Journal of Pure and Applied Algebra 24 (1982) 179–202
North-Holland Publishing Company

SPACES WITH FINITELY MANY NON-TRIVIAL HOMOTOPY GROUPS

Jean-Louis LODAY*
Institut de Recherche Mathématique Avancée, Strasbourg, 67084 France

Communicated by A. Heller
Received 20 February 1981

It is well known that the homotopy category of connected CW-complexes X whose homotopy groups $\pi_i(X)$ are trivial for $i>1$ is equivalent to the category of groups. One of the objects of this paper is to prove a similar equivalence for the connected CW-complexes X whose homotopy groups are trivial for $i>n+1$ (where n is a fixed non-negative integer). For $n=1$ the notion of crossed module invented by J.H.C. Whitehead [13], replaces that of group and gives a satisfactory answer. We reformulate the notion of crossed module so that it can be generalized to any n. This generalization is called an 'n-cat-group', which is a group together with $2n$ endomorphisms satisfying some nice conditions (see 1.2 for a precise definition). With this definition we prove that the homotopy category of connected CW-complexes X such that $\pi_i(X)=0$ for $i>n+1$ is equivalent to a certain category of fractions (i.e. a localization) of the category of n-cat-groups.

The main application concerns a group-theoretic interpretation of some cohomology groups. It is well known [10, p. 112] that the cohomology group $H^2(G;A)$ of the group G with coefficients in the G-module A is in one-to-one correspondence with the set of extensions of G by A inducing the prescribed G-module structure on A. Use of n-cat-groups gives a similar group-theoretic interpretation for the higher cohomology groups $H^n(G;A)$ and $H^n(K(C,k);A)$ where $K(C,k)$ is an Eilenberg–MacLane space with $k\geq 1$. In [8] we proved that crossed modules could be used to interpret a relative cohomology group. Here we show that the notion of n-cat-group is particularly suitable to interpret some 'hyper-relative' cohomology groups. The usefulness of this last result appears in its application to algebraic K-theory where it leads to explicit computations [4]. This was in fact our primary motivation for a generalization of crossed modules.

Section 1 contains the definitions of n-cat-groups and of n-cubes of fibrations. There are two functors:

$$\mathcal{G} : (n\text{-cubes of fibrations}) \to (n\text{-cat-groups})$$

* Partially supported by a N.S.F. grant.

and

$$\mathscr{B} : (n\text{-cat-groups}) \to (n\text{-cubes of fibrations})$$

which bear the same properties (adjointness) as the functors π_1 (= fundamental group) and B (= classifying space functor) respectively. The properties of these functors and the equivalences of categories are stated in this section.

In Section 2 we construct the functors $\mathscr{G}$ and $\mathscr{B}$ and prove their properties.

In Section 3 we define the mapping cone of non-abelian group complexes (which might be of independent interest) and use it to compute the homotopy groups of the spaces arising from n-cat-groups.

Section 4 contains the group theoretic interpretation of some cohomology groups.

In Section 5 we carry out a detailed study of the case $n=2$, and we give an application.

Unless otherwise stated all spaces are connected base-pointed CW-complexes and all maps preserve base points. A connected space S is said to be n-connected if $\pi_i(S)=0$ for $i<n$. The nerve of a discrete group G is a simplicial set denoted $\beta_* G$ where $\beta_n G = G \times \cdots \times G$ (n times). Its geometric realization $|\beta_* G|$ is the classifying space of G and is denoted BG.

1. *n*-cat-groups, definitions and results

1.1. Consider a simplicial group

$$\left(\cdots \begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} G \overset{s,\,b}{\rightrightarrows} N\right)$$

where N is identified with a subgroup of G by the degeneracy map $\sigma : N \to G$. The relations among face and degeneracy maps in a simplicial group imply $s|_N = b|_N = \mathrm{id}_N$. Moreover, as we shall see in Lemma 2.2, if the Moore complex of this simplicial group is of length one, that is

$$\cdots 1 \to 1 \cdots \quad \cdots \to 1 \to \operatorname{Ker} s \to N,$$

then the face maps s and b satisfy the following property: the group $[\operatorname{Ker} s, \operatorname{Ker} b]$ generated by the commutators $[x,y]=xyx^{-1}y^{-1}$, $x \in \operatorname{Ker} s$, $y \in \operatorname{Ker} b$ is trivial. This remark leads to the following

Definition. A *categorical group* (or *1-cat-group*) is a group G together with a subgroup N and two homomorphisms (called structural homomorphisms) $s, b : G \to N$ satisfying the following conditions:

(i) $\quad s|_N = b|_N = \mathrm{id}_N$,

(ii) $\quad [\operatorname{Ker} s, \operatorname{Ker} b] = 1$.

This 1-cat-group is denoted by $\mathfrak{G} = (G; N)$ if no confusion can arise. A morphism of

1-cat-groups $\mathfrak{G} \to \mathfrak{G}'$ is a group homomorphism $f: G \to G'$ such that $f(N) \subset N'$ and $s'f = f|_N s, b'f = f|_N b$.

The following definition is motivated by the notion of n-simplicial group.

1.2. Definition. An *n-categorical group* (or *n-cat-group* for short) $\mathfrak{G}$ is a group G together with n categorical structures which commute pairwise, that is n subgroups $N_1, \ldots, N_n$ of G and $2n$ group homomorphisms $s_i, b_i: G \to N_i$, $i = 1, \ldots, n$, such that for $1 \le i \le n$, $1 \le j \le n$,

(i) $\quad s_{i|N_i} = b_{i|N_i} = \mathrm{id}_{N_i}$,

(ii) $\quad [\mathrm{Ker}\, s_i, \mathrm{Ker}\, b_i] = 1$,

(iii) $\quad s_i s_j = s_j s_i, \quad b_i b_j = b_j b_i, \quad$ and $\quad b_i s_j = s_j b_i, \quad i \ne j$.

In (iii) and from now on the morphisms s_i and b_i are considered as endomorphisms of G by using the inclusions $N_i \to G$. When no confusion can arise $\mathfrak{G}$ is denoted by $(G; N_1, \ldots, N_n)$. A morphism of n-cat-groups $f: \mathfrak{G} \to \mathfrak{G}'$ is a group homomorphism $f: G \to G'$ such that $s_i' f = f s_i$ and $b_i' f = f b_i$ for $i = 1, \ldots, n$. By convention a 0-cat-group is just a group.

1.3. Let $\langle -1, 0, 1\rangle$ be the category associated to the ordered set $-1 < 0 < 1$. The cartesian product of n copies of $\langle -1, 0, 1\rangle$ is denoted $\langle -1, 0, 1\rangle^n$. An object of $\langle -1, 0, 1\rangle^n$ is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i = -1$ or 0 or 1.

Definition. An *n-cube of fibrations* is a functor $\mathfrak{X}$ from $\langle -1, 0, 1\rangle^n$ to the category of connected spaces such that for every i the sequence

$$\begin{aligned}\mathfrak{X}(\alpha_1, \ldots, \alpha_{i-1}, -1, \alpha_{i+1}, \ldots, \alpha_n) &\to \mathfrak{X}(\alpha_1, \ldots, \alpha_{i-1}, 0, \alpha_{i+1}, \ldots, \alpha_n) \\ &\to \mathfrak{X}(\alpha_1, \ldots, \alpha_{i-1}, 1, \alpha_{i+1}, \ldots, \alpha_n)\end{aligned}$$

is a fibration.

We will frequently write $\mathfrak{X}^\alpha$ instead of $\mathfrak{X}(\alpha)$. By convention $\langle -1, 0, 1\rangle^0$ is the category with one element and one morphism. Therefore a 0-cube of fibrations is just a connected space. For $n = 1$ a 1-cube of fibrations is an ordinary fibration of connected spaces $\mathfrak{X}^{-1} \to \mathfrak{X}^0 \to \mathfrak{X}^1$. For $n = 2$ a 2-cube of fibrations is a commutative diagram of connected spaces

$$\begin{tikzcd}
\mathfrak{X}^{-1,-1} \arrow[r] \arrow[d] & \mathfrak{X}^{0,-1} \arrow[r] \arrow[d] & \mathfrak{X}^{1,-1} \arrow[d] \\
\mathfrak{X}^{-1,0} \arrow[r] \arrow[d] & \mathfrak{X}^{0,0} \arrow[r] \arrow[d] & \mathfrak{X}^{1,0} \arrow[d] \\
\mathfrak{X}^{-1,1} \arrow[r] & \mathfrak{X}^{0,1} \arrow[r] & \mathfrak{X}^{1,1}
\end{tikzcd}$$

where each row and each column is a fibration.

A morphism $\mathfrak{X} \to \mathfrak{X}'$ of n-cubes of fibrations is a transformation of functors. It is said to be a homotopy equivalence iff for every α, the map $\mathfrak{X}^\alpha \to \mathfrak{X}'^\alpha$ is a homotopy equivalence of spaces.

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is such that $\alpha_i = -1$ or 0 for every i, it is said to be negative, written $\alpha \leq 0$.

1.4. Theorem. *There are two functors*

$$\mathscr{B} : (n\text{-cat-groups}) \to (n\text{-cubes of fibrations})$$

and

$$\mathscr{G} : (n\text{-cubes of fibrations}) \to (n\text{-cat-groups})$$

such that (a) *if* $\mathfrak{G}$ *is an n-cat-group then* $(\mathscr{B}\mathfrak{G})^\alpha$ *is an Eilenberg–MacLane space of type* $K(\pi, 1)$ *when* $\alpha \leq 0$, (b) *the composite* $\mathscr{G}\mathscr{B}$ *is the identity,* (c) *for any n-cube of fibrations* $\mathfrak{X}$ *there exists a map of n-cubes* $\mathfrak{X} \to \mathscr{B}\mathscr{G}(\mathfrak{X})$, *well-defined up to homotopy, such that for every* α, $\pi_1(\mathfrak{X}^\alpha) \to \pi_1(\mathscr{B}\mathscr{G}(\mathfrak{X})^\alpha)$ *is the identity of* $\pi_1(\mathfrak{X}^\alpha)$.

The proof is given in Section 2.

For $n = 0$ this theorem is well known: the functor $\mathscr{G}$ is π_1, the functor $\mathscr{B}$ is the classifying space functor B. Property (a) says that BG is a $K(G, 1)$, property (b) says that $\pi_1(BG) = G$ and property (c) says that for any connected space X there is a map, well-defined up to homotopy, $X \to B\pi_1 X$ inducing the identity on π_1.

The homotopy category of n-cubes of fibrations is the category whose objects are n-cubes of fibrations and whose morphisms are homotopy classes of morphisms. The following result is an immediate consequence of Theorem 1.4.

1.5. Corollary. *The category of n-cat-groups is equivalent to the homotopy category of n-cubes of fibrations* $\mathfrak{X}$ *such that* $\mathfrak{X}^\alpha$ *is a* $K(\pi, 1)$ *for every* $\alpha \leq 0$.

For $n = 0$ this corollary says that the homotopy category of $K(\pi, 1)$-spaces is equivalent to the category of groups.

For $n = 1$ it can be interpreted as an equivalence of the homotopy category of fibrations of the type $BM \to BN \to X$ (where M and N are discrete groups) with the category of crossed modules (see Section 2).

1.6. Denote the space $(\mathscr{B}\mathfrak{G})^{1,1,\ldots,1}$ by $B\mathfrak{G}$. It comes easily from Theorem 1.4 that $B\mathfrak{G}$ is connected and that $\pi_i(B\mathfrak{G})$ vanishes for $i > n + 1$. Therefore $B\mathfrak{G}$ is $(n + 1)$-co-connected (cf. 2.17). There is an algebraic device to get the homotopy groups of $B\mathfrak{G}$ from $\mathfrak{G}$. In fact there is a complex of (non-abelian) groups $C_*(\mathfrak{G})$ whose homology groups are the homotopy groups of $B\mathfrak{G}$ (see Proposition 3.4). A morphism of n-cat-groups $\mathfrak{G} \to \mathfrak{G}'$ induces a morphism of complexes. Such a morphism is called a *quasi-isomorphism* if it induces an isomorphism on homology. The set of quasi-isomorphisms is denoted by Σ.

1.7. Theorem. *The homotopy category of $(n+1)$-coconnected CW-complexes is equivalent to the category of fractions $(n\text{-cat-groups})(\Sigma^{-1})$.*

The notation $(n\text{-cat-groups})(\Sigma^{-1})$ stands for the category of fractions of (n-cat-groups) where all the quasi-isomorphisms (elements of Σ) have been inverted [3]. The proof of this theorem is in Section 3.

2. Equivalence between n-cat-groups and some n-cubes of fibrations

After some preliminaries on crossed modules we prove Theorem 1.4 for $n=1$ and then in the general case.

2.1. Definition. A *crossed module* is a group homomorphism $\mu: M \to N$ together with an action of N on M, denoted by $(n, m) \mapsto {}^n m$ and satisfying the following conditions:

(a) for all $n \in N$ and $m \in M$, $\mu({}^n m) = n\mu(m)n^{-1}$,

(b) for all m and m' in M, ${}^{\mu(m)}m' = mm'm^{-1}$.

Examples. Every normal monomorphism μ is a crossed module for the conjugation of N on M. Let M be a group and take $N = \mathrm{Aut}(M)$. Then μ sends m to the inner automorphism $m(-)m^{-1}$. This obviously is a crossed module with respect to the action of $\mathrm{Aut}(M)$ on M.

Part of the following result has already been noted by several authors (see for instance [1]).

2.2. Lemma. *The following data are equivalent:*

(1) *a crossed module $\mu: M \to N$,*

(2) *a 1-cat-group $\mathfrak{G} = (G; N)$,*

(3) *a group object in the category of categories,*

(4) *a simplicial group $(\mathfrak{G})_*$ whose Moore complex is of length one.*

Proof. (1) $\Leftrightarrow$ (2). Starting with the crossed module $\mu: M \to N$ the group G is defined as the semi-direct product $G = M \rtimes N$. The structural morphisms are $s(m, n) = n$ and $b(m, n) = \mu(m)n$, which obviously satisfy axiom (i) of 1.1. On the other hand, starting with a 1-cat-group $\mathfrak{G}$ we define $M = \mathrm{Ker}\, s$ and $\mu = b|_{\mathrm{Ker}\, s}$. The action of N on M is the conjugation in G.

It remains to prove that axiom (b) for crossed modules is equivalent to axiom (ii) for 1-cat-groups. If $x \in \mathrm{Ker}\, s$ and $y \in \mathrm{Ker}\, b$, then $x = (m, 1)$ and $y = (m'^{-1}, \mu(m'))$ with m and $m' \in M$. We have $xy = (mm'^{-1}, \mu(m'))$ and $yx = (m'^{-1}({}^{\mu(m')}m), \mu(m'))$. Therefore the equality $xy = yx$ is equivalent to $m'mm'^{-1} = {}^{\mu(m')}m$.

(2) $\Leftrightarrow$ (3). Starting with a 1-cat-group $\mathfrak{G} = (G; N)$ we construct a small category

with objects the elements of N and morphisms the elements of G. The source (resp. target) of the morphism $g\in G$ is $s(g)$ (resp. $b(g)$). The morphisms g and h are composable iff $b(g)=s(h)$ and their composite is $h\circ g=hs(h)^{-1}g$. The axioms of a category are clearly satisfied.

It remains to prove that the composition is a group homomorphism. If g' and h' are two other composable morphisms, this property reads

$$hs(h)^{-1}gh's(h')^{-1}g'=hh's(hh')^{-1}gg'.$$

After simplification use of the equality $s(h)=b(g)$ proves that it is equivalent to $b(g)^{-1}gh's(h')^{-1}=h's(h')^{-1}b(g)^{-1}g$. As any element of $\operatorname{Ker} s$ (resp. $\operatorname{Ker} b$) is of the form $h's(h')^{-1}$ (resp. $b(g)^{-1}g$) this equality is equivalent to $[\operatorname{Ker} s, \operatorname{Ker} b]=1$. In conclusion, composition in this category is a group homomorphism iff axiom (ii) for 1-cat-groups is valid.

It is obvious how to obtain the 1-cat-group from the category in view of the preceding discussion.

(3)$\Leftrightarrow$(4). Recall that if K_* is a simplicial group, the Moore complex of K_* is obtained by taking for each n the subgroup $\bigcap_{i=1}^{n}\operatorname{Ker} d_i$ of K_n; the restriction of d_0 to this subgroup is the differential of the complex. The homology groups of the Moore complex are the homotopy groups of the geometric realization $|K_*|$.

Starting from the category we obtain a simplicial set by taking the nerve. In fact this simplicial set is a simplicial group $(\mathfrak{G}_*)$ because the category is a group object in the category of categories. Its Moore complex is $\cdots 1\to 1\to M\to N$, which is of length 1.

Suppose that the Moore complex of K_* is of length one, that is

$$\cdots 1\to 1\to \operatorname{Ker} d_1\to K_0.$$

There is a 1-cat-group associated to this situation. Put $G=K_1$ and $N=$ image of K_0 in K_1 by the degeneracy map. The structural morphisms s and b are given by $s=d_1$, $b=d_0$. Axiom (i) of 1-cat-groups follows from the relations between face and degeneracy maps. To prove axiom (ii) it is sufficient to see that for $x\in\operatorname{Ker} d_1$ and $y\in\operatorname{Ker} d_0$ the element $[s_0(x), s_0(y)s_1(y)^{-1}]$ of K_2 (where s_0 and s_1 are the degeneracy maps) is in fact in $\operatorname{Ker} d_1\cap\operatorname{Ker} d_2$ and its image by d_0 is $[x,y]$. As $\operatorname{Ker} d_1\cap\operatorname{Ker} d_2=1$, it follows that $[\operatorname{Ker} d_0, \operatorname{Ker} d_1]=1$.

So $(G;N)$ is a 1-cat-group and use of the previous equivalence gives the desired category with $\mathrm{Obj}=K_0$ and $\mathrm{Mor}=K_1$. □

Proof of Theorem 1.4 for $n=1$. *The functor $\mathscr{B}$ for $n=1$.* We first construct the space $B\mathfrak{G}$ where $\mathfrak{G}=(G;N)$ is a 1-cat-group. Let $(\mathfrak{G})_*$ be the simplicial group associated to $\mathfrak{G}$ (see 2.2). If we replace each group $(\mathfrak{G})_n$ by its nerve we obtain a bisimplicial set denoted $\beta_*(\mathfrak{G})_*$, explicitly $\beta_m(\mathfrak{G})_n=(\mathfrak{G})_n\times\cdots\times(\mathfrak{G})_n$ (m times).

2.3. Definition. The *classifying space* $B\mathfrak{G}$ of the *1-cat-group* $\mathfrak{G}$ is the geometric realization of the bisimplicial set $\beta_*(\mathfrak{G})_*$, that is $B\mathfrak{G}=|\beta_*(\mathfrak{G})_*|$.

Remark. It is immediate that, if $G=N$ and $s=\mathrm{id}_N=b$, then $B\mathfrak{G}=BN$. If $G=N\rtimes N$ (semi-direct product with conjugation) and $s(n,n')=n'$, $b(n,n')=nn'$, then $B\mathfrak{G}$ is contractible.

The following lemma will be useful in the sequel.

2.4. Lemma. *Let* $1\to\mathfrak{G}'\to\mathfrak{G}\to\mathfrak{G}''\to 1$ *be an exact sequence of* 1*-cat-groups. Then* $B\mathfrak{G}'\to B\mathfrak{G}\to B\mathfrak{G}''$ *is a fibration.*

Proof. By exact sequence we mean that the maps are morphisms of 1-cat-groups and that $1\to G'\to G\to G''\to 1$ is a short exact sequence of groups.

The simplicial map $\Delta\beta_*(\mathfrak{G})_*\to\Delta\beta_*(\mathfrak{G}'')_*$ where Δ is the diagonal is a Kan fibration (see [12] for a proof when $\Delta\beta_*$ is replaced by the functor $\bar{W}$) and the exactness ensures that the fiber is $\Delta\beta_*(\mathfrak{G}')_*$. The lemma follows from the fact that the geometric realization of a bisimplicial set is homeomorphic to the geometric realization of its diagonal. □

The functors Γ^α: (1-cat-groups) $\to$ (1-cat-groups), $\alpha=-1,0,1$, are defined by

$$\Gamma^{-1}\mathfrak{G}=(M;M)\quad\text{with } s=b=\mathrm{id}_M \text{ (recall } M=\operatorname{Ker} s),$$
$$\Gamma^0\mathfrak{G}=(M\rtimes G;G)\quad\text{with } s(m,g)=g \text{ and } b(m,g)=mg,$$
$$\Gamma^1\mathfrak{G}=\mathfrak{G}.$$

There are natural transformations of functors.

$$\varepsilon:\Gamma^{-1}\mathfrak{G}\to\Gamma^0\mathfrak{G},\quad m\mapsto(1,m^{-1}b(m))$$

and

$$\iota:\Gamma^0\mathfrak{G}\to\Gamma^1\mathfrak{G},\quad (m,g)\mapsto mb(g).$$

2.5. Lemma. *Let* $\mathfrak{G}$ *be a* 1*-cat-group. Then* $B\Gamma^{-1}\mathfrak{G}\to B\Gamma^0\mathfrak{G}\to B\Gamma^1\mathfrak{G}$ *is a fibration.*

Proof. The sequence of 1-cat-groups $1\to\Gamma^{-1}\mathfrak{G}\to\Gamma^0\mathfrak{G}\to\Gamma^1\mathfrak{G}\to 1$ is exact, so it suffices to apply Lemma 2.4. □

Finally the functor $\mathscr{B}$: (1-cat-groups) $\to$ (1-cube of fibrations) is defined by

$$\mathscr{B}\mathfrak{G}=(B\Gamma^{-1}\mathfrak{G}\to B\Gamma^0\mathfrak{G}\to B\Gamma^1\mathfrak{G}).$$

2.6. *The functor* $\mathscr{G}$ *for* $n=1$. Let $\mathfrak{X}=(F\to Y\to X)$ be a fibration of connected spaces. Let

$$Z_*=(\cdots\; Y\times_X Y\times_X Y \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; Y\times_X Y \;\substack{\longrightarrow\\\longrightarrow}\; Y)$$

be the simplicial space obtained from f by taking iterated fiber products. Put $G=\pi_1(Y\times_X Y)$, $N=\pi_1 Y$, s (resp. b) being induced by the first (resp. second) projection.

2.7. Definition and Lemma. *Let $\mathfrak{X}$ be a fibration then $\mathscr{G}(\mathfrak{X})=(\pi_1(Y\times_X Y);\pi_1 Y)$ is a 1-cat-group.*

Proof. Taking π_1 dimensionwise in Z_* we get a simplicial group beginning with

$$G \overset{s,b}{\rightrightarrows} N.$$

The Moore complex of this simplicial group is $\cdots 1\to 1\to \pi_1 F\to \pi_1 Y$. Then, by Lemma 2.2, $\mathscr{G}(\mathfrak{X})=(G;N)$ is a 1-cat-group. $\square$

Remark. The fact that a fibration gives rise to a 1-cat-group, that is a crossed module, was first discovered by J.H.C. Whitehead [13].

2.8. Proof of property (a) of Theorem 1.4 for $n=1$. We must prove that $B\Gamma^{-1}\mathfrak{G}$ and $B\Gamma^0\mathfrak{G}$ are $K(\pi,1)$-spaces. We have $B\Gamma^{-1}\mathfrak{G}=B(M;M)=BM$ which is a $K(M,1)$. For $B\Gamma^0\mathfrak{G}$ we consider the 1-cat-group $\bar{\Gamma}^0\mathfrak{G}=(N;N)$. There is an exact sequence of 1-cat-groups

$$(1;1)\longrightarrow (M\rtimes M;M)\longrightarrow \Gamma^0\mathfrak{G}\xrightarrow{\theta}\bar{\Gamma}^0\mathfrak{G}\longrightarrow(1;1)$$

where $\theta: M\rtimes G\to N$ is given by $\theta(m,g)=s(g)$. By Lemma 2.4 this yields a fibration

$$B(M\rtimes M;M)\longrightarrow B\Gamma^0\mathfrak{G}\longrightarrow B\bar{\Gamma}^0\mathfrak{G}.$$

As the fiber is contractible, the last map is a homotopy equivalence and $B\Gamma^0\mathfrak{G}$ is homotopy equivalent to $B(N;N)=BN$. So the fibration $\mathscr{B}\mathfrak{G}$ is homotopy equivalent to $BM\to BN\to B\mathfrak{G}$. $\square$

Remark. There is a morphism $\zeta:\bar{\Gamma}^0\mathfrak{G}\to\Gamma^0\mathfrak{G}$ given by $n\mapsto(1,n)$ and we have $\theta\circ\zeta=\mathrm{id}$.

2.9. Proof of property (b) of Theorem 1.4 for $n=1$. We have to show that $\mathscr{G}(\mathscr{B}\mathfrak{G})=\mathfrak{G}$. To compute $\pi_1(Y\times_X Y)$ where $Y=B\Gamma^0\mathfrak{G}$ and $X=B\mathfrak{G}$ we consider the following commutative square:

$$\begin{tikzcd}
B(G;G) \arrow[r, "Bu"] \arrow[d, "Bv"] & B\Gamma^0\mathfrak{G}=Y \arrow[d, "B\iota"] \\
Y=B\Gamma^0\mathfrak{G} \arrow[r, "B\iota"] & B\mathfrak{G}=X
\end{tikzcd}$$

where $u(g)=(1,s(g)g^{-1}b(g))\in M\rtimes G$ and $v(g)=(1,s(g))\in M\rtimes G$ (u is a group homomorphism because of axiom (ii)). By Lemma 2.4 the fibers of the vertical maps are both equal to $B(M;M)$ and Bu induces the identity on them. Therefore this square is cartesian and $BG=B(G;G)=Y\times_X Y$. We have thus proved $\pi_1(Y\times_X Y)=G$. Moreover $\pi_1 Y=N$ and v (resp. u) induces s (resp. b), hence we have proved that $\mathscr{G}(\mathscr{B}\mathfrak{G})=\mathfrak{G}$. $\square$

2.10. Proof of property (c) of Theorem 1.4 for $n=1$. Let $\mathfrak{X}=(F\to Y\to X)$ be a fibration of connected spaces. First we construct a map $X\to B\mathscr{G}(\mathfrak{X})$ well-defined up to homotopy.

In 2.6 we have constructed a simplicial space Z_* associated to $Y\to X$. Let Z'_* (resp. Z''_*) be the simplicial set associated to $\mathrm{id}: X\to X$ (resp. $F\to(\mathrm{pt})$). For every n the sequence $Z''_n\to Z_n\to Z'_n$ is a fibration, therefore by the realization lemma $|Z''_*|\to|Z_*|\to|Z'_*|$ is a quasi-fibration. It is immediate that $|Z''_*|$ is contractible and that $|Z'_*|=X$, hence we get a natural homotopy equivalence $|Z_*|\xrightarrow{\sim}X$.

By definition of the functor $\mathscr{G}$ the simplicial group $\mathscr{G}(\mathfrak{X})$ is $([n]\mapsto\pi_1 Z_n)$. Up to homeomorphism the space $B\mathscr{G}(\mathfrak{X})$ can be obtained from the bisimplicial set $\beta_*([n])\mapsto\pi_1 Z_n)$ by taking the geometric realization in one direction and then in the other direction, that is $B\mathscr{G}(\mathfrak{X})=|[n]\mapsto B\pi_1 Z_n|$.

Now we replace Z_n by the homotopy equivalent space $|\mathrm{Sin}\, Z_n|$ where $\mathrm{Sin}\, Z_n$ is the reduced simplicial complex of Z_n. There is a canonical map $|\mathrm{Sin}\, Z_n|\to B\pi_1 Z_n$ which induces an isomorphism on π_1; therefore there are canonical maps

$$X\xleftarrow{\sim}\big|[n]\mapsto|\mathrm{Sin}\, Z_n|\big|\longrightarrow\big|[n]\mapsto B\pi_1 Z_n\big|\xrightarrow{\sim}B\mathscr{G}(\mathfrak{X})$$

which induce isomorphisms on π_1.

To finish the proof we put $\gamma^{-1}\mathfrak{X}=(*\to F\to F)$, $\gamma^0\mathfrak{X}=(F\to Y\times_X Y\to Y)$ and $\gamma^1\mathfrak{X}=\mathfrak{X}$. For $\alpha=-1$, 0 or 1 there is an equality $(\gamma^\alpha\mathfrak{X})^1=\mathfrak{X}^\alpha$. Moreover there are natural transformations $\gamma^{-1}\mathfrak{X}\to\gamma^0\mathfrak{X}\to\gamma^1\mathfrak{X}$ which induce $\mathfrak{X}^{-1}\to\mathfrak{X}^0\to\mathfrak{X}^1$. Applying the previous construction to the $\gamma^\alpha\mathfrak{X}$'s gives the commutative diagram:

$$\begin{tikzcd}
(\gamma^{-1}\mathfrak{X})^1 \arrow[r] \arrow[d] & (\gamma^0\mathfrak{X})^1 \arrow[r] \arrow[d] & (\gamma^1\mathfrak{X})^1 \arrow[d] \\
B\mathscr{G}\gamma^{-1}\mathfrak{X} \arrow[r] & B\mathscr{G}\gamma^0\mathfrak{X} \arrow[r] & B\mathscr{G}\gamma^1\mathfrak{X}.
\end{tikzcd}$$

Use of the identities $\mathscr{G}\gamma^\alpha\mathfrak{X}=\Gamma^\alpha\mathscr{G}\mathfrak{X}$ gives the desired map $\mathfrak{X}\to\mathscr{B}\mathscr{G}(\mathfrak{X})$.

We now proceed with the proof of the general case.

2.11. *The functor* $\mathscr{B}$: (n-cat-groups) $\to$ (n-cubes of fibrations). Let $\mathfrak{G}=(G;N_1,\dots,N_n)$ be an n-cat-group, we first construct its classifying space $B\mathfrak{G}$. Use of the first categorical structure (index 1) yields a simplicial group

$$(\cdots G\times_{N_1}G \rightrightarrows G \rightrightarrows N_1)$$

as in Lemma 2.2. The remaining $(n-1)$-categorical structures induce on each group $N_1, G, G\times_{N_1}G,\dots$ a structure of $(n-1)$-cat-group. Because of axiom (iii) the face and degeneracy operators are morphisms of $(n-1)$-cat-groups. Iterating this procedure gives an n-simplicial group $(\mathfrak{G})_\#$ such that $(\mathfrak{G})_{11\dots1}=G$, $(\mathfrak{G})_{1\dots0\dots1}=N_i$ (0 in position i and 1 otherwise). Replacing in $(\mathfrak{G})_\#$ each group by its nerve yields an $(n+1)$-simplicial set $\beta_*(\mathfrak{G})_\#$.

2.12. Definition. The *classifying space of the n-categorical group* $\mathfrak{G}$ is the geometric realization of the $(n+1)$-simplicial set $\beta_*(\mathfrak{G})_\#$, that is $B\mathfrak{G} = |\beta_*(\mathfrak{G})_\#|$.

For any $i = 1, \dots, n$ and $\alpha = -1, 0, 1$ the functor $\Gamma_i^\alpha : (n\text{-cat-groups}) \to (n\text{-cat-groups})$ is the functor Γ^α applied with respect to the ith categorical structure:

$$\Gamma_i^{-1}\mathfrak{G} = (M_i; N_1 \cap M_i, \dots, M_i, \dots, N_n \cap M_i), \quad \text{where } M_i = \operatorname{Ker} s_i,$$
$$\Gamma_i^0\mathfrak{G} = (M_i \rtimes G; (N_1 \cap M_i) \rtimes N_1, \dots, G, \dots, (N_n \cap M_i) \rtimes N_n),$$
$$\Gamma_i^1\mathfrak{G} = \mathfrak{G}.$$

As in the case of 1-cat-groups there are transformations of functors

$$\Gamma_i^{-1}\mathfrak{G} \xrightarrow{\varepsilon_i} \Gamma_i^0\mathfrak{G} \xrightarrow{\iota_i} \Gamma_i^1\mathfrak{G}$$

which give short exact sequences of n-cat-groups. For $\alpha = (\alpha_1, \dots, \alpha_n)$ we put $\Gamma^\alpha = \Gamma_1^{\alpha_1} \circ \dots \circ \Gamma_n^{\alpha_n}$. This is a functor from the category of n-cat-groups to itself.

2.13. Lemma. *Let α', α and α'' be such that $\alpha_i' = -1$, $\alpha_i = 0$, $\alpha_i'' = 1$ (i fixed) and $\alpha_j' = \alpha_j = \alpha_j''$ for $j \neq i$. Then for any n-cat-group $\mathfrak{G}$ the sequence*

$$B\Gamma^{\alpha'}\mathfrak{G} \longrightarrow B\Gamma^{\alpha}\mathfrak{G} \longrightarrow B\Gamma^{\alpha''}\mathfrak{G}$$

is a fibration.

Proof. This follows from Lemma 2.4. □

As a consequence we can give the following

Definition. The functor $\mathscr{B} : (n\text{-cat-groups}) \to (n\text{-cubes of fibrations})$ is given by $(\mathscr{B}\mathfrak{G})^\alpha = \mathscr{B}\Gamma^\alpha\mathfrak{G}$, the maps being induced by the ε_i and ι_i's. Note that $(\mathscr{B}\mathfrak{G})^{11\cdots 1} = B\Gamma^{11\cdots 1}\mathfrak{G} = B\mathfrak{G}$.

2.14. *The functor* $\mathscr{G} : (n\text{-cubes of fibrations}) \to (n\text{-cat-groups})$. Let $\mathfrak{X} : \langle 0, 1\rangle^n \to$ (connected spaces) be an n-cube of fibrations. Use of the construction of the simplicial space Z_* associated to a fibration (cf. 2.6) permits us to replace each fibration in the direction n by a simplicial space and gives a simplicial $(n-1)$-cube of fibrations. Iterating this construction we get an n-simplicial space $Z_\#$. By definition $G = \pi_1(Z_{11\dots 1})$, $N_i = \pi_1(Z_{1\dots 0 \dots 1})$ (0 in the ith place, 1 everywhere else), s_i and b_i are given by the face maps in direction i.

2.15. Lemma. *Let $\mathfrak{X}$ be an n-cube of fibrations, then $\mathscr{G}(\mathfrak{X}) = (G; N_1, \dots, N_n)$ as defined above is an n-cat-group.*

Proof. The group N_i is identified with a subgroup of G via the degeneracy map in the direction i. The verification of axioms (i) and (ii) goes back to the case $n = 1$ (Lemma 2.7).

In a multisimplicial set the faces in two different directions commute. As s_i and b_i are induced by faces in direction i they commute with s_j and b_j provided $i \neq j$. This is axiom (iii). □

2.16. Proof of property (a). Mimicking the construction of $\bar{\Gamma}^0$ introduced in 2.8 we define

$$\bar{\Gamma}_i^0 \mathfrak{G} = (N_i; N_1 \cap N_i, \ldots, N_i, \ldots, N_n \cap N_i),$$
$$\bar{\Gamma}_i^{-1} = \Gamma_i^{-1} \quad \text{and} \quad \bar{\Gamma}^\alpha = \bar{\Gamma}_1^{\alpha_1} \circ \cdots \circ \bar{\Gamma}_n^{\alpha_n}.$$

For $\alpha_i = -1$ or 0 the functor $\bar{\Gamma}_i^{\alpha_i}$ transforms any n-cat-group into an n-cat-group $(G'; N_1', \ldots, N_n')$ such that $G' = N_i'$. Therefore, if $\alpha \leq 0$, $\bar{\Gamma}^\alpha$ transforms $\mathfrak{G}$ into an n-cat-group of the form $(\pi; \pi, \ldots, \pi)$ whose classifying space is of type $K(\pi, 1)$.

As $B\bar{\Gamma}^\alpha\mathfrak{G}$ is homotopy equivalent to $B\Gamma^\alpha\mathfrak{G}$ (see 2.8) property (a) is proved. □

2.17. Corollary. *The classifying space $B\mathfrak{G}$ of the n-cat-group $\mathfrak{G}$ is $(n+1)$-coconnected.*

Proof. By induction on n, use of the fibrations

$$B\Gamma_i^{-1}\mathfrak{G} \longrightarrow B\Gamma_i^0\mathfrak{G} \longrightarrow B\Gamma_i^1\mathfrak{G}$$

proves that if 1 occurs k times in $\alpha = (\alpha_1, \ldots, \alpha_n)$ then $\pi_i B\Gamma^\alpha\mathfrak{G} = 0$ for $i > k+1$. □

2.18. Proof of property (b). To prove that $\mathscr{G}(\mathscr{B}\mathfrak{G}) = \mathfrak{G}$ we first compute $\pi_1 Z_{11\ldots1}$ where $Z_\#$ is the n-simplicial space associated to $\mathscr{B}\mathfrak{G}$. When replacing each fibration in the nth direction by a simplicial space we obtain a simplicial $(n-1)$-cube of fibrations $\mathscr{J}$ such that $(\mathscr{J}^{00\cdots0})_1 = B(G; G, \ldots, G) = BG$ (see 2.11). Finally we find the n-simplicial space $Z_\#$ with $Z_{11\ldots1} = BG$. Therefore $\pi_1(Z_{11\ldots1}) = G$. Similary we have $\pi_1(Z_{1\ldots0\ldots1}) = N_i$ and then $\mathscr{G}(\mathscr{B}\mathfrak{G}) = \mathfrak{G}$. □

2.19. Proof of property (c). Let $\mathfrak{X}$ be an n-cube of fibrations. We first construct a map $\mathfrak{X}^{11\cdots1} \to B\mathscr{G}(\mathfrak{X})$, well-defined up to homotopy, which induces an isomorphism on π_1. The n-simplicial space $Z_\#$ associated to $\mathfrak{X}$ has the property that $|Z_\#|$ is homotopy equivalent to $\mathfrak{X}^{11\cdots1}$. Then, if we replace each space in $Z_\#$ by its fundamental group, we obtain an n-simplicial group. This n-simplicial group is the same as the n-simplicial group $(\mathscr{G}(\mathfrak{X}))_\#$ obtained from $\mathscr{G}(\mathfrak{X})$ (see 2.11). Therefore there is an n-simplicial map $Z_\# \to B(\mathscr{G}(\mathfrak{G}))_\#$ which induces an isomorphism on π_1 at each level. Taking the geometric realization gives the desired map

$$\mathfrak{X}^{11\cdots1} \simeq |Z_\#| \longrightarrow |B(\mathscr{G}(\mathfrak{X}))_\#| = B\mathscr{G}(\mathfrak{X}).$$

To construct the morphism $\mathfrak{X} \to \mathscr{B}\mathscr{G}(\mathfrak{X})$ we define γ^α: (n-cubes of fibrations) $\to$ (n-cubes of fibrations) by $\gamma^\alpha = \gamma_1^{\alpha_1} \circ \cdots \circ \gamma_n^{\alpha_n}$.

Note that $\mathscr{G}\gamma^\alpha = \Gamma^\alpha\mathscr{G}$. The definition of γ^α implies $\mathfrak{X}^\alpha = (\gamma^\alpha\mathfrak{X})^{11\cdots1}$, therefore the maps $\mathfrak{X}^\alpha = (\gamma^\alpha \mathfrak{X})^{11\cdots 1} \to B\mathcal{G}(\gamma^\alpha\mathfrak{X})^{11\cdots 1} = B\Gamma^\alpha\mathcal{G}(\mathfrak{X})$ give the desired morphism $\mathfrak{X} \to \mathcal{B}\mathcal{G}(\mathfrak{X})$. $\square$

3. Complexes of non-abelian groups and $\pi_i(B\mathfrak{G})$

The object of this section is to construct a complex of groups $C_*(\mathfrak{G})$ whose homology groups are the homotopy groups of $B\mathfrak{G}$. This is analogous to the construction of the Moore complex of a simplicial group whose homology is the homotopy of the geometric realization of the simplicial group.

A complex of (non-abelian) groups (C_*, d_*) of length n is a sequence of group homomorphisms

$$C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0$$

such that $\operatorname{Im} d_{i+1}$ is normal in $\operatorname{Ker} d_i$. Therefore the homology groups $H_i(C_*) = \operatorname{Ker} d_i / \operatorname{Im} d_{i+1}$ are well defined (we assume $C_i = 1$ for $i<0$ or $i>n$). It is in general not possible to define a mapping cone of a morphism $f:(A_*, d_*) \to (B_*, d'_*)$ of two complexes (it is possible, of course, if all the groups are abelian). However we will show that it is possible if there is some extra structure.

3.1. Definition and Lemma. *Let $f:(A_*, d_*) \to (B_*, d'_*)$ be a morphism of complexes such that $f_i: A_i \to B_i$ is a crossed module (i.e. there is given an action of B_i on A_i satisfying the two conditions of 2.1) and such that the maps (d_i, d'_i) form a morphism of crossed modules. Let C_i be the semi-direct product $A_{i-1} \rtimes B_i$ where the action of B_i on A_{i-1} is obtained through d'_i and the action of B_{i-1} on A_{i-1} (crossed module structure). If we define $\delta_{i+1}: C_{i+1} \to C_i$ by*

$$\delta_{i+1}(x, y) = (d_i(x)^{-1}, f_i(x) d'_{i+1}(y)), \quad x \in A_i, \quad y \in B_{i+1},$$

then (C_*, d_*) is a complex of (non-abelian) groups which is called the mapping cone of f.

Note that if (A_*, d_*) and (B_*, d'_*) are of length n, then

$$(C_*, d_*): \quad A_n \to A_{n-1} \rtimes B_n \to \cdots \to A_{i-1} \rtimes B_i \to \cdots \to A_0 \rtimes B_1 \to B_0$$

is of length $n+1$.

Proof. To show that δ_{i+1} is a group homomorphism it is sufficient to show that

$$\delta_{i+1}(x, 1)\delta_{i+1}(x', 1) = \delta_{i+1}(xx', 1)$$

and that

$$\delta_{i+1}(1, y)\delta_{i+1}(x, 1) = \delta_{i+1}({}^{d'_{i+1}(y)}x, y).$$

We omit the indices for the computation:

$$\begin{aligned}\delta(x,1)\delta(x',1) &= (d(x)^{-1}, f(x))(d(x')^{-1}, f(x')) \\ &= (d(x)^{-1}\,{}^{d'f(x)}d(x')^{-1}, f(x)f(x')).\end{aligned}$$

As $d'f(x) = fd(x)$ and as the action of $fd(x)$ is given by conjugation of $d(x)$ (axiom (b) of crossed modules), we have

$$\delta(x,1)\delta(x',1) = (d(x')^{-1}d(x)^{-1}, f(x)f(x')) = \delta(xx',1).$$

To prove the second formula we write

$$\delta(1,y)\delta(x,1) = (1, d'(y))(d(x)^{-1}, f(x)) = (d(x)^{-1}, d'(y)f(x)),$$

(because d'^2 is trivial). On the other hand, use of the fact that (d, d') is a morphism of crossed modules gives

$$\delta({}^{d'(y)}x, y) = (d({}^{d'(y)}x)^{-1}, f({}^{d'(y)}x)d'(y)) = (d(x)^{-1}, d'(y)f(x)).$$

For the last equality we used axioms (a) and (b) for crossed modules. So we have proved that δ is a group homomorphism.

It is easily checked that $\operatorname{Im}\delta_{i+1}$ is normal in $\operatorname{Ker}\delta_i$ and therefore the homology groups of the mapping cone are well defined. □

3.2. Proposition. *Let $f:(A_*, d_*) \to (B_*, d'_*)$ be a morphism of complexes satisfying the conditions of Lemma* 3.1. *Then there is a long exact sequence*

$$\cdots \to H_i(A_*) \to H_i(B_*) \to H_i(C_*) \to H_{i-1}(A_*) \to \cdots$$

where C_* is the mapping-cone complex of f.

Proof. Two out of three of the maps are induced by morphisms of complexes. As for the third (the boundary map) we use the projection $A_{i-1} \rtimes B_i \to A_{i-1}$, $(x, y) \mapsto x^{-1}$. This is not a group homomorphism, however its restriction to the subgroup of cycles is a group homomorphism. The rest of the proof is by standard diagram chasing. □

3.3. *The (non-abelian) group complex of an n-cat-group.* The complex $C_*(\mathfrak{G})$:

$$C_n(\mathfrak{G}) \xrightarrow{\delta_n} C_{n-1}(\mathfrak{G}) \longrightarrow \cdots \xrightarrow{\delta_1} C_0(\mathfrak{G})$$

associated to an n-cat-group $\mathfrak{G}$ is constructed by induction as follows. Let (D_*, ∂_*) be a complex of groups and suppose that each group D_i has a categorical structure, that is $\mathfrak{D}_i = (D_i; B_i)$ is a categorical group and the homomorphisms ∂_i are morphisms of categorical groups. Then $(\mathfrak{D}_*, \partial_*)$ is called a complex of 1-cat-groups. As a 1-cat-group is equivalent to a crossed module (Lemma 2.2) $(\mathfrak{D}_*, \partial_*)$ gives rise to a morphism of complexes which satisfies the condition of Lemma 3.1. Hence from any complex of 1-cat-groups of length n the construction above gives a complex of groups of length $n+1$. It is immediate to remark that if $(\mathfrak{D}_*, \partial_*)$ is a complex of n-cat-groups then the new complex is a complex of $(n-1)$-cat-groups (because of axiom (iii) for n-cat-groups).

Let $\mathfrak{G}$ be an n-cat-group. Looking at it as a 0-complex of n-cat-groups and applying the preceding construction inductively (n times) we obtain an n-complex of 0-cat-groups, that is a complex of groups of length n denoted $(C_*(\mathfrak{G}), \delta_*)$.

3.4. Proposition. *For any n-cat-group $\mathfrak{G}$ the homotopy groups of the classifying space $B\mathfrak{G}$ are the homology groups of the complex $C_*(\mathfrak{G})$, i.e. $\pi_i(B\mathfrak{G}) = H_{i-1}(C_*(\mathfrak{G}))$.*

Proof. This is obvious for $n=0$. For $n=1$ it follows from the fact that the homotopy exact sequence of the homotopy fibration $BM \to BN \to B\mathfrak{G}$ (see 2.5 and 2.8) is

$$1 \longrightarrow \pi_2 B\mathfrak{G} \longrightarrow M \xrightarrow{\mu} N \longrightarrow \pi_1 B\mathfrak{G} \longrightarrow 1$$

and that the complex $C_*(\mathfrak{G})$ is $M \xrightarrow{\mu} N$.

Proceeding by induction, we suppose that the proposition is true for $n-1$ and we will prove it for n.

From any n-cat-group $\mathfrak{G}$ we constructed in 2.11 an n-simplicial group $(\mathfrak{G})_\#$. Let $\operatorname{diag}(\mathfrak{G})_\#$ be the diagonal simplicial group, then the geometric realization of $\operatorname{diag}(\mathfrak{G})_\#$ is homotopy equivalent to $\Omega B\mathfrak{G}$. Hence the homology groups of the Moore complex $C_*^M(\mathfrak{G})$ of $\operatorname{diag}(\mathfrak{G})_\#$ (see [9]) are the homotopy groups of $B\mathfrak{G}$ (up to a shift of index). We shall shortly prove that there is a natural morphism of complexes $\varepsilon(\mathfrak{G}) : C_*^M(\mathfrak{G}) \to C_*(\mathfrak{G})$. We first prove that it necessarily induces an isomorphism in homology, i.e. is a quasi-isomorphism.

First step. If $1 \to \mathfrak{G}' \to \mathfrak{G} \to \mathfrak{G}'' \to 1$ is a short exact sequence of n-cat-groups and if two of the morphisms $\varepsilon(\mathfrak{G}')$, $\varepsilon(\mathfrak{G})$, $\varepsilon(\mathfrak{G}'')$ are quasi-isomorphisms then so is the third. This is a consequence of the five lemma.

Second step. If $\mathfrak{G}$ is of the form $(M \rtimes M; N_1, \ldots, N_{n-1}, M)$ then $\varepsilon(\mathfrak{G})$ is a quasi-isomorphism. This is because, in this case, $B\mathfrak{G}$ is contractible and $C_*(\mathfrak{G})$ is acyclic.

Third step. If $\mathfrak{G}$ is of the form $(G; N_1, \ldots, N_{n-1}, G)$ then $\varepsilon(\mathfrak{G})$ is a quasi-isomorphism. This is because $C_*(\mathfrak{G}) = C_*((G; N_1, \ldots, N_{n-1}))$ and $B\mathfrak{G} = B(G; N_1, \ldots, N_{n-1})$ and the induction hypothesis.

Fourth step. From the short exact sequence of n-cat-groups

$$1 \longrightarrow (M_n \rtimes M_n; -, -, \ldots, M_n) \longrightarrow \Gamma_n^0 \mathfrak{G} \longrightarrow \Gamma_n^0 \mathfrak{G} \longrightarrow 1$$

and the steps 1, 2 and 3 it follows that $\varepsilon(\Gamma_n^0 \mathfrak{G})$ is a quasi-isomorphism.

Last step. From the short exact sequence $1 \to \Gamma_n^{-1}\mathfrak{G} \to \Gamma_n^0 \mathfrak{G} \to \mathfrak{G} \to 1$, step 1, step 3 (for $\Gamma_n^{-1}\mathfrak{G}$) and step 4 (for $\Gamma_n^0\mathfrak{G}$) it follows that $\varepsilon(\mathfrak{G})$ is a quasi-isomorphism.

It remains to show the existence of $\varepsilon(\mathfrak{G}) : C_*^M(\mathfrak{G}) \to C_*(\mathfrak{G})$. Let $U_* = (V_* \rtimes W_*, W_*)$ be a simplicial 1-cat-group. There are two kinds of complexes which can be constructed. Taking the Moore complex gives rise to a complex of 1-cat-groups (or of crossed modules) and then applying the mapping cone construction gives rise to a complex of groups

$$\cdots \longrightarrow V_2' \rtimes W_3' \longrightarrow V_1' \rtimes W_2' \longrightarrow V_0 \rtimes W_1' \longrightarrow W_0 \qquad (*)$$

where V_n' is the subgroup $\bigcap_{i=1}^n \operatorname{Ker} d_i^V$ of V_n and similary for W_n' (Here d_i^V is the ith face map $V_n \to V_{n-1}$). On the other hand, converting the categorical structure into a simplicial structure (Lemma 2.2) transforms U_* into a bisimplicial group (U_{**}). The simplicial group diag U_{**} is

$$\left(\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; (V_2 \times V_2) \rtimes W_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; V_1 \rtimes W_1 \;\substack{\longrightarrow\\\longrightarrow}\; W_0\right).$$

Its Moore complex is the second complex we are looking for. There is a morphism of complexes from this Moore complex to the mapping cone complex given by

$$(v_1, v_2, \ldots, v_n; w) \mapsto (d_n^V v_n, w)$$

in degree n.

If we start with a simplicial n-cat-group, then this construction gives a morphism of complexes of $(n-1)$-cat-groups. To get the morphism $\varepsilon(\mathfrak{G})$ we start with $\mathfrak{G}$ considered as a (trivial) simplicial n-cat-group and we apply the above construction n times. Finally we get a sequence of n morphisms of complexes. The first complex is the Moore complex of $\operatorname{diag}(\mathfrak{G})_*$, that is $C_*^M(\mathfrak{G})$ and the last one is $C_*(\mathfrak{G})$. The composition of these n morphisms gives the desired $\varepsilon(\mathfrak{G})$. □

To complete the proof of Theorem 1.7 we need the following

3.5. Lemma. *There is a functor, well-defined up to homotopy, from the category of $(n+1)$-coconnected spaces into the category of n-cubes of fibrations $\mathfrak{X}$ such that $\mathfrak{X}^\alpha$ is a $K(\pi, 1)$ then $\alpha \leq 0$ and such that the image $\mathfrak{X}$ of X verifies $\mathfrak{X}^{11\cdots1} = X$.*

Proof. Let F be the free group on the elements of $\pi_1(X)$. Then there is a fibration $BF \to X$ inducing the obvious projection on π_1.

Put $\Theta^{-1}X = \operatorname{fiber}(BF \to X)$, $\Theta^0 X = BF$ and $\Theta^1 X = X$. The spaces $\Theta^{-1}X$ and $\Theta^0 X$ have trivial homotopy groups π_i for $i > n$. Then we can define an n-cube of fibrations $\mathfrak{X}$ by $\mathfrak{X}^\alpha = \Theta^{\alpha_1} \circ \cdots \circ \Theta^{\alpha_n} X$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i = -1, 0$, or 1. As $\Theta^1 = \mathrm{id}$, we have $\mathfrak{X}^{11\cdots1} = \Theta^1 \circ \cdots \circ \Theta^1 X = X$. □

End of the proof of Theorem 1.7. The functor B (cf. 2.11) from the category of n-cat-groups to the homotopy category of $(n+1)$-coconnected spaces factors through the category of fractions (n-cat-groups)(Σ^{-1}) because of Proposition 3.4, Whitehead's theorem and the universal property of a category of fractions (cf. [3]).

Its inverse is given by the composite of the functor described in Lemma 3.5 with the functor $\mathscr{G}$ defined in 2.14. □

4. Group-theoretic interpretation of cohomology groups

A well-known theorem of Eilenberg and MacLane [10, Ch. 4, Theorem 4.1] asserts that the set of extensions of a group Q by a Q-module A is, up to congruence, isomorphic to the second cohomology group $H^2(Q;A)$. The aim of this section is to use the previous results to extend this theorem in three different directions. First we replace 2 by n and get an interpretation of $H^n(Q;A)$. In terms of topological spaces we have $H^n(Q;A)=H^n(K(Q,1);A)$, the next generalisation consists in replacing $K(Q,1)$ by an arbitrary Eilenberg-MacLane space $K(C,k)$ where C is an abelian group. Finally we give a group-theoretic interpretation of some relative and 'hyper-relative' cohomology groups.

4.1. *Interpretations of* $H^n(Q;A)$. Let Q be a group and A a Q-module. For $n\geq 2$ the set $\mathscr{S}^n(Q;A)$ consists of the triples $(\mathfrak{G},\varphi,\Psi)$ where $\mathfrak{G}$ is an $(n-2)$-cat-group

$$\varphi : A \longrightarrow \bigcap_{i=1}^{n-2} \operatorname{Ker} s_i = C_{n-2}(\mathfrak{G}) \quad \text{and} \quad \Psi : C_0(\mathfrak{G}) = \bigcap_{i=1}^{n-2} N_i \longrightarrow Q$$

are group homomorphisms subject to the following conditions
- the sequence

$$1 \longrightarrow A \xrightarrow{\varphi} C_{n-2}(\mathfrak{G}) \xrightarrow{\delta_{n-2}} C_{n-3}(\mathfrak{G}) \longrightarrow \cdots \xrightarrow{\delta_1} C_0(\mathfrak{G}) \xrightarrow{\Psi} Q \longrightarrow 1$$

  is exact,
- for any $x\in C_0(\mathfrak{G})$ and any $a\in A$, $\varphi(\Psi(x)\cdot a)=xax^{-1}$.

The complex $(C_*(\mathfrak{G}),\delta_*)$ is the complex constructed in 3.3.

Remark. By Proposition 3.4, when $n\geq 3$ the first condition implies that $\pi_1 B\mathfrak{G}=Q$, $\pi_{n-1}(B\mathfrak{G})=A$ and $\pi_i(B\mathfrak{G})=0$ for $i\neq 1$, $n-1$. The second condition asserts that the module structures on $\pi_{n-1}(B\mathfrak{G})$ and A agree. Two triples $(\mathfrak{G},\varphi,\Psi)$ and $(\mathfrak{G}',\varphi',\Psi')$ are said to be congruent if there exists a morphism of $(n-2)$-cat-groups $f:\mathfrak{G}\to\mathfrak{G}'$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, equal] & C_{n-2}(\mathfrak{G}) \arrow[r, "\delta_{n-2}"] \arrow[d, "f_*"] & \cdots \arrow[r, "\delta_1"] & C_0(\mathfrak{G}) \arrow[r] \arrow[d, "f_*"] & Q \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r, "\varphi'"] & C_{n-2}(\mathfrak{G}') \arrow[r, "\delta'_{n-2}"] & \cdots \arrow[r, "\delta'_1"] & C_0(\mathfrak{G}') \arrow[r] & Q \arrow[r] & 1
\end{tikzcd}$$

The Yoneda equivalence on $\mathscr{S}^n(Q;A)$ is the equivalence relation generated by the congruence relation.

4.2. Theorem. *There is a one-to-one correspondence between the cohomology group with coefficients* $H^n(Q;A)$ *and the set of equivalence classes* $\mathscr{S}^n(Q;A)/($*Yoneda equivalence*$)$ *for* $n\geq 2$.

Remark. For $n=2$ a triple is just a short exact sequence $1\to A\to G\to Q\to 1$ and this theorem is Eilenberg and MacLane's theorem. For $n=3$ a triple is equivalent to an extension

$$1 \longrightarrow A \longrightarrow M \xrightarrow{\mu} N \longrightarrow Q \longrightarrow 1$$

where μ is a crossed module. Under this form Theorem 4.2 was proved by several people (see [11] for references).

Proof of Theorem 4.2. We suppose $n \geq 3$. Let $(\mathfrak{G}, \varphi, \Psi)$ be an element of $\mathcal{S}^n(Q;A)$. The space $B\mathfrak{G}$ has only two non-trivial homotopy groups, and therefore it has only one non-trivial Postnikov invariant, which lies in $H^n(\pi_1 B\mathfrak{G}; \pi_{n-1} B\mathfrak{G})$. The morphisms φ and Ψ permit to identify this group to $H^n(Q;A)$. The image of the Postnikov invariant defines a map $\mathcal{S}^n(Q;A) \to H^n(Q;A)$. If $(\mathfrak{G}, \varphi, \Psi)$ and $(\mathfrak{G}', \varphi', \Psi')$ are congruent there is a map $B\mathfrak{G} \to B\mathfrak{G}'$ which induces an isomorphism on homotopy groups. Therefore this map is a homotopy equivalence and the two spaces have the same Postnikov invariant. Hence the 'Postnikov invariant' map $\mathcal{S}^n(Q;A)/\sim \to H^n(Q;A)$ is well defined.

Let $\alpha \in H^n(Q;A)$ and let X be a space (well-defined up to homotopy) such that $\pi_i X = 0$ if $i \neq 1$ and $n-1$, $\pi_1 X = Q$, $\pi_{n-1} X = A$ as a Q-module and $\alpha =$ Postnikov invariant of X. By Lemma 3.5 and Theorem 1.4 there exists an $(n-2)$-cat-group $\mathfrak{G}$ such that $B\mathfrak{G}$ is homotopy equivalent to X. Therefore we have an element $(\mathfrak{G}, \varphi, \Psi)$ where φ is the natural inclusion of $\pi_n B\mathfrak{G}$ into $C_{n-2}(\mathfrak{G})$ and Ψ is the projection of $C_0(\mathfrak{G})$ onto $\pi_1 B\mathfrak{G}$. If $\mathfrak{G}'$ is another $(n-2)$-cat-group, then there exists a homotopy equivalence $B\mathfrak{G} \to B\mathfrak{G}'$. It need not come from a morphism $\mathfrak{G} \to \mathfrak{G}'$. However, by fiber product, we can construct an $(n-2)$-cube of fibrations $\mathfrak{Y}$ (with $\mathfrak{Y}^\alpha = K(\pi, 1)$ for $\alpha \leq 0$) and morphisms $\mathcal{B}\mathfrak{G} \leftarrow \mathfrak{Y} \to \mathcal{B}\mathfrak{G}'$ such that $B\mathfrak{G} \leftarrow \mathfrak{Y}^{11\cdots 1} \to B\mathfrak{G}'$ are homotopy equivalences. Therefore there exist morphisms $\mathfrak{G} \leftarrow \mathcal{G}(\mathfrak{Y}) \to \mathfrak{G}'$ which prove that $\mathfrak{G}$ and $\mathfrak{G}'$ are Yoneda equivalent.

Thus the map $H^n(Q;A) \to \mathcal{S}^n(Q;A)/(\text{Yoneda equivalence})$ is well defined. It is immediate that this map is an inverse for the 'Postnikov invariant' map. □

Theorem 4.2 may remain valid when we replace the set $\mathcal{S}^n(Q;A)$ by some particular subset. This is the case when we impose the following condition on $\mathfrak{G}$:

- there are inclusions $N_1 \subset N_2 \subset \cdots \subset N_{n-2}$.

This will give a 'more abelian' group-theoretic interpretation of $H^n(Q;A)$ already found by several authors [5,6,7].

4.3. Lemma. *Let X be a space with only non-trivial π_1 and π_{n+1}-groups. Then there exists an n-cat-group $\mathfrak{G}$ such that $B\mathfrak{G}$ is homotopy equivalent to X and such that $N_1 \subset N_2 \subset \cdots \subset N_n$.*

Proof. The group $A = \pi_{n+1} X$ is a $\pi_1 X$-module. Let $i : \pi_{n+1} X \to I$ be an inclusion of $\pi_{n+1} X$ into an injective $\pi_1 X$-module I. The injectivity of I ensures the existence of a map $X \to K(I, n+1)$ inducing i on π_{n+1}. The fiber of this map is a space $X^{(1)}$ with only non-trivial π_1 $(= \pi_1 X)$ and π_n $(= I/A)$. Continuing this construction gives a sequence of maps

$$X^{(n-1)} \xrightarrow{g_{n-1}} X^{(n-2)} \xrightarrow{g_{n-2}} \cdots \xrightarrow{g_2} X^{(1)} \xrightarrow{g_1} X^{(0)} = X,$$

where $X^{(i)}$ has only non-trivial π_1 and π_{n+1-i}. Finally we define $g_n : X^{(n)} = BF \to X^{(n-1)}$ as in the proof of Lemma 3.5. Working up to homotopy permits us to assume that the g_i are fibrations. The n-cube of fibrations $\mathfrak{X}$ is defined by $\mathfrak{X}^{\alpha_i \cdots \alpha_{i-1} 0 1 1 \cdots 1} = X^{(i)}$, the maps being either identities or composite of g_i's. Example for $n=3$:

$$\begin{tikzcd}
X^{(3)} \arrow[rr, equal] \arrow[dr, equal] \arrow[dd, "g_3"'] & & X^{(3)} \arrow[dr, equal] \arrow[dd] & \\
& X^{(3)} \arrow[rr, equal] \arrow[dd, "g_2 g_3"] & & X^{(3)} \arrow[dd, "g_1 g_2 g_3"] \\
X^{(2)} \arrow[rr, equal] \arrow[dr, "g_2"'] & & X^{(2)} \arrow[dr, "g_1 g_2"'] & \\
& X^{(1)} \arrow[rr, "g_1"'] & & X^{(0)}
\end{tikzcd}$$

The n-cat-group $\mathscr{G}(\mathfrak{X})$ has the required properties. □

4.4. Definition. *An n-fold extension* of the group Q by the Q-module A is an exact sequence of groups

$$1 \longrightarrow A \xrightarrow{\tau_n} K_{n-1} \xrightarrow{\tau_{n-1}} K_{n-2} \longrightarrow \cdots \xrightarrow{\tau_1} K_0 \xrightarrow{\tau_0} Q \longrightarrow 1 \qquad (**)$$

where the K_i are Q-modules and the τ_i module-homomorphisms for $i > 1$ and where τ_1 is a crossed module.

A 1-fold extension is just an extension of groups and a 2-fold extension is a crossed module.

Two n-fold extensions of Q by A are said to be *congruent* if there is a morphism from one to the other inducing the identity on A and on Q. The Yoneda equivalence is the equivalence relation generated by congruence.

4.5. Corollary. (Hill [5], Holt [6], Huebschmann [7]). *The cohomology group $H^n(Q; A)$ is in one-to-one correspondence with the set of equivalence classes of $(n-1)$-fold extensions.*

Proof. From Theorem 4.2 and Lemma 4.3 it suffices to show that a triple $(\mathfrak{G}, \varphi, \Psi)$ where $\mathfrak{G}$ is an $(n-2)$-cat-group satisfying $N_1 \subset N_2 \subset \cdots \subset N_{n-2}$ is equivalent to an $(n-1)$-fold extension of Q by A.

The $(n-1)$-fold extension obtained from $(\mathfrak{G}, \varphi, \Psi)$ is

$$1 \longrightarrow A \xrightarrow{\varphi} C_{n-2}(\mathfrak{G}) \xrightarrow{\delta_{n-2}} \cdots \xrightarrow{\delta_1} C_0(\mathfrak{G}) \xrightarrow{\Psi} Q \longrightarrow 1.$$

On the other hand the $(n-2)$-cat-group is constructed from the n-fold extension $(**)$ as follows:

$$N_1 = K_0, \quad N_2 = K_1 \rtimes N_1; \quad \ldots, \quad N_{n-2} = K_{n-3} \rtimes N_{n-3}, \quad G = K_{n-2} \rtimes N_{n-2}.$$

The action of N_i on K_i is obtained via the projection of N_i onto $N_1 = K_0$ (which acts on K_i). The structural morphisms are given by

$$s_i(k_{i-1}, k_{i-2}, \ldots, k_0) = (k_{i-2}, \ldots, k_0),$$
$$b_i(k_{i-1}, k_{i-2}, k_{i-3}, \ldots, k_0) = (\tau_{i-1}(k_{i-1})k_{i-2}, k_{i-3}, \ldots, k_0).$$

The equivalence is clear. □

Example. It is well known that the group $H^n(\mathbb{Z}^n; \mathbb{Z})$ is infinite cyclic. We construct an $(n-1)$-fold extension whose invariant is a generator of this cohomology group as follows. Define

$$\begin{aligned} 1 \longrightarrow \mathbb{Z} \xrightarrow{\tau_{n-1}} \mathbb{Z}\times\mathbb{Z} \xrightarrow{\tau_{n-2}} \mathbb{Z}\times\mathbb{Z} \longrightarrow \\ \cdots \longrightarrow \mathbb{Z}\times\mathbb{Z} \xrightarrow{\tau_1} H\times\mathbb{Z}^{n-2} \xrightarrow{\tau_0} \mathbb{Z}^n \longrightarrow 1 \end{aligned}$$

by $\tau_{n-1}(a) = (a, 0)$, $\tau_i(u, v) = (v, 0)$ for $n-2 \le i \le 2$, H = Heisenberg group, i.e. $H = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ as a set and

$$(l, m, p)(l', m', p') = (l + l' + mp', m + m', p + p'),$$
$$\tau_1(u, v) = (v, 0, 0; 0, \ldots, 0)$$

and

$$\tau_0(l, m, p; a_1, a_2, \ldots, a_{n-2}) = (m, p, a_1, \ldots, a_{n-2}).$$

The group $H \subset H\times\mathbb{Z}^{n-2}$ acts trivially on all the groups $\mathbb{Z}\times\mathbb{Z}$. The ith generator of the factor $\mathbb{Z}^{n-2} \subset H\times\mathbb{Z}^{n-2}$ acts trivially on all the groups $\mathbb{Z}\times\mathbb{Z}$ but the ith one where it acts by $(a, b) \mapsto (a+b, b)$. One can verify that this is an $(n-1)$-fold extension of $\mathbb{Z}^n$ by the trivial module $\mathbb{Z}$ and that its invariant generates $H^n(\mathbb{Z}^n; \mathbb{Z})$.

Interpretation of $H^n(K(C, k); A)$. Let A and C be abelian groups. The set $\mathcal{E}^n((C, k); A)$ consists of the triples $(\mathfrak{G}, \varphi, \Psi)$ where $\mathfrak{G}$ is an $(n-2)$-cat-group, φ is an isomorphism between $H_k(C_*(\mathfrak{G}))$ and C and Ψ is an isomorphism between $H_{n-1}(C_*(\mathfrak{G}))$ and A. Moreover we assume that $H_i(C_*(\mathfrak{G})) = 0$ if $i \ne k$ and $n-1$. There is a Yoneda equivalence defined as in 4.1.

4.6. Theorem. *There is a one-to-one correspondence between the cohomology group $H^n(K(C, k); A)$ and the set $\mathcal{E}^n((C; k), A)/$(Yoneda equivalence) for $n > k$.*

Proof. The proof is similar to the proof of Theorem 4.2 and is left to the reader. □

4.7. *Interpretation of* $H^2(\mathscr{B}\mathfrak{G};A)$. Let $\mathfrak{X}:\langle 0,1\rangle^n\to$ (spaces) be an n-cube of fibrations. It can be viewed as a morphism between two $(n-1)$-cubes of fibrations $\mathfrak{Y}$ and $\mathfrak{Z}$, i.e. $\mathfrak{X}:\mathfrak{Y}\to\mathfrak{Z}$ where

$$\mathfrak{Y}^{\alpha_1\cdots\alpha_{n-1}}=\mathfrak{X}^{\alpha_1\cdots\alpha_{n-1}0}\quad\text{and}\quad\mathfrak{Z}^{\alpha_1\cdots\alpha_{n-1}}=\mathfrak{X}^{\alpha_1\cdots\alpha_{n-1}1}.$$

The cone of an n-cube of fibrations is defined by induction as follows. For $n=0$, $C\mathfrak{X}=\mathfrak{X}$ (which is merely a space). If $C\mathfrak{Y}$ (resp. $C\mathfrak{Z}$) is the mapping cone of $\mathfrak{Y}$ (resp. $\mathfrak{Z}$), then $C\mathfrak{X}$ is by definition the mapping cone of the map $C\mathfrak{Y}\to C\mathfrak{Z}$. From the connectedness of the spaces in $\mathfrak{X}$ (namely the fibers) it follows by Van Kampen's theorem that $C\mathfrak{X}$ is simply connected (for $n\geq 1$).

4.8. Definition. *The homology* (resp. *cohomology*) *groups of the n-cube of fibrations* $\mathfrak{X}$ *with trivial coefficients in* A *are* $H_i(\mathfrak{X};A)=H_{n+i}(C\mathfrak{X};A)$ (resp. $H^i(\mathfrak{X};A)=H^{n+i}(C\mathfrak{X};A)$).

From this definition it follows that there is an exact sequence

$$\cdots\longrightarrow H_i(\mathfrak{X};A)\longrightarrow H_i(\mathfrak{Y};A)\longrightarrow H_i(\mathfrak{Z};A)\longrightarrow H_{i-1}(\mathfrak{X};A)\longrightarrow\cdots$$

and similarly in cohomology.

Let $\mathfrak{G}$ (resp. A) be a fixed n-cat-group (resp. abelian group). We are now concerned with the set $\operatorname{Opext}(\mathfrak{G};A)$ of extensions of n-cat-groups of the following type

$$1\longrightarrow(A;1,1,\ldots,1)\longrightarrow\mathfrak{K}\longrightarrow\mathfrak{G}\longrightarrow 1$$

which are central, i.e. the group A maps into the center of K. Two such extensions $\mathfrak{K}$ and $\mathfrak{K}'$ are said to be congruent if there is a morphism f of n-cat-groups making the diagram

$$\begin{tikzcd}
1 \arrow[r] & (A;1,1,\ldots,1) \arrow[r] \arrow[d, equal] & \mathfrak{K} \arrow[r] \arrow[d, "f"] & \mathfrak{G} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & (A;1,1,\ldots,1) \arrow[r] & \mathfrak{K}' \arrow[r] & \mathfrak{G} \arrow[r] & 1
\end{tikzcd}$$

commutative.

4.9. Theorem. *There is a one-to-one correspondence between* $H^2(\mathscr{B}\mathfrak{G};A)$ *and* $\operatorname{Opext}(\mathfrak{G},A)/$*(congruence)*.

Proof. By Theorem 1.2 the set $\operatorname{Opext}(\mathfrak{G},A)/$(congruence) is in one-to-one correspondence with the homotopy classes of fibrations

$$\mathscr{B}(A;1,1,\ldots,1)\longrightarrow\mathscr{B}\mathfrak{K}\longrightarrow\mathscr{B}\mathfrak{G}$$

where A and $\mathfrak{G}$ are fixed. By obstruction theory these diagrams are classified, up to homotopy, by the cohomology group $H^{n+2}(C\mathscr{B}\mathfrak{G};A)=H^2(\mathscr{B}\mathfrak{G};A)$. $\square$

Example. Let $\nu: N \to Q$ be a group epimorphism with kernel V and let A be a Q-module. The inclusion $V \to N$ is a crossed module whose corresponding 1-cat-group is $(V \rtimes N; N)$. The fibration $\mathscr{B}(V \rtimes N; N)$ is $BY \to BN \to BQ$ and the group $H^2(\mathscr{B}(V \rtimes N; N); A)$ is the relative cohomology group $H^3(Q, N; A)$ which fits into the exact sequence

$$\cdots \to H^2(N; A) \to H^2(Q; A) \to H^3(Q, N; A) \to H^3(N; A) \to H^3(Q; A) \to \cdots$$

An extension of $(V \rtimes N; N)$ with kernel $(A; 1)$ is equivalent to a crossed module of the following form $1 \longrightarrow A \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\nu} Q \longrightarrow 1$, i.e. such that $N \to \operatorname{Coker} \mu$ is precisely ν. Such an object was called a relative extension in [8]. Therefore Theorem 4.9 asserts that the set of relative extensions of ν with kernel A modulo congruence is in one-to-one correspondence with $H^3(Q, N; A)$. This result was proved in [8, Theorem 1] by explicit cocycle computations.

One can combine the ideas of 4.1 and 4.3 to obtain an interpretation of the groups $H^i(\mathscr{B}\mathfrak{G}; A)$ for any i. This is left to the reader.

As a consequence of 4.9 we will prove a result which we use in [4] for $n=2$.

4.10. Proposition. *Let $\mathfrak{K} \to \mathfrak{G}$ be a central extension of n-cat-groups with kernel $(A; 1, 1, \ldots, 1)$. If $H_i(\mathscr{B}\mathfrak{G}; \mathbb{Z}) = 0$ for $i \leq 2$ and if the group K is perfect then this extension is an isomorphism, i.e. $A = 1$.*

Proof. From the hypotheses $H_1(\mathscr{B}\mathfrak{G}; A) = H_2(\mathscr{B}\mathfrak{G}; A) = 0$ and the universal coefficient theorem we get $H^2(\mathscr{B}\mathfrak{G}; A) = 0$. By Theorem 4.9 this implies that the extension is congruent to the trivial extension, and therefore splits. The extension of groups $1 \to A \to K \to G \to 1$ is central and splits, so K is isomorphic to $A \times G$. The abelianization of K is $A \times G^{\mathrm{ab}}$ and K is perfect, therefore $A = 1$. $\square$

In fact Theorem 4.9 allows one to develop a whole theory of universal central extensions of n-cat-groups in the same spirit as what was done for groups by Kervaire in [14] (resp. for crossed modules in [8]). Proposition 4.10 is part of this theory.

5. Crossed squares and 2-cat-groups

Lemma 2.2 which describes a 1-cat-group in terms of a crossed module (resp. a category, resp. a simplicial group) has an analogue for any n. We implicitly used it when we described in 2.11 the simplicial group $K_{\#}$ associated to the n-cat-group $\mathfrak{G}$. The description in terms of categories can easily be made by using the notion of n-fold category.

Finding the analogue of crossed modules for higher n is more complicated. We will give such a description for $n=2$. Another group-theoretic description of 3-co-connected spaces was obtained by Conduché [2].

5.1. Definition. A *crossed square* is a commutative square of groups

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ M' \arrow[r, "\mu'"] & P \end{tikzcd} \qquad (*)$$

together with an action of P (resp. P, resp. P, resp. M, resp. M') on L (resp. M, resp. M', resp. L, resp. L) and with a function $h: M \times M' \to L$ satisfying the following axioms

(i) the homomorphisms $\lambda, \lambda', \mu, \mu'$ and $\kappa = \mu\lambda = \mu'\lambda'$ are crossed modules and the morphisms of maps $(\lambda) \to (\kappa)$; $(\kappa) \to (\mu)$, $(\lambda') \to (\kappa)$ and $(\kappa) \to (\mu')$ are morphisms of crossed modules,

(ii) $\lambda h(m, m') = m\,{}^{\mu'(m')}m^{-1}$ and $\lambda' h(m, m') = {}^{\mu(m)}m'\,m'^{-1}$,

(iii) $h(\lambda(l), m') = l\,{}^{m'}l^{-1}$ and $h(m, \lambda'(l)) = {}^{m}l\,l^{-1}$,

(iv) $h(m_1 m_2, m') = {}^{m_1}h(m_2, m')\,h(m_1, m')$ and $h(m, m_1' m_2') = h(m, m_1')\,{}^{m_1'}h(m, m_2')$,

(v) $h({}^n m, {}^n m') = {}^n h(m, m')$,

(vi) ${}^m({}^{m'}l)\,h(m, m') = h(m, m')\,{}^{m'}({}^m l)$,

for all $m, m_1, m_2 \in M$, $m', m_1', m_2' \in M'$ and $l \in L$.

A morphism of crossed squares is a commutative diagram

$$\begin{tikzcd} L_1 \arrow[rr] \arrow[d] \arrow[ddr] & & M_1 \arrow[d] \arrow[ddr] & \\ M_1' \arrow[rr] \arrow[ddr] & & P_1 \arrow[ddr] & \\ & L_2 \arrow[rr] \arrow[d] & & M_2 \arrow[d] \\ & M_2' \arrow[rr] & & P_2 \end{tikzcd}$$

such that the oblique maps are compatible with the actions and the functions h_1 and h_2.

5.2. Proposition. *The category of 2-cat-groups is isomorphic to the category of crossed squares.*

Proof. Let $\mathfrak{G} = (G; N_1, N_2)$ be a 2-cat-group. Define $L = \operatorname{Ker} s_1 \cap \operatorname{Ker} s_2$, $M = N_1 \cap \operatorname{Ker} s_2$, $M' = \operatorname{Ker} s_1 \cap N_2$, $P = N_1 \cap N_2$ and λ = restriction of b_1 to L, λ' = restriction of b_2 to L, μ' = restriction of b_1 to M, μ = restriction of b_2 to M'. If m is in M

and m' is in M' then the commutator $[m,m']$ is in L therefore the function $h:M\times M'\to L$, $h(m,m')=[m,m']$ is well defined. The equality $\mu\lambda=\mu'\lambda'$ follows from $b_1b_2=b_2b_1$. Using the equivalence of 1-cat-groups with crossed modules we easily prove axiom (i) of 5.1. The other axioms are also easily verified: it suffices to compute in G, replacing $h(m,m')$ by the commutator and all the actions by conjugation.

We will now construct a 2-cat-group from a crossed square. First there are semi-direct products $L\rtimes M'$ and $M\rtimes P$. We define an action of $M\rtimes P$ on $L\rtimes M'$ as follows:

$$(m,p)\cdot(l,m')=(m\cdot(p\cdot l)h(m,p\cdot m'),\ p\cdot m').$$

Use of the axioms (iv), (v) and (vi) of 5.1 shows that this action is well defined. Put $G=(L\rtimes M')\rtimes(M\rtimes P)$, $N_1=M\rtimes P$, s_1 = projection on $M\rtimes P$ and define b_1 by $b_1(l,m',m,p)=(\lambda(l)\mu'(m')\cdot m,\ \mu'(m')p)$. Then $(G;N_1)$ is a 1-cat-group.

We can switch the role of M and M', that is we can define an action of $M'\rtimes P$ on $L\rtimes M$ such that G is canonically isomorphic to $(L\rtimes M)\rtimes(M'\rtimes P)$. Similary there is a 1-cat-group $(G;N_2)$ with $N_2=M'\rtimes P$. These two categorical group structures on G commute because $\mu\lambda=\mu'\lambda'$. Thus we have constructed a 2-cat-group.

These two constructions are inverses of each other. □

5.3. Application. Let $\mu:M\to N$ be a group homomorphism. It is well known that the necessary and sufficient condition for the existence of a fibration $K(M;1)\to K(N,1)\to X$ inducing μ is that there exists an action of N on M making μ into a crossed module. Similarly we have the following result.

5.4. Proposition. *Let*

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ M' \arrow[r, "\mu'"] & P \end{tikzcd} \qquad (*)$$

be a commutative square of groups. A necessary and sufficient condition for the existence of a diagram of fibrations

$$\begin{tikzcd} K(L,1) \arrow[r] \arrow[d] & K(M,1) \arrow[r] \arrow[d] & X \arrow[d] \\ K(M',1) \arrow[r] \arrow[d] & K(P,1) \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z \end{tikzcd}$$

inducing $(*)$ *is the existence of a crossed square structure on* $(*)$.

Proof. If (*) is a crossed square, then by Proposition 5.2 there is associated a 2-cat-group $\mathfrak{G}$. The 2-cube of fibrations $\mathscr{B}\mathfrak{G}$ is the desired diagram because $\mathscr{B}\mathfrak{G}^{-1,-1}$ (resp. $\mathscr{B}\mathfrak{G}^{-1,0}$, resp. $\mathscr{B}\mathfrak{G}^{0,-1}$, resp. $\mathscr{B}\mathfrak{G}^{0,0}$) is equal to $B\Gamma^{-1,-1}\mathfrak{G}=B(L;L,L)=BL$ (resp. $B\Gamma^{-1,0}\mathfrak{G}\simeq B(M';M',M')=BM'$, resp. $B\Gamma^{0,-1}\mathfrak{G}\simeq B(M;M,M)=BM$, resp. $B\Gamma^{0,0}\mathfrak{G}\simeq B(P;P,P)=BP$).

On the other hand, if we start with a 2-cube of fibrations $\mathfrak{X}$ then by 2.15 and 5.2 the commutative square

$$\begin{tikzcd}
\pi_1(\mathfrak{X}^{-1,-1}) \arrow[r] \arrow[d] & \pi_1(\mathfrak{X}^{0,-1}) \arrow[d] \\
\pi_1(\mathfrak{X}^{-1,0}) \arrow[r] & \pi_1(\mathfrak{X}^{0,0})
\end{tikzcd}$$

is a crossed square. □

Acknowledgement

I wish to thank L. Evens who spent hours to hear me and D. Guin-Waléry, M. Zisman and K. Brown for helpful conversations. I am also grateful to the Mathematics Department of Northwestern University for its hospitality while this work was being done.

References

[1] R. Brown and C.B. Spencer, *G*-groupoids, crossed modules and the fundamental groupoid of a topological group, Proc. Kon. Ned. Akad. 79 (1976) 296–302.
[2] D. Conduché, Modules croisés généralisés de longueur 2, en préparation.
[3] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory, Ergeb. Math. (Springer, Berlin, New York, 1967).
[4] D. Guin-Walery et J.-L. Loday, Obstruction à l'excision en *K*-théorie algébrique, in: Evanston Conference, 1980, Lecture Notes in Math. No. 854 (Springer, Berlin–New York, 1981) 179–216.
[5] R.O. Hill, A natural algebraic interpretation of the group cohomology group $H^n(Q,A)$, $n\geq 4$, Notices A.M.S. Vol. 25, No. 3 (1978) p. A-351.
[6] O.F. Holt, An interpretation of the cohomology groups $H^n(G,M)$, J. Algebra 60 (1979) 307–318.
[7] J. Huebschmann, Crossed *n*-fold extensions of groups and cohomology, Comment. Math. Helvetici 55 (1980) 302–314.
[8] J.-L. Loday, Cohomologie et groupe de Steinberg relatifs, J. Algebra 54 (1978) 178–202.
[9] J.P. May, Simplicial Objects in Algebraic Topology, Van Nostrand Math. Studies (New York, 1967).
[10] S. MacLane, Homology, Grund. Math. Wiss., Bd 114 (Springer, Berlin, 1963).
[11] S. MacLane, Historical Note, J. Algebra 60 (1979) 319–320 (Appendix to [6]).
[12] W. Shih, Homologie des espaces fibrés, Publ. I.H.E.S. 13 (1962) 91–175.
[13] J.H.C. Whitehead, Combinatorial homotopy II, Bull. A.M.S. 55 (1949) 453–496.
[14] M. Kervaire, Multiplicateurs de Schur et *K*-théorie, in: Essays on Topology and Related Topics; Mémoires dédiés à G. de Rham (Springer, Berlin–Heidelberg–New York, 1970) 212–225.