eta_{\bar{Q}X} : \bar{Q}X \to \bar{Q}\bar{Q}X$ are degreewise left homotopic* [4, I.1] *relative to X. This means that there are commutative diagrams*

$$\begin{array}{ccc} (\bar{Q}X)_k \cup_{X_k} (\bar{Q}X)_k & \xrightarrow{\bar{Q}_{\eta X} \cup \eta_{\bar{Q}X}} & (\bar{Q}\bar{Q}X)_k \\ \downarrow & \searrow & \uparrow \\ (\bar{Q}X)_k & \xleftarrow{\sim} & W \end{array}$$

where the left map is the codiagonal.

(e) *$\bar{Q}$ preserves strictly homotopy cartesian squares of degreewise fibrant spectra.*

Proof. (a) The acyclic fibrations $Z_{ik} \xrightarrow{\sim} \Omega Z_{i-1,k+1}$ fit into the big diagram above keeping it commutative. On colimits they induce the map $(\bar{Q}X)_k \to \Omega(\bar{Q}X)_{k+1}$ which is thus an acyclic fibration by Lemma 1.3.2 (b). If all X_k are fibrant, so are all $Z_{i,k}$ and thus also the $(\bar{Q}X)_k$ by Lemma 1.3.2 (a). Hence $\bar{Q}X$ is an Ω-spectrum.

(b) If X is a degreewise fibrant Ω-spectrum, the maps $X_k \xrightarrow{\sim} \cdots \xrightarrow{\sim} \Omega^i X_{k+i} \xrightarrow{\sim} \cdots$ are all weak equivalence, hence the maps $Z_{ik} \overset{\sim}{\rightarrowtail} Z_{i+1,k}$ are all acyclic cofibrations. By lemma 1.1.4, $X_k \overset{\sim}{\rightarrowtail} \mathrm{colim}_{i\geq 0} Z_{ik} = (\bar{Q}X)_k$ is also an acyclic cofibration.

(c) If $X \to Y$ is a strict weak equivalence between degreewise fibrant spectra, the maps on the loops $\Omega^i X_{k+i} \xrightarrow{\sim} \Omega^i Y_{k+i}$ are weak equivalences and so are the maps between the Z_{ik}'s belonging to X and Y. By Lemma 1.3.2 (a) the induced map $(\bar{Q}X)_k \xrightarrow{\sim} (\bar{Q}Y)_k$ is weak equivalence.

(d) Fix a degree k. We denote by Z'_{ik} the objects arising in the construction of $\bar{Q}\bar{Q}X$ from $\bar{Q}X$. Consider the diagram

$$\begin{array}{ccccccc} Z_k : X_k = Z_{0k} & \rightarrowtail & Z_{1k} & \rightarrowtail & \cdots & \rightarrowtail & (\bar{Q}X)_k \\ \downarrow & & \downarrow & & & & \downarrow {\scriptstyle \bar{Q}_{\eta X}} \\ Z'_k : (\bar{Q}X)_k = Z'_{0k} & \rightarrowtail & Z'_{1k} & \rightarrowtail & \cdots & \underset{\eta_{\bar{Q}X}}{\rightarrowtail} & (\bar{Q}\bar{Q}X)_k \end{array}$$

The two maps in question $(\bar{Q}X)_k \to (\bar{Q}\bar{Q}X)_k$ agree on X_k and if we restrict both to Z_{ik}, they factor through Z'_{ik}. Furthermore, one checks that both restrictions $Z_{ik} \to Z'_{ik}$ agree after composition with the acyclic fibration $Z'_{ik} \xrightarrow{\sim} \Omega^i(\bar{Q}X)_{k+i}$.

We use the model category structure for sequences in $\mathscr{C}$ introduced in Section 1.3. As indicated in the above diagram, we denote by Z_k the sequence whose colimit is $(\bar{Q}X)_k$, by Z'_k the one whose colimit is $(\bar{Q}\bar{Q}X)_k$ and we denote by $\underline{X}_k$ the constant sequence made up of X_k. Factor the codiagonal map

$$Z_k \cup_{\underline{X}_k} Z_k \rightarrowtail \bar{W} \xrightarrow{\sim} Z_k$$

in the model category $\mathscr{C}^{\mathbb{N}}$ and lift in the square

$$\begin{array}{ccccc}
Z_k \cup_{\underline{X}_k} Z_k & \longrightarrow & & \longrightarrow & Z'_k \\
\downarrow & & \nearrow & & \downarrow \sim \\
\bar{W} & \longrightarrow & Z_k & \longrightarrow & \Omega^*(\bar{Q}X)_{k+*}
\end{array}$$

The left homotopy is obtained by taking the colimit of this diagram.

(e) By part (c), $\bar{Q}$ preserves strict weak equivalences between degreewise fibrant spectra. So it remains to show that a pullback square

$$\begin{array}{ccc}
B \times_Y X & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}$$

with the right map being a strict fibration is mapped to a strictly homotopy cartesian square. We fix k and check that the degree k part is homotopy cartesian in $\mathscr{C}$. For this we need more notation. Denote by Z^X_{ik} the objects arising in the construction of $\bar{Q}X$ from X, and similarly for the other spectra involved. As in part (d), Z^X_k denotes the sequence whose colimit is $(\bar{Q}X)_k$.

Choose a factorization

$$Z^X_k \overset{\sim}{\rightarrowtail} W \twoheadrightarrow Z^Y_k$$

in the category of sequences. Lemma 1.3.2 (a) shows that applying the colimit gives a facorization

$$(\bar{Q}X)_k \overset{\sim}{\rightarrowtail} \operatorname{colim} W \twoheadrightarrow (\bar{Q}Y)_k$$

in $\mathscr{C}$ which we will use to check homotopy cartesianness. Consider the diagram

$$\begin{array}{ccccccc}
Z^B_{ik} & \longrightarrow & Z^Y_{ik} & \twoheadleftarrow & W_i & \overset{\sim}{\leftarrowtail} & Z^X_{ik} \\
\downarrow \sim & & \downarrow \sim & & & \searrow \sim & \downarrow \sim \\
\Omega^i B_{k+i} & \longrightarrow & \Omega^i Y_{k+i} & \twoheadleftarrow & & & \Omega^i X_{k+i}
\end{array}$$

where the diagonal map on the right is a chosen lifting. By the dual of the gluing Lemma (Lemma 1.1.9), the induced map $Z^B_{ik} \times_{Z^Y_{ik}} W_i \xrightarrow{\sim} \Omega^i B_{k+i} \times_{\Omega^i Y_{k+i}} \Omega^i X_{k+i}$ is a weak equivalence. The commutative triangle

$$\begin{array}{ccc} Z_{ik}^{B\times_Y X} & \longrightarrow & Z_{ik}^{B}\times_{Z_{ik}^{Y}} W_i \\ \downarrow{\scriptstyle\sim} & \swarrow{\scriptstyle\sim} & \\ \Omega^i(B_{i+k}\times_{Y_{i+k}} X_{i+k}) & & \end{array}$$

shows that $Z_k^{B\times_Y X} \xrightarrow{\sim} Z_k^B \times_{Z_k^Y} W$ is a weak equivalence of sequences and Lemma 1.3.2(c) applied to pullbacks shows that the map

$$\begin{aligned}(\bar{Q}(B\times_Y X))_k &= \mathrm{colim}_i Z_{ik}^{B\times_Y X} \xrightarrow{\sim} \mathrm{colim}_i Z_{ik}^B \times_{\mathrm{colim}_i Z_{ik}^Y} \mathrm{colim}_i W_i \\ &= (\bar{Q}B)_k \times_{(\bar{Q}Y)_k} \mathrm{colim}_i W_i\end{aligned}$$

is a weak equivalence. This shows that the square

$$\begin{array}{ccc} (\bar{Q}(B\times_Y X))_k & \longrightarrow & (\bar{Q}X)_k \\ \downarrow & & \downarrow \\ (\bar{Q}B)_k & \longrightarrow & (\bar{Q}Y)_k \end{array}$$

is homotopy cartesian and thus finishes the proof. □

The previous lemma indicates that before applying $\bar{Q}$ we should replace a spectrum X by a degreewise fibrant spectrum X^{f}: we define X_0^{f} by the (functorial) factorization

$$X_0 \overset{\sim}{\rightarrowtail} X_0^{\mathrm{f}} \twoheadrightarrow *$$

For $k > 0$, X_k^{f} is similarly defined by the factorization

$$X_k \cup_{\Sigma X_{k-1}} \Sigma X_{k-1}^{\mathrm{f}} \overset{\sim}{\rightarrowtail} X_k^{\mathrm{f}} \twoheadrightarrow *$$

By induction, $X_k \overset{\sim}{\rightarrowtail} X_k^{\mathrm{f}}$ is an acyclic cofibration, hence $X \rightarrow X^{\mathrm{f}}$ is a strict weak equivalence of spectra. Set $QX = \bar{Q}(X^{\mathrm{f}})$; there is again a natural map $X \rightarrow X^{\mathrm{f}} \rightarrow \bar{Q}(X^{\mathrm{f}}) = QX$. We set $\Omega^\infty X = (QX)_0$ and call this the *infinite loop object* of X.

Definition 2.1.4. A map of spectra $X \rightarrow Y$ is a *weak equivalence* if the map $QX \rightarrow QY$ is a strict weak equivalence. The *cofibrations* are the strict cofibrations and a map is a *fibration* if it has the right lifting property with respect to all acyclic cofibrations.

Part (c) of Lemma 2.1.3 shows that strict weak equivalences are weak equivalences. By part (a), QX is an Ω-spectrum. By part (b), for an Ω-spectrum X, the map $X \rightarrow QX$ is a strict weak equivalence, hence a map between Ω-spectra is a weak equivalence if and only if it is a strict weak equivalence. To see that $X \rightarrow QX$ is a weak equivalence of spectra, it is enough to show that for a degreewise fibrant spectrum X, the map $\eta_X : X \to \bar{Q}X$ is a weak equivalence of spectra. According to our definition of weak equivalences, we have to check that the map $\bar{Q}(\eta_X^{\mathrm{f}}) : \bar{Q}(X^{\mathrm{f}}) \to \bar{Q}((\bar{Q}X)^{\mathrm{f}})$ is a strict weak equivalence. Part (d) of Lemma 2.1.3 implies that the diagram

$$\begin{array}{ccc} \bar{Q}X & \longrightarrow & (\bar{Q}X)^{\mathrm{f}} \\ \big\downarrow & & \big\downarrow {\scriptstyle \eta_{(\bar{Q}X)^{\mathrm{f}}}} \\ \bar{Q}(X^{\mathrm{f}}) & \xrightarrow[\bar{Q}(\eta_X^{\mathrm{f}})]{} & \bar{Q}((\bar{Q}X)^{\mathrm{f}}) \end{array}$$

commutes up to degreewise left homotopy. But the three other maps in the square are strict weak equivalences by the various parts of Lemma 2.1.3, hence $\bar{Q}(\eta_X^{\mathrm{f}})$ also is a strict weak equivalence. By part (e) of Lemma 2.1.3, Q preserves strictly homotopy cartesian squares. Now we can prove:

Proposition 2.1.5. *Let $\mathscr{C}$ be a pointed proper closed simplicial model category which admits the small object argument. Then $\mathscr{C}^{\infty}$ is a closed simplicial model category. Moreover, a map $f : X \to Y$ is a fibration if and only if it is a strict fibration and the square*

$$\begin{array}{ccc} X & \longrightarrow & QX \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & QY \end{array}$$

is strictly homotopy cartesian.

Proof. We can apply Theorem A.7 of [1] to the strict model category structure. The assumptions (A.4) and (A.5) of that theorem are satisfied. The pullback part of condition (A.6) is proved as follows: if

$$\begin{array}{ccc} B \times_Y X & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow[\sim]{} & Y \end{array}$$

is a pullback square of spectra such that the lower map is a weak equivalence and the right map is a fibration, we know that the square is strictly homotopy cartesian since fibrations are strict fibrations. So if we apply Q we obtain another strictly homotopy cartesian square. But the lower map becomes a strict weak equivalence after application of Q, so the upper map also does, hence it is a weak equivalence.

The proof in [1] of the fact that the category in question is a closed model category does not use the pushout part of (A.6). This is only needed to show properness of

$\mathscr{C}^\infty$, which we have not claimed in the proposition. Thus Theorem A.7 of [1] proves that $\mathscr{C}^\infty$ is a closed model category and it also provides the characterization of the fibrations as we have stated it above.

It remains to check that $\mathscr{C}^\infty$ is simplicial. The definition of $X \otimes K$ and X^K for spectra was given at the beginning of this section. The function complexes can be defined by

$$\underline{\mathrm{Hom}}_{\mathscr{C}^\infty}(X, Y)_n \quad = \quad \mathrm{Hom}_{\mathscr{C}^\infty}(X \otimes \Delta^n, Y)$$

and the other structure of Definition 1.2.1 is obtained from the corresponding data in $\mathscr{C}$. Most of (SM7b′) is straightforward; the only nontrivial thing is to verify that for an acyclic cofibration $A \overset{\sim}{\rightarrowtail} B$ of spectra and a cofibration $K \rightarrowtail L$ of finite simplicial sets, the cofibration

$$A \otimes L \cup_{A \otimes K} B \otimes K \rightarrowtail B \otimes L$$

is again a weak equivalence.

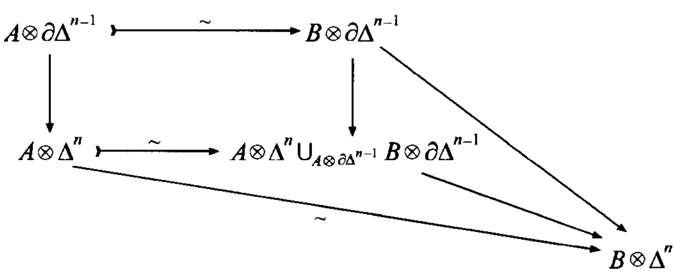
Fix a vertex of Δ^n. Then the acyclic cofibration $\Delta^0 \overset{\sim}{\rightarrowtail} \Delta^n$ induces a strict acyclic cofibration $A \otimes \Delta^n \cup_A B \overset{\sim}{\rightarrowtail} B \otimes \Delta^n$. The map $A \otimes \Delta^n \overset{\sim}{\rightarrowtail} A \otimes \Delta^n \cup_A B$ is an acyclic cofibration because it is a pushout of $A \overset{\sim}{\rightarrowtail} B$. Hence $A \otimes \Delta^n \overset{\sim}{\rightarrowtail} B \otimes \Delta^n$ is an acyclic cofibration. Now we induct on the dimension and number of the nondegenerate simplices in L which do not lie in K. Suppose

$$A \otimes L \cup_{A \otimes K} B \otimes K \overset{\sim}{\rightarrowtail} B \otimes L$$

is an acyclic cofibration whenever all additional simplices of L have dimension strictly less than n. Then $A \otimes \partial\Delta^{n-1} \overset{\sim}{\rightarrowtail} B \otimes \partial\Delta^{n-1}$ is an acyclic cofibration, hence

$$A \otimes \Delta^n \cup_{A \otimes \partial\Delta^{n-1}} B \otimes \partial\Delta^{n-1} \overset{\sim}{\rightarrowtail} B \otimes \Delta^n$$

is seen to be one by looking at the pushout square



Now if all simplices in $L - K$ have dimension less than or equal to n, the result follows by induction on the number of these additional simplices. □

Corollary 2.1.6. *Under the hypothesis of the previous proposition, the fibrant spectra in $\mathscr{C}^\infty$ are precisely the degreewise fibrant Ω-spectra.*

We conclude this section with a remark how weak equivalences of spectra can also be characterized as the maps inducing isomorphisms on homotopy groups if the objects of $\mathscr{C}$ have underlying simplicial sets. Suppose there is a 'forgetful' functor V from $\mathscr{C}$ to the category of pointed simplicial sets, commuting with inverse limits, with the following properties: a map in $\mathscr{C}$ is a fibration or weak equivalence if and only if its image is a fibration or weak equivalence of simplicial sets respectively; there is a natural isomorphism $V(\Omega X) \cong \Omega V(X)$. We can define the homotopy groups of an object of $\mathscr{C}$ as the homotopy groups of the geometric realization of its underlying simplicial set. Suppose further that for a sequence $X_0 \to \cdots \to X_k \to \cdots$ of cofibrations we have $\pi_* \mathrm{colim}\, X_i \cong \mathrm{colim}\, \pi_* X_i$. For pointed simplicial sets Y there is a natural transformation $|\Omega Y| \to \Omega |Y|$ which is a weak homotopy equivalence if Y is fibrant (where $|-|$ denotes geometric realization).

Given a spectrum X, the maps $X_n \to \Omega X_{n+1}$ induce maps

$$|V(X_n)| \to |V(\Omega X_{n+1})| \cong |\Omega V(X_{n+1})| \to \Omega |V(X_{n+1})|\,,$$

hence they induce maps $\pi_j X_n \to \pi_{j+1} X_{n+1}$. X is an Ω-spectrum if and only if these maps are all isomorphisms. We define the homotopy groups of the spectrum X as $\pi_j X = \mathrm{colim}_{i \geq 0} \pi_{j+i} X_i$ for j an integer. Then we have $\pi_j(QX)_k \cong \mathrm{colim}_{i \geq 0} \pi_j Z_{ik} \cong \mathrm{colim}_{i \geq 0} \pi_j \Omega^i X^{\mathrm{f}}_{k+i} \cong \mathrm{colim}_{i \geq 0} \pi_{j+i} X_{k+i} \cong \pi_{j-k} X$. Hence a map of spectra is a weak equivalence if and only if it induces isomorphisms on homotopy groups.

2.2. *Linear model categories*

Definition 2.2.1. We will call a pointed closed model category $\mathscr{C}$ *linear* if for any object X the adjunction map $X \to \Omega \Sigma X$ in the homotopy category of $\mathscr{C}$ is an isomorphism.

If $\mathscr{C}$ is simplicial, we have functors on $\mathscr{C}$ which give models for loop and suspension on $Ho\,\mathscr{C}$. Hence in this case linearity means that for any cofibrant object A, fibrant object X and weak equivalence $\Sigma A \xrightarrow{\sim} X$, the adjoint map $A \to \Omega X$ is also a weak equivalence. Thus in a linear closed simplicial model category, the suspension spectrum of any cofibrant object is an Ω-spectrum. One can show furthermore that for proper pointed closed model categories, linearity is equivalent to the condition that homotopy cocartesian squares are also homotopy cartesian. This explains the terminology, for a linear model category is one for which the identity functor is linear in the sense of Goodwillie's Calculus of Functors [3].

In a linear closed simplicial model category $\mathscr{C}$, the suspension spectrum functor induces an equivalence of homotopy theories of $\mathscr{C}$ and of a certain full subcategory of the spectra category. In order to characterize the spectra in this subcategory, we need a suitable notion of a 'connected object' in $\mathscr{C}$. So let us assume that there is a class of objects of $\mathscr{C}$ which we call *connected*, closed under weak equivalences, containing all suspensions, with the following property: a map $X \to Y$ between connected fibrant objects is already a weak equivalence if the map $\Omega X \to \Omega Y$ is one. Then a spectrum X will be called *connective* if for all $k \geq 1$ the objects $(QX)_k$ are connected.

If objects in $\mathscr{C}$ have underlying simplicial sets and suspension increases connectivity, the usual notion of connected simplicial sets does the job. However, if the category in question is a category of spectra, our terminology is misleading because in this case every object is a suspension (up to weak equivalence). Hence all objects are connected according to our definition and all spectra (of spectra) are connective. Thus the next lemma, which we will eventually apply to the category of simplicial modules over a simplicial ring, also indicates that the homotopy theory is really stable under stabilization, because spectra and spectra of spectra have the same homotopy theory.

Lemma 2.2.2. *Suppose $\mathscr{C}$ is a linear pointed proper closed simplicial model category which admits the small object argument. Then the total left derived functor (cf. [4, I.4]) of the suspension spectrum functor induces an equivalence of the homotopy theory of $\mathscr{C}$ and the homotopy theory of the connective spectra in $\mathscr{C}$. The quasi-inverse is given by the functor Ω^∞.*

Proof. 'Equivalence of homotopy theories' is to be understood in the sense of [4, I.4]. It comprises an equivalence of the homotopy categories that preserves cofibration sequences, fibration sequences, loop and suspension on the homotopy categories. The standard way to construct such an equivalence is to apply Theorem 3 of [4, I.4]. However, we will refer to Theorem 9.7 of [2] since it shows how the same conclusion can be achieved with fewer assumptions on the functors involved.

As we mentioned before, the suspension spectrum functor Σ^∞ is left adjoint to the functor $-_0 : \mathscr{C}^\infty \to \mathscr{C}$ which associates to a spectrum its degree zero term. Σ^∞ preserves cofibrations and $-_0$ preserves fibrations since these are always strict fibrations. Thus Theorem 9.7 (i) of [2] shows that the total derived functors $L(\Sigma^\infty) : Ho\,\mathscr{C} \to Ho\,\mathscr{C}^\infty$ and $R(-_0) : Ho\,\mathscr{C}^\infty \to Ho\,\mathscr{C}$ exist and are adjoint. The recipe for total right derived functors says that we replace an object by a weakly equivalent fibrant one and the apply the functor under consideration. Now QX is a degreewise fibrant Ω-spectrum, hence it is such a fibrant substitute for a spectrum X, and since $\Omega^\infty X = (QX)_0$, the total right derived functor of $-_0$ is Ω^∞.

Now let M be a cofibrant object of $\mathscr{C}$ and X a connective fibrant spectrum. Then $\Sigma^\infty M$ is a connective Ω-spectrum, so $L(\Sigma^\infty)$ takes values in the subcategory of connective spectra. A map between the connective spectra $\Sigma^\infty M \to X$ is a weak equivalence if and only if its adjoint map $M \to X_0$ is one. The proof of Theorem 9.7 (ii) shows that this implies that for all objects M of $Ho\,\mathscr{C}$ the adjunction morphism $1_{Ho\,\mathscr{C}} \to R(-_0) \circ L(\Sigma^\infty)$ is a isomorphism and that for all connective spectra in $Ho\,\mathscr{C}^\infty$ the adjunction morphism $L(\Sigma^\infty) \circ R(-_0) \to 1_{Ho\,\mathscr{C}^\infty}$ is an isomorphism. Hence $L(\Sigma^\infty)$ and Ω^∞ provide an equivalence of $Ho\,\mathscr{C}$ with the full subcategory of $Ho\,\mathscr{C}^\infty$ consisting of the connective spectra. □

3. Commutative simplicial rings and simplicial modules

3.1. Preliminaries

Specializing the results obtained to the examples we primarily had in mind, we will see that spectra of modules and spectra of commutative simplicial rings over a fixed commutative simplicial ring form proper closed simplicial model categories. Furthermore, the category of simplicial modules over a simplicial ring is linear, hence its homotopy theory does not change under stabilization.

All rings and algebras we consider will be commutative. For a fixed simplicial ring B we denote the category of simplicial B-modules by B-mod. The category of commutative simplicial rings will be denoted by $s\mathcal{R}$. Following Quillen, we give a definition of a closed simplicial model category structure for the category of simplicial B-modules and for the category of commutative simplicial rings.

Call a map of simplicial B-modules (resp. commutative simplicial rings) a fibration or a weak equivalence if it is a fibration or weak equivalence of the underlying simplicial sets. Call such a map a cofibration if it has the left lifting property with respect to all acyclic fibrations. If M is a simplicial B-module (resp. R a commutative simplicial ring) and K a finite simplicial set, define the simplicial B-module $M \otimes K$ (resp. the commutative simplicial ring $R \otimes K$) as the diagonal of the bisimplicial module $M \times K$ (resp. the bisimplicial ring $R \times K$; cf. Section 1.2 for the definition of $X \times K$). The simplicial mapping complex of maps of K into the underlying simplicial set of M (resp. R) is naturally endowed with the structure of a simplicial B-module (resp. commutative simplicial ring), which we define to be the object M^K (resp. R^K). Finally, the function complex simplicial sets for simplicial B-modules or commutative simplicial rings are defined as in [4, II, p. 1.7].

We say that a map $M \to M'$ of simplicial B-modules (resp. commutative simplicial rings) is a free map if there are subsets $C_n \subseteq M'_n$, stable under the degeneracy maps, such that M'_n is isomorphic (via the given map) to the direct sum of M_n and the free B_n-module generated by C_n (resp. to the polynomial ring over M_n with C_n the set of indeterminates).

Lemma 3.1.1. *With these definitions, the categories of simplicial B-modules and the category of commutative simplicial rings become closed simplicial model categories which admit the small object argument. The cofibrations are precisely the retracts of the free maps.*

Proof. For a proof, we again refer to [4]. In the case of simplicial B-modules, this is given in II.6. In II.4 (Theorem 4), Quillen gives a general criterion for a category of simplicial objects to be a closed simplicial model category. This theorem applies in the case of commutative simplicial rings. For the characterization of cofibrations cf. Remark 4 of [4, II.4].

We have to mention why the small object argument works. The point is that in both cases there exist free objects generated by simplicial sets (i.e., the forgetful functor to simplicial sets has a left adjoint). The free objects generated by the finite simplicial sets are small and as a set of test maps we can take the maps induced by the inclusions $\partial\triangle^n \to \triangle^n$ of the boundaries and $\Lambda^n_k \to \triangle^n$ of the horns into the standard simplicies. □

Lemma 3.1.2. *The model category of commutative simplicial rings is proper.*

Proof. The pullback part of the properness definition is proved using the 5-lemma because for fibrations we have long exact sequences of homotopy groups.

Now consider a cofibration $X \rightarrowtail M$ and a weak equivalence $X \xrightarrow{\sim} Y$ of commutative simplicial rings. The pushout of commutative simplicial rings is given by degreewise tensor product. Since M is also cofibrant as an X-module, the tensor product with M over X is equivalent to the derived tensor product (by the corollary [4, II, p. 6.10]). The spectral sequence Theorem 6(b) of [4, II.6] thus shows that tensoring with M over X preserves weak equivalences. So $M \xrightarrow{\sim} M \otimes_X Y$ is a weak equivalence. □

Lemma 3.1.3. *The model category of simplicial B-modules is proper and linear.*

Proof. Let $M \rightarrowtail N$ be a cofibration of simplicial B-modules. The quotient map $N \twoheadrightarrow N/M$ is a degreewise surjective homomorphism of simplicial groups, hence a Kan fibration. The cofibration, being the retract of a free map, is degreewise injective, hence M is the fibre of $N \twoheadrightarrow N/M$. So for cofibrations of simplicial B-modules we have long exact sequences of homotopy groups and the pushout part of the properness definition can be proved like the pullback part using the 5-lemma. For any B-module M, the adjunction map $M \xrightarrow{\sim} \Omega\Sigma M$ is a weak equivalence [4, II.6, Proposption 1]; since furthermore all objects in B-mod are fibrant, B-mod is linear. □

Since $s\mathscr{R}$ is a proper closed simplicial model category with small object argument, so is $s\mathscr{R}//B$, the category of commutative simplicial rings containing a simplicial ring B as a retract. Note that an object of $s\mathscr{R}//B$ is nothing but an augmented commutative simplicial B-algebra. Denote by I the augmentation ideal functor $I : s\mathscr{R}//B \to B$-mod. Then for an augmented B-algebra X, $X \cong B \oplus I(X)$ as a B-module and all morphisms in $s\mathscr{R}//B$ map the B summand of this decomposition by the identity. Hence a map in $s\mathscr{R}//B$ is a weak equivalence (resp. fibration) if and only if it is one on the augmentation ideals. If we take $I(X)$, rather than the whole X, as the 'underlying simplicial set' of an object X of $s\mathscr{R}//B$, the remark at the end of Section 2.1 applies and shows that weak equivalences of spectra in $s\mathscr{R}//B$ are the maps inducing isomorphisms on homotopy groups. The same is true for B-modules, where 'underlying simplicial set' has its usual meaning. Proposition 2.1.5 and Lemma 2.2.2 apply and show

Corollary 3.1.4. *The spectra categories $(s\mathscr{R}//B)^\infty$ and $(B\text{-mod})^\infty$ are closed simplicial model categories such that the weak equivalences are the maps inducing isomorphisms on homotopy groups of spectra. The total left derived functor of the suspension spectrum functor induces an equivalence of the homotopy theory of simplicial B-modules and the homotopy theory of connective spectra of simplicial B-modules.*

Later we will need to know that the connectivity of modules behaves under tensor product in the same way as the connectivity of spaces behaves under smash product; this follows, e.g., from the spectral sequence for the derived tensor product ([4, II.6], Theorem 6 (b) and its corollary]):

Lemma 3.1.5. *Let M be a cofibrant, m-connected simplicial B-module and N an n-connected simplicial B-module. Then $N \otimes_B M$ is $(n+m+1)$-connected.*

3.2. Equivalence of stable homotopy theories in the rational case

The next task will be to prove that for commutative simplicial $\mathbb{Q}$-algebras B the homotopy theories of spectra in $s\mathcal{R}//B$ and of spectra of B-modules are equivalent. The augmentation ideal functor $I : s\mathcal{R}//B \to B$-mod is right adjoint to $Sym : B\text{-mod} \to s\mathcal{R}//B$ which maps a B-module to its symmetric algebra. The natural map $Sym(M) \otimes K \to Sym(M \otimes K)$ is an isomorphism for any B-module M and any finite simplicial set K since Sym commutes with colimits. It makes Sym into a simplicial functor such that the natural map $\Sigma Sym(M) \to Sym(\Sigma M)$ is again an isomorphism. So we get induced adjoint functors $Sym^\infty : (B\text{-mod})^\infty \to (s\mathcal{R}//B)^\infty$ and $I^\infty : (s\mathcal{R}//B)^\infty \to (B\text{-mod})^\infty$ via degreewise application (cf. Section 2.1). These are the functors that will be shown to induce an equivalences of homotopy theories.

I preserves underlying simplicial sets (remember that in the case of augmented algebras over B we are disregarding the B summand). Hence I preserves homotopy groups, I^∞ preserves homotopy groups of spectra and so it preserves all weak equivalences of spectra.

We also need to know

Lemma 3.2.1. *I^∞ preserves fibrations of spectra.*

Proof. We use the characterization of fibrations given by Proposition 2.1.5: a map $f : X \to Y$ is a fibration of spectra if and only if it is a strict fibration and the diagram

$$\begin{array}{ccc} X & \longrightarrow & QX \\ \downarrow & & \downarrow \\ Y & \longrightarrow & QY \end{array}$$

is strictly homotopy cartesian (i.e., degreewise homotopy cartesian in $\mathcal{C}$). Since I preserves fibrations, weak equivalences and pullback, I^∞ preserves strict fibrations and strictly homotopy cartesian squares, i.e.,

$$\begin{array}{ccc} I^\infty(X) & \longrightarrow & I^\infty(QX) \\ \downarrow & & \downarrow \\ I^\infty(Y) & \longrightarrow & I^\infty(QY) \end{array}$$

is strictly homotopy cartesian. If we can replace $I^\infty(QX)$ by $QI^\infty(X)$ and $I^\infty(QY)$ by $QI^\infty(Y)$ and still obtain a strictly homotopy cartesian square we are done because then the above criterion shows that $I^\infty(X) \twoheadrightarrow I^\infty(Y)$ is a fibration.

In the commutative square

$$\begin{array}{ccc} I^\infty(X) & \longrightarrow & I^\infty(QX) \\ \downarrow & & \downarrow \\ QI^\infty(X) & \longrightarrow & QI^\infty(QX) \end{array}$$

all maps are weak equivalences. Since I^∞ preserves Ω-spectra, the lower and right map are weak equivalences between Ω-spectra, hence they are strict weak equivalences. The same is true for Y instead of X, and so two applications of the dual of Lemma 1.1.8 to the strict model category structure give the replacement we want. □

The following lemma is the key step in the proof of the equivalences of stable homotopy theories.

Lemma 3.2.2. *If B is a commutative simplicial $\mathbb{Q}$-algebra and M a cofibrant spectrum of B-modules, the adjunction morphism $M \to I^\infty Sym^\infty(M)$ is a weak equivalence of spectra.*

Proof. We first show the lemma in the case where M is the suspension spectrum of a cofibrant B-module W and then use a direct limit argument. Since $\Sigma W = (\mathbb{Z}[S^1]/\mathbb{Z}[*]) \otimes_{\mathbb{Z}} W$ and $\mathbb{Z}[S^1]/\mathbb{Z}[*]$ is a connected simplicial $\mathbb{Z}$-module, suspension increases connectivity by Lemma 3.1.5. Hence $\Sigma^k W$ is $(k-1)$-connected.

Next comes the part where we need the assumption that B is a $\mathbb{Q}$-algebra. We show that for a cofibrant m-connected B-module A, the map $A \to I(Sym(A))$ is $(2m+1)$-connected. Since A is m-connected, $A^{\otimes_B n}$ is $(nm+n-1)$-connected (Lemma 3.1.5). Let $S_n^B(A)$ denote the quotient of $A^{\otimes_B n}$ by the action of the symmetric group. Since B contains $\mathbb{Q}$, this quotient is actually a direct summand and hence it is also $(nm+n-1)$-connected. So the map

$$A \to I(Sym(A)) = A \oplus S_2^B(A) \oplus \cdots \oplus S_n^B(A) \oplus \cdots$$

is $(2m+1)$-connected.

Putting the first two parts together, we see that the degree k part of the map $\Sigma^\infty W \to I^\infty Sym^\infty(\Sigma^\infty W)$ is $(2k-1)$-connected, hence the map is an equivalence of spectra. The same argument with possibly different connectivity estimates works if M is only ultimately a suspension spectrum. By this we mean that there is a number n such that $M_k = \Sigma^{k-n} M_n$ for all $k \geq n$, the structure maps in these degrees being identity maps.

The general case follows because every spectrum of B-modules can be written as the colimit of a sequence of spectra which are ultimately suspension spectra and because homotopy groups commute with sequential colimits of spectra of B-modules. □

Now we can prove

Theorem 3.2.3. *If B is a commutative simplicial $\mathbb{Q}$-algebra, the total left derived functor of Sym^∞ is an equivalence of homotopy theories $L(Sym^\infty): Ho(B\text{-mod})^\infty \to Ho(s\mathscr{R}//B)^\infty$ with quasi-inverse $R(I^\infty)$.*

Proof. We apply Theorem 9.7 of [2] again. The functors Sym^∞ and I^∞ are adjoint. I^∞ preserves fibrations by Lemma 3.2.1 and Sym^∞ preserves cofibrations. It remains to check that for cofibrant B-module spectra M and fibrant $s\mathscr{R}//B$ spectra X a map $M \to I^\infty(X)$ is a weak equivalence if and only if the adjoint map $Sym^\infty(M) \to X$ is a weak equivalence. But the latter is true if and only if $I^\infty Sym^\infty(M) \to I^\infty(X)$ is a weak equivalence, hence the result follows since Lemma 3.2.2 states that $M \xrightarrow{\sim} I^\infty Sym^\infty(M)$ is a weak equivalence. □

3.3. Relation with André–Quillen homology

Suppose $\mathscr{C}$ is a proper closed simplicial model category admitting the small object argument, so that for every object B of $\mathscr{C}$, the category of spectra in $\mathscr{C}//B$ becomes a closed simplicial model category by the procedure of Section 2.1. Let us think of a spectrum in $\mathscr{C}//B$ as a B-module in a generalized sense. We have seen that for commutative simplicial $\mathbb{Q}$-algebras, this notion of a module coincides with the classical one (at least if one is interested in the homotopy theory). Given a map $A \to B$ in $\mathscr{C}$, we can associate to it a generalized B-module, which is the suspension spectrum of the unreduced suspension of A over B, in the following way. We consider the category $A/\mathscr{C}/B$ of objects under A and over B with respect to $A \to B$. This is a closed simplicial model category in the obvious way, analogous to the recipe for $\mathscr{C}//B$. The functor $\mathscr{C}//B \to A/\mathscr{C}/B$ induced by $A \to B$ has a left adjoint

$$-_+ : A/\mathscr{C}/B \to \mathscr{C}//B, \quad (A \to P \to B) \mapsto P_+ = B \cup_A P\,.$$

The composition with the suspension spectrum functor $\Sigma^\infty_+ : A/\mathscr{C}/B \to (\mathscr{C}//B)^\infty$ preserves weak equivalences between cofibrant objects. To obtain a construction invariant under weak equivalences, we apply the left derived functor of Σ^∞_+, to $(A \to B \xrightarrow{=} B)$ to get the suspension spectrum of the unreduced suspension of A over B.

For a commutative simplicial ring B we have constructed pairs of adjoint functors

$$Ho(B\text{-mod}) \underset{\Omega^\infty}{\overset{L(\Sigma^\infty)}{\rightleftarrows}} Ho(B\text{-mod})^\infty \underset{R(I^\infty)}{\overset{L(Sym^\infty)}{\rightleftarrows}} Ho(s\mathcal{R}//B)^\infty$$

such that the first pair is an equivalence of the homotopy theories of B-modules and of connective spectra of B-modules. If B is a $\mathbb{Q}$-algebra, the second pair is also an equivalence of homotopy theories. The last thing we show is that for commutative simplicial $\mathbb{Q}$-algebras B, the suspension spectrum of a map $A \to B$ as discussed above coincides with the cotangent complex as defined in [6] under these equivalences.

Let us recall the definition of the cotangent complex (cf. [6, 2.2, 4]). Given a morphism of commutative simplicial rings $A \to B$, consider B as an A-algebra. Abelian group objects in the category of commutative A-algebras over B can be identified with B-modules. The forgetful functor from the category of abelian group objects has a left adjoint

$$Ab_{A\text{-alg}/B} \; : \; A\text{-alg}/B \longrightarrow B\text{-mod}$$

The cotangent complex $L_{B/A}$ of $A \to B$ is defined as the value of the left derived functor of this abelianization on the object B. $L_{B/A}$ is thus an object well defined up to canonical isomorphism in the homotopy category of simplicial B-modules. (There also is an explicit formula for $Ab_{A\text{-alg}/B}$, hence for $L_{B/A}$, in terms of modules of Kähler differentials.)

Under the identification of B-modules with abelian group objects in A-alg/B, the forgetful functor corresponds to the composite

$$B\text{-mod} \xrightarrow{B\oplus -} s\mathcal{R}//B \longrightarrow A\text{-alg}/B = A/s\mathcal{R}/B,$$

where $B \oplus M$ is considered as a ring over B with trivial multiplication on the M summand. $B \oplus -$ also has a left adjoint

$$Ab_B \; : s\mathcal{R}//B \longrightarrow B\text{-mod}\;, \qquad Ab_B(X) = I(X)/I^2(X)\,.$$

Hence abelianization for A-algebras over B factors through $s\mathcal{R}//B$.

The diagram

$$\begin{array}{ccccc} A\text{-alg}/B & \xrightarrow{+} & s\mathcal{R}//B & \xrightarrow{\Sigma^\infty} & (s\mathcal{R}//B)^\infty \\ & {\scriptstyle Ab_{A\text{-alg}/B}} \searrow & \downarrow {\scriptstyle Ab_B} & & \downarrow {\scriptstyle Ab_B^\infty} \\ & & B\text{-mod} & \xrightarrow[\Sigma^\infty]{} & (B\text{-mod})^\infty \end{array}$$

commutes up to isomorphism of functors because $Ab_B(\Sigma X) \cong \Sigma Ab_B(X)$. Since $L_{B/A} = L(Ab_{A\text{-alg}/B})(B)$, it follows that the cotangent complex coincides with $L(\Sigma_+^\infty)(B)$ if we can show that $R(I^\infty)$ is isomorphic to the left derived functor of Ab_B^∞. This is the content of the last lemma (remember that I^∞ preserves all weak equivalences of spectra, so that $R(I^\infty)(X)$ is isomorphic to $I^\infty(X)$ in the homotopy category for all spectra X, not just for the fibrant ones). Projection defines a natural transformation $I \to I/I^2 = Ab_B$ and hence a natural transformation $I^\infty \to Ab_B^\infty$.

Lemma 3.3.1. *Let B be a commutative simplicial $\mathbb{Q}$-algebra. Then for all cofibrant spectra X in $(s\mathcal{R}//B)^\infty$, the map $I^\infty X \to Ab_B^\infty X$ is a weak equivalence.*

Proof. Suppose first that $X = Sym^\infty(W)$ for some cofibrant spectrum W of B-modules. Forming the symmetric algebra and then dividing its augmentation ideal by its square gives back the module one has started with. Hence $Ab_B^\infty(X) \cong W$. Furthermore, the map $Ab_B^\infty(X) \cong W \to I^\infty Sym^\infty(W) = I^\infty(X)$, which was shown in Lemma 3.2.2 to be a weak equivalence, is a right inverse to the map in question. Hence that map is also a weak equivalence.

The general case follows because, up to weak equivalence, all spectra in $(s\mathscr{R}//B)^\infty$ are of the form considered in the first part. More precisely, if X is any cofibrant spectrum of rings over B, choose a cofibrant B-module spectrum W and a weak equivalence $W \xrightarrow{\sim} I^\infty(X)$ in $(B\text{-mod})^\infty$. As we have noted in the proof of Lemma 3.2.2, this implies that the adjoint map $Sym^\infty(W) \xrightarrow{\sim} X$ is also a weak equivalence. I^∞ preserves all weak equivalences, hence we are reduced to the first part if we can show that Ab_B^∞ preserves weak equivalences between cofibrant objects.

But this is the usual argument for functors with right adjoints. The right adjoint of Ab_B^∞, which is degreewise application of $M \mapsto B \oplus M$, can be shown to preserve fibrations in the same way we showed in Lemma 3.2.1 that I^∞ has this property. Hence Ab_B^∞ preserves acyclic cofibrations and thus weak equivalences between cofibrant objects by Lemma 9.9 of [2]. □

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