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SOME PROPERTIES OF PURELY SIMPLE KRONECKER MODULES, I

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Let K be an algebraically closed field. A K^2-system is a pair of K-vector spaces (V, W) together with a K-bilinear map from $K^2 \times V$ to W. The category of systems is equivalent to the category of right modules over some K-algebra, R. Most of the concepts in the theory of modules over the polynomial ring $K[\xi]$ have analogues in Mod-R. Unlike the purely simple $K[\xi]$-modules, which are easily described, purely simple R-modules are quite complex. If M is a purely simple R-module of finite rank n then any submodule of M of rank less than n is finite-dimensional. The following corollaries are derived from this fact:

1. Every non-zero endomorphism of M is monic.
2. Every torsion-free quotient of M is purely simple.
3. An ascending union of purely simple R-modules of increasing rank is not purely simple.

It is also shown that a large class of torsion-free rank one modules can occur as the quotient of a purely simple system of rank n, n any positive integer. Moreover, starting from a purely simple system another purely simple module M' of the same rank is constructed and M' is shown to be both a submodule of M and a submodule of a rank 1 torsion-free system. Since the category of right R-modules is a full subcategory of right S-modules, where S is any finite-dimensional hereditary algebra of tame type, the paper provides a way of constructing infinite-dimensional indecomposable S-modules.

Introduction

All undefined terms involving K^2-systems and algebras in this paper may be found in [5] and [8] respectively. Under the term 'canonical pencils or matrices', Kronecker found the finite-dimensional indecomposable K^2-systems. The study of infinite-dimensional systems was initiated by Aronszajn and Fixman in [1] as part of a programme to develop a theory suitable for applications to perturbation problems. Our interest in K^2-systems is entirely algebraic. We first note that the category of K^2-systems is equivalent to the category of right modules over some K-algebra, R. Ringel has extended many of the results in [1] to tame hereditary

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finite-dimensional algebras, see [8]. Following [8] a K^2-system is sometimes called a Kronecker module.

Let V be a $K[\xi]$-module, $K[\xi]$ the polynomial ring in one variable over K. For (a,b) a fixed basis of K^2, (V,V) becomes a K^2-system when we put $av=v$, $bv=\xi v$ for all v in V. This way we get an embedding

$$T_1: \text{Mod-}K[\xi] \to \text{Mod-}R$$

with the following properties:

(1) It is exact and full.

(2) It is closed under extensions.

(3) It is closed under pure submodules, i.e. if N' is a pure submodule of $T(M)$ then $N'=T(N)$ for some N.

If S is any tame hereditary finite-dimensional K-algebra, there is a full and exact embedding $T_2: \text{Mod-}R \to \text{Mod-}S$, see [2]. The modules we study in this paper – purely simple R-modules of rank greater than one – have no analogues in Mod-$K[\xi]$. We would be able to extend the results of the paper to S if we knew that T_2 also has the property (3) of T_1. In any case, the images of the modules constructed in Theorem 2.1 are indecomposable S-modules.

The paper is in two sections. Section 1 contains properties of purely simple R-modules which are essentially corollaries of an easy result already in the literature. However, they are useful in the construction of purely simple modules. From Proposition 1.2, for instance, we know that we cannot construct a purely simple module of rank 3 starting from two infinite-dimensional modules as familiarity with the rank two case may lead one to suppose. Proposition 1.4 shows how not to go about constructing a purely simple module of infinite rank, see also Theorem 2 of [7]. In Section 2 we show the abundance of purely simple modules. Not only do many rank one torsion-free systems occur as the quotient of a purely simple module of rank n for all positive integers, n, Theorem 2.1, but we also show in Proposition 2.3 how from a given purely simple module M of rank n, one gets a submodule M' which is also purely simple of rank n. Moreover M' is a submodule of a rank one module. $\mathscr{R}$ will denote the rank one system (V,W) with $V=W=K(\xi)=$ the K-rational functions with $av=v$, $bv=\xi v$ where (a,b) is a fixed basis of K^2. For simplicity of notation we shall denote a system (V,W) by $\mathbf{V}$, (X,Y) by $\mathbf{X}$ etc. Unless otherwise stated all systems are torsion-free and of finite rank.

1. General properties

Proposition 1.1. *Let* $\mathbf{V}$ *be a purely simple system of rank n and* $\mathbf{X}$ *a subsystem of* $\mathbf{V}$. *If* rank $\mathbf{X}<$ rank $\mathbf{V}$ *then* $\mathbf{X}$ *is finite-dimensional.*

Proof. Since rank $\mathbf{X}<$ rank $\mathbf{V}$, we have that the torsion-closure of $\mathbf{X}$ in $\mathbf{V}$ is a proper torsion-closed subsystem of $\mathbf{V}$. Hence the proposition follows from Lemma 1.12 of [5]. □

Proposition 1.2. *Let* $\mathbf{V}$ *be a purely simple system of rank n. Then every torsion-free quotient of* $\mathbf{V}$ *is purely simple.*

Proof. Suppose $\mathbf{X}^2 = \mathbf{V}/\mathbf{X}^1$ is a torsion-free quotient with $\mathbf{X}^1 \neq 0$. By Theorem 2.4 of [3],

$$\text{Rank } \mathbf{V} = \text{Rank } \mathbf{X}^1 + \text{Rank } \mathbf{X}^2. \tag{1}$$

By Proposition 1.1, $\mathbf{X}^1$ is finite-dimensional.

Suppose $\mathbf{X}^2$ is not purely simple. As in [5], p. 178, $\mathbf{X}^2$ may be assumed to have a proper infinite-dimensional pure subsystem, $\mathbf{X}^3$. Let $\{x_1, x_2, \ldots, x_s\}$ be a basis of $\mathbf{X}^1$ with respect to generation and $\{x_1', \ldots, x_t'\}$ be a set of representatives of a basis of $\mathbf{X}^3$ with respect to generation. So $t < \text{rank } \mathbf{X}^2$. Now, $\mathbf{X}$ = torsion-closure of $(\emptyset, \{x_1, \ldots, x_s, x_1', \ldots, x_t'\})$ in $\mathbf{V}$ is infinite-dimensional by the assumption on $\mathbf{X}^3$. By (1), $s + t < \text{rank } \mathbf{V}$. But $s + t = \text{rank } \mathbf{X}$. Hence, by Proposition 1.1, $\mathbf{X}$ is finite-dimensional, contradiction. □

Proposition 1.3. *Every nonzero endomorphism of a purely simple system* $\mathbf{V}$ *of finite rank is monic.*

Proof. Let (φ, ψ) be a nonzero endomorphism of $\mathbf{V}$. Suppose $\mathbf{X} = \text{Ker}(\varphi, \psi) \neq 0$. $\mathbf{V}/\mathbf{X}$ is isomorphic to a subsystem of $\mathbf{V}$ hence is torsion-free. So by (1) rank $\mathbf{X} <$ rank $\mathbf{V}$ and rank $\mathbf{V}/\mathbf{X} <$ rank $\mathbf{V}$. By Proposition 1.1 both $\mathbf{X}$ and $\mathbf{V}/\mathbf{X}$ are finite-dimensional. Therefore, if $\mathbf{V}$ is infinite-dimensional, $\mathbf{X}$ must be 0. If $\mathbf{V}$ is finite-dimensional then by Theorem 4.3 of [1] (Kronecker's theorem) it is of type $\text{III}^{m_1} \oplus \cdots \oplus \text{III}^{m_r}$. Since $\mathbf{V}$ is purely simple, $r = 1$. Hence rank $\mathbf{V} = 1$. So $\mathbf{X} = 0$ by (1). □

Proposition 1.4. *An ascending union of pure simple systems of strictly increasing rank is not purely simple.*

Proof. Let $\mathbf{V} = \bigcup_{i=1}^{\infty} \mathbf{V}^i$, where $\mathbf{V}^i \subset \mathbf{V}^{i+1}$. Let $n_i = \text{rank } \mathbf{V}^i$. The hypothesis implies that $n_{i_0} > 1$ for some i_0. Since $\mathbf{V}^{i_0}$ is purely simple it must, therefore, be infinite-dimensional, as the argument in Proposition 1.3 shows. Since $\mathbf{V}^{i_0} \subset \mathbf{V}^{i_0+1} \subset \mathbf{V}$, $\mathbf{V}$ is not purely simple by Proposition 1.1. □

Proposition 1.5. *Suppose* $\mathbf{V}^2$ *is a torsion-free quotient of a purely simple system* $\mathbf{V}$. *Then any infinite-dimensional subsystem* $\mathbf{V}^3$ *of* $\mathbf{V}^2$ *with* rank $\mathbf{V}^3 \leq$ rank $\mathbf{V}^2$ *is also the quotient of a purely simple subsystem of* $\mathbf{V}$.

Proof. Let $\mathbf{V}^2 = \mathbf{V}/\mathbf{X}$. The pushout of the inclusion of $\mathbf{V}^3$ into $\mathbf{V}^2$ and the onto map from $\mathbf{V}$ to $\mathbf{V}^2$ yields an extension $\mathbf{U}$ of $\mathbf{X}$ by $\mathbf{V}^3$, and an injection of $\mathbf{U}$ into $\mathbf{V}$. By (1) and the hypothesis rank $\mathbf{U} \leq$ rank $\mathbf{V}$. Suppose $\mathbf{U}$ is not purely simple, then the argument in Proposition 1.2 shows that $\mathbf{U}$ has a proper infinite-dimensional pure subsystem $\mathbf{U}^1$ with rank $\mathbf{U}^1 <$ rank $\mathbf{U}$. Since $\mathbf{U}^1 \subsetneq \mathbf{V}$ and is infinite-dimensional, Proposition 1.1 is contradicted. □

Proposition 1.6. *If* $\mathbf{V}$ *is a purely simple system of rank* n*, then any infinite-dimensional subsystem* $\mathbf{X}$ *of rank* n *is also purely simple.*

Proof. If $\mathbf{X}$ is not purely simple it has an infinite-dimensional pure subsystem $\mathbf{X}^1$ of rank $<n$. The torsion-closure of $\mathbf{X}^1$ in $\mathbf{V}$ is infinite-dimensional and has rank $<n$. This contradicts Proposition 1.1. □

Let $\mathbf{V}^2$ be a rank one torsion-free system with a representative height function H^2. Let $P=\{\theta\in\tilde{K}: H^2_\theta\neq 0\}$.

The proof of the next lemma is similar to the proof of Lemma 1.11 in [5] and is therefore omitted.

Lemma 1.7. *Let* $\mathbf{V}$ *be an extension of a system,* $\mathbf{X}$ *of type* $\mathrm{III}^{m_1}\oplus\cdots\oplus\mathrm{III}^{m_r}$ *by a rank one torsion-free system* $\mathbf{V}^2$*. Suppose* $\mathrm{Card}(P)<\mathrm{Card}(K)$*, then* $\mathbf{V}$ *is also an extension of a system of type* $r\mathrm{III}^1$ *by a rank one torsion-free system.* □

Proposition 1.8. *Let* $\mathbf{V}$ *be as in Lemma* 1.7. *Then* $\mathbf{V}$ *is isomorphic to a subsystem of an extension of a system of type* III^m *by a rank one torsion-free system.*

Proof. Let $m=2(m_1+m_2+\cdots+m_r)$. Then there is an embedding $(\varphi,\psi):\mathbf{X}\to\mathbf{X}^1$, where $\mathbf{X}^1$ is of type III^m. The pushout of

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{X} & \longrightarrow & \mathbf{V} & \longrightarrow & \mathbf{V}^2 & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & \mathbf{X}^1 & & & & & & \end{array}$$

gives the required system of rank two with $\mathbf{V}$ a subsystem. □

The system $\mathscr{R}$ has the property that for any $0\neq e$ in K^2, $eV=W$, i.e. $\mathscr{R}$ is a divisible system as defined in [1]. In [8] an S-module Y is called divisible if and only if $\mathrm{Ext}(M,Y)=0$ for all simple regular modules M. The simple regular K^2-systems are systems of type II^1_θ, $\theta\in\tilde{K}$. As an aside we prove the equivalence of the two definitions for systems.

Proposition 1.9. *A* K^2*-system* $\mathbf{V}$ *is divisible if and only if* $\mathrm{Ext}(\mathrm{II}^1_\theta,\mathbf{V})=0$ *for all* θ *in* $\tilde{K}$.

Proof. Suppose $\mathbf{V}$ is divisible, i.e. $eV=W$ for all $0\neq e$ in K^2. Then by Lemma 7.3 of [1], $\mathrm{Ext}(\mathrm{II}^1_\theta,\mathbf{V})=0$ for all $\theta\in\tilde{K}$.

Conversely, suppose $\mathrm{Ext}(\mathrm{II}^1_\theta,\mathbf{V})=0$ for all $\theta\in\tilde{K}$. We want to show that $eV=W$ for all $0\neq e$ in K^2. Let $t(\mathbf{V})$ denote the torsion part of $\mathbf{V}$. From the exact sequence

$$0\to t(\mathbf{V})\to\mathbf{V}\to\mathbf{V}/t(\mathbf{V})\to 0$$

we get the exact sequence

$$\mathrm{Hom}(\mathrm{II}_\theta^1, \mathbf{V}/t(\mathbf{V})) \to \mathrm{Ext}(\mathrm{II}_\theta^1, t(\mathbf{V})) \to \mathrm{Ext}(\mathrm{II}_\theta^1, \mathbf{V}) \to \mathrm{Ext}(\mathrm{II}_\theta^1, \mathbf{V}/t(\mathbf{V})) \to 0.$$

$\mathrm{Hom}(\mathrm{II}_\theta^1, \mathbf{V}/t(\mathbf{V}))=0$, because $V/t(V)$ is torsion-free and II_θ^1 is torsion, while $\mathrm{Ext}(\mathrm{II}_\theta^1, \mathbf{V})=0$ by hypothesis. Therefore, $\mathrm{Ext}(\mathrm{II}_\theta^1, t(V))=0$. By Corollary 9.16 of [1],

$$t(\mathbf{V}) = \mathrm{Div}(t(\mathbf{V})) \dotplus \mathrm{Red}(t(\mathbf{V}))$$

where $\mathrm{Div}(t(\mathbf{V}))$ is a divisible subsystem of $t(\mathbf{V})$ and $\mathrm{Red}(t(\mathbf{V}))$ has no divisible subsystem. Moreover, $\mathrm{Red}(t(\mathbf{V}))$ has a primary decomposition

$$\mathrm{Red}(t(\mathbf{V})) = \sum_{\theta\in\tilde{K}}{}^{\cdot}\, \mathrm{Red}(t(\mathbf{V}))_\theta.$$

Suppose $\mathrm{Red}(t(\mathbf{V}))_\theta \neq 0$ for some θ. Then it contains a direct summand of type II_θ^n for some n, by Lemma 9.4 of [1]. Since $\mathrm{Ext}(\mathrm{II}_\theta^1, \mathrm{II}_\theta^n)\neq 0$ by the table in [4], this contradicts $\mathrm{Ext}(\mathrm{II}_\theta^1, t(\mathbf{V}))=0$. Hence $\mathrm{Red}(t(\mathbf{V}))_\theta=0$ for any θ in $\tilde{K}$, i.e. $t(\mathbf{V})$ is divisible. Since $t(\mathbf{V})$ is pure in $\mathbf{V}$, by Proposition 9.12 of [1], we have

$$\mathbf{V} = t(\mathbf{V}) \oplus \mathbf{V}/t(\mathbf{V})$$

by Theorem 9.15 of [1].

Since $t(\mathbf{V})$ is divisible, we may now suppose that $\mathbf{V}$ is torsion-free, reduced, i.e. has non nonzero divisible subsystem. We shall now show that if $\mathrm{Ext}(\mathrm{II}_\theta^1, \mathbf{V})=0$ for all $\theta\in\tilde{K}$ then $\mathbf{V}=0$. Suppose $\mathbf{V}\neq 0$. Let $\mathbf{V}^1$ be a torsion-closed subsystem of $\mathbf{V}$ of rank 1 (e.g. $\mathbf{V}^1$ = torsion-closure in $\mathbf{V}$ of $(0,\{\omega\})$, ω a nonzero element of W.) The exact sequence

$$0 \to \mathbf{V}^1 \to \mathbf{V} \to \mathbf{V}/\mathbf{V}^1 \to 0$$

gives the exact sequence

$$\mathrm{Hom}(\mathrm{II}_\theta^1, \mathbf{V}/\mathbf{V}^1) \to \mathrm{Ext}(\mathrm{II}_\theta^1, \mathbf{V}^1) \to \mathrm{Ext}(\mathrm{II}_\theta^1, \mathbf{V}).$$

Just as $\mathrm{Ext}(\mathrm{II}_\theta^1, t(\mathbf{V}))$, this implies that $\mathrm{Ext}(\mathrm{II}_\theta^1, \mathbf{V}^1)=0$. So, by the table in [4], any height function H representing $\mathbf{V}^1$ must have $H_\theta=\infty$. Since this is true for all θ in $\tilde{K}$, we conclude that $\mathbf{V}^1$ is isomorphic to $\mathscr{R}$, contradicting the hypothesis that $\mathbf{V}$ is reduced. Hence, $\mathbf{V}=0$. Since $\mathbf{V}=\mathrm{Div}(\mathbf{V})\oplus\mathrm{Red}(\mathbf{V})$, this proves that $\mathrm{Red}(\mathbf{V})=0$. □

2. Abundance

In the proof of Theorem 2.1 we borrow from the proof of Theorem 3.1 of [5] and Proposition 10 of [6]. P stands for $\{\theta\in\tilde{K}: H_\theta^2\neq 0\}$, see Lemma 1.6.

Theorem 2.1. *Let* $\mathbf{V}^2$ *be an infinite-dimensional torsion-free system of rank* 1 *with* $\mathrm{Card}(P)<\mathrm{Card}(K)$. *Then* $\mathbf{V}^2$ *is a quotient of a purely simple system of rank n, n any positive integer.*

Proof. *Let* $V=V^2\subset K(\xi)$ and $W=V\oplus[\omega_2]\oplus\cdots\oplus[\omega_n]$, where $\omega_i\neq 0$, for $i=2,3,\ldots,n$.

Given any set of $n-1$ linear functionals $\{l_2,\ldots,l_n\}$ we can make **V** a torsion-free system of rank n by setting

$$av=v,\qquad bv=\xi v+l_2(v)\omega_2+\cdots+l_n(v)\omega_n. \tag{2}$$

We shall now derive sufficient conditions on the linear functionals that ensure that **V** has no infinite-dimensional pure subsystem.

Suppose **X** is a rank 1 infinite-dimensional pure subsystem of **V**. Then **X** = torsion-closure of $(\emptyset, y)$ in Y, $0\neq y\in Y$. Therefore,

$$y=p(\xi)+\sum_{i=2}^{n}\beta_i\omega_i,\quad \beta_i\in K,\ p(\xi)\in V. \tag{3}$$

Since **X** is infinite-dimensional, either

$$H(y)_\nu=\infty \quad \text{for some } \nu \text{ in } \tilde{K} \tag{4}$$

or

$$H(y)_\theta\neq 0 \quad \text{on an infinite subset } L\subseteq K. \tag{5}$$

If (4) is satisfied, we take a partial fraction decomposition of $p(\xi)$, and apply the definition of the system operation in (2). We now proceed as on p. 177–8 of [5], with ν not possibly ∞, (see also Lemma 1 of [6],) to get that for the countable set $S=\{((\xi-\nu^{-k}: k=1,2,3,\ldots\}$ and some $i_0\in\{2,3,\ldots,n\}$.

$$\begin{array}{l}F(l_{i_0}(S)) \text{ is contained in a finitely generated}\\ \text{extension of } F(l_2(S),\ldots,\langle l_{i_0}(S)\rangle,\ldots,l_n(S)),\end{array} \tag{6}$$

where F is the prime subfield of K and $\langle l_{i_0}(S)\rangle$ means $l_{i_0}(S)$ is excluded.

If (5) is satisfied we use (G) of [6] to obtain that, for any countable infinite subset S of $\{(\xi-\theta)^{-1}: \theta\in L\}$,

$$\begin{array}{l}F(l_{i_0}(S)) \text{ is contained in a finitely generated}\\ \text{extension of } F(P_S, l_2(S),\ldots,\langle l_{i_0}(S)\rangle,\ldots,l_n(S)),\end{array} \tag{7}$$

where $P_S=\{\theta: (\xi-\theta)^{-1}\in S\}$.

So if $l_2,\ldots,l_n$ can be chosen such that (6) and (7) are impossible we would have proved that any **V** constructed as in (2) has no infinite-dimensional pure subsystem of rank 1. We now assume that **V** does not contain an infinite-dimensional pure subsystem of rank $\leq s\leq n$ and then show that it cannot contain an infinite-dimensional pure subsystem of rank $s+1<n$. Suppose **X** is such a subsystem of rank $s+1$. As in [5], p. 178, **X** is an extension of a finite-dimensional system $\mathbf{X}_0$ of rank s by an infinite-dimensional system of rank 1. Therefore, $\mathbf{X}/\mathbf{X}_0$ is a rank one infinite-dimensional torsion-free system. Let H be a representative height function of $\mathbf{X}/\mathbf{X}_0$. Let $P'=\{\theta\in K: H_\theta\neq 0\}$.

Even though one can deduce from $\mathrm{Card}(P)<\mathrm{Card}(K)$ that $\mathrm{Card}(P')<\mathrm{Card}(K)$, we sidetrack the issue by using the fact that $\mathbf{X}/\mathbf{X}_0$ contains an infinite-dimensional subsystem $\mathbf{V}^3$ of rank 1 with the property that $P'' = \{\theta \in K : H_\theta^3 \neq 0\}$, H^3 a representative height function of $\mathbf{V}^3$, has $\mathrm{Card}(P'') < \mathrm{Card}(K)$. From the pullback of

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{X}_0 & \longrightarrow & \mathbf{X} & \longrightarrow & \mathbf{X}/\mathbf{X}_0 & \longrightarrow & 0 \\ & & & & & & \uparrow & & \\ & & & & & & \mathbf{V}^3 & & \end{array}$$

we get that $\mathbf{X}$ contains a subsystem $\mathbf{X}^1$ which is an extension of $\mathbf{X}_0$ by $\mathbf{V}^3$. By Lemma 1.7, $\mathbf{X}^1$ is also an extension of a system $\mathbf{X}^0$ of type $s\mathrm{III}^1$ by a rank one torsion-free system.

Let

$$\mathbf{X}^0 = \left(0, \sum_{j=1}^{s} \left[\gamma_j p_j + \sum_{i=2}^{n} \chi_{ij}\omega_i\right]\right) \tag{8}$$

where γ_j, χ_{ij} are in K and $p_j \in K(\xi)$. Let $p + Y^0$ be a generator of $\mathbf{V}^3$. Since $\mathbf{X}^1/\mathbf{X}^0$ is infinite-dimensional, we conclude that either (4) or (5) is satisfied with $p + Y^0$ in place of y. $H(p+Y^0)_\theta \neq 0$ implies that $b_\theta(x + X^0) = p + Y^0$, for some x in X. So, $b_\theta x - p = y \in Y^0$. Hence,

$$b_\theta x - p = \beta_{1\theta} f_1 + \cdots + \beta_{s\theta} f_s \tag{9}$$

where

$$f_j = \gamma_j p_j + \sum_{i=2}^{n} \chi_{ij}\omega_i,$$

$\gamma_j, \chi_{ij} \in K$ and p_j rational functions, $j = 1, \ldots, s$.

After taking a partial fraction expansion of the rational functions in (9) we equate the coefficients of $\omega_i, i = 2, \ldots, n$. In this way we get $n-1$ equations involving $\beta_{1\theta}, \ldots, \beta_{s\theta}$. The coefficients in these equations are linear combinations of θ, coefficients of p and p_j, χ_{ij} and, of course, $\beta_{1\theta}, \ldots, \beta_{s\theta}$. By considering only the first s equations we can express $\beta_{1\theta}, \ldots, \beta_{s\theta}$ in terms of θ, χ_{ij}, the coefficients of p and $p_j, l_i(\xi-\theta)^{-1}$, $i = 2, \ldots, s$. If $(\xi-\theta)^{-1}$ occurs as a nonzero coefficient in (9) after the partial fraction development and application of (2) we get from the $(s+1)$-th equation that $l_{s+1}(\xi-\theta)^{-1}$ is contained in a finitely generated extension of F generated by θ, coefficients of p and p_j, χ_{ij} and $l_i(\xi-\theta)^{-1}$, $i = 2, \ldots, s$. If (9) is satisfied for infinitely many θ then the infinite set of x_θ's is linearly independent by Remark (a), p. 171 of [5]. If a subset of $K(\xi)$ has a bound on the orders of the pole of its elements at ∞ and has only finitely many poles then it generates a finite-dimensional subspace of $K(\xi)$, see p. 174 of [5] and the proof of Lemma 2 in [6], for instance. Since $A = \{p, f_1, \ldots, f_s\}$ is a finite set and division of linear combinations of elements in A by $\xi - \theta$ does not increase the order of the pole at ∞ we conclude that the satisfaction of (9) by infinitely many θ implies that $(\xi-\theta)^{-1}$ occurs with a nonzero coefficient in infinitely many cases. Therefore, condition (7) is satisfied with $i_0 = s+1$. On the other hand, if (4) is satisfied, we proceed in a similar manner to obtain that (6) is satisfied with $i_0 = s+1$.

Therefore, if we construct $l_2, \dots, l_n$ such that neither (6) nor (7) is satisfied we would have constructed a torsion-free K^2-system $\mathbf{V}$ of rank n which has no infinite-dimensional pure subsystem. By the argument on p. 181 of [5] such a system would be purely simple. The next step is the construction of such linear functionals.

Let $F(P)$ be the field extension of F obtained by adjoining elements of P. Since $\mathrm{Card}(P) < \mathrm{Card}(K)$, also $\mathrm{Card}(F(P)) < \mathrm{Card}(K)$. Suppose that P is infinite and $H_\theta^2 = \infty$ for all θ in P. Since K is algebraically closed, we can find a subset K_1 of K such that

$$K_1 = \dot{\bigcup_{i=2}^{n}} K_{1i},$$

$$\mathrm{Card}(K_{1i}) = \mathrm{Card}(P)$$

and (10)

K_{1i_0} is not contained in a finitely generated extension of $F(P, K_{12}, \dots, \langle K_{1i_0}\rangle, \dots, K_{1n})$ for any i_0 in $\{2, \dots, n\}$.

A one–one correspondence between P and K_{1i} enables us to define l_i on $\{(\xi-\theta)^{-1} : \theta \in P\}$.

It remains now to define the linear functionals on $A_\theta = \{(\xi-\theta)^{-k} : k = 2, \dots\}$ and on $B = (\xi^k : k = 0, 1, 2, \dots\}$. We can choose countable infinite subsets $\mathscr{A}_\theta$, $\mathscr{B}_\theta$ of K amenable to the same treatment as K_1 with $\mathrm{Card}(\mathscr{A}_{\theta i}) = \mathrm{Card}(\mathscr{B}_i) = \mathrm{Card}(A_\theta)$, i.e. property (10) is satisfied with $\mathscr{A}_{\theta i}$, $\mathscr{B}_i$ respectively replacing K_{1i}'s. l_i is now defined on A_θ and B via the respective one–one correspondences between A_θ and $\mathscr{A}_{\theta i}$ and between B and $\mathscr{B}_i$.

The property (10) and the definition of the linear functionals ensure that neither (6) nor (7) is satisfied. So, $\mathbf{V}$ constructed as in (2), using $l_2, \dots, l_n$, is purely simple. If $\mathbf{V}$ is a rank one torsion-free system with P finite or $H_\theta^2 < \infty$ for some θ in P, it may be embedded in another rank one torsion-free system $\mathbf{V}^3$ with representative height function H such that $P^3 = \{\theta \in K : H_\theta \neq 0\}$ is infinite, $\mathrm{Card}(P^3) < \mathrm{Card}(K)$ and $H_\theta = \infty$ for θ in P^3. The above argument shows that $\mathbf{V}^3$ is a quotient of a purely simple system of rank n. So is $\mathbf{V}^2$, by Proposition 1.5. □

Lemma 2.2. *If* $\mathbf{V}$ *is a purely simple system described by linear functionals* $l_2, \dots, l_n$ *as in* (2) *then the set* $\{l_2, \dots, l_n\}$ *is linearly independent. In fact if* $\alpha_2 l_2 + \alpha_3 l_3 + \dots + \alpha_n l_n$ *vanishes on an infinite subset of* $\{(\xi-\theta)^{-1} : \theta \in K\}$ *then* $\alpha_2 = \dots = \alpha_n = 0$.

Proof. Suppose $\alpha_2 l_2 + \dots + \alpha_n l_n \equiv 0$ on an infinite set $\{(\xi-\theta)^{-1} : \theta \in T_0\}$, $T_0 \subseteq K$ and $\alpha_i \neq 0$ for some i.

Let $S = \{(-\alpha_j/\alpha_i)\omega_i + \omega_j : j \neq i\} \cup \{1\}$. S has $n-1$ elements. The torsion-closure (X, Y) of $(\emptyset, S)$ in $\mathbf{V}$ is infinite-dimensional: for a fixed θ in T_0, let

$$l_i(\xi-\theta)^{-1} = \gamma_i; \qquad b_\theta(\xi-\theta)^{-1} = 1 + \gamma_2\omega_2 + \dots + \gamma_n\omega_n.$$

Since $\alpha_2\gamma_2 + \cdots + \alpha_n\gamma_n = 0$,

$$\gamma_i = \frac{-\alpha_2}{\alpha_i}\gamma_2 - \cdots - \frac{\alpha_n}{\alpha_i}\gamma_n.$$

Therefore,

$$\gamma_2\omega_2 + \cdots + \gamma_n\omega_n = \gamma_2\left(\frac{-\alpha_2}{\alpha_1}\omega_i + \omega_2\right) + \gamma_3\left(\frac{-\alpha_3}{\alpha_i}\omega_i + \omega_3\right) + \cdots + \gamma_n\left(\frac{-\alpha_n}{\alpha_i}\omega_i + \omega_n\right).$$

Therefore, $1 + \gamma_2\omega_2 + \cdots + \gamma_n\omega_n \in Y$ and $(\xi - \theta)^{-1} \in X$ for all θ in T_0. Hence (X, Y) is infinite-dimensional. Since rank $\mathbf{X} \le n - 1 <$ rank $\mathbf{V}$, a contradiction to Proposition 1.1 is obtained. A similar contradiction is obtained if $H_\theta^2 = \infty$. □

Proposition 2.3. *Every infinite-dimensional torsion-free system of rank one contains a purely simple subsystem of rank n for any positive integer n.*

Proof. Let $\mathbf{V}^2$ be a torsion-free system of rank one and $\mathbf{V}$ a purely simple system of rank n with $\mathbf{V}^2$ as quotient, described in (2).

By Lemma 2.2, $\{l_2, \ldots, l_n\}$ is a linearly independent set. Hence $X_{\mathcal{L}} = \bigcap_{i=2}^n \operatorname{Ker} l_i$ is a subspace of V^2 of codimension $n - 1$.

$(X_{\mathcal{L}}, W^2)$ is a system with system operation defined in (2). However, for $v \in X_{\mathcal{L}}$, $l_2(v) = \cdots = l_n(v) = 0$. Hence, the system operation in $(X_{\mathcal{L}}, W^2)$ is the same as in (V^2, W^2). Therefore $\mathbf{V}^2$ and $\mathbf{V}$ have $(X_{\mathcal{L}}, W^2)$ as a common subsystem. Since $\mathbf{V}$ is purely simple and $(X_{\mathcal{L}}, W^2)$ is infinite-dimensional, $\operatorname{rank}(X_{\mathcal{L}}, W^2) \ge n$ by Proposition 1.1. If $\operatorname{rank}(X_{\mathcal{L}}, W^2) = n$ it would be purely simple by Proposition 1.6 and we would be done. Suppose $\operatorname{rank}(X_{\mathcal{L}}, W^2) > n$.

Let $\{\omega_1, \omega_2, \ldots, \omega_n\}$ be part of a basis of $(X_{\mathcal{L}}, W^2)$ with respect to generation. Let (X^1, Y^1) be the torsion-closure in $(X_{\mathcal{L}}, W^2)$ of $(\emptyset, \{\omega_1, \omega_2, \ldots, \omega_n\})$. If $\mathbf{X}^1$ is infinite-dimensional it would be purely simple, by Proposition 1.6, and since its rank is n, we would be done.

If $V^2 = W^2$ then $X_{\mathcal{L}}$ is also of codimension $n - 1$ in W^2. Suppose $\mathbf{X}^1$ is finite-dimensional. Then it would be of type $\mathrm{III}^{m_1} \oplus \cdots \oplus \mathrm{III}^{m_n}$ and, hence, $\mathbf{X}^1$ would be of codimension n in Y^1. Since $X^1 \subset X_{\mathcal{L}}$ and $Y^1 \subset W^2$, this is a contradiction. Therefore, if $V^2 = W^2$ the proof is complete. Suppose $V^2 \ne W^2$. From the description of torsion-free rank one systems, e.g. Theorem 1.7 of [5], this is equivalent to the existence of a bound on the orders of the pole at ∞ of elements in W^2. In that case, V^2 is of codimension 1 in W^2, hence $X_{\mathcal{L}}$ is of codimension n in W^2. As above, we conclude that if $\mathbf{X}^1$ is finite-dimensional then X^1 is of codimension n in Y^1. Since $X_{\mathcal{L}} \subset W^2$ and $X^1 \subset Y^1$ and $\mathbf{X}^1$ is torsion-closed in $(X_{\mathcal{L}}, W^2)$ we have the following vector space decompositions:

$$X_{\mathcal{L}} = X^1 \oplus X^3, \qquad W^2 = Y^1 \oplus Y^3,$$

with $X^3 \subset Y^3$. Since X^1 is of codimension n in Y^1 and X_2 is of codimension n in W^2, we conclude that $X^3 = Y^3$. Therefore

$$H^{\mathbf{X}^3}(y)_\infty = \infty \tag{11}$$

for y any nonzero element of Y^3. Proceeding as on p. 179 of [5] and using Remark (a), p. 171 of [5], the fact that X^1 is finite-dimensional, we are led from (11) to the conclusion that there is no bound on the order of the pole at ∞ of elements in W^2. This is a contradiction. Hence, $\mathbf{X}^1$ is infinite-dimensional. This proves the proposition for rank 1 torsion-free systems satisfying the hypothesis of Theorem 2.1, i.e. $\text{Card}(P) < \text{Card}(K)$. If $\mathbf{V}^2$ is a rank one torsion-free system with $\text{Card}(P) = \text{Card}(K)$ then it contains a rank one torsion-free subsystem with the corresponding $\text{Card}(P) < \text{Card}(K)$. □

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