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# Solenoids

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Circles are relatively simple geometric objects, although the theory of *functions on them*, Fourier series, is already complicated enough to have upset people in the 19<sup>th</sup> century, and to have precipitated the creation of set theory by Cantor *circa* 1880. But we postpone the function theory to another time. Here, we just consider circles and simple mappings among circles.

Surprisingly, automorphism groups of *families* of circles connected by simple sorts of maps bring to light structures and objects that are invisible when looking at a single circle rather than the aggregate.

In particular, we discover *p*-adic numbers  $\mathbb{Q}_p$  inside automorphism groups of families of circles, and even the *adeles*  $\mathbb{A}$ . These appearances are far more important than *ad hoc* definitions as completions with respect to metrics (recalled later). That is, *p*-adic numbers and the adeles appear *inevitably* in the study of modestly complicated structures, and play a *dynamic* role as parts of automorphism groups.

This discussion of automorphisms of families of circles is a warm-up to the more complicated situation of automorphisms of families of *higher-dimensional* objects<sup>[1]</sup> acted upon by *non-abelian* groups.

In all cases, an underlying theme is that when a group  $G$  acts<sup>[2]</sup> *transitively*<sup>[3]</sup> on a set  $X$ , then  $X$  is in bijection with  $G/G_x$ , where  $G_x$  is the isotropy subgroup<sup>[4]</sup> in  $G$  of a chosen base point  $x$  in  $X$ , by  $gG_x \rightarrow gx$ . The point is that such sets  $X$  are really *quotients of the group*  $G$ . Topological and other structures also correspond, under mild hypotheses. Isomorphisms  $X \approx G/G_x$  are informative and useful, as we will see later.

- The 2-solenoid
- Automorphisms of solenoids
- A cleaner viewpoint
- Automorphisms of solenoids, again
- Appendix: uniqueness of projective limits
- Appendix: topology of  $X \approx G/G_x$

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## 1. The 2-solenoid

As reported by MacLane in his autobiography, around 1942 Eilenberg talked to MacLane (in Michigan) about families of circles related to each other by repeated *windings*, for example, double coverings, and trying to understand the *limiting object*. We make this precise and repeat some of the relevant discussion. The point is that a surprisingly complicated physical object can be made from families of *circles* related in simple ways.

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[1] The next example in mind is *modular curves*, which are two-dimensional. Their definition is considerably more complicated than that of a circle, and requires commensurately more preparation.

[2] Of course, the spirit of the notion of *action* of  $G$  on  $X$  is that  $G$  moves around elements of the set  $X$ . But a little more precision is needed. Recall that an action of  $G$  on a set  $X$  is a map  $G \times X \rightarrow X$  such that  $1_G \cdot x = x$  for all  $x \in X$ , and  $(gh)x = g(hx)$  for  $g, h \in G$  and  $x \in X$ .

[3] Recall that a group  $G$  acts *transitively* on a set  $X$  if, for all  $x, y \in X$ , there is  $g$  in  $G$  such that  $gx = y$ .

[4] Recall that the *isotropy subgroup*  $G_x$  of a point  $x$  in a set  $X$  on which  $G$  acts is the subgroup of  $G$  *fixing*  $x$ , that is,  $G_x$  is the subgroup of  $g \in G$  such that  $gx = x$ .

As a handy model for the circle  $S^1$  we take  $S^1 = \mathbb{R}/\mathbb{Z}$ .<sup>[5]</sup> Eilenberg (and MacLane) considered a family of circles and maps

$$\dots \xrightarrow{\times 2} \mathbb{R}/\mathbb{Z} \xrightarrow{\times 2} \mathbb{R}/\mathbb{Z} \xrightarrow{\times 2} \mathbb{R}/\mathbb{Z}$$

where each circle mapped to the next by *doubling* itself onto the smaller circle.<sup>[6]</sup> More precisely, this is literally multiplication by 2 on the quotients  $\mathbb{R}/\mathbb{Z}$ , namely

$$x + \mathbb{Z} \rightarrow 2x + \mathbb{Z}$$

for  $x \in \mathbb{R}$ . Since  $2\mathbb{Z} \subset \mathbb{Z}$  this is well-defined. That is, each circle is a *double cover* of the circle to its immediate right in the sequence. This sequence of circles with doubling maps is the **2-solenoid**.<sup>[7]</sup> We might ask *what is the limiting object*

$$??? \quad \dots \xrightarrow{\times 2} \mathbb{R}/\mathbb{Z} \xrightarrow{\times 2} \mathbb{R}/\mathbb{Z} \xrightarrow{\times 2} \mathbb{R}/\mathbb{Z}$$

Part of the issue is to say what we might *mean* by this question.

A different but topologically equivalent model is a little more convenient for our discussion. Consider the sequence

$$\dots \xrightarrow{\varphi_{43}} \mathbb{R}/8\mathbb{Z} \xrightarrow{\varphi_{32}} \mathbb{R}/4\mathbb{Z} \xrightarrow{\varphi_{21}} \mathbb{R}/2\mathbb{Z} \xrightarrow{\varphi_{10}} \mathbb{R}/\mathbb{Z}$$

where each map  $\varphi_{n,n-1} : \mathbb{R}/2^n\mathbb{Z} \rightarrow \mathbb{R}/2^{n-1}\mathbb{Z}$  is induced from the identity map on  $\mathbb{R}$  in the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\text{id}} & \mathbb{R} \\ \text{mod } 2^n \downarrow & & \downarrow \text{mod } 2^{n-1} \\ \mathbb{R}/2^n\mathbb{Z} & \xrightarrow{\varphi_{n,n-1}} & \mathbb{R}/2^{n-1}\mathbb{Z} \end{array}$$

That is, this is the map

$$\varphi_{n,n-1} : x + 2^n\mathbb{Z} \rightarrow x + 2^{n-1}\mathbb{Z}$$

This second model has the advantage that the maps  $\varphi_{n,n-1}$  are locally distance-preserving on the circles. In the first model each map *stretches* the circle by a factor of 2. In the second model as we move to the left in the sequence of circles the circles get larger. Also, in the second model there is the single copy of  $\mathbb{R}$  lying over (or *uniformizing*) all the circles.

Again we would like to ask *what is the limit of these circles?* Note that this use of *limit* is ambiguous, and we cannot be sure *a priori* that there is any potential sense to be made of this. Presumably we expect the limit to be a *topological space*.

**[1.0.1] Remark:** A plausible, naive guess would be that since  $\bigcap_n 2^n\mathbb{Z} = \{0\}$ , that perhaps the limit is  $\mathbb{R}/\{0\} \approx \mathbb{R}$ .

<sup>[5]</sup> One could also take the unit circle in  $\mathbb{C}$ . The map  $x \rightarrow e^{2\pi ix}$  mapping  $\mathbb{R} \rightarrow \mathbb{C}$  factors through the quotient  $\mathbb{R}/\mathbb{Z}$ , showing that these two things are the same.

<sup>[6]</sup> The integer 2 could be replaced with other integers, and, for that matter, the *sequence*  $2, 2, 2, \dots$  could be replaced with other sequences of integers. Qualitatively, these other choices give similar things, though the details are significantly different.

<sup>[7]</sup> The name is by analogy with the wiring in electrical motors and inductance circuits, where things are made by repeated winding. Since the limiting object here is allegedly created by repeated *unwinding*, it might be more apt to call it an *anti-solenoid*.

A precise version of the question is the following. Consider

$$\cdots \xrightarrow{\varphi_{n+1,n}} X_n \xrightarrow{\varphi_{n,n-1}} X_{n-1} \xrightarrow{\varphi_{n-1,n-2}} \cdots \xrightarrow{\varphi_{21}} X_1 \xrightarrow{\varphi_{10}} X_0$$

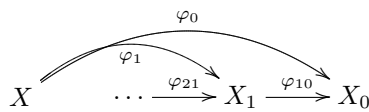
with topological spaces  $X_i$  and continuous **transition maps**  $\varphi_{i,i-1}$ . The **(projective) limit**  $X$  of the  $X_n$ , written

$$X = \lim_i X_i \quad (\text{dangerously suppressing reference to transition maps } \varphi_{i,i-1})$$

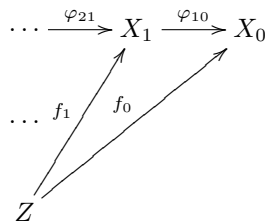
is a topological space  $X$  and maps  $\varphi_n : X \rightarrow X_n$  **compatible with** the transition maps  $\varphi_{n,n-1} : X_n \rightarrow X_{n-1}$  in the sense that

$$\varphi_{n-1} = \varphi_{n,n-1} \circ \varphi_n$$

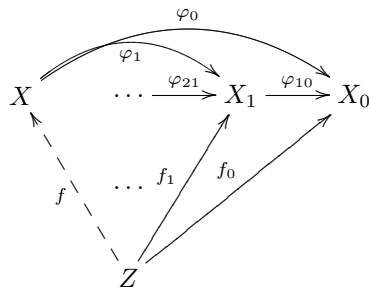
and such that, for any *other* space  $Z$  with maps  $f_n : Z \rightarrow X_n$  **compatible with** the maps  $\varphi_{n,n-1}$  (that is,  $f_{n-1} = \varphi_{n,n-1} \circ f_n$ ), there is a unique  $f : Z \rightarrow X$  through which all the maps  $f_n$  factor. That is, in pictures, first, all the (curvy) triangles commute in



and, for all families of maps  $f_i : Z \rightarrow X_i$  such that all triangles commute in



there is a unique map  $f : Z \rightarrow X$  such that all triangles commute in



**[1.0.2] Remark:** Note that the definition of the limit definitely *does* depend on the *transition maps* among the objects of which we take the limit, not just on the *objects*.

As usual with mapping-property definitions:

**[1.0.3] Theorem:** If a (projective) limit exists it is unique up to unique isomorphism. (*Proof in appendix: it works for the usual abstract reasons.*)

A little more concretely, we can prove *existence* of limits from existence of *products* (at least for topological spaces):

[1.0.4] **Proposition:** A limit  $X$  (and maps  $\varphi_i : X \rightarrow X_i$ ) of a family

$$\cdots \xrightarrow{\varphi_{n+1,n}} X_n \xrightarrow{\varphi_{n,n-1}} X_{n-1} \xrightarrow{\varphi_{n-1,n-2}} \cdots \xrightarrow{\varphi_{21}} X_1 \xrightarrow{\varphi_{10}} X_0$$

is a subset  $X$  (with the subset topology)<sup>[8]</sup> of the product<sup>[9]</sup>  $Y = \prod_i X_i$  (with projections  $p_i : Y \rightarrow X_i$ ) on which the projections are compatible with the transition maps  $\varphi_{i,i-1}$ , that is,

$$X = \{x \in Y : \varphi_{i,i-1}(p_i(x)) = p_{i-1}(x) \text{ for all } i\}$$

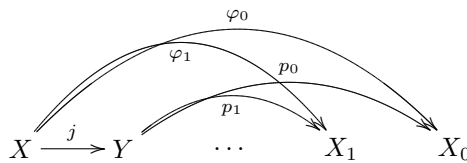
with maps  $\varphi_i$  obtained by restriction of the projection maps  $p_i$  from the whole product to  $X$ , namely

$$\varphi_i = p_i|_X : X \rightarrow X_i$$

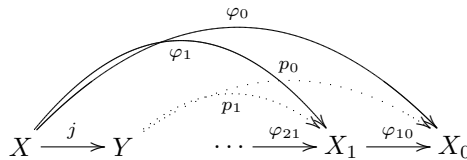
*Proof:* First, let  $j : X \rightarrow Y$  be the inclusion of  $X$  into  $Y$ , and let  $\varphi_i : X \rightarrow X_i$  be the restriction of the projection  $p_i : Y \rightarrow X_i$  to the subset  $X$  of the product  $Y$ . That is,

$$\varphi_i = p_i \circ j$$

Then, before thinking about any other space  $Z$  and other maps, we do have a diagram



with commuting (curvy) *solid* triangles. While the maps from  $Y$  do *not* respect the transition maps  $\varphi_{i,i-1} : X_i \rightarrow X_{i-1}$ , by the very definition of the subset  $X$  of  $Y$ , the restrictions  $\varphi_i = p_i \circ j$  of the projections  $p_i$  to  $X$  *do* respect the transition maps. Thus, the *solid* triangles commute in the diagram



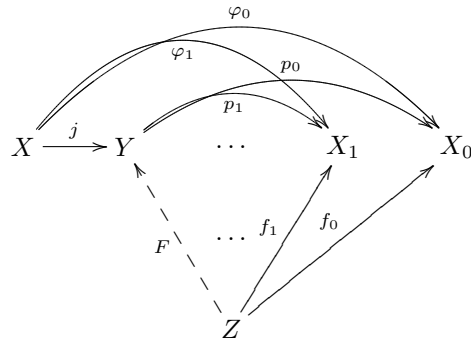
but *not* necessarily any triangle involving dotted arrows.

Now consider another space  $Z$ . By the mapping properties of the product, for *any* collection of maps  $f_i : Z \rightarrow X_i$  (not only those meeting the compatibility condition  $\varphi_{i,i-1} \circ f_i = f_{i-1}$ ) there is a unique

[8] The *subset topology* on a subset  $X$  of a topological space  $Y$  can be characterized as the topology on  $X$  such that the inclusion map  $j : X \rightarrow Y$  is continuous, and such that every continuous map  $f : Z \rightarrow Y$  from another space  $Z$  such that  $f(Z) \subset X$  factors through the inclusion. That is, there a continuous  $F : Z \rightarrow X$  such that  $f = j \circ F$ . This does not prove existence. Also, one can show that the subspace topology is the coarsest topology on  $X$  such that the inclusion  $X \rightarrow Y$  is continuous. Finally, the *construction* of this topology, which proves *existence*, is that a set  $U$  in  $X$  is open if and only if there exists an open  $V$  in  $Y$  such that  $U = X \cap V$ .

[9] By now we know that the usual product can be characterized *intrinsically*, and that intrinsic characterization is all we use in this proposition.

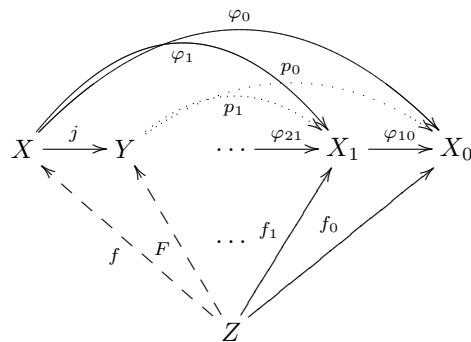
$F : Z \rightarrow Y$  through which all the projections  $p_i : Y \rightarrow X_i$  factor. That is, we have a unique  $F$  such that the triangles commute in the diagram



Note that we cannot include the transition maps in the diagram since the projections  $p_i : Y \rightarrow X_i$  do not respect them. But, since the maps  $f_i$  are compatible with the maps  $\varphi_{i,i-1}$ , we could suspect that the image  $F(Z) \subset Y$  is a smaller subset of the product  $Y$ . Indeed, for  $z \in Z$ , using the compatibility

$$p_{i-1}(F(z)) = f_{i-1}(z) = \varphi_{i,i-1}(f_i(z)) = \varphi_{i,i-1}(p_i(F(z)))$$

we see that  $F(Z) \subset X$ , as claimed. That is,  $F$  factors through the inclusion map  $j : X \rightarrow Y$ , and the composites  $p_i \circ F$  factor through  $j : X \rightarrow Y$ , giving a picture with commuting *solid or dashed (but not dotted)* triangles



(Again, the projections from  $Y$  do not respect the transition maps.) That is, with the compatibility conditions, the maps from  $Z$  *do* factor through the subset  $X$  of the product. ///

This general argument gives some surprising *qualitative* information about projective limits:

[1.0.5] **Corollary:** The projective limit of a family  $X_i$  of *compact*,<sup>[10]</sup> spaces is *compact*.

[1.0.6] **Remark:** In particular, the (projective) limit of circles is *compact*, since circles (with their usual topologies) are compact. In particular, it cannot be  $\mathbb{R}$ , which is non-compact!

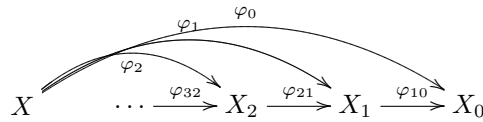
*Proof:* (of corollary) The product  $Y$  of a family of compact spaces is compact. This is exactly the content of *Tychonoff's theorem*. The compatibility conditions  $\varphi_{i,i-1}(p_i(x)) = p_{i-1}(x)$  are *closed* conditions in the sense that

$$\{x \in Y : \varphi_{i,i-1}(p_i(x)) = p_{i-1}(x)\} = \text{closed set in } Y$$

<sup>[10]</sup> We need a better definition of *compact* than the metric-space definition that *every sequence contains a convergent subsequence*. Instead, we need the definition that both applies to general topological spaces *and* is more useful. That is, first in words, a set  $E$  inside a topological space is *compact* if *every open cover admits a finite subcover*. That is, for  $E \subset \bigcup_{i \in I} U_i$  with opens  $U_i$ , there is a finite subset  $I_o$  of  $I$  such that still  $E \subset \bigcup_{i \in I_o} U_i$ . It is not obvious that this definition is superior to the sequence definition.

since the maps  $p_i$  and  $\varphi_{i,i-1}$  are continuous and since the circles are *Hausdorff*.<sup>[11]</sup> <sup>[12]</sup> <sup>[13]</sup> The intersection of an arbitrary family of closed sets is closed,<sup>[14]</sup> so the (projective limit)  $X$  of points  $x$  meeting this condition for *all*  $i$ , is closed. And in a compact space  $Y$ , closed subsets are compact.<sup>[15]</sup> ///

**[1.0.7] Remark:** By paraphrasing the assertion of the proposition, we now have a concrete (if not perfectly useful) model of the limit  $X$  in a diagram



with spaces indexed by non-negative integers, namely, the collection of all sequences  $x_0, x_1, x_2, \dots$  such that the transition maps  $\varphi_{n,n-1}$  map them to each other, that is,

$$\varphi_{n,n-1}(x_n) = x_{n-1}$$

for all indices  $n$ . This follows from the usual model of the *product* as Cartesian product, which for countable products can be written as the collection of *all* sequences  $x_0, x_1, x_2, \dots$  with  $x_i \in X_i$ . We may choose to write a *compatible* family of elements as

$$\dots \rightarrow x_3 \rightarrow x_2 \rightarrow x_1 \rightarrow x_0$$

This description of the limit as a set of sequences is *deficient* in several regards (for example, it does not tell us a *topology*), but it is occasionally useful, certainly as a heuristic.

[11] Recall that a topological space is *Hausdorff* if any two points have disjoint neighborhoods. It is useful to know that  $Z$  is Hausdorff if and only if the diagonal  $Z^\Delta = \{(z, z) \in Z \times Z : z \in Z\}$  is *closed* in  $Z \times Z$ . Indeed, for  $Z$  Hausdorff, points  $x \neq y$  in  $Z$  have disjoint neighborhoods  $U$  and  $V$ . Then  $U \times V$  is open in the product topology in  $Z \times Z$ , contains  $x \times y$ , and since  $U \cap V = \emptyset$  the set  $U \times V$  does not meet the diagonal  $\{(z, z) : z \in Z\}$  in  $Z \times Z$ . Thus, the diagonal is the complement of the union of all such opens  $U \times V$ , so is closed. The converse reverses the argument: for closed diagonal, given  $x \neq y$  in  $Z$ , there is an open  $U \times V$  containing  $x \times y$  and not meeting the diagonal, since the product topology has sets  $U \times V$  as a basis. Since  $U \times V$  does not meet the diagonal,  $U$  and  $V$  are disjoint neighborhoods of  $x, y$  in  $Z$ .

[12] As a critical auxiliary point, we should note that for any topological space  $X$  the diagonal imbedding  $\delta : X \rightarrow X \times X$  by  $\delta(x) = (x, x)$  is a homeomorphism (topological isomorphism) to the image, with the subspace topology. Certainly  $\delta$  is a set bijection. For a neighborhood  $U$  of  $x$  in  $X$ , the open  $U \times U$  in  $X \times X$  meets  $\delta(X)$  at  $\delta(U)$ . On the other hand, given opens  $U, V$  in  $X$ , the basis open  $U \times V$  in  $X \times X$  meets  $\delta(X)$  in  $\delta(U \cap V)$ , and, indeed,  $U \cap V$  is open. Thus, the images by  $\delta$  of opens are open, and vice-versa.

[13] Characterizing Hausdorff-ness by the closed-ness of the diagonal is useful to show that for continuous maps  $f : X \rightarrow Z$  and  $g : X \rightarrow Z$  with  $Z$  Hausdorff, the set  $\{x \in X : f(x) = g(x)\}$  is *closed*, as follows. The map  $(f \times g)(x, y) = f(x) \times g(y)$  from  $X \times X$  to  $Z \times Z$  is continuous, that is, inverse images of opens are open. Then inverse images of closed sets are closed, and the inverse image of  $Z^\Delta$  under  $f \times g$  is closed. The intersection of the diagonal with the inverse image of  $Z^\Delta$  by  $f \times g$  is  $\{(x, x) : f(x) = g(x)\}$ . Closed-ness in  $X \times X$  gives closedness in the diagonal (with the subspace topology), and we just noted that the diagonal is homeomorphic (topologically isomorphic) to  $X$ .

[14] That an arbitrary intersection of closed sets is closed is equivalent to the defining property that an arbitrary union of open sets is open, since a set is closed if and only if its complement is open.

[15] This important fact is easy to prove: let  $E$  be a closed subset of a compact space  $Y$ , and let  $\{U_i : i \in I\}$  be an open cover of  $E$ . Let  $U = Y - E$ . Then  $\{U_i : i \in I\} \cup \{U\}$  is a cover of the entire space  $Y$ . By the compactness of  $Y$ , there is a finite subcover  $U_1, \dots, U_n, U$ . (If  $E \neq Y$  the subcover must use  $U$ .) Then  $U_1, \dots, U_n$  is a finite cover of  $E$ .

## 2. Automorphisms of solenoids

Even without trying to imagine what meaning to attach to a solenoid  $X$  or other limit object, we can *directly* make sense of *automorphisms* of  $X$  by looking at automorphisms<sup>[16]</sup> of the *diagram*. Then, with a large-enough group  $G$  of automorphisms to act *transitively*<sup>[17]</sup> on  $X$ , we can write  $X$  as a *quotient*

$$X \approx G/G_x = \{G_x\text{-cosets in } G\} = \{gG_x : g \in G\}$$

of  $G$ , where  $G_x$  is the *isotropy subgroup*<sup>[18]</sup> (in  $G$ ) of a point  $x$  in  $X$ .

One virtue of identifying automorphisms is that this might be done piece-by-piece, identifying subgroups of the whole group, then assembling them at the end. And it is important to note that this is an isomorphism of  $G$ -spaces, meaning (topological) spaces  $A, B$  on which  $G$  acts continuously. As expected, a map of  $G$ -spaces is a set map  $\psi : A \rightarrow B$  such that

$$\psi(g \cdot a) = g \cdot \psi(a)$$

for  $a \in A, g \in G$ , where on the left the action is of  $G$  on  $A$ , and on the right it is the action on  $B$ .

True, the solenoid is itself a group already, being a projective limit of groups, so this approach might seem silly. However, we can present the solenoid as a quotient of *more familiar* (and simpler) objects. In any case, since we'll consider an *abelian* group  $G$  of automorphisms of the solenoid, any group quotient  $G/G_x$  is again a group,<sup>[19]</sup> and, incidentally,  $G_x$  and the quotient are independent of  $x$ .

Thus, even without thinking of projective limits, one kind<sup>[20]</sup> of **automorphism**  $f$  of the 2-solenoid is a collection of maps  $f_n : \mathbb{R}/2^n\mathbb{Z} \rightarrow \mathbb{R}/2^n\mathbb{Z}$  such that all squares commute in the diagram

$$\begin{array}{ccccccc} \dots & \xrightarrow{\varphi_{43}} & \mathbb{R}/8\mathbb{Z} & \xrightarrow{\varphi_{32}} & \mathbb{R}/4\mathbb{Z} & \xrightarrow{\varphi_{21}} & \mathbb{R}/2\mathbb{Z} & \xrightarrow{\varphi_{10}} & \mathbb{R}/\mathbb{Z} \\ & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ \dots & \xrightarrow{\varphi_{43}} & \mathbb{R}/8\mathbb{Z} & \xrightarrow{\varphi_{32}} & \mathbb{R}/4\mathbb{Z} & \xrightarrow{\varphi_{21}} & \mathbb{R}/2\mathbb{Z} & \xrightarrow{\varphi_{10}} & \mathbb{R}/\mathbb{Z} \end{array}$$

Without being too extravagant<sup>[21]</sup> we want to think of some obvious families of maps  $f_n$ . Since all our circles are quotients of  $\mathbb{R}$  in a compatible fashion, we can certainly create a simple sort of family of maps  $f_n$  by letting  $r \in \mathbb{R}$  act, by

$$f_n(x_n + 2^n\mathbb{Z}) = x_n + r + 2^n\mathbb{Z}$$

with the same real number  $r$  for every index.<sup>[22]</sup>

[16] Note that this discussion is different from the argument that objects defined by mapping properties have no endomorphisms that leave the other objects unmoved. Here we *are moving* the objects in the diagram.

[17] Again, for a group to act *transitively* means that  $G$  moves any point of  $X$  to any other point, that is, for  $x, y \in X$  there is  $g \in G$  such that  $gx = y$ .

[18] Again, with a group  $G$  acting on a set  $X$ , the *isotropy subgroup*  $G_x$  of an element  $x \in X$  is the subgroup *not moving*  $x$ , that is,  $G_x = \{g \in G : gx = x\}$ . It is straightforward to see that this is a *subgroup* of  $G$ , not merely a *subset*.

[19] We will attend to topological details slightly later.

[20] It is not *a priori* clear that any useful collection of maps would necessarily send each  $\mathbb{R}/2^n\mathbb{Z}$  to itself, but this will suffice for now.

[21] But sometimes extravagance can have a simplicity that is hard to achieve otherwise.

[22] And the  $x_i \in \mathbb{R}/2^i\mathbb{Z}$  are chosen compatibly in the first place, that is, such that  $(x_i + 2^i\mathbb{Z}) + 2^{i-1}\mathbb{Z} = x_{i-1} + 2^{i-1}\mathbb{Z}$ , for all indices  $i$ .

[2.0.1] **Remark:** We are neglecting continuity, but will return to this point when we recapitulate this discussion of automorphisms in a form that is better suited to discussion of the topology. This copy of  $\mathbb{R}$  does act continuously. One may verify that it is *not* transitive, and that the isotropy groups in  $\mathbb{R}$  of points in the solenoid are *trivial*. Thus, overlooking the failure of the action to be transitive, one might naively imagine that the limit is a copy of  $\mathbb{R}$ . True, the orbit  $\mathbb{R} \cdot x$  of any given point is *dense*<sup>[23]</sup> in the solenoid, but it is not *closed*.<sup>[24]</sup>

Another relatively simple family of maps is created by taking a sequence of *integers*  $y_n$  and maps

$$f_n(x_n + 2^n\mathbb{Z}) = x_n + y_n + 2^n\mathbb{Z}$$

and *requiring* that the sequence  $y_n$  be chosen so that the squares in the diagram commute. That is, we must have

$$(x_n + y_n + 2^n\mathbb{Z}) + 2^{n-1}\mathbb{Z} = x_{n-1} + y_{n-1} + 2^{n-1}\mathbb{Z}$$

Since already

$$(x_n + 2^n\mathbb{Z}) + 2^{n-1}\mathbb{Z} = x_{n-1} + 2^{n-1}\mathbb{Z}$$

it is necessary and sufficient that

$$(y_n + 2^n\mathbb{Z}) + 2^{n-1}\mathbb{Z} = y_{n-1} + 2^{n-1}\mathbb{Z}$$

That is, the compatible sequence of integers  $y_n$  gives an element in another projective limit, the **2-adic integers**<sup>[25]</sup>  $\mathbb{Z}_2$ .

$$\dots \xrightarrow{\text{mod } 8} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\text{mod } 4} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\text{mod } 1} \mathbb{Z}/\mathbb{Z}$$

Each of the limit objects is *finite*, so certainly *compact*. Thus, this projective limit is *compact*, whatever other features it may have.

Still without worrying about the topology, we claim

[2.0.2] **Proposition:** The product group  $\mathbb{R} \times \mathbb{Z}_2$  acts transitively on the 2-solenoid. The point  $\rightarrow 0 \rightarrow 0 \rightarrow 0$  in the solenoid has isotropy group which is the diagonally imbedded copy of the integers

$$\mathbb{Z}^\Delta = \{(\ell, -\ell) \in (\mathbb{Z} \times \mathbb{Z}) \subset \mathbb{R} \times \mathbb{Z}_2 : \ell \in \mathbb{Z}\}$$

*Proof:* Given a compatible family

$$\dots \rightarrow x_3 + 8\mathbb{Z} \rightarrow x_2 + 4\mathbb{Z} \rightarrow x_1 + 2\mathbb{Z} \rightarrow x_0 + \mathbb{Z}$$

of elements  $x_n + 2^n\mathbb{Z} \in \mathbb{R}/2^n\mathbb{Z}$ , act by  $r \in \mathbb{R}$  as above such that  $x_0 + r = 0 \in \mathbb{R}/\mathbb{Z}$ . Since the  $x_n$ 's are compatible, it must be that  $(r + x_1) \bmod 1 = (x_0 + r) = 0$ ,  $(x_2 + r) \bmod 2 = (x_1 + r)$ ,  $(x_3 + r) \bmod 4 = x_2 + r$ , and so on. That is, every  $x_n + r \in \mathbb{Z}$ , and the sequence  $y_n = x_n + r$  gives a compatible family

$$\dots \rightarrow y_3 + 8\mathbb{Z} \rightarrow y_2 + 4\mathbb{Z} \rightarrow y_1 + 2\mathbb{Z} \rightarrow y_0 + \mathbb{Z}$$

[23] Recall that a subset  $E$  of a topological space  $X$  is *dense* if every non-empty open set in  $X$  has non-empty intersection with  $E$ .

[24] This highly-wound copy of  $\mathbb{R}$  may be the thing that earned the name *solenoid*.

[25] Replacing 2 by another prime  $p$  throughout gives a  $p$ -solenoid and  $p$ -adic integers  $\mathbb{Z}_p$ . This approach is not the most conventional way to present the  $p$ -adic integers, but *does* illustrate the role that  $\mathbb{Z}_p$  plays in situations that are not obviously number-theoretic. We will review a more conventional description of  $\mathbb{Z}_p$  later, for comparison.



which gives an element in  $\mathbb{Z}_2$ . That is, the further action by  $-y_n$  on the solenoid will send every element to

$$(x_n + r) - y_n = (x_n + r) - (x_n + r) = 0$$

This proves the transitivity.

To determine the isotropy group of a point, suppose that  $r$  is a real number and the  $y_n$  is an integer modulo  $2^n$ , such that the 0-element

$$\dots 0 \rightarrow 0 \rightarrow 0 \rightarrow 0$$

is mapped to itself. That is, require that

$$0 + r + y_n \in 0 + 2^n \mathbb{Z}$$

for all  $n$ . First, this implies that  $r \in \mathbb{Z}$ . Then  $y_n$ , which is only determined modulo  $2^n$  anyway, is completely determined modulo  $2^n$  by

$$y_n + 2^n \mathbb{Z} = -r + 2^n \mathbb{Z}$$

That is,  $y_n = -r \pmod{2^n}$ . And these conditions are visibly *sufficient*, as well, to fix the 0. Thus, the isotropy group truly is the diagonal copy of  $\mathbb{Z}$ . ///

[2.0.3] **Corollary:** (Still not worrying about the topology) the 2-solenoid is isomorphic to the quotient

$$(\mathbb{R} \times \mathbb{Z}_2) / \mathbb{Z}^\Delta$$

*Proof:* Notably ignoring the topology, whenever a group  $G$  acts transitively on a set  $X$  containing a chosen element  $x$ , there is a bijection

$$X \longleftrightarrow G/G_x = \{gG_x : g \in G\}$$

by

$$gx \longleftrightarrow gG_x$$

A map from  $G$  to  $X$  by  $g \rightarrow gx$  is a *surjection*, since  $G$  is transitive. This map factors through  $G/G_x$  and is *injective*, since  $gx = hx$  if and only if  $h^{-1}gx = x$ , if and only if  $h^{-1}g \in G_x$ , if and only if  $gG_x = hG_x$ . ///

[2.0.4] **Remark:** We need a somewhat better set-up to keep track of the topologies.

---

### 3. A cleaner viewpoint

Having run through an informative heuristic about the structure of the solenoid as a quotient  $G/G_x$ , we can redo things more elegantly, and thereby *not* lose sight of the topological features of the situation.

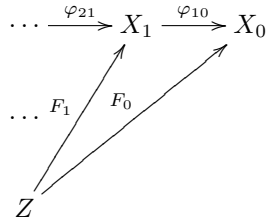
First, in any projective limit, families of maps<sup>[26]</sup>  $f_n : X_n \rightarrow X_n$  such that all squares commute in

$$\begin{array}{ccccc} \dots & \xrightarrow{\varphi_{32}} & X_2 & \xrightarrow{\varphi_{21}} & X_1 & \xrightarrow{\varphi_{10}} & X_0 \\ & & \uparrow f_2 & & \uparrow f_1 & & \uparrow f_0 \\ \dots & \xrightarrow{\varphi_{32}} & X_2 & \xrightarrow{\varphi_{21}} & X_1 & \xrightarrow{\varphi_{10}} & X_0 \end{array}$$

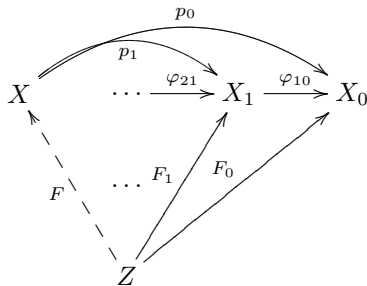
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[26] Again, *maps* here are continuous maps, but the arguments do not use this explicitly.

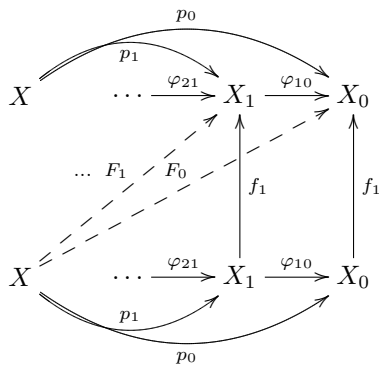
do give rise to a map  $f : X \rightarrow X$  of the projective limit  $X = \lim_n X_n$  to itself, as follows. Again, from the definition of the projective limit  $X$  of the  $X_n$ , to give a map  $F : Z \rightarrow X$  is to give a compatible family of maps  $F_n : Z \rightarrow X_n$ , meaning that all triangles commute in



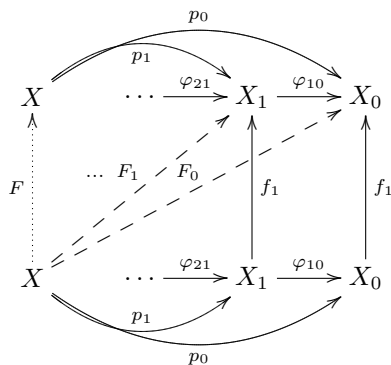
This induces a unique map  $F : Z \rightarrow X$ , making a commutative diagram



In particular, for a compatible family of maps  $f_n : X_n \rightarrow X_n$  we can take  $Z = X$  and  $F_n = f_n \circ p_n$ , giving a commutative



which then yields a unique  $F : X \rightarrow X$  in a commutative diagram



That is, automorphisms of *diagrams* (in the sense of the previous section) do give automorphisms of the projective limit objects attached to the diagrams.

We also observe that we can identify *points* in the projective limit as *compatible sequences*

$$\dots \rightarrow x_3 \rightarrow x_2 \rightarrow x_1 \rightarrow x_0$$

with  $x_n \in X_n$  (and the compatibility  $\varphi_{n,n-1}(x_n) = x_{n-1}$ ) *without* using the Cartesian product model of the product and identifying the projective limit inside that Cartesian product. To do so, recall the trick that for any set  $Y$  and for  $\{s\}$  a set with a single element, we have a natural bijection

$$\mu_Y : Y = \{\text{elements of } Y\} \longleftrightarrow \{\text{maps } \{s\} \rightarrow Y\}$$

by

$$\mu_Y : y \rightarrow f \text{ with } f(s) = y$$

These maps  $\mu_Y$  are *natural* in the precise sense that for a set map  $f : Y \rightarrow Z$ , we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\mu_Y} & \{\text{maps } \{s\} \rightarrow Y\} \\ f \downarrow & & \downarrow f \circ - \\ Z & \xrightarrow{\mu_Z} & \{\text{maps } \{s\} \rightarrow Z\} \end{array}$$

where  $f \circ -$  is *post* composition with  $f$ , that is  $\varphi \rightarrow f \circ \varphi$ . And since maps to a projective limit  $X = \lim X_n$  are given exactly by compatible family of maps to the  $X_n$ , maps of  $S = \{s\}$  to  $X$  are given by compatible families of maps to the  $X_n$  as in

$$\begin{array}{ccccc} & & p_0 & & \\ & & \curvearrowright & & \\ & & p_1 & & \\ X & \xrightarrow{\quad} & X_1 & \xrightarrow{\varphi_{10}} & X_0 \\ & \cdots & \xrightarrow{\varphi_{21}} & & \\ & & \uparrow & & \\ & & \{s\} & & \end{array}$$

That is, *elements* of  $X$  are given by compatible families of elements of the  $X_n$ , as claimed. This will be useful in proving *transitivity* of a group action.

A **topological group** is a group  $G$  which has a topology such that multiplication  $g \times h \rightarrow gh$  and inversion  $g \rightarrow g^{-1}$  are continuous maps  $G \times G \rightarrow G$  and  $G \rightarrow G$ , and  $G$  is *locally compact*<sup>[27]</sup> and *Hausdorff*.<sup>[28]</sup> Further, it is often necessary or wise to require that a topological group have a *countable basis*.<sup>[29]</sup> An action of a topological group  $G$  on a topological space  $X$  is **continuous** if the map

$$G \times X \rightarrow X \text{ by } g \times x \rightarrow gx \text{ is continuous}$$

Now we see how to get an action of a topological group  $G$  on a projective limit.

**[3.0.1] Claim:** Let  $a_n : G \times X_n \rightarrow X_n$  be continuous group actions of a topological group  $G$  on topological spaces  $X_n$ , and suppose that these actions are *compatible* in the sense that squares commute in the diagram

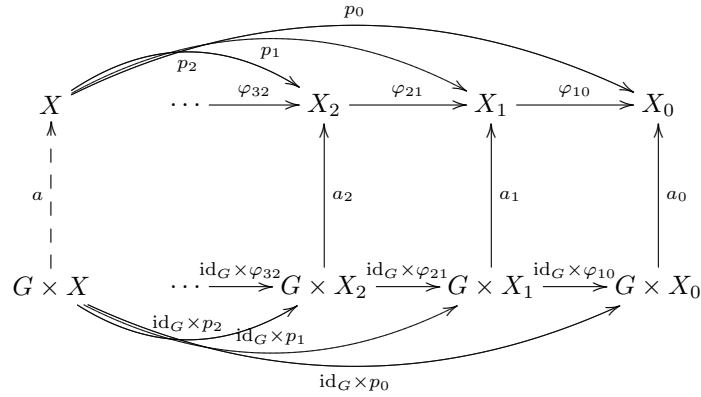
$$\begin{array}{ccccccc} \dots & \xrightarrow{\varphi_{32}} & X_2 & \xrightarrow{\varphi_{21}} & X_1 & \xrightarrow{\varphi_{10}} & X_0 \\ & & \uparrow a_2 & & \uparrow a_1 & & \uparrow a_0 \\ \dots & \xrightarrow{\text{id}_G \times \varphi_{32}} & G \times X_2 & \xrightarrow{\text{id}_G \times \varphi_{21}} & G \times X_1 & \xrightarrow{\text{id}_G \times \varphi_{10}} & G \times X_0 \end{array}$$

[27] A topological space is *locally compact* if there is a basis of (open) subsets each having *compact closure*.

[28] A topological space is *Hausdorff* if any two distinct points have neighborhoods disjoint from each other.

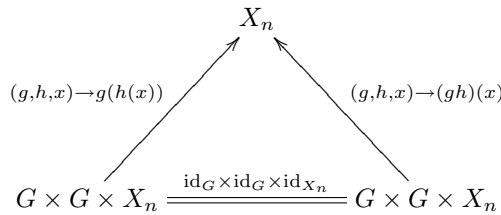
[29] A topological space  $X$  has a *countable basis* if, as suggested by the terminology, it has a basis that is countable.

Then there is a unique continuous group action  $a : G \times X \rightarrow X$  on the projective limit  $X$  such that we have a commutative diagram<sup>[30]</sup>



*Proof:* Composing the maps  $\text{id}_G \times p_n : G \times X \rightarrow G \times X_n$  with the action map  $a_n : G \times X_n \rightarrow X_n$  gives a compatible family of maps  $G \times X \rightarrow X_n$ . By definition of the projective limit  $X$ , we have a unique map  $G \times X \rightarrow X$  making the diagram commute, as claimed.

But we should really check the associativity property  $(gh)x = g(hx)$  required of a group action, with  $g, h \in G$  and  $x \in X$ , not to mention the condition  $e_G x = x$ . (Unsurprisingly, it turns out fine.) We need to rewrite the associativity in terms of maps. In a diagram, the associativity of the action on  $X_n$  asserts the commutativity of the triangle



That is, associativity is equivalent to the equality of two maps  $G \times G \times X_n \rightarrow X_n$ . Thus, by the uniqueness of the induced map on the projective limit  $X$ , we obtain the same limit maps  $G \times G \times X \rightarrow X$ . This gives the associativity on the projective limit from the known associativities. ///

In a similar vein, thinking of our glib presumption that the projective limit  $\mathbb{Z}_2$  of the groups  $\mathbb{Z}/2^n\mathbb{Z}$  was a group, not to mention a *topological* group, we should verify these things.

**[3.0.2] Claim:** Projective limits of topological groups, with all but finitely many *compact*, are topological groups.<sup>[31]</sup> Further, *countable* projective limits<sup>[32]</sup> of *countably-based* topological groups have countable bases.

**[3.0.3] Remark:** The proof has several parts, which show somewhat more than the claim asserts. For example, it becomes clear that arbitrary projective limits of groups exist. Arbitrary projective limits of

<sup>[30]</sup> In this diagram, there is no claim that  $G \times X$  is the projective limit of the objects on the bottom row, only that the maps to the top row exist as indicated.

<sup>[31]</sup> That is, such projective limits of topological groups *exist*, as topological groups.

<sup>[32]</sup> All our diagrams have implicitly used only countably-many objects in the family from which the projective limit is formed. Nevertheless, this countability is not mandated in a more general notion of projective limit, so should be explicitly noted when it matters.

Hausdorff spaces are Hausdorff. Projective limits of families of locally compact Hausdorff spaces  $X_i$ , with all but finitely many  $X_i$  compact, are locally compact. And countable limits of countably-based topological spaces are countably-based.

*Proof:* Let

$$\begin{array}{c}
 \xrightarrow{p_0} \\
 \begin{array}{ccc}
 G & \xrightarrow{p_1} & G_1 \\
 & \dots \xrightarrow{\varphi_{21}} & \dots \xrightarrow{\varphi_{10}} & G_0
 \end{array}
 \end{array}$$

be a projective limit of topological groups, where each transition map  $\varphi_{i,i-1}$  is a continuous group homomorphism, and the  $p_i$  are continuous maps from the projective limit object  $G$ . All that we truly know about  $G$  and the  $p_i$  at the outset is that  $G$  is a topological space and that the  $p_i$  are continuous. We must prove that  $G$  is a group, in fact a topological group, and that the  $p_i$  are group homomorphisms.

First, we need to find the very definition of the alleged group operation  $G \times G \rightarrow G$  on the limit object, much as we defined the group action on a limiting object above. Of course, this must be some sort of limit of the multiplication maps  $\mu_n : G_n \times G_n \rightarrow G_n$  by  $\mu_n : g \times h \rightarrow gh$ . At the same time, to make a map  $G \times G \rightarrow G$  is to make a compatible family of maps  $f_n : G \times G \rightarrow G_n$ . Indeed, let

$$f_n = \mu_n \circ (p_n \times p_n) : G \times G \rightarrow G_n$$

That is, there is a unique  $\mu : G \times G \rightarrow G$  induced by the  $f_n$  making a commutative diagram

$$\begin{array}{ccccc}
 & & \xrightarrow{p_0} & & \\
 & & \begin{array}{ccc}
 \xrightarrow{p_1} & & \\
 \dots \xrightarrow{\varphi_{21}} & \xrightarrow{\varphi_{10}} & \\
 \end{array} & & \\
 & & \xrightarrow{\varphi_{10}} & & \\
 \begin{array}{c} G \\ \uparrow \mu \\ G \times G \end{array} & \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_0} \end{array} & \begin{array}{c} G_1 \\ \uparrow \mu_1 \\ G_1 \times G_1 \end{array} & \begin{array}{c} \xrightarrow{\varphi_{10}} \\ \xrightarrow{\varphi_{10} \times \varphi_{10}} \\ G_0 \end{array} & \begin{array}{c} G_0 \\ \uparrow \mu_0 \\ G_0 \times G_0 \end{array} \\
 & & \begin{array}{ccc}
 \xrightarrow{\varphi_{21} \times \varphi_{21}} & \xrightarrow{\varphi_{10} \times \varphi_{10}} & \\
 \dots & & \\
 \end{array} & & \\
 & & \begin{array}{ccc}
 \xrightarrow{p_1 \times p_1} & \xrightarrow{p_0 \times p_0} & \\
 \dots & & \\
 \end{array} & & \\
 & & \xrightarrow{p_0} & & 
 \end{array}$$

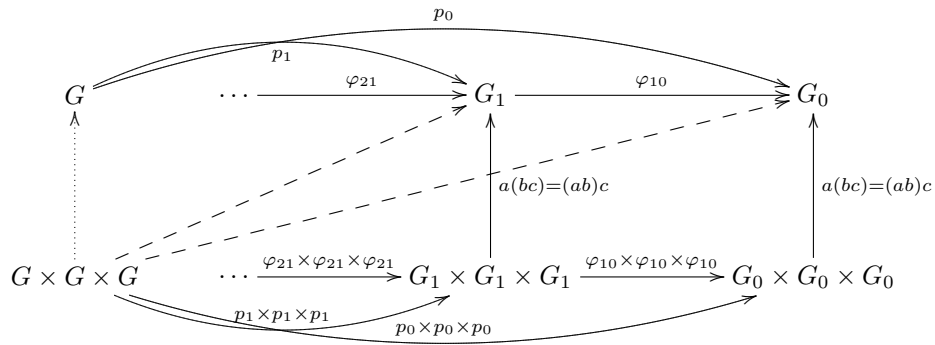
The associativity  $a(bc) = (ab)c$  of the alleged<sup>[33]</sup> group operation comes (much as in the discussion of group actions on limits), first from the commutativity of the diagrams

$$\begin{array}{c}
 \begin{array}{ccc}
 & G_n & \\
 (a,b,c) \rightarrow a(bc) & \nearrow & \nwarrow (a,b,c) \rightarrow (ab)c \\
 G_n \times G_n \times G_n & \xrightarrow{\text{id}_{G_n} \times \text{id}_{G_n} \times \text{id}_{G_n}} & G_n \times G_n \times G_n \\
 \uparrow p_n \times p_n \times p_n & & \uparrow p_n \times p_n \times p_n \\
 G \times G \times G & \xrightarrow{\text{id}_G \times \text{id}_G \times \text{id}_G} & G \times G \times G
 \end{array}
 \end{array}$$

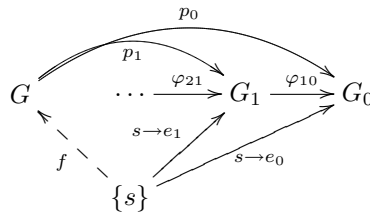
which proves that the two different maps  $G \times G \times G \rightarrow G_n$  are the same, and, second, the uniqueness of the

<sup>[33]</sup> In fact, it is slightly dangerous to use this notation, since it makes it too easy to lose track of what we truly know, versus what must be shown. The associativity we want to prove is properly written as  $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$ .

dotted induced map in



The identity element  $e$  in the limit is specified as a sort of limit of the identities  $e_n$  in  $G_n$ , specifically, as the image  $f(s)$  of the induced map  $f$  in the diagram



Existence of an inversion map (and its property) is a further exercise in this technique, which we leave to the reader. Thus, the map  $\mu : G \times G \rightarrow G$  does have the properties of a group operation on  $G$ .

To show that the projections  $p_n : G \rightarrow G_n$  are group homomorphisms, we note that to say that  $f : A \rightarrow B$  is a group homomorphism for groups  $A, B$  is to require the commutativity of the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \mu_A \uparrow & & \uparrow \mu_B \\ A \times A & \xrightarrow{f \times f} & B \times B \end{array}$$

where  $\mu_A$  and  $\mu_B$  are the multiplication maps belonging to  $A, B$ , respectively. In the case at hand, we would want the commutativity of

$$\begin{array}{ccc} G & \xrightarrow{p_n} & G_n \\ \mu \uparrow & & \uparrow \mu_n \\ G \times G & \xrightarrow{p_n \times p_n} & G_n \times G_n \end{array}$$

Happily, the commutativity of these squares is part of the commutativity of the diagram defining the multiplication  $\mu : G \times G \rightarrow G$ . That is, the fact that the projections are group homomorphisms is a by-product of the construction of the multiplication on  $G$ .

The Hausdorff-ness of the limit will follow from the earlier observation that a limit  $\lim_i X_i$  is a subspace of the corresponding product  $\prod_i X_i$ . An arbitrary product of Hausdorff spaces is Hausdorff.<sup>[34]</sup> And arbitrary

<sup>[34]</sup> That products of Hausdorff spaces are Hausdorff has a natural proof, as follows. Given  $x \neq y$  in the product of spaces  $X_i$ , there is at least one index  $j$  such that the projection  $p_j$  of the product to  $X_j$  distinguishes  $x$  and  $y$ , that is, such that  $p_j x \neq p_j y$ . (This assertion itself can be proven as an exercise using the mapping property definition of product, as suggested in an earlier handout.) Since  $X_j$  is Hausdorff, there are disjoint neighborhoods  $U_j$  and  $V_j$  of  $p_j x$  and  $p_j y$ . Perhaps using the explicit construction of products as cartesian products, let  $U = U_j \times \prod_{i \neq j} X_i$  and  $V = V_j \times \prod_{i \neq j} X_i$ . These are disjoint neighborhoods of  $x$  and  $y$ .

subspaces of Hausdorff spaces, given the subspace topology, are Hausdorff. <sup>[35]</sup> Thus, limits of Hausdorff spaces are Hausdorff.

Similarly, the local compactness of limits of locally compact topological spaces  $X_i$ , with all but finitely many compact, will follow from the analogous assertion for *products*. First, observe that a basic open  $\prod_i U_i$  in a product has *closure* the product of the closures  $\overline{U_i}$  of the factors  $U_i$ . <sup>[36]</sup> When all the  $\overline{U_i}$  are compact, the product is compact, by Tychonoff's theorem. Since only *finitely-many*  $X_i$  are not in fact compact themselves, but in any case are still *locally* compact, inside such factors the product topology allows us to take compact-closure neighborhoods of any point. Thus, every point in the product has a neighborhood (in fact, a *basic* neighborhood) with compact closure.

Finally, to discuss countable-based-ness, it suffices to prove that countable *products* of countably-based topological spaces  $X_i$  are countably-based, since limits are subspaces of products. A basis for a product topology consists of products  $\prod_i U_i$  where for each  $i$  the set  $U_i$  is in a countable basis for  $X_i$ , and for all but finitely-many indices  $i$  the set  $U_i$  is just  $X_i$ . To count these possibilities, note first that there are only countably-many *finite* subsets of a countable set. Next, for each finite subset  $\{i_1, \dots, i_n\}$  of the countable indexing set, there are only countably-many choices of

$$U_{i_1}, \dots, U_{i_n} \quad (\text{with } U_{i_j} \text{ a basis element in } X_{i_j})$$

The sum of countably many countables is countable. ///

**[3.0.4] Remark:** The *box* topology behaves worse than the genuine product topology with regard to preservation of countable-based-ness: since there are *uncountably* many *not-necessarily-finite* subsets of a countable index set, a product of (infinitely) countably-many countably-based spaces, with the *box* topology, will *not* be countably-based.

Having verified that things work as hoped, especially that topological aspects and group-theoretic aspects are captured, we return to the solenoid.

## 4. Automorphisms of solenoids, again

Having cleaned up our viewpoint, we can give an economical and rigorous treatment of the automorphisms found earlier of the solenoid

$$\begin{array}{c}
 \begin{array}{ccc}
 & \xrightarrow{p_0} & \\
 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & & \\
 X & \xrightarrow{p_1} & \mathbb{R}/2\mathbb{Z} \xrightarrow{\text{mod } 1} \mathbb{R}/\mathbb{Z} \\
 & \xrightarrow{\text{mod } 2} & 
 \end{array}
 \end{array}$$

Our aim is to prove that we have a transitive (continuous) group action of  $\mathbb{R} \times \mathbb{Z}_2$ , with an isotropy group a diagonal copy  $\mathbb{Z}^\Delta$  of the integers  $\mathbb{Z}$ , and, thus, that

$$\text{2-solenoid} \approx (\mathbb{R} \times \mathbb{Z}_2) / \mathbb{Z}^\Delta$$

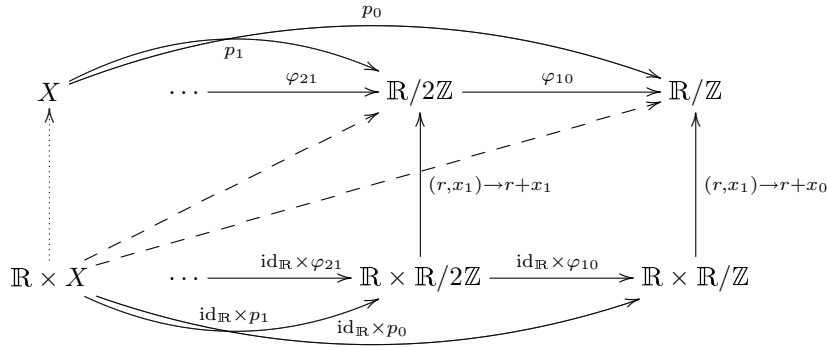
as  $G$ -spaces. <sup>[37]</sup>

<sup>[35]</sup> That subspaces  $Y$  of Hausdorff spaces  $X$  are Hausdorff is straightforward: given  $x \neq y$  in  $Y$ , let  $U, V$  be disjoint neighborhoods of  $x, y$  in  $X$ . Then  $U \cap Y$  and  $V \cap Y$  are disjoint neighborhoods of  $x, y$  in  $Y$ .

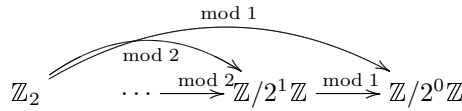
<sup>[36]</sup> The product of the closures  $\overline{U_i}$  of the opens  $U_i$  is a closed set (from the definition of product topology) and *contains* the product of the  $U_i$ . On the other hand, let  $x$  be in the product of the closures. Then every basic neighborhood  $\prod_i V_i$  of  $x$  (with  $V_i$  open in  $X_i$ ) has the property that  $V_i$  meets  $U_i$ , since  $p_i x$  is in the closure of  $U_i$ . That is,  $\prod_i V_i$  meets  $\prod_i U_i$ , so  $x$  is in the closure of the product. This proves the equality.

<sup>[37]</sup> Again, the notion of  $G$ -space is the reasonable one, of topological spaces acted upon continuously by a topological group  $G$ . A map of  $G$ -spaces  $\psi : A \rightarrow B$  is a continuous map of topological spaces which respects the action of  $G$ , in the sense that  $\psi(g \cdot a) = g \cdot \psi(a)$ .

First, we have an induced continuous group action  $\mathbb{R} \times X \rightarrow X$  (the dotted arrow below) induced by the compatible family of (dashed arrow) maps  $\mathbb{R} \times X \rightarrow \mathbb{R}/2^n\mathbb{Z}$  created by composition of the actions  $\mathbb{R} \times \mathbb{R}/2^n\mathbb{Z} \rightarrow \mathbb{R}/2^n\mathbb{Z}$  with the projection  $\mathbb{R} \times X \rightarrow \mathbb{R} \times \mathbb{R}/2^n\mathbb{Z}$ , in

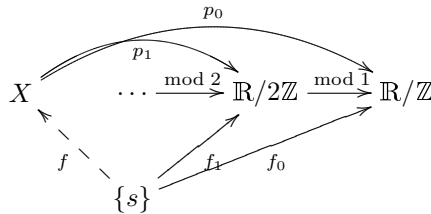


The diagrammatic form of the action of the projective limit (countably-based topological) group



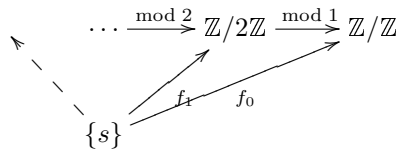
on the solenoid  $X$  is nearly identical, with the minor complication that the action of  $\mathbb{Z}_2$  on  $\mathbb{R}/2^n\mathbb{Z}$  is via the image group  $\mathbb{Z}/2^n\mathbb{Z}$  action on  $\mathbb{R}/2^n\mathbb{Z}$ , by definition.

Next, we want to prove *transitivity* of the joint action  $\mathbb{R} \times \mathbb{Z}_2$  on the solenoid. Specify a point  $x$  on the solenoid by a compatible family of maps (and the induced map  $f$  to  $X$ )



The action of  $\mathbb{R}$  is transitive on the rightmost circle  $\mathbb{R}/\mathbb{Z}$ , so is transitive on maps  $f_0$  from  $\{s\}$  to that circle. Thus, given a point  $f$  on the solenoid (given by a family  $\{f_n\}$  of maps from  $\{s\}$ ), we adjust it by  $\mathbb{R}$  so that  $f_0(s) = 0$  in  $\mathbb{R}/\mathbb{Z}$ .

Then the compatibility condition on the images  $f_n(s)$  requires that, given  $f_0(s) = 0$ , all  $f_n(s)$  are inside  $\mathbb{Z}/2^n\mathbb{Z} \subset \mathbb{R}/2^n\mathbb{Z}$ . That is, the family of maps  $f_n$  gives a compatible family



which is exactly our definition of  $\mathbb{Z}_2$ . Thus, visibly this  $\mathbb{Z}_2$  maps all these points to 0. This proves the transitivity.

Thus, certainly as *sets*,

$$2\text{-solenoid} \approx (\mathbb{R} \times \mathbb{Z}_2)/\mathbb{Z}^\Delta$$



The surprising result proved in the appendix will imply that this is a *topological* isomorphism, if we are sure that  $\mathbb{R} \times \mathbb{Z}_2$  has a countable basis. It is standard<sup>[38]</sup> that  $\mathbb{R}$  has a countable basis. It is less standard, but still standard in light of our earlier discussion, that  $\mathbb{Z}_2$  has a countable basis, since it is a countable projective limit of countably-based spaces.<sup>[39]</sup>

## 5. Appendix: uniqueness of projective limits

As an exercise in proving the uniqueness-up-to-unique-isomorphism (assuming *existence*) of things specified by universal mapping properties, we carry out the proof of uniqueness of projective limits. Part of the point of the exercise is reiteration of the inessentialness of the details of the situation. In particular, as above, a mapping-property approach provides a very useful packaging for topological details that might otherwise be burdensome.

Thus, given topological spaces  $X_i$  with continuous maps

$$\cdots \xrightarrow{\varphi_{21}} X_1 \xrightarrow{\varphi_{10}} X_0$$

let  $X$  and projections  $p_i$  and  $Y$  and projections  $q_i$  fit into diagrams

$$\begin{array}{ccc} & p_0 & \\ & \curvearrowright & \\ X & \xrightarrow{p_1} & X_1 \xrightarrow{\varphi_{10}} X_0 \\ & \cdots \xrightarrow{\varphi_{21}} & \\ & & \end{array} \quad \begin{array}{ccc} & q_0 & \\ & \curvearrowright & \\ Y & \xrightarrow{q_1} & X_1 \xrightarrow{\varphi_{10}} X_0 \\ & \cdots \xrightarrow{\varphi_{21}} & \\ & & \end{array}$$

such that, for all families of maps  $f_i : Z \rightarrow X_i$  such that all triangles commute in

$$\begin{array}{ccc} & \varphi_{21} & \\ & \xrightarrow{\quad} & \\ & \nearrow f_1 & \\ \cdots & & X_1 \xrightarrow{\varphi_{10}} X_0 \\ & \searrow f_0 & \\ Z & & \end{array}$$

there are unique maps  $f : Z \rightarrow X$  and  $g : Z \rightarrow Y$  such that all triangles commute in both diagrams

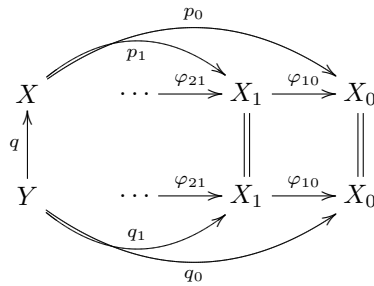
$$\begin{array}{ccc} & p_0 & \\ & \curvearrowright & \\ X & \xrightarrow{p_1} & X_1 \xrightarrow{\varphi_{10}} X_0 \\ & \cdots \xrightarrow{\varphi_{21}} & \\ & & \\ & \nearrow f & \\ & \searrow f_1 & \\ & & Z \end{array} \quad \begin{array}{ccc} & q_0 & \\ & \curvearrowright & \\ Y & \xrightarrow{q_1} & X_1 \xrightarrow{\varphi_{10}} X_0 \\ & \cdots \xrightarrow{\varphi_{21}} & \\ & & \\ & \nearrow g & \\ & \searrow f_1 & \\ & & Z \end{array}$$

Then

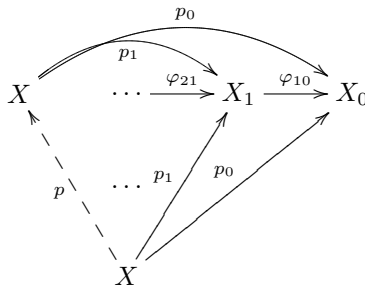
[38] The space  $\mathbb{R}$  has a countable basis consisting of open balls with rational radii centered at rational points.

[39] Again, it is easy to see that a countable product of countably-based spaces is countably-based, and the projective limit can be realized as a *subspace* of the product, so is countably-based.

[5.0.1] **Claim:** There is a unique isomorphism  $q : Y \rightarrow X$  such that we have a commutative diagram

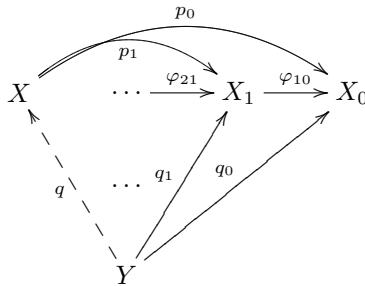


*Proof:* First, we prove that the only map of a projective limit to itself compatible with all projections is the identity map. That is, using  $p_i : X \rightarrow X_i$  itself in the role of  $f_i : Z \rightarrow X_i$ , we find a unique map  $p : X \rightarrow X$  such that all triangles commute in

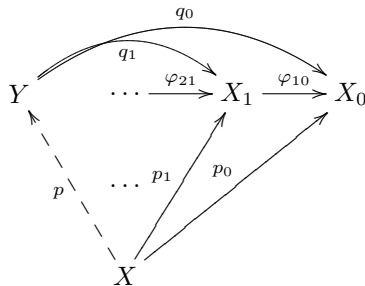


Since the identity map  $\text{id}_X$  fits the role of  $p$ , by uniqueness  $p$  can *only* be the identity on  $X$ .

Now we can do the main part of the proof. Let  $q_i : Y \rightarrow X_i$  take the role of  $f_i : Z \rightarrow X_i$ . Then there is a unique  $q : Y \rightarrow X$  such that all triangles commute in



To show that  $q$  is an isomorphism, reverse the roles of  $X$  and  $Y$ . Then there is a unique  $p : X \rightarrow Y$  such that all triangles commute in



Then  $p \circ q : Y \rightarrow Y$  and  $q \circ p : X \rightarrow X$  are maps compatible with projections, so must be the identities, by the first point of this argument. That is, these are mutually inverse maps, so  $q$  is an isomorphism. ///

**[5.0.2] Remark:** As usual in these categorical arguments, any *continuity* or other requirements on the maps are packaged (or hidden) in the quantification over all families of maps  $f_i : Z \rightarrow X_i$ . That is, the *implicit* specification that  $Z$  be a topological space and  $f_i$  be continuous are what make the result relevant to topological spaces and continuous maps. Thus, despite the lack of overt references to topology, the uniqueness proven above yields *topological* isomorphisms, not merely *set* isomorphisms.

**[5.0.3] Remark:** As in our earlier discussion of the point that a projective limit of *groups* is a *group*, the additional structure that must be demonstrated to have a group, as opposed to merely a *set*, is hidden in the proof of *existence* of a projective limit. That is, in any case there is at most one, regardless of details, but proof of existence invariable requires somewhat greater detail.

## 6. Appendix: topology of $X \approx G/G_x$

The point of this appendix is to prove that, with mild hypotheses, a topological space  $X$  acted upon transitively by a *topological* group  $G$  is homeomorphic to the quotient  $G/G_x$ , where  $G_x$  is the isotropy group of a chosen point  $x$  in  $X$ .

By the way, since we are *not* wanting to assume a pre-existing mastery of point-set topology, much less a mastery of ideas about topological *groups*, several basic ideas will need to be developed in the course of the proof. Everything here is completely standard and widely useful. The discussion includes a form of the *Baire Category Theorem* <sup>[40]</sup> for locally compact Hausdorff spaces.

**[6.0.1] Remark:** Ignoring the topology, that is, as *sets*, the bijection  $G/G_x \approx X$  is easy to see, and the proof needs nothing. The *topological* aspects are not trivial, by contrast, and it should come as a surprise that the topology of the group  $G$  completely determines the topology of the set  $X$  on which it acts.

**[6.0.2] Proposition:** Let  $G$  be a locally compact, Hausdorff topological group <sup>[41]</sup> and  $X$  a Hausdorff topological space with a continuous transitive action of  $G$  upon  $X$ . <sup>[42]</sup> Suppose that  $G$  has a *countable basis*. <sup>[43]</sup> Let  $x$  be any fixed element of  $X$ , and  $G_x$  the isotropy group <sup>[44]</sup> The natural map

$$G/G_x \rightarrow X \text{ by } gG_x \rightarrow gx$$

is a homeomorphism.

<sup>[40]</sup> The more common form of the Baire Category Theorem asserts that a *complete metric space* is *not* a countable union of closed sets each containing no non-empty open set.

<sup>[41]</sup> As expected, this means that  $G$  is a group and is a topological space, the group multiplication is a continuous map  $G \times G \rightarrow G$ , and inversion is continuous. The *local compactness* is the requirement that every point has an open neighborhood with compact closure. The Hausdorff requirement is that any two distinct points  $x \neq y$  have open neighborhoods  $U \ni x$  and  $V \ni y$  that are disjoint, that is,  $U \cap V = \emptyset$ .

<sup>[42]</sup> As expected, continuity of the action means that  $G \times X \rightarrow X$  by  $g \times x \rightarrow gx$  is continuous. The transitivity means that for any  $x \in X$  the set of images of  $x$  by elements of  $G$  is the whole set  $X$ , that is,  $\{gx : g \in G\} = X$ .

<sup>[43]</sup> That is, there is a countable collection  $B$  (the basis) of open sets in  $G$  such that *any* open set is a union of sets from the basis  $B$ .

<sup>[44]</sup> As usual, the isotropy (sub-) group of  $x$  in  $G$  is the subgroup of group elements fixing  $x$ , namely,  $G_x := \{g \in G : gx = x\}$ .

*Proof:* We must do a little systematic development of the topology of topological groups in order to give a coherent argument.

**[6.0.3] Claim:** In a locally compact Hausdorff space  $X$ , given an open neighborhood  $U$  of a point  $x$ , there is a neighborhood  $V$  of  $x$  with compact closure  $\bar{V}$  and  $\bar{V} \subset U$ .

*Proof:* By local compactness,  $x$  has a neighborhood  $W$  with compact closure. Intersect  $U$  with  $W$  if necessary so that  $U$  has compact closure  $\bar{U}$ . Note that the compactness of  $\bar{U}$  implies that the *boundary* <sup>[45]</sup>  $\partial U$  of  $U$  is compact. Using the Hausdorff-ness, for each  $y \in \partial U$  let  $W_y$  be an open neighborhood of  $y$  and  $V_y$  an open neighborhood of  $x$  such that  $W_y \cap V_y = \emptyset$ . By compactness of  $\partial U$ , there is a finite list  $y_1, \dots, y_n$  of points on  $\partial U$  such that the sets  $U_{y_i}$  cover  $\partial U$ . Then  $V = \bigcap_i V_{y_i}$  is open and contains  $x$ . Its closure is contained in  $\bar{U}$  and in the complement of the open set  $\bigcup_i W_{y_i}$ , the latter containing  $\partial U$ . Thus, the closure  $\bar{V}$  of  $V$  is contained in  $U$ . ///

**[6.0.4] Claim:** The map  $gG_x \rightarrow gx$  is a continuous bijection of  $G/G_x$  to  $X$ .

*Proof:* First,  $G \times X \rightarrow X$  by  $g \times y \rightarrow gy$  is continuous by definition of the continuity of the action. Thus, with fixed  $x \in X$ , the restriction to  $G \times \{x\} \rightarrow X$  is still continuous, so  $G \rightarrow X$  by  $g \rightarrow gx$  is continuous. The quotient topology on  $G/G_x$  is the unique topology on the *set* (of cosets)  $G/G_x$  such that any continuous  $G \rightarrow Z$  constant on  $G_x$  cosets factors through the quotient map  $G \rightarrow G/G_x$ . That is, we have a commutative diagram

$$\begin{array}{ccc} G & \longrightarrow & Z \\ \downarrow & \nearrow & \\ G/G_x & & \end{array}$$

Thus, the induced map  $G/G_x \rightarrow X$  by  $gG_x \rightarrow gx$  is continuous. ///

**[6.0.5] Remark:** We need to show that  $gG_x \rightarrow gx$  is *open* to prove that it is a homeomorphism.

**[6.0.6] Claim:** For a given point  $g \in G$ , every neighborhood of  $g$  is of the form  $gV$  for some neighborhood  $V$  of 1.

*Proof:* First, again,  $G \times G \rightarrow G$  by  $g \times g \rightarrow gh$  is continuous, by assumption. Then, for fixed  $g \in G$ , the map  $h \rightarrow gh$  is continuous on  $G$ , by restriction. And this map has a continuous inverse  $h \rightarrow g^{-1}h$ . Thus,  $h \rightarrow gh$  is a homeomorphism of  $G$  to itself. In particular, since  $1 \rightarrow g \cdot 1 = g$ , neighborhoods of 1 are carried to neighborhoods of  $g$ , as claimed. ///

**[6.0.7] Claim:** Given an open neighborhood  $U$  of 1 in  $G$ , there is an open neighborhood  $V$  of 1 such that  $V^2 \subset U$ , where

$$V^2 = \{gh : g, h \in V\}$$

*Proof:* The continuity of  $G \times G \rightarrow G$  assures that, given the neighborhood  $U$  of 1, the inverse image  $W$  of  $U$  under the multiplication  $G \times G \rightarrow G$  is open. Since  $G \times G$  has the product topology,  $W$  contains an open of the form  $V_1 \times V_2$  for opens  $V_i$  containing 1. With  $V = V_1 \cap V_2$ , we have  $V^2 \subset V_1 \cdot V_2 \subset U$  as desired. ///

**[6.0.8] Remark:** Similarly, but more simply, since inversion  $g \rightarrow g^{-1}$  is continuous and is its own (continuous) inverse, for an open set  $V$  the image  $V^{-1} = \{g^{-1} : g \in V\}$  is open. Thus, for example, given

<sup>[45]</sup> As usual, the *boundary* of a set  $E$  in a topological space is the intersection  $\bar{E} \cap \overline{E^c}$  of the closure  $\bar{E}$  of  $E$  and the closure  $\overline{E^c}$  of the complement  $E^c$  of  $E$ .

a neighborhood  $V$  of 1, replacing  $V$  by  $V \cap V^{-1}$  replaces  $V$  by a smaller *symmetric* neighborhood, meaning that the new  $V$  satisfies  $V^{-1} = V$ .

The following result is not strictly necessary, but sheds some light on the nature of topological groups.

[6.0.9] **Claim:** Given a set  $E$  in  $G$ ,

$$\text{closure } E = \bigcap_U E \cdot U$$

where  $U$  runs over open neighborhoods of 1. <sup>[46]</sup>

*Proof:* A point  $g \in G$  is in the closure of  $E$  if and only if every neighborhood of  $g$  meets  $E$ . That is, from just above, every set  $gU$  meets  $E$ , for  $U$  an open neighborhood of 1. That is,  $g \in E \cdot U^{-1}$  for every neighborhood  $U$  of 1. We have noted that inversion is a homeomorphism of  $G$  to itself (and sends 1 to 1), so the map  $U \rightarrow U^{-1}$  is a bijection of the collection of neighborhoods of 1 to itself. Thus,  $g$  is in the closure of  $E$  if and only if  $g \in E \cdot U$  for every open neighborhood  $U$  of 1, as claimed. ///

[6.0.10] **Remark:** This allows us to give another proof, for topological groups, of the fact that, given a neighborhood  $U$  of 1 in  $G$ , there is a neighborhood  $V$  of 1 such that  $\bar{V} \subset U$ . (We did prove this above for locally compact Hausdorff spaces generally.)

*Proof:* First, from the continuity of  $G \times G \rightarrow G$ , there is  $V$  such that  $V \cdot V \subset U$ . From the previous claim,  $\bar{V} \subset V \cdot V$ , so  $\bar{V} \subset V \cdot V \subset U$ , as claimed. ///

[6.0.11] **Remark:** We can improve the conclusion of the previous remark using the local compactness of  $G$ , as follows. Given a neighborhood  $U$  of 1 in  $G$ , there is a neighborhood  $V$  of 1 such that  $\bar{V} \subset U$  and  $\bar{V}$  is *compact*. Indeed, local compactness means exactly that there is a local basis at 1 consisting of opens with compact closures. Thus, given  $V$  as in the previous remark, shrink  $V$  if necessary to have the compact closure property, and still  $\bar{V} \subset V \cdot V \subset U$ , as claimed.

[6.0.12] **Corollary:** For an open subset  $U$  of  $G$ , given  $g \in U$ , there is a neighborhood  $V$  of  $1 \in G$  with compact closure  $\bar{V}$  such that  $gV^2 \subset U$ .

*Proof:* The set  $g^{-1}U$  is an open containing 1, so there is an open  $W \ni 1$  such that  $W^2 \subset g^{-1}U$ . Using the previous claim and remark, there is a compact neighborhood  $V$  of 1 such that  $V \subset W$ . Then  $V^2 \subset W^2 \subset g^{-1}U$ , so  $gV^2 \subset U$  as desired. ///

[6.0.13] **Claim:** Given an open neighborhood  $V$  of 1, there is a countable list  $g_1, g_2, \dots$  of elements of  $G$  such that  $G = \bigcup_i g_i V$ .

*Proof:* To see this, first let  $U_1, U_2, \dots$  be a countable basis. For  $g \in G$ , by definition of a basis,

$$gV = \bigcup_{U_i \subset gV} U_i$$

Thus, for each  $g \in G$ , there is an index  $j(g)$  such that  $g \in U_{j(g)} \subset gV$ . Do note that there are only countably many such indices. For each index  $i$  appearing as  $j(g)$ , let  $g_i$  be an element of  $G$  such that  $j(g_i) = i$ , that is,

$$g_i \in U_{j(g_i)} \subset g_i \cdot V$$

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<sup>[46]</sup> This characterization of the closure of a subset of a topological group is very different from anything that happens in general topological spaces. To find a related result we must look at more restricted classes of spaces, such as *metric* spaces. In a metric space  $X$ , the closure of a set  $E$  is the collection of all points  $x \in X$  such that, for every  $\varepsilon > 0$ , the point  $x$  is within  $\varepsilon$  of some point of  $E$ .

Then, for every  $g \in G$  there is an index  $i$  such that

$$g \in U_{j(g)} = U_{j(g_i)} \subset g_i \cdot V$$

This shows that the union of these  $g_i \cdot V$  is all of  $G$ . ///

A subset  $E$  of a topological space is **nowhere dense** if its closure contains no (non-empty) open set. <sup>[47]</sup>

**[6.0.14] Claim:** (*Variant of Baire Category theorem*) A locally compact Hausdorff topological space is not a countable union of nowhere dense sets. <sup>[48]</sup>

*Proof:* Let  $W_n$  be closed sets containing no non-empty open subsets. Thus, any non-empty open  $U$  meets the complement of  $W_n$ , and  $U - W_n$  is a *non-empty* open. Let  $U_1$  be a non-empty open with compact closure, so  $U_1 - W_1$  is *non-empty* open. From the discussion above, there is a non-empty open  $U_2$  whose *closure* is contained in  $U_1 - W_1$ . Continuing inductively, there are non-empty open sets  $U_n$  with compact closures such that

$$U_{n-1} - W_{n-1} \supset \bar{U}_n$$

Certainly

$$\bar{U}_1 \supset \bar{U}_2 \supset \bar{U}_3 \supset \dots$$

Then  $\bigcap \bar{U}_i \neq \emptyset$ , by compactness. <sup>[49]</sup> <sup>[50]</sup> Yet this intersection fails to meet any  $W_n$ . In particular, it *cannot* be that the union of the  $W_n$ 's is the whole space. ///

Now we can prove that  $G/G_x \approx X$ , using the viewpoint we've set up.

Given an open set  $U$  in  $G$  and  $g \in U$ , let  $V$  be a compact neighborhood of 1 such that  $gV^2 \subset U$ . Let  $g_1, g_2, \dots$  be a countable set of points such that  $G = \bigcup_i g_i V$ . Let  $W_n = g_n V x \subset X$ . By the transitivity,  $X = \bigcup_i W_i$ .

We observed at the beginning of this discussion that  $G \rightarrow X$  by  $g \rightarrow gx$  is continuous, so  $W_n$  is compact, being a continuous image of the compact set  $g_n V$ . So  $W_n$  is closed since it is a compact subset of the Hausdorff space  $X$ .

By the (variant) Baire category theorem, some  $W_m = g_m V x$  contains a non-empty open set  $S$  of  $X$ . For  $h \in V$  so that  $g_m h x \in S$ ,

$$gx = g(g_m h)^{-1}(g_m h)x \in gh^{-1}g_m^{-1}S$$

Every group element  $y \in G$  acts by homeomorphisms of  $X$  to itself, since the continuous inverse is given by  $y^{-1}$ . Thus, the image  $gh^{-1}g_m^{-1}S$  of the open set  $S$  is open in  $X$ . Continuing,

$$gh^{-1}g_m^{-1}S \subset gh^{-1}g_m^{-1}g_m V x \subset gh^{-1}V x \subset gV^{-1} \cdot V x \subset Ux$$

<sup>[47]</sup> The union of all open subsets of a given set is its *interior*. Thus, a set is nowhere dense if its closure has empty interior.

<sup>[48]</sup> The more common version of the Baire category theorem asserts the same conclusion for *complete metric* spaces. The argument is structurally identical.

<sup>[49]</sup> In Hausdorff topological spaces  $X$  compact sets  $C$  are closed, proven as follows. Fixing  $x$  not in  $C$ , for each  $y \in C$ , there are opens  $U_y \ni x$  and  $V_y \ni y$  with  $U_y \cap V_y = \emptyset$ , by the Hausdorff-ness. The  $U_y$ 's cover  $C$ , so there is a finite subcover,  $U_{y_1}, \dots, U_{y_n}$ , by compactness. The finite *intersection*  $W_x = \bigcap_i V_{y_i}$  is open, contains  $x$ , and is disjoint from  $C$ . The union of all  $W_x$ 's for  $x \notin C$  is open, and is exactly the complement of  $C$ , so  $C$  is closed.

<sup>[50]</sup> The intersection of a nested sequence  $C_1 \supset C_2 \supset \dots$  of non-empty compact sets  $C_n$  in a Hausdorff space  $X$  is non-empty. Indeed, the complements  $C_n^c = X - C_n$  are open (since compact sets are closed in Hausdorff spaces), and if the intersection were empty, then the union of the opens  $C_n^c$  would cover  $C_1$ . By compactness of  $C_1$ , there is a finite subcollection  $C_1^c, \dots, C_n^c$  covering  $C_1$ . But  $C_1^c \subset \dots \subset C_n^c$ , and  $C_n^c$  omits points in  $C_n$ , which is non-empty, contradiction.

Therefore,  $gx$  is an interior point of  $Ux$ , for all  $g \in U$ .

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