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Some infinite-dimensional simple Lie algebras in characteristic 0 related to those of Block

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Abstract

Given a nontrivial torsion-free abelian group $(A,+,0)$, a field F of characteristic 0, and a nondegenerate bi-additive skew-symmetric map $\varphi: A\times A \to F$, we study the Lie algebra $\mathscr{L}(A,\varphi)$ over F with basis $\{e_x\colon x\in A\backslash\{0\}\}$ and multiplication $[e_x,e_y]=\varphi(x,y)e_{x+y}$. We show that $\mathscr{L}(A,\varphi)$ is simple, determine its derivations, and show that the locally finite derivations D have the form $D(e_x)=\mu(x)e_x$, $\mu\in \mathrm{Hom}(A,F)$. We describe all isomorphisms between two such algebras. Finally, we compute $H^2(\mathscr{L},F)$. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let F be a field of characteristic 0 and A an abelian group. Let L be the vector space over F with basis consisting of all symbols $e_x, x\in A$. Define a bilinear multiplication in L by

$$(e_x,e_y)\to[e_x,e_y]:=f(x,y)e_{x+y},$$

where $x,y\in A$ are arbitrary and

$$f(x,y)=\varphi(x,y)+\alpha(x-y)$$

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for some skew-symmetric bi-additive function $\varphi : A \times A \to F$ and some additive function $\alpha : A \to F$. We denote the vector space L with this algebra structure by $L(A, \varphi, \alpha)$.

When $\varphi \neq 0$ and $\alpha \neq 0$, this algebra was studied by Block [2] and in our previous paper [4]. In that case L is a Lie algebra if and only if $\varphi = \alpha \wedge \beta$ for some additive function $\beta : A \to F$, i.e.,

$$\varphi(x, y) = \alpha(x)\beta(y) - \alpha(y)\beta(x).$$

Assume that $\ker(\alpha) \cap \ker(\beta) = 0$ and $A \neq 0$. In that case the Lie algebra $L = L(A, \varphi, \alpha)$ is close to being simple. More precisely, the derived algebra $L^2 = [L, L]$ is either equal to L or has codimension 1 in L, the center Z of L is either 0 or has dimension 1, $Z \subset L^2$, and the quotient algebra $\mathscr{L}(A, \varphi, \alpha) := L^2/Z$ is simple. The algebras $\mathscr{L}(A, \varphi, \alpha)$ are called generalized Block algebras. In [4] we have determined the derivation algebra of $\mathscr{L}(A, \varphi, \alpha)$, described its automorphism group and computed its second cohomology group with coefficients in F.

In the special case when $\varphi = \alpha \wedge \beta$ and $\beta(A) = \mathbf{Z}$ one can define a proper simple subalgebra of $\mathscr{L}(A, \varphi, \alpha)$. These subalgebras were studied in detail in our paper [5].

If $\varphi = 0$ and $\alpha \neq 0$, then L is automatically a Lie algebra. In fact it is a special case of so called generalized Witt algebras. In this case L is simple if and only if α is injective. For the properties of generalized Witt algebras (in characteristic 0), we refer the reader to our paper [6].

In the present paper we study the remaining case where $\varphi \neq 0$ and $\alpha = 0$. Again $L(A, \varphi, 0)$ is a Lie algebra, and we simplify the notation by writing just $L(A, \varphi)$ instead of $L(A, \varphi, 0)$. Hence, we have

$$[e_x, e_y] = \varphi(x, y)e_{x+y} \tag{1.1}$$

for all $x, y \in A$.

Let K_φ be the kernel of φ, i.e., K_φ is the subgroup of A consisting of all $x \in A$ such that $\varphi(x, y) = 0$ for all $y \in A$. The subspace $Z \subset L$ spanned by all e_x with $x \in K_\varphi$ is the center of $L = L(A, \varphi)$. Let $\bar{A} = A/K_\varphi$ and let $\bar{\varphi} : \bar{A} \times \bar{A} \to F$ be the (skew-symmetric) bilinear map induced by φ. It is easy to check that

$$L(A, \varphi)/Z \simeq L(\bar{A}, \bar{\varphi})/Fe_{\bar{0}},$$

where $\bar{0} = 0 + Z \in \bar{A}$ and $Fe_{\bar{0}}$ is the center of $L(\bar{A}, \bar{\varphi})$. Since we are interested only in studying the quotient algebra $L(A, \varphi)/Z$, the above isomorphism shows that, without any loss of generality, it suffices to consider the case where $K_\varphi = 0$.

Hence, we assume from now on that φ is non-degenerate (i.e., $K_\varphi = 0$). Since F has characteristic 0, this assumption implies that A is torsion-free. To avoid the trivial case, we assume also that $A \neq 0$. The condition $K_\varphi \neq 0$ implies that the rank of A is at least 2.

The one-dimensional subspace Fe_0 is the center of $L(A, \varphi)$. The subspace

$$\mathscr{L}(A, \varphi) = \sum_{x \in A \setminus \{0\}} Fe_x$$

is an ideal of $L(A,\varphi)$ and we have

$$L(A,\varphi) = Fe_0 \oplus \mathscr{L}(A,\varphi).$$

In Section 2 we show that the Lie algebra $\mathscr{L}(A,\varphi)$ is simple. In particular, it follows that $\mathscr{L}(A,\varphi)$ is the derived algebra of $L(A,\varphi)$. We mention that the finite-dimensional version of the simple Lie algebra $\mathscr{L}(A,\varphi)$, but now over a field of prime characteristic, has been introduced long ago by Albert and Frank in their paper [1]. The algebras $L(\mathbf{Z}^n,\varphi)$ in characteristic 0 were studied by Koepp in his Ph.D. thesis [7]. He showed that $\mathscr{L}(\mathbf{Z}^n,\varphi)$ is simple under an additional condition on φ. It follows from our simplicity theorem (Theorem 2.1) that the additional condition used by Koepp is not needed.

Note that $\mathscr{L}(A,\varphi)$ and $L(A,\varphi)$ are A-graded Lie algebras: the homogeneous component of $L(A,\varphi)$ of degree x is Fe_x. In Section 3 we describe the derivations of $\mathscr{L}(A,\varphi)$. In particular, we show that the derivations of degree $x \neq 0$ are inner, and that the derivations of degree 0 have the form $e_x \mapsto \mu(x)e_x$ where $\mu \in \mathrm{Hom}(A,F)$. The main result of that section is that the locally finite derivations of $\mathscr{L}(A,\varphi)$, $\mathrm{rank}(A) < \infty$, are precisely the derivations of degree 0.

In Section 4 we describe all isomorphisms between two simple algebras $\mathscr{L}(A,\varphi)$ and $\mathscr{L}(B,\psi)$ when A and B have finite ranks. As a consequence we obtain a description of the automorphism group of $\mathscr{L}(A,\varphi)$ when A has finite rank.

Finally in Section 5 we compute the second cohomology group $H^2(\mathscr{L},F)$ for the simple Lie algebra $\mathscr{L} = \mathscr{L}(A,\varphi)$.

More general Lie algebras (in characteristic 0) than the algebras studied in the present paper and [4] can be constructed by analogy with Block algebras in characteristic p described in [3].

2. Simplicity of $\mathscr{L}(A,\varphi)$

As mentioned before, we assume that A is a nonzero torsion-free abelian group and $\varphi: A \times A \to F$ is a nondegenerate skew-symmetric bi-additive map.

Theorem 2.1. *The Lie algebra $\mathscr{L}(A,\varphi)$ is simple.*

Proof. Let I be a nonzero ideal of $\mathscr{L} = \mathscr{L}(A,\varphi)$. Let

$$u = a_1 e_{x_1} + \cdots + a_n e_{x_n}$$

be a nonzero element of I, where $x_1,\ldots,x_n \neq 0$ and $a_1,\ldots,a_n \in F$, and assume that u is chosen so that n is minimal. It follows that the x_i's are distinct and the a_i's are all nonzero.

Assume that $n > 1$. Let $y \in A$ be arbitrary and let $v = [u, e_y]$. Thus,

$$v = \varphi(x_1,y)e_{x_1+y} + \cdots + \varphi(x_n,y)e_{x_n+y} \in I. \tag{2.1}$$

We claim that

$$\varphi(x_1 - x_2, y) = 0. \tag{2.2}$$

If $\varphi(x_1, y) = 0$, then (2.1) and the minimality of n imply that also $\varphi(x_2, y) = 0$, and so (2.2) holds. In particular, by taking $y = x_1$, we conclude that $\varphi(x_1, x_2) = 0$.

If $\varphi(x_1, y) \neq 0$, then $v \neq 0$ and the minimality of n implies that $\varphi(x_i, y) \neq 0$ for all i's. By replacing u with v, we conclude that $\varphi(x_1 + y, x_2 + y) = 0$. Since also $\varphi(x_1, x_2) = 0$ and φ is skew-symmetric and bi-additive, we conclude that (2.2) holds.

Since φ is nondegenerate and (2.2) holds for all $y \in A$, we conclude that $x_1 = x_2$, a contradiction. Hence $n = 1$, i.e., $e_{x_1} \in I$.

We claim that $e_y \in I$ for all $y \neq 0$. If $\varphi(y, x_1) \neq 0$, then $y - x_1 \neq 0$ and the claim follows from

$$[e_{y-x_1}, e_{x_1}] = \varphi(y, x_1) e_y \in I.$$

Assume now that $\varphi(y, x_1) = 0$, $y \neq 0, x_1$. Choose $z \in A$ such that $\varphi(z, x_1) \neq 0$ and $\varphi(y, z) \neq 0$. Since $\varphi(z, x_1) \neq 0$, we infer that $e_z \in I$. As $y \neq z$ and $[e_{y-z}, e_z] = \varphi(y, z) e_y \in I$, we conclude again that $e_y \in I$. Thus our claim is proved.

So, we have $I = \mathscr{L}$, and $\mathscr{L}$ is simple. □

In the case $A = \mathbf{Z}^n$, $n \geq 2$, the above theorem was proved by Koepp in his thesis [7], under the additional hypothesis:

(H) If $x_1, \ldots, x_k \in A$ are independent and $1 \leq k < n$, then there exists $y \in A$ such that $x_1, \ldots, x_k, y$ are also independent and $\varphi(x_i, y) \neq 0$ for some $i \in \{1, \ldots, k\}$.

Since φ is assumed to be nondegenerate, the hypothesis (H) is automatically satisfied. Indeed, let $x_1, \ldots, x_k \in A$ be independent and $1 \leq k < n$. Assume that $\varphi(x_i, y) = 0$ for all $i = 1, \ldots, k$ whenever y is chosen so that $x_1, \ldots, x_k, y$ are independent. Now assume that $x_1, \ldots, x_k, y$ are dependent and choose $z \in A$, such that $x_1, \ldots, x_k, z$ are independent. Then $\varphi(x_i, z) = 0$ and $\varphi(x_i, y + z) = 0$ for all i. We conclude that $\varphi(x_i, y) = 0$ for all $i = 1, \ldots, k$ and all $y \in A$. This means that $x_1, \ldots, x_k \in K_\varphi$, which contradicts the nondegeneracy of φ.

We conclude this section with an example of a simple Lie algebra $\mathscr{L}(\mathbf{Z}^3, \varphi)$.

Example 1. Let $A = \mathbf{Z}^n$, $n \geq 2$. A bi-additive skew-symmetric map $\varphi : A \times A \to F$ is given by a skew-symmetric n by n matrix over F, say the matrix M. Then φ is nondegenerate (in our sense) if and only if

$$Mv = 0 \Rightarrow v = 0$$

for all $v \in \mathbf{Z}^n$. Hence, φ can be nondegenerate even if $\det(M) = 0$.

For instance, if $n = 3$ and

$$M = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

with $a, b, c \in F$ linearly independent over $\mathbf{Q}$, then φ is nondegenerate. In that case the Lie algebra

$$\mathscr{L}(a,b,c) := \mathscr{L}(\mathbf{Z}^3, \varphi)$$

is simple.

3. Derivations of $\mathscr{L}(A, \varphi)$

Let D be a derivation of $\mathscr{L} = \mathscr{L}(A, \varphi)$. We extend D to a derivation of $L = L(A, \varphi)$, and denote the extension again by D, by setting $D(e_0) = 0$. For arbitrary $y \in A$ we have

$$D(e_y) = \sum_{x \in A} c(x, y) e_{x+y} \tag{3.1}$$

for some scalars $c(x, y) \in F$. The scalars $c(x, y)$ satisfy the following condition:

(F) for each $y \in A$ there are only finitely many $x \in A$ such that $c(x, y) \neq 0$.

For each $x \in A$ we define the linear map $D_x : L \to L$ by

$$D_x(e_y) = c(x, y) e_{x+y}, \quad y \in A. \tag{3.2}$$

It is easy to verify that each D_x is a derivation of L. Furthermore,

$$D = \sum_{x \in A} D_x \tag{3.3}$$

in the sense that for each $y \in A$ only finitely many terms $D_x(e_y)$ are nonzero and

$$D(e_y) = \sum_{x \in A} D_x(e_y).$$

Since $D(e_0) = 0$, we have

$$c(x, 0) = 0, \quad \forall x \in A. \tag{3.4}$$

Since $D(\mathscr{L}) \subset \mathscr{L}$, we also have

$$c(x, -x) = 0, \quad \forall x \in A. \tag{3.5}$$

Lemma 3.1. *If $x \neq 0$, then D_x is an inner derivation, i.e., $D_x = \lambda \operatorname{ad}(e_x)$ for some $\lambda \in F$.*

Proof. As x is fixed, we shall write c_y instead of $c(x, y)$. By applying D_x to $[e_y, e_z] = \varphi(y, z) e_{y+z}$, we obtain

$$c_{y+z} \varphi(y, z) = c_y \varphi(x + y, z) + c_z \varphi(y, x + z). \tag{3.6}$$

By replacing z with ky, $k \in \mathbf{Z}$, we obtain

$$\varphi(x, y) \cdot [c_{ky} - k c_y] = 0.$$

Hence, if $\varphi(x,y) \neq 0$, then

$$c_{ky} = kc_y, \quad k \in \mathbf{Z}. \tag{3.7}$$

We now choose $y, z \in A$ such that $\varphi(x,y)$, $\varphi(x,z)$, and $\varphi(x,y+z)$ are all nonzero. By replacing y with ky and z with kz in (3.6), we obtain that

$$k^3\varphi(y,z)\cdot[c_{y+z} - c_y - c_z] = k^2[\varphi(x,z)c_y - \varphi(x,y)c_z] \tag{3.8}$$

holds for all integers k. We deduce that

$$\frac{c_y}{\varphi(x,y)} = \frac{c_z}{\varphi(x,z)} \tag{3.9}$$

holds. We claim that (3.9) remains valid when we remove the restriction $\varphi(x,y+z) \neq 0$.

Thus assume that $\varphi(x,y+z) = 0$. We can choose $u \in A$ such that the numbers $\varphi(x,u)$, $\varphi(x,y+u)$, and $\varphi(x,z+u)$ are nonzero. Consequently, we have

$$\frac{c_y}{\varphi(x,y)} = \frac{c_u}{\varphi(x,u)} = \frac{c_z}{\varphi(x,z)},$$

and so (3.9) holds.

Let λ be the common value of all numbers $c_y\varphi(x,y)^{-1}$ with $\varphi(x,y)\neq 0$. Let $D' = D_x - \lambda\,\mathrm{ad}(e_x)$. For $y \in A$ such that $\varphi(x,y) \neq 0$ we have

$$D'(e_y) = D_x(e_y) - \lambda[e_x, e_y] = [c_y - \lambda\varphi(x,y)]e_{x+y} = 0.$$

The elements e_y such that $\varphi(x,y) \neq 0$ generate $\mathscr{L}$ as a Lie algebra, and so $D' = 0$, i.e., $D_x = \lambda\,\mathrm{ad}(e_x)$. □

In the next lemma we determine the derivation D_0. By (3.2) we have

$$D_0(e_x) = \mu(x)e_x, \quad x \in A,$$

where $\mu(x) = c(0,x)$.

Lemma 3.2. *The map $\mu : A \to F$ is additive.*

Proof. We have to show that

$$\mu(x+y) = \mu(x) + \mu(y) \tag{3.10}$$

holds for all $x, y \in A$. If $\varphi(x,y) \neq 0$, this follows by applying D_0 to (1.1). Since $\mu(0) = c(0,0) = 0$ by (3.4), it follows that (3.10) also holds if $x = 0$ or $y = 0$.

Now let $y = -x \neq 0$. Choose $z \in A$ such that $\varphi(x,z) \neq 0$. Then we have

$$\mu(z) = \mu(z-x) + \mu(x) = \mu(z) + \mu(-x) + \mu(x).$$

Hence, $\mu(x) + \mu(-x) = 0$, i.e., (3.10) holds also when $x + y = 0$.

Finally, let $x,\ y,\ x+y \neq 0$ and $\varphi(x,y)=0$. We choose $z \in A$ such that $\varphi(x,z), \varphi(y,z)$, and $\varphi(x+y,z)$ are all nonzero. It follows that also $\varphi(x+z,y-z) \neq 0$. Hence, we can apply (3.10) to each of the pairs $(x+z,y-z),(x,z)$, and $(y,-z)$. So, we obtain that

$$\mu(x+y)=\mu(x+z)+\mu(y-z)=\mu(x)+\mu(y)+\mu(z)+\mu(-z).$$

Since $\mu(z)+\mu(-z)=0$, (3.10) is proved. □

Let $\eta : A \to \mathrm{Hom}(A,F)$ be the map such that $\eta(x)(y)=\varphi(x,y)$ for all $x,y \in A$. Since φ is non-degenerate, the homomorphism η is injective. We denote by $\langle \eta(A) \rangle$ the F-subspace of $\mathrm{Hom}(A,F)$ spanned by the subgroup $\eta(A)$.

Lemma 3.3. *If* $\dim_F \langle \eta(A) \rangle = n < \infty$, *then* $D' := D - D_0$ *is an inner derivation.*

Proof. By (3.3) and Lemma 3.1 we have

$$D' = \sum_{x \neq 0} \lambda_x \ \mathrm{ad}\,(e_x)$$

for some $\lambda_x \in F$. Let $B \subset A$ consist of all $x \neq 0$ such that $\lambda_x \neq 0$.

Choose $a_1,\ldots,a_n \in A$ such that their images under η form a basis of $\langle \eta(A) \rangle$ over F. Let B_i consist of all $x \in B$ such that $\varphi(x,a_i) \neq 0$. Since $c(x,a_i) = \lambda_x \varphi(x,a_i)$, the finiteness condition (F) implies that B_i is a finite set.

Assume that there exists an $x \in B$ such that $x \notin B_i$ for all $i=1,\ldots,n$. Thus, $\varphi(x,a_i)=0$ for all i's. For arbitrary $y \in A$ there exist $t_1,\ldots,t_n \in F$ such that

$$\eta(y) = t_1\eta(a_1)+\cdots+t_n\eta(a_n).$$

It follows that

$$\varphi(y,x)=\sum_{i=1}^{n} t_i\eta(a_i)(x)=\sum_{i=1}^{n} t_i\varphi(a_i,x)=0$$

for all $y \in A$. As φ is non-degenerate, we conclude that $x=0$. As $x \in B$, we have a contradiction.

Hence, we have shown that B is the union of the B_i's, and so B is a finite set. Consequently, D' is an inner derivation. □

Proposition 3.4. *Suppose that* $\mathrm{rank}\,(A) < \infty$. *If D is a locally finite derivation of* $\mathscr{L}$, *then there exists* $\mu \in \mathrm{Hom}(A,F)$ *such that* $D(e_x)=\mu(x)e_x$ *for all* x.

Proof. By (3.3) and Lemma 3.1, we have

$$D = D_0 + \sum_{x \neq 0} \lambda_x \mathrm{ad}(e_x)$$

for some scalars $\lambda_x \in F$. By Lemma 3.3, the set $B=\{x \in A \setminus \{0\} : \lambda_x \neq 0\}$ is finite. Assume that B is not empty. We can choose a total ordering "$\geq$" on A, compatible with its group structure, and such that the maximal element u of B is > 0. Choose $z \in A$ such that $\varphi(u,z) \neq 0$. By induction on $k \geq 1$, it is easy to show that

$$D^k(e_z) = \lambda_u^k \varphi(u,z)^k e_{z+ku} + v_k,$$

where v_k is a linear combination of e_x's with $x < z+ku$. It follows that D is not locally finite.

Hence, if D is locally finite, then $B = \emptyset$ and so $D = D_0$. It remains to apply Lemma 3.2. □

We do not know whether or not the restriction on the rank of A can be removed from the above proposition.

Corollary 3.5. *A simple Lie algebra $\mathscr{L}(A,\varphi)$ (with no restriction on the rank of A) is not isomorphic to any generalized Block algebra or simple generalized Witt algebra.*

Proof. It follows from the proof of Proposition 3.4 that $\mathscr{L}(A,\varphi)$ has no ad-semisimple elements except 0. On the other hand, all generalized Block algebras and simple generalized Witt algebras have non-trivial tori. □

4. The isomorphism theorem

We shall determine all isomorphisms

$$\theta : \mathscr{L}(A,\varphi) \to \mathscr{L}(B,\psi) \tag{4.1}$$

between two simple algebras $\mathscr{L}(A,\varphi)$ and $\mathscr{L}(B,\psi)$, assuming that A and B have finite ranks. Clearly, θ extends to an isomorphism, again denoted by θ, of the Lie algebras $L(A,\varphi)$ and $L(B,\psi)$ by defining $\theta(e_0) = e_0$.

Theorem 4.1. *The Lie algebra isomorphisms* (4.1) *are precisely the linear maps θ such that*

$$\theta(e_x) = a\chi(x)e_{\sigma(x)}, \quad \forall x \in A\backslash\{0\}, \tag{4.2}$$

where $\chi \in \mathrm{Hom}(A,F^*), \sigma : A \to B$ is an isomorphism, and the constant $a \in F^*$ satisfies

$$\varphi(x,y) = a\psi(\sigma(x),\sigma(y)), \quad \forall x,y \in A. \tag{4.3}$$

Proof. Assume that the map (4.1) is an isomorphism of Lie algebras. For every $\mu \in \mathrm{Hom}(A,F)$, the linear map $D_\mu : \mathscr{L}(A,\varphi) \to \mathscr{L}(A,\varphi)$ defined by

$$D_\mu(e_x) = \mu(x)e_x, \quad x \in A\backslash\{0\},$$

is a derivation of degree 0 (with respect to the A-gradation of $\mathscr{L}(A,\varphi)$).

By Proposition 3.1 we know that the derivations D_μ are exactly the locally finite derivations of $\mathscr{L}(A,\varphi)$. Furthermore, the vectors e_x, $x \in A$, are the only common eigenvectors (up to scalar multiple) of all D_μ's. Analogous statements are of course valid for $\mathscr{L}(B,\psi)$. Consequently, there is a bijection $\sigma : A \to B$ such that

$$\theta(e_x) = c_x e_{\sigma(x)}, \quad x \in A$$

for some scalars $c_x \in F^*$. Clearly, $\sigma(0) = 0$.

By applying θ to (1.1) we obtain

$$c_{x+y}\varphi(x,y)e_{\sigma(x+y)} = c_x c_y \psi(\sigma(x),\sigma(y))e_{\sigma(x)+\sigma(y)}. \tag{4.4}$$

If $\varphi(x,y) \neq 0$, we derive that

$$\sigma(x+y) = \sigma(x) + \sigma(y). \tag{4.5}$$

Let $x \neq 0$ and choose $y \in A$ such that $\varphi(x,y) \neq 0$. By (4.5) we have

$$\sigma(y) = \sigma(x) + \sigma(y-x) = \sigma(x) + \sigma(y) + \sigma(-x).$$

Consequently, (4.5) also holds for $y = -x$.

Obviously, (4.5) holds if $x = 0$ or $y = 0$. Assume now that $x \neq 0, y \neq 0$, while $\varphi(x,y) = 0$. We choose $z \in A$ such that the numbers $\varphi(x,z), \varphi(y,z)$, and $\varphi(x+y,z)$ are all nonzero. Then we can apply (4.5) to each of the pairs $(x-z, y+z), (x,-z)$, and (y,z). So, we obtain that

$$\sigma(x+y) = \sigma(x-z) + \sigma(y+z) = \sigma(x) + \sigma(-z) + \sigma(y) + \sigma(z).$$

As $\sigma(z) + \sigma(-z) = 0$, we infer that (4.5) holds also for the pair (x,y).

Hence we have shown that $\sigma : A \to B$ is a homomorphism, and consequently an isomorphism.

Eq. (4.4) now implies that

$$c_{x+y}\varphi(x,y) = c_x c_y \psi(\sigma(x),\sigma(y)) \tag{4.6}$$

holds for all $x, y \in A$.

We claim that the ratio

$$\lambda = \frac{\psi(\sigma(x),\sigma(y))}{\varphi(x,y)} \tag{4.7}$$

is independent of x and y. Of course, we have to assume that $\varphi(x,y) \neq 0$, and so, by (4.6), also $\psi(\sigma(x),\sigma(y)) \neq 0$.

By replacing x with $2x$ in (4.6) we obtain that

$$c_{2x+y}\varphi(x,y) = c_{2x}c_y\psi(\sigma(x),\sigma(y)).$$

By replacing y with $x+y$ in (4.6), we obtain that

$$c_{2x+y}\varphi(x,y)^2 = c_x^2 c_y \psi(\sigma(x),\sigma(y))^2.$$

The above two equations imply that $\lambda = c_{2x}c_x^{-2}$. Since the expression (4.7) is symmetric in x and y, we also have $\lambda = c_{2y}c_y^{-2}$. Hence, we have shown that

$$c_{2x}c_x^{-2} = c_{2y}c_y^{-2} \tag{4.8}$$

if $\varphi(x,y) \neq 0$. The restriction $\varphi(x,y) \neq 0$ can easily be removed, i.e., (4.8) holds for all nonzero x and y. In particular, our claim is proved.

If $a = \lambda^{-1}$, then (4.6) shows that

$$a \cdot c_{x+y} = c_x c_y \tag{4.9}$$

holds whenever $\varphi(x,y) \neq 0$.

Suppose that $x, y, x+y \neq 0$ while $\varphi(x,y) = 0$. Choose $z \in A$ such that the numbers $\varphi(x,z), \varphi(y,z)$, and $\varphi(x-y,z)$ are all nonzero. We can apply (4.9) to each of the pairs $(x+z,-z), (x,z), (y,-z)$, and $(x+z, x-z)$.

Hence, we have

$$a^2 c_x = a c_{x+z} c_{-z} = c_x c_z c_{-z}$$

and

$$a^3 c_{x+y} = a^2 c_{x+z} c_{y-z} = c_x c_z c_y c_{-z}.$$

Consequently, (4.9) holds whenever $x, y, x+y \neq 0$.

If we define $\chi : A \to F^*$ by $\chi(0) = 1$ and $\chi(x) = \lambda c_x$ for $x \neq 0$, then (4.9) implies that χ is a homomorphism. Furthermore, (4.2) and (4.3) hold.

The converse is straightforward. □

We now apply Theorem 4.1 to obtain a description of the automorphism group of $\mathscr{L} = \mathscr{L}(A,\varphi)$, assuming that A has finite rank. Every character $\chi \in \mathrm{Hom}(A, F^*) = X(A)$ determines an automorphism θ_χ of $\mathscr{L}$ by

$$\theta_\chi(e_x) = \chi(x) e_x, \quad x \neq 0.$$

The map $\chi \mapsto \theta_\chi$ is an injective homomorphism $X(A) \to \mathrm{Aut}(\mathscr{L})$ and we shall identify the character group $X(A)$ of A with its image in $\mathrm{Aut}(\mathscr{L})$.

Let $\mathscr{A} = \mathscr{A}(\mathscr{L})$ be the subgroup of $\mathrm{Aut}(A)$ consisting of all automorphisms σ of A for which there is a constant $a_\sigma \in F^*$ such that

$$\varphi(\sigma(x), \sigma(y)) = a_\sigma \varphi(x,y), \quad \forall x, y \in A. \tag{4.10}$$

Clearly, such constant a_σ is unique.

Each $\sigma \in \mathscr{A}$ determines an automorphism θ_σ of $\mathscr{L}$ by

$$\theta_\sigma(e_x) = a_\sigma^{-1} e_{\sigma(x)}, \quad x \neq 0.$$

The homomorphism sending $\sigma \mapsto \theta_\sigma$ is injective and we identify $\mathscr{A}$ with its image in $\mathrm{Aut}(\mathscr{L})$.

The following corollary follows immediately from Theorem 4.1.

Corollary 4.2. *If* $\mathscr{L} = \mathscr{L}(A,\varphi)$ *is simple and* $\mathrm{rank}\,(A) < \infty$, *then*

$$\mathrm{Aut}(\mathscr{L}) = X(A) \rtimes \mathscr{A}(\mathscr{L})$$

(*semidirect product, with* $X(A)$ *normal in* $\mathrm{Aut}(\mathscr{L})$).

Example 2. Let $A = \mathbf{Z}^2$ and let $e_1 = (1,0)$ and $e_2 = (0,1)$ be the standard basis vectors. A bi-additive skew-symmetric map $\varphi : A \times A \to F$ is uniquely determined by the scalar $\alpha = \varphi(e_1, e_2) \in F$. We shall write φ_α for this φ. Clearly, φ_α is nondegenerate if and only if $\alpha \neq 0$. We set

$$\mathscr{L}_\alpha := \mathscr{L}(\mathbf{Z}^2, \varphi_\alpha), \quad \alpha \neq 0.$$

If $\alpha\beta \neq 0$, then the linear map $\theta : \mathscr{L}_\alpha \to \mathscr{L}_\beta$ defined by $\theta(e_x) = \alpha\beta^{-1}e_x, x \in \mathbf{Z}^2 \setminus \{0\}$ is an isomorphism of Lie algebras. Hence, in the case $A = \mathbf{Z}^2$, there is only one (up to isomorphism) simple Lie algebra $\mathscr{L}(\mathbf{Z}^2, \varphi)$.

Assume now that $\varphi = \varphi_1$, i.e., $\varphi(e_1, e_2) = 1$. We claim that $\mathscr{A}(\mathscr{L}) = \mathrm{GL}_2(\mathbf{Z})$ holds in this case. A simple computation shows that if

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}), \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then $\sigma' J \sigma = \det(\sigma) J$, where σ' is the transpose of σ. Hence, (4.10) holds with $a_\sigma = \det(\sigma) = \pm 1$. This proves our claim.

Consequently, $\mathrm{Aut}(\mathscr{L}) \simeq X(\mathbf{Z}^2) \rtimes \mathrm{GL}_2(\mathbf{Z})$.

5. Computation of $H^2(\mathscr{L}, F)$

In this section we compute the second cohomology group $H^2(\mathscr{L}, F)$ of the simple Lie algebra $\mathscr{L} = \mathscr{L}(A, \varphi)$.

Let $\psi : \mathscr{L} \times \mathscr{L} \to F$ be an arbitrary 2-cocycle, i.e., a skew-symmetric bilinear form satisfying the identity

$$\psi([u,v],w) + \psi([v,w],u) + \psi([w,u],v) = 0. \tag{5.1}$$

We set

$$\lambda(x,y) = \psi(e_x, e_y) \tag{5.2}$$

for $x, y \neq 0$. By setting $u = e_x, v = e_y, w = e_z$ in (5.1), we obtain that

$$\varphi(x,y)\lambda(x+y,z) + \varphi(y,z)\lambda(y+z,x) + \varphi(z,x)\lambda(z+x,y) = 0 \tag{5.3}$$

holds for $x, y, z \neq 0$. If $x + y = 0$, then $\lambda(x+y, z)$ is not defined. In that case $\varphi(x,y) = 0$ and the first term in (5.3) should be interpreted as 0. Similar interpretations should be used for the second and third terms if $y + z = 0$ and $z + x = 0$, respectively.

Since ψ is skew-symmetric, it follows from (5.2) that

$$\lambda(x,y) + \lambda(y,x) = 0. \tag{5.4}$$

For $u \in A$ define $\lambda_u(x) = \lambda(x, u - x)$ for $x \neq 0, u$. From (5.4) we deduce that

$$\lambda_u(u - x) = -\lambda_u(x), \quad x \neq 0, u. \tag{5.5}$$

By setting $z = u - x - y$ in (5.3), we obtain that

$$\varphi(x,y)[\lambda_u(x+y) - \lambda_u(x) - \lambda_u(y)] - \varphi(y,u)\lambda_u(x) - \varphi(u,x)\lambda_u(y) = 0 \tag{5.6}$$

holds for $x, y \neq 0$ and $x + y \neq u$.

Assume that $u \neq 0$. By setting $y = 2x$ in (5.6), we obtain that

$$\lambda_u(2x) = 2\lambda_u(x) \tag{5.7}$$

holds if $\varphi(u,x) \neq 0$.

If $\varphi(u,x), \varphi(u,y)$, and $\varphi(u,x+y)$ are all nonzero, then by replacing x and y in (5.6) with $2x$ and $2y$, respectively, and by using (5.7), we obtain that

$$\varphi(x,y)[\lambda_u(x+y) - \lambda_u(x) - \lambda_u(y)] = 0 \tag{5.8}$$

and

$$\varphi(u,x)\lambda_u(y) = \varphi(u,y)\lambda_u(x). \tag{5.9}$$

If $\varphi(u,x), \varphi(u,y) \neq 0$ and $\varphi(u,x+y) = 0$ then $\varphi(u,x+2y) \neq 0$ and so (5.9) is valid if we replace y with $2y$. By invoking (5.7), we conclude that (5.9) is valid as written.

It follows from (5.9) that the ratio

$$a_u = \frac{\lambda_u(x)}{\varphi(u,x)}$$

is independent of x, provided that $\varphi(u,x) \neq 0$. In other words, there is a constant $a_u \in F$ such that

$$\lambda_u(x) = a_u\varphi(u,x) \tag{5.10}$$

holds whenever $\varphi(u,x) \neq 0$.

Let $x \neq 0, u$ and $\varphi(u,x) = 0$. Choose $y \in A$ such that $\varphi(x,y)$ and $\varphi(u,y)$ are both nonzero. By replacing x in (5.8) with $x - y$, we infer that

$$\lambda_u(x) = \lambda_u(x-y) + \lambda_u(y) = a_u[\varphi(u,x-y) + \varphi(u,y)] = 0.$$

Hence, (5.10) is valid for all $x \neq 0, u$.

Let $l : \mathscr{L} \to F$ be the linear function defined by $l(e_x) = a_x$ for $x \neq 0$. Let $\tilde{\psi}$ be the 2-cocycle defined by

$$\tilde{\psi}(u,v) = \psi(u,v) + l([u,v]).$$

If $x, y, x+y \neq 0$, then we have

$$\tilde{\psi}(e_x, e_y) = \lambda(x,y) + \varphi(x,y)a_{x+y} = \lambda_{x+y}(x) - a_{x+y}\varphi(x+y,x) = 0.$$

Hence, by replacing ψ with the cohomologous 2-cocycle $\tilde{\psi}$, we may assume that $\lambda_u = 0$ for all $u \neq 0$.

It remains to determine λ_0. For $u = 0$, (5.6) becomes

$$\varphi(x,y) \cdot [\lambda_0(x+y) - \lambda_0(x) - \lambda_0(y)] = 0.$$

Hence,

$$\lambda_0(x+y) = \lambda_0(x) + \lambda_0(y) \tag{5.11}$$

holds if $\varphi(x,y) \neq 0$.

Now assume that $x, y, x+y \neq 0$ while $\varphi(x,y) = 0$. We choose $z \in A$ such that the numbers $\varphi(x,z), \varphi(y,z)$, and $\varphi(x+y,z)$ are all nonzero. Then we have

$$\begin{aligned} \lambda_0(x+y) &= \lambda_0(x+z) + \lambda_0(y-z) \\ &= \lambda_0(x) + \lambda_0(y) + \lambda_0(z) + \lambda_0(-z) \end{aligned}$$

and

$$\lambda_0(x) = \lambda_0(x+z) + \lambda_0(-z) = \lambda_0(x) + \lambda_0(z) + \lambda_0(-z).$$

Consequently, (5.11) holds whenever $x, y, x+y \neq 0$.

Let $\mu : A \to F$ be defined by $\mu(x) = \lambda_0(x)$ if $x \neq 0$ and $\mu(0) = 0$. It follows from (5.11) that $\mu \in \mathrm{Hom}(A,F)$.

Hence, we have proved the following result.

Theorem 5.1. *For the simple Lie algebra $\mathscr{L} = \mathscr{L}(A,\varphi)$, $H^2(\mathscr{L},F)$ is spanned by the cohomology classes $[\psi_\mu]$ where $\mu \in \mathrm{Hom}(A,F)$ and the 2-cocycle ψ_μ is defined by*

$$\psi_\mu(e_x, e_y) = \delta_{x+y,0}\mu(x), \quad x, y \neq 0.$$

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