Some Results in the Theory of Nonnegative Matrices

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1. INTRODUCTION

The theory of nonnegative irreducible matrices, which was initiated by Perron [11] and Frobenius [4], is of fundamental importance in the theory of the iterative solution of matrix equations (cf. [7, 15]), derived from the discretization of elliptic boundary value problems. This is not only valid for the standard discretizations, such as those described in [15], but also for more refined methods (cf. [1, 2, 12, 14]).

We shall not be concerned here explicitly with such applications. However, our results give some criteria for deciding whether a matrix (or its inverse) is a nonnegative irreducible matrix (cf. Theorems 1, 2, 4, and 5) and as such, they might be of interest in view of possible applications to numerical analysis.

Theorem 1 as well as the second part of Theorem 4 are extensions of similar results proved in the case of positive stochastic matrices in [13], and in the case of positive matrices in [8] and [9]. In this paper we extend those results to the case of nonnegative irreducible matrices, among other things.

For completeness, we mention that Fiedler and Pták (cf. [4]) have recently given a necessary and sufficient condition for a matrix to be monotone with a positive inverse, although their methods are different in essence.

2. A NECESSARY AND SUFFICIENT CONDITION FOR A LINEAR OPERATOR TO BE REPRESENTABLE BY A NONNEGATIVE IRREDUCIBLE MATRIX

We first recall a few definitions: An $n \times n$ real matrix $A = (a_{ij})$ is said to be *nonnegative*, or *positive*, iff $a_{ij} \geqslant 0$, or > 0, respectively, for

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all $1 \leqslant i, j \leqslant n$. An $n \times n$ matrix A is *reducible* iff there exists an $n \times n$ permutation matrix P such that

$$PAP^T = \begin{vmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{vmatrix},$$

where A_{11} is an $r \times r$ submatrix and A_{22} is an $(n-r) \times (n-r)$ submatrix, for some $1 \leqslant r \leqslant n-1$. If no such permutation exists, then A is *irreducible* (for detailed accounts on the theory of irreducible nonnegative matrices, we refer to [6], [7], or [15]).

The *spectral radius* $\rho(A)$ of a matrix A is the greatest modulus of its eigenvalues.

A collection of $(n+1)$ points $\mathbf{p}_i$ in a vector space E_n of dimension n forms the *vertices* of an *n-simplex* S_n if and only if the $(n+1) \times (n+1)$ determinant whose first n rows are formed with the coordinates of the vectors $\mathbf{p}_i$ over a basis in E_n and whose last row is composed of 1's is different from zero; the *n-simplex* S_n itself is the collection of all vectors of the form $\mathbf{x} = \sum_{i=1}^{n+1} \alpha_i \mathbf{p}_i$, $0 \leqslant \alpha_i \leqslant 1$, $1 \leqslant i \leqslant n+1$, $\sum_{i=1}^{n+1} \alpha_i = 1$ (i.e., is the convex hull of the vertices $\mathbf{p}_i$). A *face* of the n simplex S_n is any m-simplex S_m formed with a subcollection of m of the vertices $\mathbf{p}_i (1 \leqslant m < n)$ of the n-simplex S_n. For details, we refer to [10].

A *stochastic* matrix A is a nonnegative matrix such that the sum of the elements of each row of A is 1 (cf. [6, p. 83]).

Let there be given a real Euclidean space E_{n+1}, of dimension $n+1$. Let $\mathscr{A}$ be a linear operator acting from E_{n+1} into itself. We denote by $\{\mathbf{x}_i, 1 \leqslant i \leqslant n+1\}$ a canonical basis in E_{n+1}, and by A the $(n+1) \times (n+1)$ real matrix which represents the linear operator in the above basis $\{\mathbf{x}_i, 1 \leqslant i \leqslant n+1\}$. We begin with

LEMMA 1. *Let the matrix A be nonnegative and irreducible. Then the space E_{n+1} can be written as the direct sum $E_{n+1} = E_n \oplus E_1$, where both the subspaces E_n and E_1 are invariant under the operator $\mathscr{A}$, and E_n and E_1 have the dimensions n and 1, respectively.*

Proof. Since A is a nonnegative irreducible matrix, its spectral radius $\rho(A)$ is a simple eigenvalue (cf. [15, p. 30]); hence by a standard result in matrix theory (cf. [6]), the space E_{n+1} can be written as the direct sum of the subspace E_1, spanned by the eigenvector $\mathbf{e}$ corresponding to the eigenvalue $\rho(A)$, and of the subspace E_n which is a subspace of dimension n of E_{n+1}, also invariant under $\mathscr{A}$. Q.E.D.

In what follows, we shall assume that the eigenvector $\mathbf{e}$ is chosen so as to have all its coordinates positive; that this is indeed possible follows from [15, p. 30].

LEMMA 2. *Let the matrix A be nonnegative and irreducible. Denote by C the cone generated by the basis vectors $\{\mathbf{x}_i, 1 \leqslant i \leqslant n+1\}$, i.e., $C = \{\mathbf{x} \in E_{n+1}; \mathbf{x} = \sum_{i=1}^{n+1} \beta_i \mathbf{x}_i, \beta_i \geqslant 0, 1 \leqslant i \leqslant n+1\}$. Then $E_n \cap C = \mathbf{0}$, where E_n is the subspace introduced in Lemma 1, and $\mathbf{0}$ denotes the zero vector of E_{n+1}.*

Proof. Assume the conclusion of Lemma 2 is false, i.e., let $\tilde{C} = \{\mathbf{x} \in E_{n+1}; \mathbf{x} \neq \mathbf{0}, \mathbf{x} \in E_n, \mathbf{x} \in C\}$ be nonempty. Given any vector $\tilde{\mathbf{x}} \in \tilde{C}$, $A\tilde{\mathbf{x}} \in C$ since A is nonnegative, $A\tilde{\mathbf{x}} \neq \mathbf{0}$ since A is irreducible, and finally $A\tilde{\mathbf{x}} \in E_n$ since E_n is an invariant subspace. Therefore $A\tilde{C} \subset \tilde{C}$. Using the Brouwer's fixed point theorem as in [3], there exists an eigenvector, say $\tilde{\mathbf{e}}$, of A in $\tilde{C}$, hence also in C. However, since the matrix A is nonnegative and irreducible, the only eigenvector of A in C is the vector $\mathbf{e}$ introduced in Lemma 1 (cf. [15, p. 34]). This is a contradiction, since $\mathbf{e} \in E_1$. Q.E.D.

LEMMA 3. *Given any basis vector $\mathbf{x}_i$, $1 \leqslant i \leqslant n+1$, we can write $\mathbf{x}_i$ in a unique way as $\mathbf{x}_i = \mu_i \mathbf{e} + \mathbf{p}_i$, where $\mu_i > 0$,* and $\mathbf{p}_i \in E_n$.

Proof. That the above decomposition is possible in a unique fashion follows from the decomposition $E_{n+1} = E_n \oplus E_1$ of Lemma 1. Hence it remains to prove that $\mu_i > 0$. Let us first observe that $\mu_i \neq 0$, since $\mathbf{x}_i$ is not in the subspace E_n, by Lemma 2.

Since E_n is a subspace of dimension n of E_{n+1}, it can be written as $E_n = \{\mathbf{x} \in E_{n+1}; F_n(\mathbf{x}) = 0\}$, where $F_n(\mathbf{x})$ is a linear and homogeneous expression in the coordinates of $\mathbf{x}$. Since $\mathbf{x}_i = \mu_i \mathbf{e} + \mathbf{p}_i$, it follows by linearity that $F_n(\mathbf{x}_i) = \mu_i F_n(\mathbf{e}) + F_n(\mathbf{p}_i) = \mu_i F_n(\mathbf{e})$, since $\mathbf{p}_i \in E_n$. Finally, the vectors $\mathbf{x}_i$ and $\mathbf{e}$ are in the same half-space determined by E_n (since both are in C and $E_n \cap C = \mathbf{0}$ by Lemma 2). Hence $F_n(\mathbf{x}_i)$ and $F_n(\mathbf{e})$ are of the same sign, i.e., $\mu_i > 0$, for any i. Q.E.D.

For convenience, we henceforth assume that all the coefficients μ_i are equal to 1. This is no loss of generality: it amounts to performing a positive scalar multiplication on each basis vector.

LEMMA 4. *With the above assumption, each basis vector $\mathbf{x}_i$ can be written as $\mathbf{x}_i = \mathbf{e} + \mathbf{p}_i$, $1 \leqslant i \leqslant n+1$. Then, the $(n+1)$ points $\mathbf{p}_i$ are the vertices of an n-simplex S_n in E_n.*

Proof. To show that the $(n+1)$ points $\mathbf{p}_i$ are actually the vertices of an n-simplex, it suffices to prove that the $(n+1) \times (n+1)$ determinant, where the n first elements of each column represent the coordinates of the $\mathbf{p}_i$'s over any given basis in E_n, and whose last row is composed of 1's, is different from zero. But this is nothing else than the determinant of the basis vectors $\mathbf{x}_i$, $1 \leqslant i \leqslant n+1$, expressed over a basis of $E_n \oplus E_1$: hence, it is different from zero. Q.E.D.

Consider now any vector $\mathbf{x}$ in $C \cap (\mathbf{e} + E_n)$. Since it is in C, it can be written as $\mathbf{x} = \sum_{i=1}^{n+1} \alpha_i \mathbf{x}_i$, all the α_i's being $\geqslant 0$, and since it is in $(\mathbf{e} + E_n)$, it can also be written as $\mathbf{x} = \mathbf{e} + \mathbf{x}_n$, for some $\mathbf{x}_n \in E_n$. On the other hand, by Lemma 3, we have

$$x = \sum_{i=1}^{n+1} \alpha_i \mathbf{x}_i = \left(\sum_{i=1}^{n+1} \alpha_i \right) \mathbf{e} + \sum_{i=1}^{n+1} \alpha_i \mathbf{p}_i.$$

Hence, by the uniqueness of the decomposition of the vector $\mathbf{x}$, we must have $\sum_{i=1}^{n+1} \alpha_i = 1$, and $\mathbf{x}_n = \sum_{i=1}^{n+1} \alpha_i \mathbf{p}_i$. Since the α_i's are all $\geqslant 0$, it follows that $\mathbf{x}_n$ belongs to the n-simplex S_n. Conversely, given any point $\mathbf{x}_n \in S_n$, any point of the form $\mathbf{e} + \mathbf{x}_n$ belongs to $C \cap (\mathbf{e} + E_n)$ and it is clear that this correspondence is one-to-one.

As a consequence let us observe that the n-simplex S_n contains the origin $\mathbf{0}$ strictly in its interior. We have thus proved

LEMMA 5. *There exists a bijection between the n-simplex S_n and the set $C \cap (\mathbf{e} + E_n)$.*

We now achieve the series of lemmas with the key result:

LEMMA 6. *Let the linear operator $\mathscr{A}$ in E_{n+1} be represented by a nonnegative and irreducible matrix A over the basis $\{\mathbf{x}_i,\ 1 \leqslant i \leqslant n\}$. Then, the restriction of $\rho(A)^{-1}\mathscr{A}$ to the subspace E_n maps the n-simplex S_n into itself, i.e.,*

$$\rho(A)^{-1} \mathscr{A} S_n \subset S_n. \tag{1}$$

Moreover, it maps the n-simplex S_n strictly into its interior if and only if the matrix A is positive.

Proof. Since the matrix A is nonnegative and irreducible, it easily follows from Lemma 3 that $A\mathbf{x} \in C$ whenever $\mathbf{x} \in C \cap (\mathbf{e} + E_n)$. Such a vector $\mathbf{x}$ can be written (Lemma 5) as $\mathbf{x} = \mathbf{e} + \mathbf{x}_n$, where $\mathbf{x}_n \in S_n$. Recalling that $\rho(A) > 0$, we can write: $A\mathbf{x} = A\mathbf{e} + A\mathbf{x}_n = \rho(A)(\mathbf{e} + \rho(A)^{-1}A\mathbf{x}_n)$.

Thus, the vector $\rho(A)^{-1}A\mathbf{x}_n$ belongs to the n-simplex S_n, since $A\mathbf{x} \in C$.

The last statement of Lemma 6 follows by observing that the matrix A is positive if and only if $A\mathbf{x}_i$ is a vector with all its components strictly positive, for any $1 \leqslant i \leqslant n + 1$. Q.E.D.

We are now able to prove

THEOREM 1. *In the Euclidean space E_{n+1}, let $\mathscr{A}$ be a linear operator represented by a nonnegative irreducible matrix A. Then,*

(1) *the space E_{n+1} can be decomposed as the direct sum $E_{n+1} = E_n \oplus E_1$, where both E_n and E_1 (of dimensions n and 1, respectively) are invariant under A (E_1 is spanned by the eigenvector $\mathbf{e}$ corresponding to the simple eigenvalue $\rho(A) > 0$);*

(2) *in the subspace E_n, there exists an n-simplex S_n containing the origin $\mathbf{0}$ strictly in its interior, which is mapped inside itself under the restriction of $\mathscr{A}/\rho(A)$ to the subspace E_n (and inside its interior if the matrix A is positive);*

(3) *no face of the n-simplex S_n is invariant under this transformation.*

Conversely, let there be given a linear operator $\mathscr{A}$ in the Euclidean space E_{n+1}. Assume that:

(4) *the space E_{n+1} can be decomposed as the direct sum $E_{n+1} = E_n \oplus E_1$, where both the subspaces E_n and E_1 (of dimensions n and 1, respectively) are invariant under $\mathscr{A}$. The subspace E_1 is spanned by an eigenvector $\mathbf{e}$ corresponding to a positive eigenvalue λ. Moreover, in the subspace E_n, there exists an n-simplex S_n of vertices $\mathbf{p}_i$, $1 \leqslant i \leqslant n + 1$, and containing the origin $\mathbf{0}$, which is mapped inside itself under the restriction of $\lambda^{-1}\mathscr{A}$ to the subspace E_n, and finally,*

(5) *no face of the n-simplex S_n is invariant under the restriction of $\lambda^{-1}\mathscr{A}$ to the subspace E_n.*

Then, the operator $\mathscr{A}$ is representable by a nonnegative irreducible matrix A, with spectral radius $\rho(A) = \lambda$, in any basis of the form $\{\mathbf{x}_i = \mathbf{e} + \mu\mathbf{p}_i, 1 \leqslant i \leqslant n + 1\}$, where μ is an arbitrary positive scalar.

Proof. The first part follows readily from Lemmas 1 to 6.

Conversely, let there be given a basis of the form $\mathbf{x}_i = \mathbf{e} + \mu\mathbf{p}_i$, $1 \leqslant i \leqslant n + 1$, where μ is an arbitrary positive scalar. Consider a nonzero vector $\mathbf{x}$ in the cone C generated by the $\mathbf{x}_i$'s. Then a simple computation using hypothesis 4 alone shows that $A\mathbf{x} \in C$. Finally, condition 5 will guarantee irreducibility. In particular, it implies that the moduli of the eigenvalues of the operator $\mathscr{A}$ corresponding to eigenvectors contained in the subspace E_n are less than or equal to λ (cf. [15, p. 30]). Q.E.D.

Remark. In our geometrical interpretation, the concept of a p-cyclic nonnegative irreducible matrix (cf. [15, p. 35]) can be formulated as follows:

In addition to properties 1, 2, and 3, there exists a partition $\{\mathbf{p}_1^1, \ldots, \mathbf{p}_{i_1}^1; \mathbf{p}_1^2, \ldots, \mathbf{p}_{i_2}^2; \ldots; \mathbf{p}_1^p, \ldots, \mathbf{p}_{i_p}^p\}$ of the vertices of the n-simplex S_n such that the associated faces $\mathscr{P}_k$, $1 \leqslant k \leqslant p$ (the face $\mathscr{P}_k$ being generated by the vertices $\{\mathbf{p}_1^k, \ldots, \mathbf{p}_{i_k}^k\}$) satisfy

$$\mathscr{A}\mathscr{P}_k \subset \mathscr{P}_{k+1}(\text{mod.}\, p+1), \qquad 1 \leqslant k \leqslant p.$$

Remark. It is clear that condition 4 alone is sufficient to guarantee the representation of the operator $\mathscr{A}$ by a nonnegative matrix, which is not necessarily irreducible. However, nothing can be said in general about the converse, since the results of Lemmas 1 and 2 depended essentially on the assumption of irreducibility.

Example. Let the operator $\mathscr{A}$ be represented in E_4 by the matrix

$$A = \begin{vmatrix} 2 & -2 & -1 & 0 \\ -2 & 2 & 0 & -1 \\ -1 & 0 & 2 & -2 \\ 0 & -1 & -2 & 2 \end{vmatrix}.$$

The eigenvectors and eigenvalues of the operator $\mathscr{A}$ are respectively

$$\begin{aligned} \mathbf{e} &= \{1, -1, -1, 1\}, & \rho(A) &= 5, \\ \mathbf{e}_1 &= \{1, 1, 1, 1\}, & \lambda_1 &= -1, \\ \mathbf{e}_2 &= \{-1, -1, 1, 1\}, & \lambda_2 &= 1, \\ \mathbf{e}_3 &= \{1, -1, 1, -1\}, & \lambda_3 &= 3. \end{aligned}$$

Hence, we let $E_1 = \text{span}\{\mathbf{e}\}$, and $E_3 = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. In E_3, the following are the vertices of a 3-simplex S_3 (in fact, a regular tetrahedron):

$$\mathbf{p}_1 = -\mathbf{e}_2 - \frac{1}{\sqrt{2}}\mathbf{e}_3,$$

$$\mathbf{p}_2 = \mathbf{e}_2 - \frac{1}{\sqrt{2}}\mathbf{e}_3,$$

$$\mathbf{p}_3 = \mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_3,$$

$$\mathbf{p}_4 = -\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_3.$$

As a new basis, we choose $\mathbf{x}_i = \mathbf{p}_i + (1/\sqrt{2})\mathbf{e}$, $1 \leqslant i \leqslant 4$. Then in that basis, the operator $\mathscr{A}$ is represented by the positive matrix

$$A' = \tfrac{1}{2}\begin{Vmatrix} 5 & 3 & 1 & 1 \\ 3 & 5 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 5 & 3 \end{Vmatrix}.$$

It can be readily checked that each vertex $\mathbf{p}_i$, $1 \leqslant i \leqslant 4$, is mapped strictly in the interior of the 3-simplex S_3 under the restriction of $\mathscr{A}/5$ to E_3.

As a complement to Theorem 1, we have

Corollary 1. *Assume that the linear operator $\mathscr{A}$ is representable by a nonnegative (or positive) irreducible matrix A'. Then the operator $\mathscr{A}$ is also representable by the transpose of a stochastic (or positive stochastic) matrix A, up to a multiplicative factor equal to the spectral radius of the operator $\mathscr{A}$.*

Proof. By Theorem 1, the operator $\mathscr{A}$ can be represented as follows, after we have chosen, once and for all, a basis in E_n:

$$A' = \rho(A)\begin{Vmatrix} & & & 0 \\ & & & 0 \\ & A_n & & \cdot \\ & (n \times n) & & \cdot \\ & & & \cdot \\ & & & 0 \\ 0 & 0 & \cdots\ 0 & 1 \end{Vmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \end{matrix}\right\} \text{components on } E_n, \\ \} \text{ components on } E_1. \end{matrix}$$

The vertices $\mathbf{p}_i$ of the n-simplex S_n in E_n can accordingly be represented by the column vectors

$$\mathbf{p}_i = \begin{vmatrix} p_{i1} \\ \cdot \\ \cdot \\ \cdot \\ p_{in} \\ 0 \end{vmatrix}, \qquad 1 \leqslant i \leqslant n+1, \text{ in the same basis.}$$

By Theorem 1, it follows that

$$A'\mathbf{p}_i = \rho(A) \sum_{j=1}^{n+1} \alpha_{ij}\mathbf{p}_j, \qquad 1 \leqslant i \leqslant n+1, \tag{2}$$

where $\alpha_{ij} \geqslant 0$, $1 \leqslant j \leqslant n+1$, and

$$\sum_{j=1}^{n+1} \alpha_{ij} = 1, \qquad 1 \leqslant i \leqslant n+1 \tag{3}$$

(all the α_{ij}'s being positive if A is a positive matrix).

Let the matrix P be defined as

$$P = \begin{vmatrix} p_{11} & p_{21} \cdots p_{n+1,1} \\ & \cdot \\ & \cdot \\ & \cdot \\ p_{1n} & p_{2n} \cdots p_{n+1,n} \\ 1 & 1 \cdots 1 \end{vmatrix}.$$

Then the above relations (2) and (3) can be rewritten in matrix form as

$$A'P = \rho(A)PA, \tag{4}$$

where $A = (\alpha_{ij})$ is the transpose of a stochastic matrix. The proof is achieved by observing that the matrix P is nonsingular (cf. Lemma 4). Q.E.D.

Remark. Corollary 1 is a generalization of a result of [10].

3. A NECESSARY AND SUFFICIENT CONDITION FOR AN IRREDUCIBLE MATRIX TO BE MONOTONE

An $(n + 1) \times (n + 1)$ matrix A is said to be *monotone* if and only if its inverse A^{-1} exists and is a nonnegative matrix. Monotone matrices theory plays a fundamental role in the derivation of finite difference schemes for elliptic operators (cf. [1, 2, 7, 12, 14, 15]).

Before stating Theorem 2, let us observe the obvious fact that a nonsingular matrix A is irreducible if and only if its inverse A^{-1} is irreducible.

THEOREM 2. *Let there be given an* $(n+1) \times (n+1)$ *irreducible and monotone matrix* A, *representing a linear operator* $\mathscr{A}$ *in a basis* $\{\mathbf{x}_i, 1 \leqslant i \leqslant n+1\}$ *of the real Euclidean space* E_{n+1}. *Then,*

(1) *the space* E_{n+1} *can be written as the direct sum* $E_{n+1} = E_n \oplus E_1$, *where both* E_n *and* E_1 *are invariant under* A, *and the subspace* E_1 *is spanned by the eigenvector* $\mathbf{e}$ *corresponding to the simple eigenvalue* $\sigma(A) = \rho(A^{-1})$;

(2) $E_n \cap C = 0$, *where* C *is the cone generated by the basis vectors* $\{\mathbf{x}_i, 1 \leqslant i \leqslant n+1\}$.

(3) *Let the positive numbers* α_i *be uniquely determined by the condition that* $\alpha_i \mathbf{x}_i - \mathbf{e} \in E_n$, $1 \leqslant i \leqslant n+1$. *Then, in the subspace* E_n, *the n-simplex* S_n *of vertices* $\mathbf{p}_i = \alpha_i \mathbf{x}_i - \mathbf{e}$, $1 \leqslant i \leqslant n+1$, *is contained in its image under the restriction of the operator* $\sigma(A)A$ *to* E_n.

Conversely, let there be given an irreducible matrix A *in the basis* $\{\mathbf{x}_i, 1 \leqslant i \leqslant n+1\}$ *of the space* E_{n+1}. *Assume that*

(4) *the space* E_{n+1} *can be decomposed as the direct sum* $E_{n+1} = E_n \oplus E_1$, *where both the subspaces* E_n *and* E_1 *(of dimensions* n *and* 1, *respectively) are invariant under* A. *Moreover, the subspace* E_1 *is generated by an eigenvector* $\mathbf{e}$ *corresponding to a positive eigenvalue* $\sigma(A)$ *of the matrix* A. *Further,*

(5) *condition* 2 *holds, and finally,*

(6) *condition* 3 *holds.*

Then the matrix A *is monotone.*

Proof. It is an immediate consequence of Theorem 1 applied to the matrix A^{-1} (conditions 4, 5, and 6 imply that A^{-1} exists). Q.E.D.

Example. Let

$$A = \begin{vmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{vmatrix}.$$

One can easily check that $\sigma(A) = \rho(A^{-1}) = 1$, and $\mathbf{e} = \{1, 1, 1\}$. The subspace E_2 is the plane $\{\mathbf{x};\, x_1 + x_2 + x_3 = 0\}$ and the restriction of A to E_3 amounts to a scalar multiplication by -5. The 2-simplex S_3 of vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ is an equilateral triangle and clearly $S_3 \subset -5S_3$. Since it is even strictly contained in S_3, it follows that A^{-1} is a positive matrix and, actually, the inverse matrix A^{-1} is given by

$$A^{-1} = \tfrac{1}{5}\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}.$$

In Section 4, we shall give some sufficient criteria allowing us to use the results of Theorems 1 and 2. However, we need first to describe the "regular" n-simplex and this is the purpose of the first part of Section 4.

4. THE REGULAR n-SIMPLEX AND ITS APPLICATIONS

Let E_{n+1} be metrized by the usual Euclidean metric: $d(\mathbf{x}, \mathbf{y}) = \{\sum_{i=1}^{n+1} |x_i - y_i|^2\}^{1/2}$.

In any space of dimension $\geqslant n$, an n-simplex T_n of vertices $\mathbf{p}_i$, $1 \leqslant i \leqslant n+1$, is said to be *regular* with *center at the origin* if and only if the following conditions are satisfied:

$$d(\mathbf{p}_j, \mathbf{0}) = d(\mathbf{p}_j, \mathbf{0}) = \delta_n, \qquad 1 \leqslant i, j \leqslant n+1, \quad i \neq j, \tag{5}$$

$$d(\mathbf{p}_i, \mathbf{p}_j) = d(\mathbf{p}_k, \mathbf{p}_l) = \mu_n, \qquad 1 \leqslant i, j \leqslant n+1, \quad i \neq j, \tag{6}$$
$$1 \leqslant k, l \leqslant n+1, \quad k \neq l.$$

Observe that a regular n-simplex is the generalization of an equilateral triangle in E_k, $k \geqslant 2$, or of a regular tetrahedron in E_k, $k \geqslant 3$.

The previous results will allow us to construct such an n-simplex. This is the purpose of

THEOREM 3. *There exists a regular n-simplex T_n with center at the origin in an n-dimensional subspace E_n of the Euclidean space E_{n+1}. Moreover, the following metric properties hold:*

$$\mu_n = \left[\frac{2(n+1)}{n}\right]^{1/2} \delta_n. \tag{7}$$

The radii R_n and r_n of the circumscribed and inscribed spheres, respectively, are given by

$$R_n = \delta_n \qquad \textit{and} \qquad r_n = \frac{\delta_n}{n}. \tag{8}$$

Finally, we have that

$$\alpha T_n \subset T_n \qquad \textit{if and only if } -\frac{1}{n} \leqslant \alpha \leqslant 1, \tag{9}$$

and

$$\alpha T_n \subset \operatorname{int} T_n \qquad \textit{if and only if } -\frac{1}{n} < \alpha < 1. \tag{10}$$

Proof. In the Euclidean space E_{n+1} provided with the usual canonical *basis* $\{\mathbf{x}_i;\, 1 \leqslant i \leqslant n\}$, consider the $(n+1) \times (n+1)$ matrix

$$A(\alpha) = \frac{1}{n+1} \begin{vmatrix} 1+n\alpha & 1-\alpha & \cdots & 1-\alpha \\ 1-\alpha & 1+n\alpha & \cdots & 1-\alpha \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ 1-\alpha & 1-\alpha & \cdots & 1+n\alpha \end{vmatrix} \tag{11}$$

for any real α. It is readily seen that we can decompose the space E_{n+1} as $E_{n+1} = E_n \oplus E_1$, where both the subspaces E_1 and E_{n+1} are invariant under A; E_1 is spanned by $\mathbf{e} = \{1, 1, \ldots, 1\}$ and E_n is its orthogonal complement: $E_n = \{\mathbf{x};\, \sum_{i=1}^{n+1} x_i = 0\}$. The vector $\mathbf{e}$ is an eigenvector corresponding to the eigenvalue 1, and the restriction of $A(\alpha)$ to E_n merely amounts to a scalar multiplication by α.

Clearly, the following properties hold:

$$\begin{aligned} \alpha > 1:&\quad A \text{ has alternate signs among its coefficients,} \\ \alpha = 1:&\quad A = I \text{ (hence } A \text{ is a reducible nonnegative matrix),} \\ -\frac{1}{n} < \alpha < 1:&\quad A \text{ is a positive matrix (hence irreducible),} \\ \alpha = -\frac{1}{n}:&\quad A \text{ is a nonnegative irreducible matrix,} \\ \alpha < -\frac{1}{n}:&\quad A \text{ has alternate signs among its coefficients.} \end{aligned} \tag{12}$$

Call T_n the n-simplex associated with the matrix A along the lines of Section 2 (notice that it is independent of α). As an immediate consequence of Theorem 1 and relations (12), it follows that the inclusion relations (9) and (10) are valid.

Next, it is easily verified that each vertex $\mathbf{p}_i$ of T_n has coordinates $\mathbf{p}_i = \{-1, -1, \ldots, -1, n, -1, \ldots, -1\}$, i.e., $\mathbf{p}_i = (n+1)\mathbf{x}_i - \mathbf{e}$. From this, the formulas (5), (6), and (7) directly follow, proving that the n-simplex T_n is regular.

Finally, we compute the radii of the circumscribed and inscribed spheres, respectively: each point $\mathbf{x}_\partial$ on the boundary ∂T_n of T_n can be written as

$$\mathbf{x}_\partial = \sum_{i=1}^{n+1} \alpha_i \mathbf{p}_i, \qquad \sum_{i=1}^{n+1} \alpha_i = 1, \qquad \alpha_i \geqslant 0, \quad 1 \leqslant i \leqslant n+1, \tag{13}$$

where *at most* n of the coefficients α_i are different from zero. Clearly then, $R_n = \sup_{\mathbf{x}_\partial \in \partial T_n} d(\mathbf{x}_\partial, \mathbf{0})$ and $r_n = \inf_{x_\partial \in \partial T_n} d(\mathbf{x}_\partial, \mathbf{0})$. An easy computation yields that $d(\mathbf{x}_\partial, \mathbf{0}) = -(n+1) + (n+1)^2 \sum_{i=1}^{n+1} \alpha_i^2$, from the expression of x_∂ as given in (13). A simple argument will then give the formulas (8).

Remark. For $\alpha \neq 0$, the inverse of the matrix $A(\alpha)$ as given in (11) is explicitly given by

$$[A(\alpha)]^{-1} = \frac{1}{(n+1)\alpha} \begin{vmatrix} \alpha+n & \alpha-1 & \cdots & \alpha-1 \\ \alpha-1 & \alpha+n & \cdots & \alpha-1 \\ & & \vdots & \\ \alpha-1 & \alpha-1 & \cdots & \alpha+n \end{vmatrix}. \tag{14}$$

Moreover, the following properties hold:

$$\begin{aligned} \alpha > 1: &\quad [A(\alpha)]^{-1} \text{ is a positive matrix,} \\ \alpha = 1: &\quad [A(\alpha)]^{-1} = I \text{ is a reducible nonnegative matrix,} \\ -n < \alpha < 1: &\quad [A(\alpha)]^{-1} \text{ has alternate signs among its coefficients,} \quad (15) \\ \alpha = -n: &\quad [A(\alpha)]^{-1} \text{ is a nonnegative irreducible matrix,} \\ \alpha < -n: &\quad [A(\alpha)]^{-1} \text{ is a positive matrix,} \end{aligned}$$

all properties that could have been derived directly from the results of Section 3 coupled with those of Theorem 3.

As applications, we now state, without proofs, the two following results, which are easy consequences of Theorems 1, 2, and 3:

THEOREM 4 (*Sufficient condition for a diagonalizable matrix to be similar to a nonnegative matrix*). *Let the diagonal matrix D be of the form*

$$D = \operatorname{diag}\{1, \lambda_1, \lambda_2, \ldots, \lambda_n\}, \tag{16}$$

where all the λ_i's are real numbers. Then,

(1) *if $|\lambda_i| \leqslant 1/n$, $1 \leqslant i \leqslant n$, the diagonal matrix D is similar to a nonnegative matrix;*

(2) *if $|\lambda_i| < 1/n$, $1 \leqslant i \leqslant n$, the diagonal matrix D is similar to a positive matrix.*

THEOREM 5 (*Sufficient condition for a symmetric matrix to be monotone*). *Let A be a symmetric matrix with eigenvalue* 1 *corresponding to the eigenvector $\mathbf{e} = \{1, 1, \ldots, 1\}$, and let λ_i, $1 \leqslant i \leqslant n$, be its other eigenvalues. Then,*

(1) *if $|\lambda_i| \geqslant n$, $1 \leqslant i \leqslant n$, the matrix A is monotone;*

(2) *if $|\lambda_i| > n$, $1 \leqslant i \leqslant n$, the matrix A is monotone; moreover the inverse matrix A^{-1} is positive.*

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