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SOME CONSTRUCTIONS OF RINGS

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Introduction

If R is a Noetherian ring and I is an ideal of R, then the Rees ring $R(I) = R \oplus I \oplus \cdots I^n \oplus \cdots$, is nothing but the graded ring associated to the I-adic filtration of R. The use of such rings in commutative algebra usually relates to the Artin-Rees property or properties of filtered R-modules expressed in terms of the associated graded modules over the Rees ring. Starting from a Dedekind domain R and a fractional ideal I of R one may constuct the so-called generalised Rees ring as follows:

$$\check{R}(I) = \cdots \oplus I^{-n} \oplus \cdots \oplus I^{-1} \oplus R \oplus I \oplus \cdots \oplus I^n \oplus \cdots$$

The so-called arithmetically graded rings have been studied in [26], [27]. The concept of a generalised Rees ring then underwent consequent generalisations both in the commutative case and in the non-commutative case and it turned out that these constructions of graded nature also have very nice ungraded properties. Moreover, in constructing those rings we obtain new examples of Goldie rings, maximal orders, tame orders in the sense of R. Fossum cf. [6], Krull orders in the P.I. case or the general case, and vHC-orders in the sense of H. Fujita, cf. [10], or H. Marubayashi cf. [18]. It may be apparent from the foregoing list of classes of rings that the constructions are also very effective in the non-Noetherian case if the ground ring is a Krull domain or just an integrally closed domain. The organization of this survey is as follows. There are three major classes considered here: strong Rees rings, divisorial Rees rings, scaled Rees rings; in each of these classes we pay particular attention to the commutative case, the P.I. case and the *general case*. In the list of references I included several recent papers where generalised Rees rings have been applied or studied. The reader who cares to check out these references will find that the term *generalised Rees ring* is used in many ways; I hope that the terminology used in this note becomes the standard terminology for the subject. Finally let me point out that many rings constructed here turn out to be nice graded subrings of certain crossed products, twisted grouprings or grouprings, where the embedding is described in some sense (to be explicited in the sequel) by the Picard-group or the class group of the part of degree zero. Due to the fact that the rings dealt with in this paper are usually $\mathbb{Z}$-graded the crossed product aspect of the theory is sometimes hidden; nevertheless I shall occasionally point out results concerning G-graded rings for a torsion free abelian group G. For a rather extensive treatment of graded ring theory we refer to the book by C. Năstăsescu and the author, cf. [20].

1. Strong Rees rings

Let G be an arbitrary group. A ring R is said to be a *G-graded* ring if there exist additive subgroups R_σ of R, for $\sigma \in G$, such that $R = \bigoplus_{\sigma \in G} R_\sigma$, $R_\sigma R_\tau \subset R_{\sigma\tau}$. For all $\sigma, \tau \in G$. We say that the ring R is *strongly G-graded* if it is G-graded and such that $R_\sigma R_\tau = R_{\sigma\tau}$. For all $\sigma, \tau \in G$. If e is the neutral element of G, then R_e is a subring of R and it is one of our main aims to derive properties of R from given properties of R_e. An R-module M is a *graded R-module* if there are additive subgroups M_σ of $M, \sigma \in G$, such that: $M = \bigoplus_{\sigma \in G} M_\sigma$, $R_\tau M_\tau \subset M_{\tau\sigma}$ (all modules will be left modules unless otherwise specified, if G is not abelian the order of τ and σ in the foregoing definition has to be respected!). The set $h(M) = \bigcup_\sigma M_\sigma$ is called the set of *homogeneous elements* of M. If M and N are graded R-modules, then an R-linear morphism $f : M \to N$ is said to be *graded of degree* $\sigma \in G$ if $f(M_\tau) \subset N_{\tau\sigma}$ for all $\tau \in G$ (note the order!). Then we define

$$\mathrm{HOM}_R(M, N) = \bigoplus_\sigma \mathrm{HOM}_R(M, N)_\sigma,$$

where each term consist of the morphisms which are graded of degree $\sigma \in G$. We write

$$\mathrm{Hom}_{R\text{-gr}}(M, N) = \mathrm{HOM}_R(M, N)_e.$$

The category R-gr consisting of graded R-modules with morphisms being the graded morphisms of degree zero, is a Grothendieck category (but R is not a generator for R-gr). Full detail on graded rings and modules may be found in [20]; let us just mention here that every graded module M over a strongly graded ring R is strongly graded in the sense that $R_\sigma M_\tau = M_{\sigma\tau}$ for all $\sigma, \tau \in G$, and that consequently the categories R-gr and R_e-mod are equivalent if R is strongly graded. (The equivalence being given by the functors $R \otimes_{R_e} -$ and $(-)_e$, as is easily seen.) From the definition of a strongly graded ring one easily deduces that each R_σ, $\sigma \in G$, is an invertible R_0-module because

$$R_\sigma \otimes_{R_e} R_{\sigma^{-1}} \cong R_\sigma R_{\sigma^{-1}} = R_e = R_{\sigma^{-1}} R_\sigma = R_{\sigma^{-1}} \otimes_{R_e} R_\sigma.$$

From $R_\sigma \otimes_{R_e} R_\tau \cong R_{\sigma\tau}, \sigma, \tau \in G$, it follows that a strongly graded ring R may be constructed starting from R_e by giving a group homomorphism $G \to \mathrm{Pic}(R_e)$ (one should check that changing the representative for $[R_\sigma] \in \mathrm{Pic}(R_e)$, one actually defines the same strongly graded ring up to graded isomorphism). If R_e is a commutative domain, then we say that R is *fractionally graded* if each $R_\sigma, \sigma \in G$, is isomorphic to a fractional ideal of R_e. It is clear how this notion may be extended to the non-commutative case if R_e is some 'nice' order over an integrally closed domain in some central simple algebra or in more general cases where the notion of fractional ideal (two-sided!) makes sense. Now we define a *strong Rees ring* to be a *fractionally strongly graded ring*; this high-brow terminology may be clearified by looking at some concrete situations, (as explained in the introduction):

1.1. *The commutative case*

In this case a strong Rees ring may be written as $R=\sum_{\sigma\in G} I_\sigma X_\sigma$, where $I_\sigma, \sigma\in G$, is an invertible ideal of R_e and where $I_\sigma I_\tau = I_{\sigma\tau}$ for all $\sigma,\tau\in G$, $X_\sigma X_\tau = X_{\sigma\tau}$ for all $\sigma,\tau\in G$. Let K be the field of fractions of R_e (note that we have assumed here that R_e is a domain). Then R is a subring of the group ring $KG=\sum_{\sigma\in G} KX_\sigma$ completely described (up to graded isomorphism) by the group morphism $G\to\mathrm{Pic}(R_e)$, $\sigma\mapsto[I_\sigma]$. In case $G=\mathbb{Z}$, we obtain $R=\sum_{n\in\mathbb{Z}} I^n X^n$, for some invertible ideal I of R_e. If G is an ordered group, with positive part G_+ say, then we may define the positively graded ring:

$$R_+ = \sum_{\sigma\in G_+} I_\sigma X_\sigma$$

(and similarly for R_-).

Properties

1.1.1. Suppose $G=\mathbb{Z}$. Then R is Noetherian if and only if R_0 is Noetherian, if and only if R_- and R_+ are Noetherian rings, cf. Lemma A.II.3.7, Proposition A.II.3.4 of [20]. A similar statement holds if G is polycyclic by finite, cf. Theorem A.II.3.8 of [20].

1.1.2. Suppose $G=\mathbb{Z}$. If R is a gr-Dedekind domain (a graded domain such that graded ideals are projective), then R_0 is a Dedekind domain, cf. Theorem B.II.2.7 in [20].

If R' is a gr-Dedekind domain, then there is an $n\in\mathbb{N}$ such that $R'_{(n)}$ is a strong Rees ring, where $(R'_{(n)})_i = R'_{in}$ for all $i\in\mathbb{Z}$; cf. Theorem B.II.2.12 in [20].

1.1.3. If G is torsion free abelian, then R is integrally closed if and only if R_e is integrally closed, cf. Proposition 7.1 in [21].

If R is a Krull domain, then R_e is a Krull domain and the kernel of the class group map $\mathrm{Cl}(R_e)\to\mathrm{Cl}(R)$ is $\mathrm{Im}\,\Pi$ where Π is the group homomorphism $\Pi: G\to\mathrm{Pic}(R_e)\to\mathrm{Cl}(R_e)$ deriving from the structure of R, cf. Proposition 7.2 of [21].

1.1.4. If $G=\mathbb{Z}$, then a combination of foregoing properties states that R is a Noetherian integrally closed domain if and only if R_0 is Noetherian integrally closed. Expounding on A.1.3 one immediately deduces that in case $G=\mathbb{Z}$, R is a Krull domain if and only if R_0 is a Krull domain, cf. Note 7.4 of [21]. Actually there it is stated that the latter equivalence is still true when G is a torsion free abelian group satisfying the ascending chain condition on cyclic subgroups.

1.1.5. If R is a $\mathbb{Z}$-graded Krull domain, then $\mathrm{Pic}^g(R)=\mathrm{Pic}(R)$, $\mathrm{Cl}^g(R)=\mathrm{Cl}(R)$ where Pic^g and Cl^g are defined to be the subgroups of Pic resp. Cl consisting of classes which may be represented by a graded element; cf. Lemma B.II.1.13 of [20].

$\mathrm{Br}^g R=\mathrm{Br}\, R_0$, for $G=\mathbb{Z}$; cf. [28]. For full detail on graded Brauer groups cf. [34]. Moreover, $\mathrm{Br}^g R=\mathrm{Br}\, R_0=\mathrm{Br}^g(R_+,\kappa_+)$; this is a rephrasing in the terminology of relative Brauer groups, cf. [34], of a result mentioned in [28], here κ_+ stands for the kernel functor on R-gr associated to the filter of powers of $\sum_{n>0} R_n$.

Remark. Strong Rees rings R over Dedekind domains R_0 turn out to be the key tools in obtaining a structure theorem for $\mathbb{Z}$-graded gr-Dedekind rings, see Section 3. Surprisingly (if one looks at $\mathrm{Br}^g R=\mathrm{Br}\, R_0$) this is also true for the structure of the graded Brauer group of a gr-Dedekind ring. Here one will use the trick of creating holes in the gradation i.e. considering $R_{(n)}$ for variable n, cf. [34].

1.2. *The P.I.-case*

For generalities concerning the theory of P.I. rings we refer to C. Procesi's book [22]; some extra results in combination with a graded structure have been obtained in [30], cf. also section C.I.2 of [20]. In this section we consider strong Rees rings R over 'nice' orders R_0 in some central simple algebra Q_0. There are two types of results mentioned here. First we have results for arbitrary strongly $\mathbb{Z}$-graded rings over the orders considered, these are obtained by restricting to the P.I.-case some results of H. Marubayashi, E. Nauwelaerts, F. Van Oystaeyen, [19]. Secondly we have results for a restricted kind of strong Rees rings graded by a group G which is abelian torsion free and which satisfies the ascending chain condition for cyclic subgroups, these are obtained by restricting to the strong Rees case some results of L. Le Bruyn, F. Van Oystaeyen [16] (the latter reappear in Section 2).

We have used a rather cheap boutade when extending the term 'fractionally graded' to the non-commutative case; so let us be a little more careful in explaining what is really meant here now. First consider a strongly G-graded ring R over a prime P.I. ring R_e with ring of fractions Q_e, a central simple algebra with center $Z(Q_e)=Q(Z(R_e))$, (see Posner's theorem in [22]). Since R_e is a prime Goldie ring every graded essential left ideal of R will contain a regular element of degree zero (using the fact that R is strongly graded!) and so the regular homogeneous elements of R from a (left) Ore set of R with (left)ring of fractions Q^g a gr-simple gr-Artinian ring in the sense of [20]. It is easy to see that the regular elements of degree e also form on Ore set, S_e say, and that:

$$Q^g=\{s^{-1}r,\, s\in S_e, r\in R\},\qquad (Q^g)_e=Q_e.$$

The structure theorem for gr.c s.a. (cf. Corollary A.I.4.3, Theorem A.I.5.8 of [20]), yields that $Q^g = Q_0[X, X^{-1}, \gamma]$ for some $\gamma \in \mathrm{Aut}(Q_0)$ in case $G = \mathbb{Z}$ and more generally one easily deduces from the crossed product theorem in [20] that $Q^g = Q_e * G$. Clearly it makes sense to write $R = \sum_{\sigma \in G} I_\sigma X_\sigma$ now; we always assume that γ_σ restricts to an automorphism of R_e for each $\sigma \in G$, where γ_σ is the automorphism of Q_e corresponding to the action of X_σ. If R has to be an P.I. ring, it follows from a result of G. Cauchon, cf. [3], that each γ_σ has to have the property: $\gamma_\sigma^{n_\sigma}$ is inner for some $n_\sigma \in \mathbb{N}$. If $G = \mathbb{Z}$ there is no condition on the cocycle describing the crossed product $Q_e * G$ for the latter to be a P.I. ring, if G is torsion free abelian such a condition may be readily written down, cf. [16]. In the sequel of this section *R will be a P.I. strong Rees ring* as explained above. We will say that R is a *normalizing strong Rees ring* if for all $\sigma \in G$, γ_σ is defined as conjugation by some normalizing element $a_\sigma \in Q_e$ (i.e. $a_\sigma R_e = R_e a_\sigma$).

Properties

1.2.1. If P is a prime ideal of R lying over zero in R_0, then $C(P)$ is a regular (left and right) Ore set of R; the localization $Q_P(R)$ is a principal left and right ideal ring with unique maximal ideal $Q_P(R)P$, cf. Proposition 2.9 of [19].

1.2.2. Let $G = \mathbb{Z}$. If R_0 is a (classical) maximal order in Q_0, then R is a maximal order in $Q = Q_{\mathrm{Cl}}(R)$, cf. Theorem 3.1 of [19].

1.2.3. Let $G = \mathbb{Z}$. If R_0 is a Krull order (in the P.I. case the concepts of Krull orders introduced in [4], [13], [17] all coincide, and there is no confusion possible here), then R is a Krull order.

1.2.4. Let $G = \mathbb{Z}$. If R_0 is a Dedekind prime ring, then R is prime Noetherian maximal order in Q such that graded left (and right) ideals of R are projective left (right) R-modules. In this case R is also a graded Asano order in Q^g; cf. Proposition 3.9 in [19].

1.2.5. Let $G = \mathbb{Z}$. If R_0 is an HNP ring with enough invertible ideals, then R is a υHC-order with enough υ-invertible ideals in the sense of [10], [18], cf. Theorem 4.12 of [19]. In this case the divisor group satisfies: $D(R) \cong D_g(R) \oplus D(Q^g)$, where $D_g(R)$ is the graded divisor subgroup of $D(R)$.

1.2.6. If R_0 is a prime Noetherian Asano order, then R is in particular a Krull order, cf. Note 4.15 in [19].

1.2.7. Let G be torsion free abelian. If R_0 is a relative maximal order in the sense of [15], then so is R if R is a normalizing Rees ring over R_0; cf. Theorem 4.5 in [16].

1.2.8. Let G be torsion free abelian. A normalizing strong Rees ring R over a maximal order (in the usual sense) is again a maximal order; cf. Corollary 4.6 of [16].

1.2.9. Let G be a torsion free abelian group satisfying the ascending chain condition with respect to cyclic subgroups. If R_0 is a Krull order and R is a normalizing strong *Rees ring over* R_e, then R is a Krull order; cf. Corollary 4.8 in [16].

1.2.10. Let G be as in 1.2.9. If R_e is an HNP ring, then a normalizing strong Rees ring R over R_e is a tame order of Q, (Theorem 4.12 of [16]). If R_e is a tame order (cf. R. Fossum [6] for details on tame orders), then so is R, cf. Corollary 4.14 of [16].

1.2.11. Let G be as in 1.2.9. Let R_e be a Krull order and let R be a normalizing strong Rees ring over R_e. The central class group, of the Krull orders involved will be denoted by CCl, CCl_g. For the definition and further properties of central classgroups we refer to [14] or to [16]. We have: $\mathrm{CCl}(R)=\mathrm{CCl}_g(R)\oplus\mathrm{CCl}(Q^g)$, where $Q^g=Q_e*G$; cf. Theorem 5.6 of [16].

Remark. A P.I. υHC-order is the same as a tame order. In more general cases, υHC-orders may be considered to be the generalizations of tame orders.

1.3. *The general case*

We consider a strongly $\mathbb{Z}$-graded ring R over a prime left Goldie ring R_0. All the results mentioned here stem from [19] unless otherwise mentioned. Lifting and descent of the left Noetherian property, similar to 1.1.1., again follows from general graded ring theory, cf. [20].

Properties

1.3.1. If for all prime ideals P of R such that $P\cap R_0=0$ we have that R/P is a left Goldie ring, then the left version of 1.2.1 holds but still $Q_P^1(R)$ is a bounded prime principal left and *right* ideal ring with unique maximal ideal. Let $\mathscr{P}$ be the set of prime ideals of R such that $P\cap R_0=0$, then $\mathscr{P}$ corresponds bijectively to the set of proper prime ideals Q^g and

$$Q^g=\bigcap_{P\in\mathscr{P}} Q_P^1(R)\cap S(Q^g)$$

where $S(Q^g)$ is a simple Noetherian ring.

1.3.2. The statements similar to 1.2.2, ..., 1.2.6 are valid in this general situation if we assume that R_0 is a prime left and right Goldie ring (where this should not follow from the assumptions made in the phrasing of the statements).

1.3.6. The author verified that the above results may be generalized to the case where $G = \mathbb{Z}^n$, $n \in \mathbb{N}$, thus yielding a restricted generalization of the statements 1.2.7, ..., 1.2.11 in the non-P.I. case. It is conjectured that all these results are still valid if G is an abelian torsion free group satisfying the ascending chain condition on cyclic subgroups.

2. Divisorial Rees rings

Whereas strongly graded rings of type G over a given ring R_e may be parametrized, up to graded isomorphism, by the group homomorphisms $G \to \mathrm{Pic}(R_e)$, the divisorially graded rings over R_e will be paramerized by $\mathrm{Cl}(R_e)$ or a convenient non-commutative equivalent. Actually any group of suitably defined fractional ideals could be used in the construction e.g. the central class roup of an Ω-Krull ring, cf. [14], the normalizing class group of certain orders over Krull rings forms a variant of the K-theoretically flavored LFP groups of Fröhlich, Reiner, Ullom, [9], ... Here we have chosen to use a relative Picard group with respect to the prime ideals of height one, for the theory of relative Picard groups we refer to [34], [35], or [16]. This $\mathrm{Pic}(R_e, \kappa)$ reduces to the usual class group whenever R_e is a Krull domain, so the term divisorial in the title of this section will make sense.

Let A be an arbitrary ring and let κ be any kernel functor (cf. [11]) in A-mod. An A-bimodule P is said to be *κ-flat* if for every exact sequence $0 \to K \to M \to N$ in A-mod, where $\kappa(K) = K$, i.e. K is a κ-torsion module, we have that the kernel of the induced map $P \otimes_A M \to P \otimes_A N$ is a κ-torsion module too. An A-bimodule P is *κ-invertible* if $P \otimes_A -$ maps κ-torsion modules to κ-torsion modules and if there exists an A-bimodule Q with the same property such that

$$Q_\kappa(P \otimes_A Q) \cong Q_\kappa(A) \cong Q_\kappa(Q \otimes_A P)$$

(isomorphisms are A-bimodule isomorphisms and Q_κ is the localization functor associated to κ). Each κ-invertible module is κ-flat, cf. [34]. The κ-invertible modules form a group under the operation

$$(M, N) \mapsto Q_\kappa(M \otimes_A N);$$

this group is called the *relative Picard group* for κ. Roughly said a κ-divisorially graded ring R is a G-graded ring such that

$$Q_\kappa(R_\sigma R_\tau) = R_{\sigma\tau}$$

for all $\sigma, \tau \in G$. A very interesting case is obtained by taking for κ the kernel functor σ which is the infimum of the kernel functors associated to the prime ideals of height one. In the 'nice cases' taking the bidual of a module is the same as localizing the module at σ, i.e. $Q_\sigma(M) = M^{**}$. It will now not be surprising to see that most results concerning strong Rees rings which are of a 'Krull type' will have equivalents in the case of divisorial Rees rings.

2.1. *The commutative case*

In this case a divisorial Rees ring may be written as

$$R=\sum_{\gamma\in G} I_\gamma X_\gamma$$

where each I_γ is a divisorial ideal and such that

$$I_\gamma * I_\tau=(I_\gamma I_\tau)^{**}=Q_\sigma(I_\gamma I_\tau)=I_{\gamma\tau} \quad \text{for all } \gamma,\tau\in G.$$

We assume that R_e is a Krull domain here. It is clear that R is a graded subring of KG, where K is the field of fractions of R_e, and that R is up to graded isomorphism completely determined by the group homomorphism

$$G\to \mathrm{Pic}(R_e,\sigma)=\mathrm{Cl}(R_e), \qquad \gamma\mapsto [I_\gamma].$$

Properties

2.1.1. If G is abelian torsion free and if G satisfies the ascending chain condition for cyclic subgroups, then R is a Krull domain. cf. Note 7.4 in [21].

2.1.2. If M is a graded reflexive ($M=M^{**}$) R-module, then there is a canonical graded isomorphism of degree e,

$$Q_\sigma(R\otimes_{R_e} M_e)\cong M,$$

cf. Lemma 3.2 in [21]. From Theorem 3.5 in loc. cit. it follows that R is a generalized crossed product with respect to a suitably defined cocycle; so to obtain the most general definition of a commutative divisorial Rees ring we might allow a cocycle in the definition of the multiplication i.e.

$$X_\gamma X_\tau=C_{\gamma,\tau}X_{\gamma,\tau} \quad \text{for all } \gamma,\tau\in G.$$

Nevertheless we will assume here that C is the trivial cocycle (results will be valid in the more general situation as well).

2.1.3. Let $G=\mathbb{Z}$. An application of some results from [34] yields that the reflexive Brauer group $\beta(R_e)$ equals the graded reflexive Brauer group, $\beta^g(R)$, of R.

2.2. *The P.I. case*

Here we assume that R_e is a relative maximal order in a c.s.a. Q_e; let σ be the kernel functor associated with R_e in the sense of [15]. Examples of relative maximal orders are: maximal orders (σ is trivial), Krull orders (here σ is the central kernel functor associated to the height one prime ideals of the center which is a Krull domain), HNP rings (σ is trivial), tame orders (cf. [16]).

Properties

The divisorial equivalents of 1.2.7, …, 1.2.11, hold, because in [16] these properties were proved in this generality. In the $\mathbb{Z}$-graded case the divisorial equivalents of 1.2.2, 1.2.3, 1.2.5, are very likely to be true but I have not checked them completely as yet. In the P.I. case, the non-commutative version of 2.1.2 holds if σ has the property that $\mathscr{L}(\sigma)$ is G-invariant (the latter is the case in each of the examples given). *In the general case*, i.e. R_e is a relative maximal order in a simple Artinian ring but not necessarily P.I., the divisorial versions of the results in 1.3 are still conjectures (although some parts are obvious in the $\mathbb{Z}$-graded case).

3. Scaled Rees rings

Consider the ring $A = \mathbb{C}[X, -]$ of skew polynomials over $\mathbb{C}$ with respect to conjugation in $\mathbb{C}$. Put $I = (X)$ and consider the strong Rees ring

$$R = \sum_{n \in \mathbb{Z}} I^n Y^n \quad \text{over } A.$$

It is not hard to verify that R is an Azumaya algebra with center:

$$\cdots + I^{-2}Y^{-3} + I^{-2}Y^{-2} + \mathbb{C}Y^{-1} + \mathbb{C} + I^2Y + I^2Y^2 + I^4Y^3 + \cdots$$

This ring looks like a strong Rees ring but it is doubled. We restrict attention to *commutative $\mathbb{Z}$-graded* rings here and then we may generalize the above situation in the following definition. Let $t > 0$ in $\mathbb{N}$; a *scaled Rees ring of level t* is a ring of the form

$$R = \sum_{n \in \mathbb{Z}} I^{\alpha_n(t)} X^n$$

where I is an invertible ideal of R and

$$\alpha_n(t) = [n/t]$$

is the integral part of n/t. It is clear that R is Noetherian if and only if R_0 is Noetherian. Let us assume that R_0 is a Dedekind ring (if R_0 is a Krull domain then one can introduce divisorial scaled Rees rings but we do not go into this here).

3.1. If $t = 1$, then R is integrally closed.

If $t > 1$, then R is integrally closed if and only if I is a semiprime ideal (Van den Bergh, Van Oystaeyen).

In the proof of 3.1 it is essential that R contains the gr-Dedekind ring:

$$\bigoplus_{n \in \mathbb{Z}} P_{nt}.$$

This observation actually leads to a complete description of gr-Dedekind rings (first attempts to this effect appeared in [26]):

3.2. If R is any gr-Dedekind ring, then

$$R=\sum_{n\in\mathbb{Z}} I^{\alpha n}X^n$$

where $\alpha\in Q$ and I is an invertible ideal of R_0 (M. Van den Bergh).

In the proof of 3.2 it is essential that one can define rational powers of an invertible ideal in an unambiguous way such that these powers satisfy the expected rules like: $I^\alpha I^\beta \subset I^{\alpha+\beta}$ etc. (not necessarily equality though!). In general a ring of the form

$$R=\sum_{n\in\mathbb{Z}} I^{\alpha n}X^n$$

where $\alpha\in Q$, and I an invertible ideal of R_0, is called a *lepidopterous Rees ring*. So a lepidopterous Rees ring is a scaled Rees ring if and only if the ideal I is semiprime. Actually one may refine these techniques in order to obtain structure results concerning certain Noetherian graded integrally closed domains. Non-commutative scaled and lepidopterous Rees rings are the topic of some recent research, so out of the scope of this survey.

Finally, the property of the class group map $\mathrm{Cl}(R_e)\to\mathrm{Cl}(R)$ mentioned in 1.1.3 allows an interesting type of applications. Indeed this property allows on many occasions to construct a strong Rees ring R over R_e such that $\mathrm{Cl}(R)=1$, the consequences of this fact (oriality) in the particular situation one is considering may then be pulled back to degree e by graded methods in order to obtain results about R_e. This idea has been used in [23] in order to investigate diagonalization of matrices over Dedekind domains.

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