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SPACE OF MINIMAL PRIME IDEALS OF A RING WITHOUT NILPOTENT ELEMENTS

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Hull–kernel topology on the set $\Sigma(R)$ of prime ideals of a ring R with unity and without nilpotent elements is discussed. The restriction of this topology to the set $\pi(R)$ of minimal prime ideals of R has been investigated in detail. The compactness of $\pi(R)$ has been characterized in several ways. An interesting characterization of Baer rings is given.

A functorial correspondence between the category of rings having the property that every prime ideal contains a unique minimal prime ideal and their minimal spectra is established.

Introduction

Throughout let R denote an associative ring with unity. By an ideal of R we shall always mean a two-sided ideal. The set of all prime ideals of R will be denoted by $\Sigma(R)$. A nonzero element of $\Sigma(R)$ will be called a minimal prime ideal of R if it does not contain any other nonzero element of $\Sigma(R)$. The set of all minimal prime ideals of R will be denoted by $\pi(R)$.

In literature an extensive study of $\Sigma(R)$ and $\pi(R)$, where R is a commutative ring with unity, has been carried out. In particular, the study of $\Sigma(R)$ and $\pi(R)$ when equipped with what is called Zariski topology (equivalently, Jacobson's topology or Stone's topology or hull–kernel topology) has been carried out in detail; see Atiyah and MacDonald [1], Lambek [6]; Simons [11] et al. As against it, there is a scarce literature on these aspects when R is a noncommutative ring with unity. However, there is some discussion of such a topology for noncommutative rings in Koh [5]. The purpose of this paper is to see how far it is possible to push the concepts of prime and minimal spectra in the context of noncommutative rings.

The detailed discussion is, however, postponed to respective sections.

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1. Preliminaries

There is an interesting proposition that runs as follows.

"Let T be any subset of a ring R with unity. Then any ideal B of R which has no element in common with T except possibly 0 is contained in an ideal M which is maximal with respect to this property". See Lambek [6, p. 16].

We shall, however, need a variant of this, which can be put in the following nice form. For the sake of completeness, we outline its proof.

Theorem 1.1. *If R is a ring with unity, I is an ideal of R and M is an m-system of R with $M \cap I = \emptyset$ then there exists a maximal ideal Q of R such that $I \subsetneq Q$ and $M \cap Q = \emptyset$. This ideal Q is prime.*

Proof. The existence of a maximal ideal Q containing I and having no element in common with M follows by Zorn's lemma.

We show that this ideal Q is prime. Let J, K be two ideals of R with $JK \subsetneq Q$. By maximality of Q, there exist x and y in R such that $x \in (J+Q) \cap M$ and $y \in (K+Q) \cap M$. Then $x, y \in M$ and for some r in R, $xry \in M$. Also $x = a+b$; $y = c+d$ where $a \in J$, $c \in K$, $b, d \in Q$, clearly $xry \in Q$; a contradiction. □

This separation theorem plays crucial role in all our analysis.

The next result is a folk theorem; and its proof uses usual techniques; see McCoy [7].

Theorem 1.2. *Let A be an ideal of R and M be an m-system of R with $M \cap A = \emptyset$. Then M is contained in a maximal m-system N with $N \cap A = \emptyset$.*

If P is a nonempty subset of R, let $C(P)$ denote the complement of P in R. It is highly known that an ideal P in R is prime if and only if $C(P)$ is an m-system; see McCoy [8]. We shall use this fact repeatedly to obtain several interesting results.

For a ring R with unity, a prime ideal P is called a minimal prime ideal belonging to an ideal I if and only if $I \subsetneq P$ and there is no prime ideal Q such that $I \subsetneq Q \subsetneq P$.

The following characterization essentially uses the separation Theorem 1.1 and Theorem 1.2 alongwith the above mentioned characterization of prime ideals.

Theorem 1.3. *A nonempty subset P of R is a minimal prime ideal belonging to the ideal I if and only if $C(P)$ is an m-system, maximal with respect to the property of not meeting I.*

As every maximal ideal in a ring with unity is prime, the next two results are immediate consequences of Theorems 1.2 and 1.3.

Corollary 1.4. *Any prime ideal containing the ideal I contains a minimal prime ideal belonging to I.*

Corollary 1.5. *Every minimal prime ideal belonging to an ideal A is contained in a maximal ideal belonging to A.*

We use, in the proof of the next theorem, a minor but interesting observation of Herstein [3, p. 4].

Lemma 1.6. *For a ring R with unity and without nilpotent elements, the following are equivalent.*

(i) *M is a maximal m-system of R.*

(ii) *For any nonzero $a \notin M$; there exists a $b \in M$ with $ab = 0$.*

Proof. (i)$\Rightarrow$(ii): Let M be a maximal m-system for which (ii) does not hold. Let $a \notin M$ be such that $ab \neq 0$ for all $b \in M$. Let K be the multiplicative system generated by $M \cup \{a\}$. We claim that $0 \notin K$. For, if $0 \in K$; let $m_1 m_2, \ldots, m_k = 0$ where $m_i \in K$ $(1 \leq i \leq k)$. Since $C(M)$ is a prime ideal, for some $x_1, x_2, \ldots, x_{k-1}$ in R, $m_1 x_1 m_2 x_2 \cdots x_{k-1} m_k \neq 0$. However, $m_1 m_2 \cdots m_k x_1 x_2 \cdots x_k = 0$ and as R does not have nilpotent elements we conclude that $m_1 x_1 m_2 x_2 \cdots x_{k-1} m_k = 0$, a contradiction. Hence $0 \notin K$. Since K is an m-system containing M; this contradicts the maximality of M.

(ii)$\Rightarrow$(i): Let M be an m-system satisfying (ii) but which is not maximal. By Theorem 1.2, M is contained in a maximal m-system, say N. Let $x \in N - M$; by (ii) there exists $y \in M$ with $xy = 0$. Since R has no nilpotent elements, we have $(x)(y) = 0$ which contradicts the primeness of $C(N)$ and we are through. □

2. Hull-kernel topology

The concepts of hulls, kernels and hull–kernel topology in commutative rings, lattices, semilattices, semigroups have been studied by several authors. See for example, Atiyah and Macdonald [1], Lambek [6], Speed [12], Pawar and Thakare [9, 10], Kist [4], DeMarco and Orsatti [2], Simmons [11] et al.

In this section we shall consider such a topology for a ring with unity and without nilpotent elements.

For any nonempty subset θ of $\Sigma(R)$ we define the kernel of θ, denoted by $k(\theta)$, as the set $k(\theta) = \bigcap \{P \mid P \in \theta\}$. For an ideal I of R we define its hull; denoted by $h(I)$, as the set $h(I) = \{P \mid P \in \Sigma(R); I \subseteq P\}$.

For any x in R, $\{P\}_x$ denotes the set $\{P \mid P \in \Sigma(R), x \notin P\}$. Since $x \in P$ if and only if $(x) \subseteq P$, we shall not make distinction between $\{P\}_x$ and $\{P\}_{(x)}$ and similarly between $h(x)$ and $h((x))$.

It can be shown that the sets of the form $\{\{P\}_x \mid x \in R\}$ have the following properties and hence they form a basis for the open sets of the topology on $\Sigma(R)$.

(1) $\bigcup_{i \in I} \{P\}_{x_i} = \{P\}_{\{x_i \mid i \in I\}}$.

(2) $\{P\}_{A \cap B} = \{P\}_A \cap \{P\}_B$ for any ideals A and B of R, where $\{P\}_A$ denotes the

set $\{P \mid P \in \Sigma(R) \text{ and } A \nsubseteq P\}$.

(3) $\{P\}_0 = \emptyset$.

(4) $\{P\}_R = \{P\}_1 = \Sigma(R)$.

It can be shown that for a set $\{A_i \mid i \in I\}$ of ideals of R,

(5) $\bigcup_{i \in I} \{P\}_{A_i} = \{P\}_{\sum A_i}$ where $\sum_i A_i$ denotes the ideal sum of the ideals A_i.

We denote $\Sigma(R)$, together with this topology called the hull–kernel topology, again by $\Sigma(R)$. $\Sigma(R)$ with this topology will be called the prime spectrum of R.

In the next theorem we show that the prime spectrum of R is compact.

Theorem 2.1. *$\Sigma(R)$ is a compact space.*

Proof. Let $\Sigma(R) = \bigcup_{i \in I} \{P\}_{A_i}$ where A_i; $i \in I$; are ideals of R. Clearly $\Sigma(R) = \{P\}_{\sum_{i \in I} A_i}$ and this shows that $\sum_{i \in A} A_i$ is an ideal of R not contained in any prime ideal of R and so it must contain 1. Thus $1 = \sum_{i \in I} a_i$ where $a_i \in A_i$ and the sum has only finitely many nonzero terms. Thus, we may assume that $a_i \neq 0$ for $i \in F$, where F is a finite subset of I. Thus $1 \in \sum_{i \in F} A_i$ and $\Sigma(R) = \{P\}_{\sum_{i \in F} A_i} = \bigcup_{i \in F} \{P\}_{A_i}$. $\square$

Let P be a prime ideal of a ring R without nilpotent elements. Let $0(P) = \{r \in R \mid ra = 0 \text{ for some } a \notin P\}$.

Since R has no nonzero nilpotent elements, it follows that $0(P)$ is an ideal of R and that $0(P) \subseteq P$.

Theorem 2.2. *The prime spectrum $\Sigma(R)$ of a ring R without nilpotent elements is Hausdorff if and only if P is the unique prime ideal containing $0(P)$.*

Proof. Suppose that $\Sigma(R)$ is Hausdorff. Let Q, S be distinct prime ideals of R such that $0(Q) \subseteq S$. Let $\{P\}_x$, $\{P\}_y$ be disjoint neighbourhoods of Q and S respectively, where $x \in S - Q$, $y \in Q - S$. Hence there is no prime ideal P with $x \notin P$ and $y \notin P$. Thus, every prime ideal contains either x or y. Hence $xy \in \bigcap \{P \mid P \in \Sigma(R)\}$. It can be shown that $\bigcap \{P \mid P \in \Sigma(R)\} = 0$. Hence $xy = 0$. Thus $y \in 0(Q)$ a contradiction to the choice of y.

Conversely, let Q be the unique prime ideal containing $0(Q)$. If $S \neq Q$ is a prime ideal, then there exists $x \in 0(Q) - S$. But $xy = 0$ for some $y \notin Q$. Since R has no nilpotent elements, $xy = 0$ if and only if $(x)(y) = 0$; and hence it follows that $y \in S$. Consider the open neighbourhoods $\{P\}_x$, $\{P\}_y$ in $\Sigma(R\}$ of S and Q respectively. Further,

$$\{P\}_x \cap \{P\}_y = \{P\}_{(x)} \cap \{P\}_{(y)} = \{P\}_{(x)(y)} = \{P\}_0 = \emptyset.$$

and so we are through. $\square$

Next, we give a sufficient condition for a particular subset of $\Sigma(R)$ to be totally ordered.

Theorem 2.3. *If any two incomparable elements of $\Sigma(R)$ have disjoint neighbourhoods, then for any Q in $\Sigma(R)$ the set $\{P \mid P \in \Sigma(R); Q \subseteq P\}$ is a chain.*

Proof. Let A, B be two incomparable elements of $\{P \mid P \in \Sigma(\dot{R}), Q \subsetneq P\}$. By assumption, there exist disjoint neighbourhoods $\{P\}_x, \{P\}_y$ of A and B respectively, where $x \notin A$, $y \notin B$. $\{P\}_x \cap \{P\}_y = \emptyset$ implies $(x)(y) = (0)$ and so either $x \in Q$ or $y \in Q$. In either case this contradicts the choice of x and y and we are through. □

We now glue together our considerations of Section 1 and 2.

Theorem 2.4. *Let M be a nonempty subset of a ring R without nilpotent elements. Then the following are equivalent:*

(1) *$C(M)$ is a maximal m-system.*
(2) *M is a minimal prime ideal.*
(3) *For any $a \in M$ there exists $b \in C(M)$ with $ab = 0$.*

Proof. The proof is immediate from our Theorem 1.2 and from Theorem 2.4 of Koh [5]. □

3. The minimal spectrum

In this section we concentrate our attention on the minimal spectrum. In the beginning we list a few characterizations of minimal prime ideals of R, where R is a ring without nilpotent elements and with unity 1.

The notation A^* for a nonempty subset A of R stands for the set $A^* = \{x \in R \mid xa = 0$ for all $a \in A\}$. It can be shown that A^* is an ideal of R. Clearly $(x)^* = \{x\}^*$ and they will be used interchangeably.

Theorem 3.1. *A prime ideal M is minimal prime if and only if $(x)^* - M \neq \emptyset$ for any x in M.*

Proof. Let M be a minimal prime ideal and $x \in M$. As $C(M)$ is a maximal m-system, by Lemma 1.6 there exists y in $C(M)$ with $xy = 0$. Thus $y \in \{x\}^* - M$.

The converse follows by just retracing the steps. □

Next, we have one more characterization of minimal prime ideals.

Theorem 3.2. *A prime ideal M is minimal prime if and only if it contains precisely one of $(x), (x)^*$.*

Proof. Let M be a minimal prime ideal. if $x \in M$ then by Theorem 3.1, $(x)^* \nsubseteq M$. On the other hand, if $(x)^* \subsetneq M$ and $(x) \subsetneq M$ then by Lemma 1.6 we are led to a contradiction.

Conversely, let a prime ideal M satisfy the given condition. Choose any $x \notin C(M)$ then $x \in M$ and so $(x)^* - M \neq \emptyset$. By Lemma 1.6 and Theorem 1.3 we are done. □

In the next result, we establish a relation between annihilators and hulls and kernels of an annihilator.

Theorem 3.3. *For any ideal I, $I^* = k(\pi(R) - h(I))$.*

Proof. Since $II^* = (0)$, it follows that whenever $I \not\subseteq M \in \pi(R)$ then $I^* \subseteq M$. Thus

$$I^* \subseteq \bigcap \{M \in \pi(R) \mid I \not\subseteq M\}.$$

Let $x \in \bigcap \{M \in \pi(R) \mid I \not\subseteq M\}$ and $x \notin I^*$. Then for some $y \in I, xy \neq 0$. Consider the multiplicative system $T = \{(xy)^i \mid i = 1, 2, \dots\}$. Clearly $0 \notin T$ and so by Theorem 1.2, T is contained in a maximal m-system say F in R. Clearly $xy \notin C(F)$ and hence $x \notin C(F), y \notin C(F)$. Thus $I \not\subseteq C(F)$ and so $\bigcap \{M \in \pi(R) \mid I \not\subseteq M\} \subseteq C(F)$; but then $x \in C(F)$, a contradiction. $\square$

This readily leads to the following sequence:

Corollary 3.4. *For any x in R, $(x)^* = k(\{M\}_x)$.*

A more useful consequence of our considerations is stated in the next result.

Corollary 3.5. *For any x in R, $h(k(\{M\}_x)) = h((x)^*) = \{M\}_x$.*

In particular, $h(x)$ and $h((x)^*)$ are both open and closed sets in $\pi(R)$ that are disjoint.

The next corollary is immediate from Theorem 3.3 and Corollary 3.5.

Corollary 3.6. *For any x in R, $h(x) = h((x)^{**})$.*

Theorem 3.7. *For each element r and prime ideal P of a ring R without nilpotent elements the following are equivalent:*

(i) *$(r)^* \subseteq P$.*

(ii) *There is some $Q \in \pi(R)$ with $Q \subseteq P$ and $r \notin Q$.*

Proof. Let $(r)^* \subseteq P$ where $r \in R, P \in \Sigma(R)$. Then $(r)^* \cap C(P) = \emptyset$ and so by Theorem 1.2, $C(P)$ is contained in a maximal m-system say T with $T \cap (r)^* = \emptyset$. But then $C(T) \in \pi(R)$ is a prime ideal contained in P and $(r)^* \subseteq C(T)$ hence by Theorem 3.2 we are through. $\square$

If $\{M\}_x \subseteq \{M\}_y$ then $k(\{M\}_y) \subseteq k(\{M\}_x)$ and by Corollary 3.4 it follows that $(y)^* \subseteq (x)^*$ and hence $(x)^{**} \subseteq (y)^{**}$. Conversely from Corollaries 3.4 and 3.5 we can get $(x)^{**} \subseteq (y)^{**}$ implies $\{M\}_x \subseteq \{M\}_y$. Thus, we have proved:

Theorem 3.8. $\{M\}_x \subseteq \{M\}_y$ *if and only if* $(x)^{**} \subseteq (y)^{**}$.

One readily notes that $\{M\}_x$ are clopen sets of $\pi(R)$. In fact, we have:

Theorem 3.9. *The hull–kernel topology on $\pi(R)$ is Hausdorff. The base sets $\{M\}_x$ of which are open as well as closed.*

Proof. Let A, B be two distinct minimal prime ideals in R. Let $x \in A - B$. As $x \notin C(A)$ and $C(A)$ is a maximal m-system, by Lemma 1.6 there exists $y \notin A$ with $xy = 0$. Clearly $B \in \{M\}_x$ and $A \in \{M\}_y$. From $xy = 0$ it follows that $\pi(R)$ is Hausdorff. □

An ideal I is said to be *dense* if $I^* = (0)$ and *normal* if $I = I^{**}$.

Theorem 3.10. *Any non-dense ideal is contained in a minimal prime ideal.*

Proof. Clearly I is non-dense if and only if $I^* \neq (0)$. For $x \in I^*$, $T = \{x^i \mid i = 1, 2, \ldots\}$ is a multiplicative system of R not meeting I, which must be contained in a maximal m-system, say F. We then get $I \subseteq C(F)$ where $C(F)$ is a minimal prime ideal. □

Now, we discuss normal ideals.

Theorem 3.11. *Any normal ideal of R is the intersection of all minimal prime ideals containing it.*

Proof. Clearly $I^{**} = \bigcap \{M \in \pi(R) \mid I^* \nsubseteq M\}$. By Theorem 3.2 and normality of I, we have

$$I = \bigcap \{M \in \pi(R) \mid I \subseteq M\}. \qquad \square$$

Here is an immediate consequence.

Corollary 3.12. *An ideal I is normal if and only if I is the intersection of all minimal prime ideals containing it.*

A ring R is called a Baer ring if the right annihilator of any element is a right ideal generated by an idempotent.

Since, for a ring R without nilpotent elements, the right annihilator of an element is equal to its left annihilator, for such rings we may say, R is a Baer ring if and only if $(r)^* = eR$ for any r in R and for some idempotent e in R.

We give a characterization of Baer rings in the next theorem.

Theorem 3.13. *For a ring R without nilpotent elements, the following are equivalent:*

(i) *R is Baer.*

(ii) *For each r in R there is some idempotent e such that for any P in $\pi(R)$, $r \in P$ if and only if $e \in P$.*

Proof. (i)$\Rightarrow$(ii): Let $r\in P$; $P\in\pi(R)$. By Theorem 3.2, $(r)^*\not\subseteq P$. But $(r)^*=eR$ for some idempotent e and so $e\notin P$. But then $(e)^*=(1-e)R\subsetneq P$. Thus $1-e\in P$ where $1-e$ is an idempotent. Thus $r\in P$ implies there is an idempotent $f=1-e$ in P. On the other hand if $f=1-e$ is in a minimal prime ideal Q then, as $(r)\subseteq(r)^{**}$ it follows that $r\in(1-e)R$ and so $r\in Q$.

(ii)$\Rightarrow$(i): Let $r\in R$. if $r\notin M$ for any $M\in\pi(R)$ then trivially $(r)^*=0R$ and R is Baer.

If $r\in M\in\pi(R)$ then there exists an idempotent e with $r\in M$ if and only if $e\in M$. Then $\{M\}_r=\{M\}_e$ and by Theorem 3.8, $(r)^*=(e)^*$. But $x\in(e)^*$ if and only if $x\in(1-e)R$ and thus we are through. □

Let us concentrate on $\pi(R)$ as a space. We noted earlier that $\pi(R)$ is a topological space under the hull–kernel topology. It can also be shown that the sets of the form

$$\{h(x)\mid x\in R\}\quad\text{where}\quad h(x)=\{P\in\Sigma(R)\mid x\in P\}$$

have the following properties.

(1) $h(0)=\Sigma(R)$.

(2) $h(R)=\emptyset$.

(3) If $(E_i)_{i\in I}$ is any family of subsets of R then

$$h\left(\bigcup_{i\in I}E_i\right)=\bigcap_{i\in I}h(E_i)$$

(4) $h(A\cap B)=h(A)\cup h(B)$ for any ideals A,B of R.

Hence it follows that the sets $\{h(x)\mid x\in R\}$ form a basis for closed sets. We call this topology as the dual hull–kernel topology on $\Sigma(R)$, as against the earlier mentioned hull–kernel topology. We shall denote the hull–kernel topology on $\pi(R)$ by T^{h} and T^{d} will indicate the dual hull–kernel topology on $\pi(R)$ induced by the dual hull–kernel topology of $\Sigma(R)$.

One observes that open sets in $(\pi(R),T^{\mathrm{d}})$ are also open in $(\pi(R),T^{\mathrm{h}})$:

Theorem 3.14. *The hull–kernel topology T^{h} is finer than the dual hull–kernel topology T^{d}.*

Proof. The sets $\{h(x)\mid x\in R\}$ form a basis for T^{d} and $h(x)=\pi(R)-\{M\}_x$ for any x in R. As $\{M\}_x$ is closed in $(\pi(R),T^{\mathrm{h}})$ we are through. □

In fact, the reverse inclusion is valid under some restriction.

Theorem 3.15. *$T^{\mathrm{h}}=T^{\mathrm{d}}$ if for any x in R there exists y such that $(x)^{**}=(y)^*$.*

Now we are in a position to obtain the main result:

Theorem 3.16. *The statements given below are equivalent in R.*

(1) *$(\pi(R),T^{\mathrm{h}})$ is compact.*

(2) *Finite unions of* $\{\{M\}_x \mid x \in R\}$ *form a Boolean lattice.*

(3) *For any* x *in* R *there exists* t_i $(1 \le i \le n)$ *in* R *such that* $t_i \in (x)^*$, $1 \le i \le n$, *and* $(x)^* \cap \bigcap_{i=1}^n (t_i)^* = (0)$.

(4) *For any* x *in* R *there exist* t_i $(1 \le i \le n)$ *in* R *such that* $(x)^{**} = \bigcap_{i=1}^n (t_i)^*$.

(5) $T^{\mathrm{h}} = T^{\mathrm{d}}$.

(6) $\{h(x) \mid x \in R\}$ *is a subbasis for the open sets of* $(\pi(R), T^{\mathrm{d}})$.

(7) $\{\{M\}_x \mid x \in R\}$ *is a subbasis for the open sets of* $(\pi(R), T^{\mathrm{h}})$.

Proof. The equivalence of (5), (6) and (7) is trivial, because topologies are completely determined by any of their subbases. The theorem would be proved if we show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): By Theorem 3.9, $\pi(R)$ is Hausdorff and as $h(x)$ is a closed subset of $\pi(R)$ we conclude that $h(x)$ is compact in its relative topology. By Theorem 3.2 it follows that $h(x) \cap h((x)^*) = \emptyset$ and so $h(x) \cap \{h(t) \mid t \in (x)^*\} = \emptyset$. By compactness of $h(x)$, there exist t_i $(1 \le i \le n)$ in $(x)^*$ such that

$$h(x) \cap \{h(t_i) \mid t_i \in (x)^*, 1 \le i \le n\} = \emptyset.$$

On taking complements in $\pi(R)$ we get

$$\pi(R) = \{M\}_x \cup \{M\}_{t_1} \cup \{M\}_{t_2} \cup \cdots \cup \{M\}_{t_n}.$$

As $t_i \in (x)^*$ for $1 \le i \le n$, by Theorem 3.2 we conclude that

$$\{M\}_x \cap \bigcup_{i=1}^n \{M\}_{t_i} = \emptyset.$$

Thus $\bigcup_{i=1}^n \{M\}_{t_i}$ is a complement of $\{M\}_x$. Since $\{\{M\}_x \mid x \in R\}$ is a semilattice we conclude that finite unions of $\{\{M\}_x \mid x \in R\}$ form a Boolean lattice; see Varlet [13].

(2) $\Rightarrow$ (3): Let $\bigcup_{i=1}^n \{M\}_{t_i}$ be the complement of $\{M\}_x$. Clearly we have $xt_i = 0$, $1 \le i \le n$, and so $t_i \in (x)^*$. Furthermore,

$$k\left(\{M\}_x \cup \bigcup_{i=1}^n \{M\}_{t_i}\right) = k(\pi(R)),$$

that is

$$k(\{M\}_x) \cap \bigcap_{i=1}^n k(\{M\}_{t_i}) = k(\pi(R)).$$

Hence by Corollary 3.4 and using $k(\pi(R)) = (0)$ we have $(x)^* \cap \bigcap_{i=1}^n (t_i)^* = (0)$.

(3) $\Rightarrow$ (4): As $xt_i = 0$, $1 \le i \le n$, we have $x \in (t_i)^*$. Hence $(x)^{**} \subseteq (t_i)^{***} = (t_i)^*$ for $1 \le i \le n$. Thus $(x)^{**} \subseteq \bigcap_{i=1}^n (t_i)^*$. Let $a \in \bigcap_{i=1}^n (t_i)^*$ then $at_i = 0$, $1 \le i \le n$. If $y \in (x)^*$ then $xy = 0$. Also $yat_i = 0$ and $yax = 0$. Hence, by (3) $ya = 0$ and so $a \in (x)^{**}$. hence the implication.

(4) $\Rightarrow$ (5): In view of Theorem 3.14, we need only prove that the basic open sets $\{\{M\}_x \mid x \in R\}$ in T^{h} are open in T^{d}. For $x \in R$, there exist t_i, $1 \le i \le n$, in R such that $(x)^{**} = \bigcap_{i=1}^n (t_i)^*$. Hence, using Corollaries 3.5 and 3.6 we have $h(x) = \bigcup_{i=1}^n \{M\}_{t_i}$. Taking complements in $\pi(R)$ we get $\{M\}_x = \bigcap_{i=1}^n h(t_i)$. Thus $\{M\}_x$ is a finite

intersection of open sets in T^d and so is open and we are through.

(5)$\Rightarrow$(1): By (5), $\{\{M\}_x \mid x \in R\}$ will also be a base for closed sets in $(\pi(R), T^h)$. To prove (1), we shall show that every family of closed sets with the finite intersection property has nonempty intersection. Let $\{\{M\}_x \mid x \in J\}$ be a family of closed sets having the finite intersection property. This implies that $\bigcap_{x\in F}\{M\}_x \neq \emptyset$ whenever $F \subseteq J$ is finite and so $\prod_{x\in F}(x) \neq (0)$ where $\prod(x)$ denotes the product of ideals $(x), x \in F$. As R has no nilpotents, this further implies that $\prod_{x\in F} x \neq 0$ where $\prod x$ is the product of x; $x \in F$. Let A be the multiplicative semigroup generated by J. Clearly $0 \notin A$ and so by Theorem 1.2, A is contained in a maximal m-system say T not containing 0. But then $C(T)$ is a minimal prime ideal not meeting J. Thus, $C(T) \in \bigcap_{x\in J}\{M\}_x$. □

The proof of the next theorem is immediate.

Theorem 3.17. *Let R and S be rings with unity. Let $f: R \to S$ be a ring epimorphism. Then $f^{-1}(P)$ is a prime ideal of R for every prime ideal P of S.*

A ring R is said to be *normal* if every prime ideal contains a unique minimal prime ideal.

Let R and S be normal rings with unity and without nilpotent elements. Let $\pi(R)$ and $\pi(S)$ denote the minimal prime spectra of R and S respectively. For any epimorphism $f: R \to S$ define the map $f^*: \pi(S) \to \pi(R)$ by $f^*(M) = [f^{-1}(M)]^m$ where $M \in \pi(S)$ and $[f^{-1}(M)]^m$ denotes the unique minimal prime ideal contained in $f^{-1}(M)$.

Theorem 3.18. *The map f^* defined as above is a continuous map.*

Proof. By Theorem 3.17 and by normality of R it follows that the map f^* is well defined.

We first observe that $f^{*-1}(\{M\}_x\} = \{M\}_{f(x)}$. Let $Q \in \{M\}_{f(x)}$, then $x \notin f^{-1}(Q)$ and so $x \notin f^*(Q)$. Hence $Q \in f^{*-1}(\{M\}_x)$. Thus $\{M\}_{f(x)} \subseteq f^{*-1}(\{M\}_x)$.

Now, let $Q \in f^{*-1}(\{M\}_x)$ then $f^*(Q) \in \{M\}_x$ and so $[f^{-1}(Q)]^m \in \{M\}_x$ we claim that $x \notin f^{-1}(Q)$. Suppose to the contrary. Then $f(x) \in Q$. Since Q is a minimal prime ideal by Lemma 1.6 there exists $f(y)$ in $C(Q)$ with $f(x)f(y) = 0$. As R and S have no nilpotent elements, it follows that $f((x)(y)) = 0$ and so $(x)(y) \subseteq f^{-1}(0)$. Since $[f^{-1}(Q)]^m$ is a minimal prime ideal containing $f^{-1}(0)$ it follows that $(y) \subseteq [f^{-1}(Q)]^m$ which is a contradiction. Hence $x \notin f^{-1}(Q)$ implies $Q \in \{M\}_{f(x)}$. Hence $f^{*-1}(\{M\}_x) = \{M\}_{f(x)}$.

Thus f^* pulls back open sets onto open sets and so it is continuous. □

This theorem establishes a functorial correspondence between the category of normal rings and their minimal spectra.

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