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SOME SUFFICIENT CONDITIONS FOR THE GONALITY OF A SMOOTH CURVE

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Introduction

In this paper, a smooth curve means an integral, complete, non-singular one-dimensional scheme of finite type over $\mathbb{C}$. One can also consider a Riemann-surface. I say that a smooth curve C is d-gonal if there exists a surjective morphism $\phi : C \to \mathbb{P}^1$ of degree d.

Let C be a smooth curve of genus g and let W_d^r be as defined in Notation 4. Clifford's Theorem says that a special effective divisor D on C satisfies

$$2\dim(|D|) \le \deg(D).$$

Moreover, equality

$$2\dim(|D|) = \deg(D)$$

is only possible if C is hyperelliptic or if D is a canonical divisor on C (see e.g. [6, Chapter IV, Theorem (5.4)]). In [11] it is proved that, if $0 \le r \le s \le g-1$, then

$$\dim(W_{r+s}^r) \le s - r.$$

Here, $\dim(X)$ is used to denote the maximum of the dimensions of the irreducible components of X, while $\underline{\dim}(X)$ is the minimum of the dimensions of the irreducible components of X. Moreover, if $0 \le r \le s \le g-2$, then

$$\underline{\dim}(W_{r+s}^r) = s - r$$

if and only if C is hyperelliptic. In the appendix of [12], it is proved that, if there exists $d \in \mathbb{Z}$ with $2 \le d \le g-2$ such that

$$\dim(W_d^1) \ge d - 3,$$

then C is of one of the following types of curves:

a hyperelliptic curve;

a trigonal curve;
a two-sheeted covering of an elliptic curve;
a smooth plane curve of degree 5.

Assume that C is not a covering of degree at least two of a smooth curve C' of genus at least one. In this paper, we give sufficient conditions, concerning $\dim(W^r_d)$, for C to be f-gonal (see Theorem 6(a)).

Let C be a non-hyperelliptic smooth curve of genus g. In [10] and in [8], it is proved that C is $(g-1)$-gonal. In [10], it is proved that if a smooth curve of genus $g \geq 15$ is not hyperelliptic and is not a two-sheeted covering of a smooth curve of genus two, then it is $(g-2)$-gonal. In [9], it is proved that if a smooth curve of genus $g \geq 12$ is a two-sheeted covering of a smooth curve of genus two, then it is $(g-2)$-gonal. Assume that C is not a covering, of degree at least two, of a smooth curve of genus at least one. In this paper, we give sufficient conditions for C to be $(g-f)$-gonal (see Theorem 6(b)).

In Example 12, I show how the results of Theorem 4 can be sharpened. In particular, if a smooth curve of genus $g \geq 13$ is not a hyperelliptic curve, not a trigonal curve, and not a covering of degree at least two of a smooth curve of genus at least one, then it is $(g-2)$-gonal. This result is sharper than G. Martens' result mentioned before.

Let X be a scheme of finite type over $\mathbb{C}$. If we write $x \in X$, then we mean that x is a $\mathbb{C}$-rational point on X. If E is a locally free O_X-module, then we write E^D for $\mathrm{Hom}_{O_X}(E, O_X)$.

Let C be a smooth curve. We say that C has a g^r_d, if C has a linear system of dimension r and of degree d. We also use the notation g^r_d to indicate a particular linear system of dimension r and of degree d on C.

If $A \subset \mathbb{P}^r$, then we write $\langle A \rangle$ for the linear span of A. Assume that C is embedded in $\mathbb{P}^r$ and that D is an effective divisor on C. We define

$$\langle D \rangle := \bigcap \{H, \text{ a hyperplane in } \mathbb{P}^r : H \cdot C \geq D\}.$$

If C is non-hyperelliptic and if C is canonically embedded, then the geometric Riemann–Roch Theorem says that

$$\dim(\langle D \rangle) = \deg(D) - \dim(|D|) - 1$$

(see e.g. [4, p. 248]). We define

$$\mathrm{Supp}(D) := \{x \in C : D - x \geq 0\}.$$

We write K_C for the canonical linear system on C. A divisor D on C is called a special divisor on C if and only if $|K_C - D| \neq \emptyset$. A linear system g^r_d on C is called a special linear system on C if the elements of g^r_d are special divisors on C. Let D_1 and D_2 be two effective divisors on C. Then

$$A = \mathrm{Inf}(D_1, D_2)$$

is the effective divisor on C defined by the conditions that

$$D_i - A \geq 0 \quad \text{for } i \in \{1, 2\}$$

and for each $\mathbb{C}$-rational point x on C satisfying $D_1 - A - x \geq 0$, the divisor $D_2 - A - x$ is not effective. Let g_1 and g_2 be two linear systems on C. Assume that D is the divisor on C, defined by the condition that there exist linear systems g_1' and g_2' on C, having no common fixed points, so that $g_i = g_i' + D$ for $i \in \{1, 2\}$. Then we say that g_1 and g_2 are compounded by the same involution on C if and only if for each $x \in C$ and for any two divisors $D_i \in g_i'$ ($i \in \{1, 2\}$), so that $D_i - x \geq 0$, we have

$$\deg(\mathrm{Inf}(D_1, D_2)) \geq 2.$$

If D is a divisor on C, then we write $O_C(D)$ for the sheaf on C associated to D, as it is defined in [6, p. 144].

Results

In this paper, we are going to consider smooth curves having infinitely many g_d^1's. Let $g, d \in \mathbb{Z}_{\geq 2}$. If

$$g \leq 2d - 3,$$

then every smooth curve of genus g has infinitely many g_d^1's (see [5]).

Lemma 1. *Let $g, d \in \mathbb{Z}_{\geq 2}$ so that $d \leq g - 1$. Let $e \in \mathbb{Z}_{\geq 2}$ so that*

$$g > (e-2)d - \binom{e-1}{2}.$$

Let C be a smooth curve of genus g and let $g_1, \ldots, g_e$ be complete linear systems on C without fixed points and each one of them has dimension 1 and degree d. Assume that for each $i, j \in \{1, \ldots, e\}$, with $i \neq j$, the linear systems g_i and g_j are not compounded by the same involution on C. Then

$$\dim(|g_1 + \cdots + g_e|) \geq \binom{e+1}{2}.$$

To prove this lemma, we use the following lemma.

Lemma 2. *Let C be a non-hyperelliptic smooth curve of genus g and let D be a special divisor of degree d on C such that $|D|$ has no fixed points. Let g_e^1 and g_f^1 be complete linear systems on C without fixed points, which are not compounded of the same involution on C. Moreover, assume that $|D + g_e^1|$ and $|D + g_f^1|$ are special linear systems on C. If*

$$\dim(|D|) = b, \qquad \dim(|D + g_e^1|) = a_1, \quad \dim(|D + g_f^1|) = a_2,$$

then

$$\dim(|D+g_e^1+g_f^1|)\geq a_1+a_2+1-b.$$

Moreover, if $|D|$ and g_e^1 (resp. $|D|$ and g_f^1) are not compounded of the same involution on C, then

$$a_1\geq b+2 \qquad (\textit{resp. } a_2\geq b+2).$$

If

$$a_1=b+2+a_1' \quad \textit{and} \quad a_2=b+2+a_2',$$

then

$$\dim(|D+g_e^1+g_f^1|)\geq a_1'+a_2'+b+5.$$

Proof. Assume that C is canonically embedded. Let $x\in C$ and $D_1\in g_e^1$ and $D_2\in g_f^1$ such that

$$\operatorname{supp}(D_1)\cap\operatorname{supp}(D)=\emptyset, \qquad \operatorname{supp}(D_2)\cap\operatorname{supp}(D)=\emptyset,$$

$$x=\inf(D_1, D_2), \qquad x\notin\langle D\rangle\cap\langle D_2\rangle.$$

Then

$$\langle D_1+D_2+D-x\rangle=\langle\langle D+D_1\rangle U\langle D_2\rangle\rangle,$$

hence

$$\dim(\langle D_1+D_2+D-x\rangle)=\dim(\langle D+D_1\rangle)+\dim(\langle D_2\rangle)$$
$$-\dim(\langle D+D_1\rangle\cap\langle D_2\rangle).$$

Because of the geometric Riemann-Roch Theorem, one has

$$\dim(\langle D\rangle)=d-b-1,$$

$$\dim(\langle D+D_1\rangle)=d+e-a_1-1,$$

$$\dim(\langle D+D_2\rangle)=d+f-a_2-1,$$

$$\dim(\langle D_2\rangle)=f-2.$$

Hence,

$$\dim(\langle D\rangle\cap\langle D_2\rangle)=a_2-b-2.$$

Because

$$\langle D_2\rangle\cap\langle D+D_1\rangle\supset\langle(\langle D\rangle\cap\langle D_2\rangle)U\{x\}\rangle,$$

one has

$$\dim(\langle D_2\rangle\cap\langle D+D_1\rangle)\geq a_2-b-1.$$

Hence,

$$\dim(\langle D_1+D_2+D-x\rangle)\leq d+e+f-a_1-a_2+b-2.$$

Therefore, because of the geometric Riemann-Roch Theorem, one has

$$\dim(|D_1+D_2+D-x|)\geq a_1+a_2-b.$$

But x is not a fixed point of $|D_1+D_2+D|$, hence

$$\dim(|D_1+D_2+D|)\geq a_1+a_2-b+1.$$

If $|D|$ and g^1_e are not compounded of the same involution on C, then there exist $x \in C$ and divisors $D_1 \in |D|$ and $D_2 \in g^1_e$, such that $\inf(D_1, D_2) = x$. Using the geometric Riemann–Roch Theorem, one finds that

$$\dim(|D + D_1 - x|) \geq b + 1.$$

But x is not a fixed point of $|D + D_1|$, hence,

$$\dim(|D + D_1|) \geq b + 2.$$

Proof of Lemma 1. See also [1], but Lemma 2 provides a more geometric proof. Because, for $i \neq j$, the linear systems g_i and g_j are not compounded of the same involution on C, the curve C is non-hyperelliptic (see e.g. [15, Chapter VI, Theorem (7.1)]). Using Lemma 2 and induction on n, one can prove, for $d' \in \mathbb{Z}_{\geq 3}$ and if $|g_1 + \cdots + g_{d'-2}|$ is special, that

$$\dim(|g_1 + \cdots + g_{d'}|) \geq \binom{d'+1}{2}.$$

But, because of the induction hypothesis, one finds that

$$\dim(|g_1 + \cdots + g_{d'-2}|) \geq \binom{d'-1}{2}$$

and therefore $|g_1 + \cdots + g_{d'-2}|$ is special if

$$g > d'(d'-2) - \binom{d'-1}{2}.$$

This proves Lemma 1.

Remark. If the assumptions made in the statement of Lemma 1 are fulfilled, then, because of the Riemann–Roch Theorem, one has

$$g \geq ed - \binom{e+1}{2}.$$

Proposition 3. *Let $g \in \mathbb{Z}_{\geq}$ and $d \in \mathbb{Z}_{\geq 3}$. Let C be a smooth curve of genus g. Assume that C has infinitely many complete g^1_d's, so that any two of them are not compounded by the same involution on C, then*

$$g \leq \tfrac{1}{2}d(d-1).$$

Proof. Assume that

$$g > \tfrac{1}{2}d(d-1).$$

Let $g_1, \ldots, g_{d-1}$ be complete linear systems of degree d and dimension 1 on C, without fixed points, so that any two of them are not compounded by the same involution on C. We can use Lemma 1 to conclude that

$$\dim(|g_1+\cdots+g_{d-1}|)\geq \tfrac{1}{2}d(d-1).$$

On the other hand,

$$\deg(g_1+\cdots+g_{d-1})=d(d-1).$$

From Clifford's Theorem (see e.g. [6, p. 343, Theorem (5..4)]), it follows that

$$|g_1+\cdots+g_{d-1}|=K_C.$$

Let g'_{d-1} be a complete linear system of dimension 1 and degree d on C without fixed points, so that $g'_{d-1}\notin\{g_1,\dots,g_{d-1}\}$ and so that g'_{d-1} is not compounded by the same involution on C with any of the linear systems $g_1,\dots,g_{d-1}$. In this case, taking $g_1,\dots,g_{d-2},g'_{d-1}$ instead of $g_1,\dots,g_{d-1}$, we find that

$$|g_1+\cdots+g_{d-2}+g'_{d-1}|=K_C,$$

hence

$$|g'_{d-1}|=|g_{d-1}|.$$

This gives a contradiction with the existence of infinitely many complete g_d^1's on C. This completes the proof of the proposition.

Notation 4. Let C be a smooth curve of genus $g\geq 2$. In what follows, we assume that we have choosen a base point P_0 on C. Let $\operatorname{Pic}^0(C)$ be the Picard variety of C whose $\mathbb{C}$-rational points correspond to isomorphism classes of invertible O_C-modules of degree 0.

If L is an invertible O_C-module of degree 0, then we write $[L]$ for the corresponding $\mathbb{C}$-rational point on $\operatorname{Pic}^0(C)$. Consider the morphisms

$$I(d): C^{(d)}\to \operatorname{Pic}^0(C): D\mapsto [O_C(D-dP_0)].$$

The morphism $I(1)$ is a closed immersion and it is also the Albanese mapping of C (see e.g. [14, (0.5)]). We call $\operatorname{Pic}^0(C)$ the Jacobian variety of C and we denote it by $J(C)$. Let $r,d\in\mathbb{Z}_{\geq 1}$. Let

$$W_d^r:=\{x\in J(C): \dim([I(d)]^{-1}(x))\geq r\}.$$

Those are Zariski closed subsets of $J(C)$. We also write W_d instead of W_d^0. We write k for the $\mathbb{C}$-rational point $[O_C(K-(2g-2)P_0)]$, on $J(C)$ where K is a canonical divisor on C.

Lemma 5. *Let C be a smooth curve of genus g. Assume that Y is an irreducible component of W_d^1 such that $Y\not\subset W_{d-1}^1+W_1$. Let $e=\dim Y$ and let $D\in C^{(d)}$ such that $[I(d)](D)\in Y$. Then,* $\dim(|2D|)\geq 2+e$.

Proof. Let $M_{g,n}$ be the fine moduli space of the smooth curves of genus g with a level-n-structure and let $\pi : \Gamma_{g,n} \to M_{g,n}$ be the universal family. Let $x \in M_{g,n}$ such that $C \cong \pi^{-1}(x)$. Let $\phi : H_{g,n,d} \to M_{g,n}$ be the fine moduli space of the couples (x, f) where $x \in M_{g,n}$ and f is a covering $C_x \to \mathbb{P}^1$ of degree d.

Let $D \in C^{(d)}$ such that

$$[I(d)](D) \in Y \setminus (W^1_{d-1} + W_1).$$

Let $f : C \to \mathbb{P}^1$ be a covering of degree d such that D is a fibre of f. Let x be a point on $M_{g,n}$ such that $C_x \cong C$ and let y be the point on $H_{g,n,d}$ corresponding to (x, f). Consider the exact sequence

$$0 \to T_C \to f^*(T_{\mathbb{P}^1}) \to N_f \to 0$$

Because of the deformation theory of holomorphic maps (see [7]), there exists a canonical isomorphism

$$T_y(H_{g,n,d}) \cong H^0(C, N_f)$$

and the tangent map $d_y(\phi)$ corresponds to the coboundary map

$$\delta : H^0(C, N_f) \to H^1(C, T_C).$$

But the dimension of the irreducible component of $\phi^{-1}(x)$ containing y equals $e+3$. Moreover,

$$\dim(\ker(d_y(\phi))) = \dim(\ker(\delta)) = \dim(H^0(C, f^*(T_{\mathbb{P}^1})))$$

and $f^*(T_{\mathbb{P}^1}) \cong O_C(2D)$, hence

$$\dim(H^0(C, O_C(2D))) \geq e+3.$$

This proves the lemma.

Assumption A. Let $f \in \mathbb{Z}_{\geq 3}$ and let C be a smooth curve of genus g. Assume that there exists no covering $\phi : C \to \tilde{C}$ such that $1 < \deg(\phi) < f$.

In Theorem 6 and Lemma 7-10, we assume that assumption A holds.

Theorem 6. *Let $f' = 2(f-1)$ and assume that $g > \frac{1}{2}f'(f'-1)$. One has:*

(a) *If there exist $r, d \in \mathbb{Z}_{\geq 1}$ so that*

$$d - 2r - f + 2 \geq 0 \quad \textit{and} \quad d - r \leq g - f$$

and

$$\dim(W^r_d) \geq d - 2r - f + 2,$$

then

$$\dim(W^r_d) = d - 2r - f + 2,$$

and C is f-gonal,

(b) *C is d-gonal for each $d \geq g - f + 2$.*

Proof. Let $f=3$. Assertion (b) is proved in [10] and in [8]. Assertion (a) is proved in the Appendix of [12].

Assume from now on that $f>3$ and assume that the theorem is true for $f-t$, with $t\in\mathbb{Z}$ and $1\le t\le f-2$, instead of f.

Let $r,d\in\mathbb{Z}_{\ge 1}$ so that

$$d-2r-f+2\ge 0 \quad \text{and} \quad d-r\le g-f.$$

Because C is not $(f-1)$-gonal, we know that

$$\dim(W_d^r)<d-2r-f+3.$$

Assume that $r\in\mathbb{Z}_{\ge 2}$ and

$$\dim(W_d^r)=d-2r-f+2.$$

Lemma 7. *Let $r,d\in\mathbb{Z}_{\ge 1}$ and assume that, for $1\le e\le r-1$, one has*

$$\dim(W_{d-e}^{r-e})\le (d-e)-2(r-e)-f+2.$$

If

$$\dim(W_d^r)\ge d-2r-f+2,$$

then,

$$\dim(W_{d-r+1}^1)=d-r+1-f.$$

Proof. The lemma trivially holds for $r=1$. Assume that $r\ge 2$ and that the lemma holds for $r-1$ instead of r. Because of Proposition 3 in [8], one has

$$\dim(W_{d-1}^{r-1})>\dim(W_d^r).$$

Hence

$$\dim(W_{d-1}^{r-1})\ge d-2r-f+3.$$

But, because of the assumptions, one has

$$\dim(W_{d-1}^{r-1})\le d-2r-f+3.$$

Hence

$$\dim(W_{d-1}^{r-1})=d-2r-f+3=(d-1)-2(r-1)-f+2$$

and the lemma follows.

Continuation of the proof of Theorem 6. Because of Lemma 7, we can assume that there exist $d'\in\mathbb{Z}_{\ge 1}$ such that

$$d'-f\ge 0 \quad \text{and} \quad d'\le g-f+1$$

and

$$\dim(W_{d'}^1)=d'-f.$$

Lemma 8. *Let $g\ge 4f-4$. Assume that, for $d,r\in\mathbb{Z}_{\ge 1}$ such that*

$$d-2r-f+3\ge 0 \quad \textit{and} \quad d-r\le g-f+1,$$

one has

$$\dim(W_d^r) \leq d - 2r - f + 3.$$

Let $d' \in \mathbb{Z}_{\geq 2f-2}$ *such that*

$$d' - f \geq 0 \quad \text{and} \quad d' \leq g - f + 1.$$

Assume that Y *is an irreducible component of* $W_{d'}^1$ *of dimension* $d' - f$. *Then,*

$$Y \subset W_{2f-2}^1 + W_{d'-2f+2}.$$

Proof. The lemma trivially holds for $d' = 2f - 2$. Assume that $d' - 2f + 2 \geq 0$ and that the lemma holds for $d' - 1$ instead of d'. Assume that $Y \not\subset W_{d'-1}^1 + W_1$.

If $x \in Y$ so that $x \notin W_{d'-1}^1 + W_1$, then

$$2x \in W_{2d'}^{d'-f+2} \qquad \text{(see Lemma 4).}$$

(a) If $d' \leq g - 2f + 3$, then

$$\dim(W_{2d'}^{d'-f+2}) < f - 1$$

(because of our assumption), hence

$$d' - f \leq f - 2 \quad \text{and} \quad d' \leq 2f - 2.$$

(b) If $d' > g - 2f + 3$, then

$$W_{2d'}^{d'-f+2} = k - W_{2g-2-2d'}^{g-d'-f+1}.$$

If $d' \leq g - f$, then we obtain that

$$\dim(W_{2d'}^{d'-f+2}) < f - 1$$

(again because of our assumption), hence

$$d' - f \leq f - 2 \quad \text{and} \quad d' \leq 2f - 2.$$

If $d' = g - f + 1$, then we obtain

$$W_{2g-2f+2}^{g-2f+3} = k - W_{2f-4},$$

hence

$$\dim(W_{2g-2f+2}^{g-2f+3}) = 2f - 4,$$

hence $g \leq 4f - 5$, which is a contradiction with the assumption.

Continuation of the proof of Theorem 6. Because of Lemma 8, one can assume that there exist d'' such that $f \leq d'' \leq 2f - 2$ and Y, an irreducible component of $W_{d''}^1$ such that $\dim(Y) = d'' - f$. In Lemma 9, we prove that there exists $\varrho \in W_f^1$ such that $Y = \{\varrho\} + W_{d''-f}$.

Lemma 9. *Let* $d \in \mathbb{Z}$ *such that* $f \leq d \leq 2f - 2$ *and assume that* $g > \binom{d}{2}$. *Assume that* Y

is an irreducible component of W_d^1 *of dimension* $d-f$. *Then, there exists* $\varrho \in W_f^1$ *such that*

$$Y=\{\varrho\}+W_{d-f}.$$

Proof. Lemma 9 holds for $d=f$. Assume that $d>f$ and that Lemma 9 holds for $d-1$ instead of d. Assume that

$$Y \not\subset W_{d-1}^1+W_1.$$

Because there exists no covering $\phi: C \to \tilde{C}$ with $1 \le \deg(\phi) < f$, we know that C has infinitely many g_d^1's so that any two of them are not compounded by the same involution on C. This is a contradiction with $g>\binom{d}{2}$ (see Proposition 3). Hence, if Y is an irreducible component of W_d^1 so that

$$\dim(Y)=d-f,$$

then

$$Y \subset W_{d-1}^1+W_1$$

and therefore, W_{d-1}^1 has an irreducible component Y' of dimension $d-1-f$ such that $Y=Y'+W_1$. Because of our induction hypothesis, there exists $\varrho \in W_f^1$ such that

$$Y'=\{\varrho\}+W_{d-f-1},$$

hence $Y=\{\varrho\}+W_{d-f}$.

Continuation of the proof of Theorem 6. Now, because $W_f^1 \ne \emptyset$, C is f-gonal and assertion (a) is proved.

Because of our inductionhypothesis, we can assume that C is d-gonal for each $d \ge g-f+3$. Because $W_{f-1}^1=\emptyset$, we know that

$$\dim(W_{g-f+2}^1) \le g-2f+2.$$

But, because of [5],

$$\underline{\dim}(W_{g-f+2}^1) \ge g-2f+2.$$

Hence, W_{g-f+2}^1 is equidimensional of dimension $g-2f+2$. If C is not f-gonal, then

$$\dim(W_{g-f+1}^1) \le g-2f$$

and therefore

$$W_{g-f+2}^1 \ne W_{g-f+1}^1+W_1,$$

which proves that C is $(g-f+2)$-gonal.

Lemma 10. *Assume that*

$$\underline{\dim}(W^1_{g-f+2}) = \dim(W^1_{g-f+2}) = g-2f+2.$$

Assume that, for $d \in Z_{\geq 1}$ such that $d \leq g-f+1$, one has the following property: if Y is an irreducible component of W^1_d of dimension $d-f$, then there exists $\varrho \in W^1_f$ such that $Y = \{\varrho\} + W_{d-f}$. Then, the curve C is $(g-f+2)$-gonal.

Proof. Assume that

$$W^1_{g-f+2} = W^1_{g-f+1} + W_1.$$

For each irreducible component Y' of W^1_{g-f+2}, there exists an irreducible component Y of W^1_{g-f+1} of dimension $g-2f+1$ so that

$$Y' = Y + W_1.$$

Because of the assumption, there exist $\varrho \in W^1_f$ such that

$$Y = \{\varrho\} + W_{g-2f+1},$$

and therefore

$$Y' = \{\varrho\} + W_{g-2f+2}.$$

Hence, from our assumptions, it follows that

$$W^1_{g-f+2} = W^1_f + W_{g-2f+2}.$$

Let $f < d < g-f+2$ and assume that

$$W^1_d \neq W^1_{d-1} + W_1.$$

Let R be an effective divisor on C of degree d with $\dim(|R|) = 1$ and such that $|R|$ has no fixed points. Assume that C is canonically embedded. Then

$$\dim(\langle R \rangle) = d-2.$$

There exist $\lambda = g-f-d-2$ points $x_1, \dots, x_\lambda$ on C, so that

$$\dim(\langle R + x_1 + \cdots + x_\lambda \rangle) = g-f,$$

hence

$$\dim(|R + x_1 + \cdots + x_\lambda|) = 1$$

because of the geometric Riemann-Roch Theorem. But there exists a linear system g^1_f on C and $\lambda' = g-2f+2$ points $y_1, \dots, y_{\lambda'}$ on C so that

$$R + x_1 + \cdots + x_\lambda \in g^1_f + y_1 + \cdots + y_{\lambda'}.$$

This gives a contradiction to $f < d$. This means that, for $f < \mu < g-f+1$, one has

$$W^1_\mu = W^1_{\mu-1} + W_1.$$

Hence, there exist exactly $f-2$ integers $a < g$, such that there exists a morphism

$p: C \to \mathbb{P}^1$ of degree a. From the results of [5], it follows that

$$g < (f-1)^2$$

(see e.g. [10], p. 69(v)). Hence,

$$W^1_{g-f+2} \neq W^1_{g-f+1} + W_1$$

and therefore, C is $(g-f+2)$-gonal.

Continuation of the proof of Theorem 6. Because of Lemmas 8 and 9, the assumptions of Lemma 10 are fulfilled. This proves assertion (b).

In order to get better results than those obtained in Theorem 6, it would be interesting if one knows more about curves having infinitely many g^1_d's. In some examples, we are going to give refinements of Theorem 6.

Theorem 11. *Let* $g \in \mathbb{Z}_{\geq 2}$, $d \in \mathbb{Z}_{\leq g}$ *and* $a \in \mathbb{Z}_{\geq 1}$. *If*

$$\dim(W^1_d) \geq 2a \quad \textit{and} \quad W^2_d = \emptyset,$$

then $W^1_{d-a} \neq \emptyset$.

Proof. Consider $J(C) \times C$ and the projection morphisms

$$p': J(C) \times C \to J(C) \quad \text{and} \quad q': J(C) \times C \to C.$$

Let $x \in J(C)$ and let F be an $O_{J(C) \times C}$-Module. The fibre of p' over x gives a closed immersion $C \to J(C) \times C$. We write F_x for the inverse image of F by this closed immersion. Consider the universal invertible sheaf L' on $J(C) \times C$ so that

$$L'_x \cong O_C(D - dP_0)$$

if $D \in C^{(d)}$ so that $[I(d)](D) = x$. Let

$$L := L' \otimes (q'^* O_C(dP_0)).$$

Let Y be an irreducible component of W^1_d so that $\dim(Y) \geq 2a$. Let L_Y be the inverse image of L by the closed immersion $Y \times C \subset J(C) \times C$. Let p and q be the restriction of p' resp. q' to $Y \times C$. The O_Y-Module

$$E := p_*(L_Y)$$

is a locally free O_Y-Module of rank 2. From [13, Proposition 7], it follows that

$$\mathbb{P}(E) \to Y$$

is obtained from $I(d)$ by making the base extension $Y \subset J(C)$. Here, $\mathbb{P}(E)$ is the projectivized geometric bundle on Y associated to E, i.e. it equals $\mathbb{P}(E^D)$ as it is defined in [2, II.4.1]. Let

$$M := p_*(q^*(O_{aP_0}) \otimes L_Y).$$

It is a locally free O_Y-Module of rank a. On $Y\times C$, we have a natural $O_{Y\times C}$-Module homomorphism

$$L_Y \to q^*(O_{aP_0})\otimes L_Y$$

which gives rise to an O_Y-Module homomorphism

$$\sigma : E \to M.$$

From [3, Lemma (2.7)], it follows that $E^D\otimes M$ is ample, hence from [3, Theorem II], it follows that

$$D_0(\sigma)=\{y\in Y : \mathrm{rk}(\sigma(y))\leq 0\}$$

is not empty. Here, if y is a $\mathbb{C}$-rational point on Y, we write $\sigma(y)$ for the morphism obtained from σ by making the base extension $\mathrm{Spec}(\mathbb{C})\to Y$ corresponding to y. This proves that $W^1_{d-a}\neq\emptyset$.

Examples 12. (a) Let C be a smooth curve of genus $g\geq 13$. Assume that there exists no covering $\phi : C\to\tilde{C}$ with $1<\deg(\phi)<4$.

We prove that the assertions (a) and (b) of Theorem 4, with $f=4$, are true under these assumptions.

If there exist $r, d\in\mathbb{Z}_{\geq 1}$ so that

$$\dim(W^r_d)\geq d-2r-2\geq 0 \quad\text{and}\quad d-r\leq g-4$$

then, because of Lemmas 7 and 8,

$$\dim(W^1_6)=2.$$

If we prove that for each 2-dimensional irreducible component Y of W^1_6 one has $Y\subset W^1_4+W_2$, then, because of Lemma 9, C is $(g-2)$-gonal. Hence, it is enough to prove that for each 2-dimensional irreducible component Y of W^1_6 one has $Y\subset W^1_4+W_2$. Because there exist no covering $\phi : C\to\tilde{C}$ with $1<\deg(\phi)<4$, the assumptions of Proposition 3 – with $d=5$ – are fulfilled if there exists a one-dimensional irreducible component Y' of W^1_5 such that $Y'\not\subset W^1_4+W_1$. Hence, for each one-dimensional irreducible component Y' of W^1_5 one has $Y'\subset W^1_4+W_1$.

Assume that there exists a 2-dimensional irreducible component Y of W^1_6 so that $Y\not\subset W^1_4+W_2$. It follows that there exists a non-empty Zariski open subset U of Y so that each $\mathbb{C}$-rational point on U corresponds to a complete linear system g^1_6 on C without fixed points. Moreover, because of Lemma 5,

$$\dim(|2g^1_6|)\geq 4.$$

Applying Theorem 11, we know that C has a g^1_5 which is complete. Let g_1 be a complete g^1_5 on C and let g_2 be a complete g^1_6 without fixed points on C satisfying $\dim(|2g_2|)\geq 4$. Because of Lemma 2, one has

$$\dim(|g_1+g_2|)\geq 3.$$

Hence, because of Lemma 2, one has

$$\dim(|g_1 + 2g_2|) \geq 7.$$

Hence $\dim(W_{17}^7) \geq 2$. Because of Lemma 7, we obtain

$$\dim(W_{12}^1) \geq 8.$$

It follows from [12] that there exists a covering $\phi: C \to \tilde{C}$ with $1 < \deg(\phi) < 4$. This is a contradiction with the assumptions. Hence, there exists no 2-dimensional irreducible component Y of W_6^1 unless $Y \subset W_4^1 + W_2$.

(b) Let C be a smooth curve of genus $g \geq 21$. Assume that there exists no covering $\phi: C \to \tilde{C}$ with $1 < \deg(\phi) < 5$. We prove that the assertions (a) and (b) of Theorem 4, with $f = 5$, are true under these assumptions.

If there exist $r, d \in \mathbb{Z}_{\geq 1}$ so that

$$\dim(W_d^r) \geq d - 2r - 3 \geq 0 \quad \text{and} \quad d - r \leq g - 5$$

then, because of Lemmas 7 and 8,

$$\dim(W_8^1) = 3.$$

If we prove that for each 3-dimensional irreducible component Y of W_8^1 one has $Y \subset W_5^1 + W_3$, then, because of lemma 10, C is $(g-3)$-gonal. Hence it is enough to prove that for each 3-dimensional irreducible component Y of W_8^1 one has $Y \subset W_5^1 + W_3$. Assume that there exists a 3-dimensional irreducible component Y of W_8^1.

Assume that there exists a non-empty Zariski open subset U of Y so that each $\mathbb{C}$-rational point on U corresponds to a complete linear system g_8^1 on C without fixed points. Hence, because of Lemma 5,

$$\dim(|2g_8^1|) \geq 5.$$

Applying Theorem 11, we know that C has a g_7^1, which has to be complete because there exist no coverings $\phi: C \to \tilde{C}$ with $1 < \deg(\phi) < 5$. Let g_1 be a complete g_7^1 on C and let g_2 and g_3 be two different complete linear systems g_8^1 on C without fixed points so that $\dim(|2g_i|) \geq 5$ for $i \in \{2, 3\}$. Because of Lemma 2, one has

$$\dim(|g_1 + g_2|) \geq 3.$$

Hence, because of Lemma 2, one has

$$\dim(|g_1 + 2g_2|) \geq 8,$$

and

$$\dim(|g_1 + g_2 + g_3|) \geq 6.$$

Again, because of Lemma 2, one has

$$\dim(|g_1 + 2g_2 + g_3|) \geq 12.$$

Also

$$\dim(|g_1+g_2+2g_3|)\geq 12.$$

Hence, because of lemma 2, one has

$$\dim(|g_1+2g_2+2g_3|)\geq 19.$$

Hence $\dim(W^{19}_{39})\geq 3$. Because of Lemma 7,

$$\dim(W^1_{21})\geq 21.$$

This is impossible because $g\geq 21$. Hence, there exists no 3-dimensional irreducible component Y of W^1_8 unless $Y\subset W^1_7+W_1$.

Assume that $g\geq 22$. Assume that Y' is an irreducible component of W^1_7 of dimension 2. Because of Lemma 9, one has $Y'\subset W^1_5+W_2$. Hence, if $g\geq 22$, then every irreducible component Y of W^1_8 of dimension 3 satisfies $Y\subset W^1_5+W_3$. On the other hand, assume that $g=21$. If $W^2_8\neq\emptyset$, then C is isomorphic to a plane curve of degree 8. Hence, $\dim(W^1_7)=1$. Therefore, if Y is an irreducible component of W^1_7 of dimension 2, then $W^2_8=\emptyset$. Assume that there exists a 2-dimensional irreducible component Y' of W^1_7 so that

$$Y'\not\subset W^1_6+W_1.$$

But, if $d,g\in\mathbb{Z}_{\geq 2}$ so that $g=\frac{1}{2}d(d-1)$ and if C is a smooth curve of genus g having infinitely many g^1_d's so that any two of them are not compounded of the same involution on C, then C is isomorphic to a smooth plane curve of degree $d+1$. This will be proved in the author's Ph.D. Thesis. A proof can also be found in E.K. Hoff's Master of Science thesis entitled "Polygonal curves", following ideas of G. Martens. But, if Y is a 2-dimensional irreducible component of W^1_7 such that $Y\subset W^1_6+W_1$, then, because of Proposition 3, $Y\subset W^1_5+W_2$. Hence, if $g=21$ and if Y is an irreducible component of W^1_8 of dimension 3, then $Y\subset W^1_5+W_3$.

In Example 6(c), we are going to give another argument that can be used to examine the case $g=21$.

(c) Let C be a smooth curve of genus $g\geq 33$. Assume that there exists no covering $\phi:C\to\tilde{C}$ with $1<\deg(\phi)<6$. We prove that the assertions (a) and (b) of Theorem 4, with $f=6$, are true under these assumptions.

If there exist $r,d\in\mathbb{Z}_{\geq 1}$ so that

$$\dim(W^r_d)\geq d-2r-4\geq 0 \quad\text{and}\quad d-r\leq g-6$$

then, because of Lemmas 7 and 8,

$$\dim(W^1_{10})=4.$$

If we prove that for each 4-dimensional irreducible component Y of W^1_{10} one has $Y\subset W^1_6+W_4$, then, because of Lemma 10, C is $(g-4)$-gonal. Hence, it is enough to prove that for each 4-dimensional irreducible component Y of W^1_{10} one has $Y\subset W^1_6+W_4$. Let Y be a 4-dimensional irreducible component of W^1_{10}.

Assume that there exists a non-empty Zariski open subset U of Y so that each $\mathbb{C}$-rational point on U corresponds to a complete linear system g^1_{10} on C without fixed points. Because of Lemma 5,

$$\dim(|2g^1_{10}|) \geq 6.$$

Applying Theorem 5, we know that C has a g^1_8, which has to be complete because there exists no covering $\phi: C \to \tilde{C}$ such that $1 < \deg(\phi) < 6$. Let g_1 be a complete g^1_8 on C and let g_2 and g_3 be two different complete linear systems g^1_{10} on C without fixed points and so that $\dim(|2g_i|) \geq 6$ for $i \in \{2, 3\}$. Using Lemma 2 in exactly the same manner as we did in Example 6(b), one can prove that

$$\dim(|g_1 + 2g_2 + 2g_3|) \geq 21.$$

Hence

$$\dim(W^{21}_{48}) \geq 4 = 48 - 42 - 4 + 2.$$

We can apply Theorem 4 to conclude that C is 4-gonal. This gives a contradiction with the fact that there exists no covering $\phi: C \to \tilde{C}$ with $1 < \deg(\phi) < 6$. Hence, $Y \subset W^1_9 + W_1$.

Assume that $g \geq 37$. Assume that Y' is an irreducible component of W^1_9 of dimension 3. Because of Lemma 9 one has $Y' \subset W^1_6 + W_3$. Hence, if $g \geq 37$, then every irreducible component Y of W^1_{10} of dimension 4 satisfies $Y \subset W^1_6 + W_4$. If $g = 36$, then one can repeat the arguments of the case $g = 22$ in Example (b). Let $g = 35$. In the author's Ph.D. Thesis, the following statement will be proved. If $d, g \in \mathbb{Z}_{\geq 2}$ so that $g = \frac{1}{2}d(d-1) - 1$ and if C is a smooth curve of genus g having infinitely many g^1_d's so that any two of them are not compounded of the same involution on C, then C is birationally equivalent to a plane curve of degree $d+1$. Using this, one can again argue as in the case $g = 36$.

One can also argue as follows. Let Y be an irreducible component of W^1_9 of dimension 3 such that $Y \not\subset W^1_6 + W_3$. If $Y \subset W^2_9$, then, because of Theorem 6, one has that C is trigonal, which is a contradiction to the fact that there exist no morphisms $\phi: C \to \tilde{C}$ such that $1 < \deg(\phi) < 6$. Because of Lemma 9, if $Y \subset W^1_8 + W_1$, then $Y \subset W^1_6 + W_3$. Hence, Y contains a non-empty Zariski open subset V of Y' so that each $\mathbb{C}$-rational point on V corresponds to a complete linear system g^1_9 without fixed point. Moreover, because of Lemma 5,

$$\dim(|2g^1_9|) \geq 5.$$

Because of Theorem 10, it follows that C has a g^1_8 which is complete because $g \geq 22$ and there exist no coverings $\phi: C \to \tilde{C}$ with $1 < \deg(\phi) < 6$. Let g_1 be a complete g^1_8 on C and let g_2, g_3 and g_4 be three different complete linear systems g^1_9 on C without fixed points so that $\dim(|2g_i|) \geq 5$ for $i \in \{2, 3, 4\}$. Using Lemma 2, one can prove that

$$\dim(|g_1 + 2g_2 + 2g_3 + g_4|) \geq 24,$$

hence

$$\dim(W^{24}_{53}) \geq 3,$$

and because $g \geq 33$, we conclude that C is 4-gonal which is a contradiction with the fact that there exists no covering $\phi: C \to \tilde{C}$ with $1 < \deg(\phi) < 6$. Hence, if $g \geq 33$ and if Y is an irreducible component of W^1_{10} of dimension 4, then $Y \subset W^1_6 + W_4$.

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Note added in proof

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