Small Diameter Interchange Graphs of Classes of Matrices of Zeros and Ones

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ABSTRACT

Let $\mathfrak{A}(R,S)$ denote the class of all $m\times n$ matrices of 0's and 1's having row sum vector R and column sum vector S. The interchange graph $G(R,S)$ is the graph where the vertices are the matrices in $\mathfrak{A}(R,S)$ and where two matrices are joined by an edge provided they differ by an interchange. We characterize those $\mathfrak{A}(R,S)$ for which the graph $G(R,S)$ has diameter at most 2 and those $\mathfrak{A}(R,S)$ for which $G(R,S)$ is bipartite.

1. INTRODUCTION

Let m and n be positive integers, and let $R=(r_1,\dots,r_m)$ and $S=(s_1,\dots,s_n)$ be nonnegative integral vectors. Denote by $\mathfrak{A}(R,S)$ the set of all $m\times n$ matrices $A=[a_{ij}]$ of 0's and 1's with row sum vector R and column sum

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vector S:

$$a_{ij}=0 \text{ or } 1 \qquad (i=1,\dots,m, \quad j=1,\dots,n),$$

$$\sum_{j=1}^{n} a_{ij}=r_i \qquad (i=1,\dots,m),$$

$$\sum_{i=1}^{n} a_{ij}=s_j \qquad (j=1,\dots,n).$$

The set $\mathfrak{A}(R,S)$ has many remarkable properties and was the subject of a recent survey paper [1], to which we refer the reader for additional references. In particular, necessary and sufficient conditions are known for $\mathfrak{A}(R,S)$ to be nonempty. Throughout we assume that $\mathfrak{A}(R,S)$ is nonempty.

An *interchange* is a transformation which replaces the 2×2 submatrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.1}$$

of a matrix A of 0's and 1's with the 2×2 submatrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{1.2}$$

or vice versa. If the submatrix (1.1) [or (1.2)] lies in rows k,l and columns u,v, then we refer to a $(k,l;u,v)$*-interchange*. Clearly, an interchange does not alter the row and column sum vector of a matrix, and thus replaces a matrix $A\in\mathfrak{A}(R,S)$ with a matrix $B\in\mathfrak{A}(R,S)$. Ryser [2, p. 68] proved that given $A,B\in\mathfrak{A}(R,S)$ there is finite sequence of interchanges which transforms A into B.

In [1] the *interchange graph* $G(R,S)$ is defined as follows: The vertices are the matrices in $\mathfrak{A}(R,S)$ where two matrices A and B are joined by an edge if and only if B can be obtained from A by one interchange. Ryser's theorem quoted above is equivalent to the statement that $G(R,S)$ is a connected graph.

Before continuing we review briefly some ideas from graph theory. Let G be a connected graph. Then the distance $\delta(x,y)$ between two vertices is the smallest length of a chain joining x and y. The *diameter* of G is the greatest distance between a pair of vertices of the graph. The *eccentricity* of a vertex x is the largest of the distances from x to each vertex of G. The *radius* of G is the smallest eccentricity of its vertices. A vertex whose eccentricity equals the radius of G is called *central*. It follows that the diameter is the largest eccentricity of a vertex.

Now consider the interchange graph $G(R,S)$. For $A, B \in \mathfrak{A}(R,S)$, the distance $\delta(A,B)$ between A and B in $G(R,S)$ is the minimum number of interchanges which can transform A into B. We denote the diameter of $G(R,S)$ by $d(R,S)$. Suppose A is a matrix in $\mathfrak{A}(R,S)$ which has a $k \times k$ submatrix ($k \geq 2$) of the form

$$P\begin{bmatrix} 0 & 1 & * & \cdots & * & * \\ * & 0 & 1 & \cdots & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & * & \cdots & 0 & 1 \\ 1 & * & * & \cdots & * & 0 \end{bmatrix}Q, \tag{1.3}$$

where P and Q are permutation matrices. Let B be the matrix obtained from A by replacing the specified 0's of the submatrix (1.3) with 1's and the specified 1's with 0's. Then B is also a matrix in $\mathfrak{A}(R,S)$, and it follows from [1, Corollary 3.5] that $\delta(A,B) = k-1$. Hence A has eccentricity at least $k-1$, so that $d(R,S) \geq k-1$.

In investigating the set $\mathfrak{A}(R,S)$ and the graph $G(R,S)$ there is no loss in generality in assuming that $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ are *monotone* in the sense that $r_1 \geq \cdots \geq r_m$ and $s_1 \geq \cdots \geq s_n$. This is because we may rearrange rows and rearrange columns without affecting the combinatorial structure of $\mathfrak{A}(R,S)$ and the isomorphism type of $G(R,S)$. *We now make the assumption that R and S are monotone.* Let $1 \leq k \leq m$ and $1 \leq l \leq n$. The position (k,l) is an *invariant 1-position* of $\mathfrak{A}(R,S)$ provided each matrix in $\mathfrak{A}(R,S)$ has a 1 in the (k,l)-position. An *invariant 0-position* is defined in a similar way. The union of the invariant 1-positions and invariant 0-positions comprises the *invariant positions* of $\mathfrak{A}(R,S)$.

Let p and q be nonnegative integers. Then $J_{p,q}$ denotes the matrix of all 1's with p rows and q columns, while $O_{p,q}$ denotes the matrix of all 0's with p rows and q columns. If either p or q equals 0, then these matrices are vacuous. From a theorem of Ryser [2, p. 69] we obtain the following:

$\mathfrak{A}(R,S)$ *has an invariant 1-position if and only if there exist integers e and f with $1 \leq e \leq m$ and $1 \leq f \leq n$ such that every matrix $A \in \mathfrak{A}(R,S)$ has the form*

$$\begin{bmatrix} J_{e,f} & A_1 \\ A_2 & O_{m-e,n-f} \end{bmatrix}. \tag{1.4}$$

$\mathfrak{A}(R, S)$ has an invariant 0-position if and only if there exist integers e and f with $0 \leqslant e \leqslant m-1$ and $0 \leqslant f \leqslant n-1$ such that every matrix $A \in \mathfrak{A}(R, S)$ has the form (1.4). In (1.4) the positions occupied by $J_{e,f}$ are invariant 1-positions while those occupied by $O_{m-e,n-f}$ are invariant 0-positions. Note that in (1.4) both $J_{e,f}$ and $O_{m-e,n-f}$ are vacuous if and only if $e=0$ and $f=n$, or $e=m$ and $f=0$.

Suppose $\mathfrak{A}(R, S)$ has an invariant position, so that there exist integers e and f with $0 \leqslant e \leqslant m$ and $0 \leqslant f \leqslant n$ such that every matrix A in $\mathfrak{A}(R, S)$ has the form (1.4). For $i=1,2$, let the row sum vector of A_i be R_i and let the column sum vector of A_i be S_i, so that $A_i \in \mathfrak{A}(R_i, S_i)$. Suppose first $1 \leqslant e \leqslant m-1$ and $1 \leqslant f \leqslant n-1$. Then the graph $G(R, S)$ is isomorphic to the *cartesian product* $G(R_1, S_1) \times G(R_2, S_2)$ of $G(R_1, S_1)$ and $G(R_2, S_2)$. The vertices of $G(R_1, S_1) \times G(R_2, S_2)$ are the pairs (A_1, A_2) with $A_i \in G(R_i, S_i)$ $(i=1,2)$; there is an edge between vertices (A_1', A_2') and (A_1'', A_2'') if and only if $A_1' = A_1''$ and A_2' and A_2'' are joined by an edge in $G(R_2, S_2)$ or $A_2' = A_2''$ and A_1' and A_1'' are joined by an edge in $G(R_1, S_1)$. It follows easily that

$$d(R,S) \geqslant d(R_1, S_1) + d(R_2, S_2), \tag{1.5}$$

and $G(R, S)$ is bipartite if and only if both $G(R_i, S_i)$ $(i=1,2)$ are bipartite. If $e=0$ or $f=n$, then A_1 is vacuous and $G(R, S)$ is isomorphic to $G(R_2, S_2)$. If $e=m$ or $f=0$, then A_2 is vacuous and $G(R, S)$ is isomorphic to $G(R_1, S_1)$.

In [1, Problem 3.6] the question of investigating the diameter $d(R, S)$ of $G(R, S)$ was raised. In this note we characterize those R and S for which $\mathfrak{A}(R, S)$ has no invariant positions and $d(R, S) \leqslant 2$. We also characterize those R and S for which $\mathfrak{A}(R, S)$ has no invariant positions and $G(R, S)$ is bipartite. A characterization of those classes $\mathfrak{A}(R, S)$ satisfying $d(R, S) \leqslant 2$ or $G(R, S)$ bipartite but having invariant positions then follows readily from the above argument and a general theorem about matrices in $\mathfrak{A}(R, S)$ having invariant positions [1, theorem 5.10, p. 190].

Finally we note the construction [2, pp. 63–65; 1, p. 165] of the following special matrix $\tilde{A} \in \mathfrak{A}(R, S)$. For each $k = 1, \ldots, n$, the leading $m \times k$ submatrix $\tilde{A}_k$ of $\tilde{A}$ has a monotone row sum vector, and the 1's in column k of $\tilde{A}_k$ are in those rows with the largest row sums, preference given to the bottommost positions in case of ties. The matrix $\tilde{A}$ plays an important role in the inductive proofs given.

2. SMALL DIAMETER GRAPHS

Throughout this section we assume $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ are monotone, nonnegative integral vectors for which $\mathfrak{A}(R, S) \neq \varnothing$. We also let

$\tau = \Sigma_{i=1}^{n} r_i = \Sigma_{j=1}^{n} s_j$. The diameter of the interchange graph $G(R,S)$ satisfies $d(R,S)=0$ exactly when there is a unique matrix in $\mathfrak{A}(R,S)$. Suppose that $\mathfrak{A}(R,S)$ contains a unique matrix $\bar{A}$. Then every position must be an invariant position. Hence it follows from a theorem of Haber [1, p. 186] that row i of $\bar{A}$ is the vector $(1,\ldots,1,0,\ldots,0)$ of r_i 1's followed by $n-r_i$ 0's $(i=1,\ldots,m)$. This means that

$$s_j = |\{i : r_i \geqslant j, i=1,\ldots,m\}| \qquad (j=1,\ldots,n),$$

and R and S are conjugate partitions of τ. Conversely, if R and S are conjugate partitions of τ, $\mathfrak{A}(R,S)$ contains a unique matrix.

THEOREM 2.1. *$d(R,S)=0$ if and only if R and S are conjugate partitions of τ.*

A connected graph has diameter 1 exactly when it is a *complete graph* with more than one vertex (every pair of distinct vertices is joined by an edge).

THEOREM 2.2. *Suppose $\mathfrak{A}(R,S)$ has no invariant positions. Then $d(R,S)=1$ if and only if one of the following holds:*

$$(2.3) \qquad m=2\leqslant n, \qquad R=(n-1,1), \quad \textit{and} \quad S=(1,\ldots,1),$$

$$(2.4) \qquad n=2\leqslant m, \qquad R=(1,\ldots,1), \quad \textit{and} \quad S=(m-1,1).$$

Proof. First assume that (2.3) holds. Then $\mathfrak{A}(R,S)$ contains exactly n matrices, namely

$$A_i = \begin{bmatrix} 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix},$$

where the unique 1 in the second row occurs in column i $(i=1,\ldots,n)$. For $i \neq j$, A_i can be transformed to A_j by one interchange, so that $G(R,S)$ is complete and $d(R,S)=1$. It follows in a similar way that $d(R,S)=1$ when (2.4) holds.

Now, suppose $d(R,S)=1$. It suffices to assume that $m \leqslant n$ and to prove that (2.3) holds. Clearly, $m \geqslant 2$. Let $A=[a_{ij}] \in \mathfrak{A}(R,S)$, and let $1 \leqslant k \leqslant m$, $1 \leqslant u \leqslant n$. Since $\mathfrak{A}(R,S)$ has no invariant positions, there is a matrix $B=[b_{ij}] \in \mathfrak{A}(R,S)$ such that $a_{kl} \neq b_{kl}$. Since $\delta(A,B)=1$, it follows that there exist l

and v such that a $(k, l; u, v)$-interchange can be applied to A (transforming A to B). First suppose $m \geqslant 3$. By the above with $k=1$, $u=1$ there exist l and v such that a $(1, l; 1, v)$ interchange can be applied to A to give a matrix B. Let p be an integer with $1 \leqslant p \leqslant m$, $p \neq 1, l$, and let q be an integer with $1 \leqslant q \leqslant n$, $q \neq 1, v$. Then there also exist integers w, t such that a $(p, w; q, t)$-interchange can be applied to A to yield a matrix C. The matrices B and C differ in at least 6 positions, implying $\delta(B, C) > 1$. This contradiction means that $m = 2$. Since there are no invariant positions, $S = (1, \dots, 1)$ and $R = (n - a, a)$ where $1 \leqslant a \leqslant n - a$. Suppose $a \geqslant 2$. There there exist matrices A, B in $\mathfrak{A}(R, S)$ of the form

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & X \\ 0 & 0 & 1 & 1 & \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 & 1 & X \\ 1 & 1 & 0 & 0 & \end{bmatrix}$$

for some $2 \times (n-4)$ matrix X. Since $\delta(A, B) = 2$, this is a contradiction. Hence $a = 1$ and the theorem follows. ■

We now consider interchange graphs of diameter 2. The conclusions depend on the magnitudes of m and n. The following lemma will be useful in inductive arguments.

Lemma 2.5. *Let $n > 2$, and suppose $\mathfrak{A}(R, S)$ has no invariant positions. Let $\tilde{A}_{n-1}$ be the matrix obtained from the special matrix $\tilde{A} \in \mathfrak{A}(R, S)$ by eliminating its last column. Suppose $\tilde{A}_{n-1} \in \mathfrak{A}(R_{n-1}, S_{n-1})$. Then one of the following holds*:

(2.6) $\mathfrak{A}(R_{n-1}, S_{n-1})$ *has no invariant positions.*

(2.7) $\mathfrak{A}(R_{n-1}, S_{n-1})$ *has invariant 0-positions but no invariant 1-positions. Thus there exists e with $1 \leqslant e \leqslant m-1$ such that every matrix in $\mathfrak{A}(R_{n-1}, S_{n-1})$ has the form*

$$\begin{bmatrix} A' \\ O_{m-e, n-1} \end{bmatrix},$$

where $A' \in \mathfrak{A}(R', S')$ and $\mathfrak{A}(R', S')$ has no invariant positions.

(2.8) $\mathfrak{A}(R_{n-1}, S_{n-1})$ *has invariant 1-positions but no invariant 0-positions. Thus there exists e with $1 \leqslant e \leqslant m-1$ such that every matrix in*

$\mathfrak{A}(R_{n-1}, S_{n-1})$ *has the form*

$$\begin{bmatrix} J_{e,n-1} \\ A' \end{bmatrix},$$

where $A' \in \mathfrak{A}(R', S')$ *and* $\mathfrak{A}(R', S')$ *has no invariant positions.*

Proof. We need only show that if $\mathfrak{A}(R_{n-1}, S_{n-1})$ has invariant 1-positions, it cannot have invariant 0-positions. So suppose $\mathfrak{A}(R_{n-1}, S_{n-1})$ has invariant 1-positions. Then we can choose integers e and f with $1 \leqslant e \leqslant m-1$ and $1 \leqslant f \leqslant n-1$ such that every matrix in $\mathfrak{A}(R_{n-1}, S_{n-1})$ has the form

$$\begin{bmatrix} J_{ef} & A'' \\ A' & 0 \end{bmatrix},$$

where $A' \in \mathfrak{A}(R', S')$ and $\mathfrak{A}(R', S')$ has no invariant 1-positions. Let $R' = (r'_{e+1}, \dots, r'_m)$ so that $r'_{e+1} < f$. Suppose $f < n-1$. Then

$$\tilde{A} = \begin{bmatrix} J_{ef} & \tilde{A}'' & \alpha_1 \\ \tilde{A}' & 0 & \alpha_2 \end{bmatrix},$$

where $s_n \leqslant s_{n-1} \leqslant e$. Since $\mathfrak{A}(R, S)$ has no invariant positions, α_2 is not a column of 0's. Hence there exists a positive integer i such that $r_{e+i} = r'_{e+i} + 1 \leqslant r'_{e+1} + 1 \leqslant f$. Since $s_n \leqslant e$, α_1 is not a column of 1's. From the construction of $\tilde{A}$ and the monotonicity of R, it now follows that $r_{e+i} = f$ and there exists an integer $t < e$ such that $r_{t+1} = \cdots = r_{e+i} = f$, the first t entries of α_1 are 1 while the last $e-t$ entries are 0, and the last $e-t$ rows of $\tilde{A}''$ contain only 0's. But then $s_n \geqslant t+1$ while $s_{n-1} \leqslant t$, contradicting $s_{n-1} \leqslant s_n$. Thus $f = n-1$, and every matrix in $\mathfrak{A}(R_{n-1}, S_{n-1})$ has the form

$$\begin{bmatrix} J_{e,n-1} \\ A' \end{bmatrix}$$

where $A' \in \mathfrak{A}(R', S')$ and $\mathfrak{A}(R', S')$ has no invariant 1-positions. Suppose $\mathfrak{A}(R', S')$ has invariant 0-positions. It then follows that R' or S' has at least one coordinate equal to 0. Thus $\tilde{A}$ has one of the forms

$$\tilde{A}=\left[\begin{array}{c|c} J_{e,n-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline * & \begin{matrix}*\\ \vdots\\ *\end{matrix} \\ \hline 0 & \begin{matrix}1\\ \vdots\\ 1\end{matrix} \end{array}\right]$$

or

$$\tilde{A}=\left[\begin{array}{c|c} J_{e,n-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline B \mid 0 & \beta \end{array}\right],$$

where, since $\mathfrak{A}(R,S)$ has no invariant 1-positions, β is not a column of 0's, and since $\mathfrak{A}(R',S')$ has no invariant 1-positions, no row of B contains only 1's. In the first case, $r_1=n-1\geqslant 2$ while $r_n=1$, so that the last column violates the construction of $\tilde{A}$. In the second case, $r_1=n-1$ while $r_i\leqslant n-2$ for $i=e+1,\dots,m$, and again the last column violates the construction of $\tilde{A}$. Hence $\mathfrak{A}(R',S')$ has no invariant 0-positions and (2.8) holds. The lemma now follows. ■

THEOREM 2.9. *Suppose* $\mathfrak{A}(R,S)$ *has no invariant positions and* $\min\{m,n\}\leqslant 3$. *Then* $d(R,S)=2$ *if and only if one of the following holds*:

(2.10) $m=2$, $n\geqslant 4$, *and* $R=(n-2,2)$, $S=(1,\dots,1)$.

(2.11) $n=2$, $m\geqslant 4$, *and* $R=(1,\dots,1)$, $S=(m-2,2)$.

(2.12) $m=3$, $n=3$; *or* $m=3$, $n=4$; *or* $m=4$, $n=3$.

(2.13) $m=3$, $n\geqslant 5$; *and either* $R=(n-1,n-1,2)$, $S=(2,\dots,2)$, *or* $R=(n-2,1,1)$, $S=(1,\dots,1)$, *or* $R=(n-1,n-l,1)$, $S=(2,\dots,2,1,\dots,1)$ *for some* l *with* $1\leqslant l\leqslant n$.

(2.14) $n=3$, $m\geq 5$; *and either* $R=(2,\dots,2)$, $S=(m-1,m-1,2)$, *or* $R=(1,\dots,1)$, $S=(m-2,1,1)$, *or* $R=(2,\dots,2,1,\dots,1)$, $S=(m-1,m-l,1)$ *for some* l *with* $1\leq l\leq m-1$.

Proof. If $\mathfrak{A}(R,S)$ has no invariant positions, then $n>r_i>0$ $(i=1,\dots,m)$ and $m>s_j>0$ $(j=1,\dots,n)$. We prove that the latter conditions suffice in that the conclusions of the theorem hold. Note that $d(R,S)=2$ implies that $\min\{m,n\}\geq 2$.

First suppose $m=2$. Then $R=(n-a,a)$ and $S=(1,\dots,1)$, where $1\leq a \leq n-a$. If $a=1$, then $d(R,S)=1$ by Theorem 2.2. So $a\geq 2$. If $a\geq 3$, then there exist matrices A,B in $\mathfrak{A}(R,S)$ of the form

$$A=\begin{bmatrix}1&1&1&0&0&0&X\\0&0&0&1&1&1&\end{bmatrix},\qquad B=\begin{bmatrix}0&0&0&1&1&1&X\\1&1&1&0&0&0&\end{bmatrix}$$

for some $2\times(n-6)$ matrix X. Since $\delta(A,B)=3$, we conclude that if $d(R,S)=2$, then $a=2$ and (2.10) holds. Conversely, it follows easily that (2.10) implies $d(R,S)=2$. Similar arguments establish the case $n=2$ in (2.11).

We now assume $\min\{m,n\}=3$. First suppose $\max\{m,n\}\leq 4$. It follows from Theorem 2.2 that $d(R,S)\geq 2$. Let A and B be distinct matrices in $\mathfrak{A}(R,S)$. Suppose $m=3$. Then $s_i=1$ or 2 for $i=1,\dots,n$ and it follows that A and B have the same entry in at least one position of each column. Hence A and B differ in at most $2n\leq 8$ positions. Hence A and B differ in 4, 6, or 8 positions. If A and B differ in 4 positions, $\delta(A,B)=1$. If A and B differ in 6 positions, then A contains a 3×3 submatrix of the form (1.3) where B has 1's (respectively, 0's) in the positions designated by 0's (respectively, 1's) in A. It follows that $\delta(A,B)=2$. Finally, if A and B differ in 8 positions, then two interchanges on these 8 positions transform A to B and $\delta(A,B)=2$. Hence $d(R,S)=2$. Similarly, $d(R,S)=2$ when $n=3$, and case (2.12) is established.

We now suppose that $m=3$ and $n\geq 5$. We first show that $d(R,S)=2$ when R,S are one of the three pairs in (2.13). It follows from Theorem 2.2 that $d(R,S)\geq 2$. Let $R=(n-1,n-1,2)$ and $S=(2,\dots,2)$. Then each matrix in $\mathfrak{A}(R,S)$ has exactly one 0 in each of rows 1 and 2. It now follows that each pair A,B of matrices in $\mathfrak{A}(R,S)$ agree on all but at most 4 columns, and as above $\delta(A,B)\leq 2$. Now let $R=(n-2,1,1)$ and $S=(1,\dots,1)$. Then each matrix is $\mathfrak{A}(R,S)$ has exactly one 1 in each of rows 2 and 3, and it follows again that $\delta(A,B)\leq 2$. Finally, let $R=(n-1,n-l,1)$ and $S=(2,\dots,2,1,\dots,1)$ for some l with $1\leq l\leq n-1$. Then S has l coordinates equal to 1. Let $A \in \mathfrak{A}(R,S)$. Then each of the first $n-l$ columns of A is one of

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \begin{bmatrix}0\\1\\1\end{bmatrix}$$

with all but at most one equal to the first (since $r_3 = 1$). Each of the last l columns of A is one of

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

with all but at most one equal to the first (since $r_1 = n-1$). It follows that each pair of matrices A, B in $\mathfrak{A}(R,S)$ agree on all but at most 4 columns, and again $\delta(A,B) \leqslant 2$. Hence $d(R,S) = 2$ when (2.13) holds.

We now assume that $m = 3$, $n \geqslant 5$ and $d(R,S) = 2$ and show that R, S is one of the 3 pairs of (2.13). First suppose that $n = 5$. Then the possibilities for $R = (r_1, r_2, r_3)$ that are not included in (2.13) are the following [note that $S = (s_1, s_2, s_3, s_4, s_5)$ is determined by R, since $1 \leqslant s_i \leqslant 2$ $(i = 1, \ldots, 5)$ and $s_1 + \cdots + s_5 = r_1 + r_2 + r_3$]:

(i) $R = (4,3,3)$, (i′) $R = (2,2,1)$,

(ii) $R = (4,3,2)$, (ii′) $R = (3,2,1)$,

(iii) $R = (4,2,2)$, (iii′) $R = (3,3,1)$,

(iv) $R = (3,3,3)$, (iv′) $R = (2,2,2)$,

(v) $R = (3,3,2)$, (v′) $R = (3,2,2)$.

We show by example that in each of these cases $d(R,S) > 2$ by exhibiting a pair of matrices in $\mathfrak{A}(R,S)$ for which no two corresponding columns are identical:

$$(2.15) \qquad \begin{bmatrix}0&1&1&1&1\\1&1&1&0&0\\1&0&0&1&a\end{bmatrix}, \quad \begin{bmatrix}1&1&1&1&0\\1&0&0&1&1\\0&1&1&0&a\end{bmatrix} \quad \begin{matrix}\text{(i)} & (a=1)\\ \text{(ii)} & (a=0)\end{matrix}$$

$$(2.16)\qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(iii)}$$

$$(2.17)\qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & b & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & b & 0 \end{bmatrix}$$

(iv) $(b=1)$

(v) $(b=0)$

The remaining examples are obtained by switching 0's and 1's in the above examples and reordering rows and columns to achieve monotonicity of the row sum and column sum vectors. We now let $n>5$ and proceed by induction.

First we prove $s_n=1$. Suppose $s_n=2$. Then $n-1\geqslant r_1\geqslant r_2\geqslant r_3\geqslant 2$. If $R\neq(n-1,n-1,2)$, then $r_3\geqslant 3$ and it is easy to see that there are two matrices $A,B\in\mathfrak{A}(R,S)$ with leading 3×5 submatrices given by (2.15) with $a=1$, a contradiction.

Consider the matrix $\tilde{A}\in\mathfrak{A}(R,S)$. Let $\tilde{A}_{n-1}$ be the leading $3\times(n-1)$ submatrix of $\tilde{A}$, so that $\tilde{A}_{n-1}\in\mathfrak{A}(R_{n-1},S_{n-1})$, where R_{n-1} and S_{n-1} are both monotone. First suppose that $\mathfrak{A}(R_{n-1},S_{n-1})$ has no invariant positions. Then it follows from Theorem 2.2 that $d(R_{n-1},S_{n-1})=2$. So by induction R_{n-1} is one of (i) $(n-2,n-2,2)$, (ii) $(n-3,1,1)$, or (iii) $(n-2,n-l,1)$ for some l with $2\leqslant l\leqslant n-1$. Since $s_n=1$, we conclude that R is one of (i) $(n-1,n-2,2)$, (ii) $(n-2,1,1)$, or (iii) either $(n-2,n-2,1)$, or $(n-1,n-l,1)$ for some l with $2\leqslant l\leqslant n-1$. In case (i) it is easy to see that there are two matrices $A,B\in\mathfrak{A}(R,S)$ with leading 3×5 submatrices given by (2.15) with $a=0$. Since $\delta(A,B)>2$, it follows that $R\neq(n-1,n-2,2)$. In the first instance of (iii), there are two matrices $A,B\in\mathfrak{A}(R,S)$ with leading 3×5 submatrices obtained from (2.16) by switching 0's and 1's. Since $\delta(A,B)>2$, $R\neq(n-2,n-2,1)$. The remaining cases in (i), (ii), and (iii) are included in (2.13).

We now suppose $\mathfrak{A}(R_{n-1},S_{n-1})$ has invariant positions. Then by Lemma 2.5 either (2.7) or (2.8) holds. Suppose (2.7) holds. Then since $m=3$, we have $e=2$ and $S=(1,\dots,1)$. Since $n\geqslant 5$, we have $r_1\geqslant 2$, and since $s_n=1$, we violate the construction of $\tilde{A}$. Suppose (2.8) holds. Then $e=1$. If $r_2\leqslant n-2$, then the first row of $\tilde{A}$ contains only 1's, contradicting the assumption $\mathfrak{A}(R,S)$ has no invariant positions. Hence $r_2=n-1$, but $r_3>1$ implies that $\tau>2(n-1)+1=2n-1$ and $s_1=3$, a contradiction. Thus $r_1=r_2=n-1$ and $r_3=1$, a case included in (2.13). So the induction is complete and the case (2.13) is established. The case (2.14) is established in a similar way, and the theorem follows. ∎

We now suppose $\min\{m, n\} \geqslant 4$ and characterize those R and S for which $\mathfrak{A}(R, S)$ has no invariant positions and $G(R, S)$ has diameter 2. The proof of this characterization is by induction. We first prove two lemmas, the second of which verifies the initial step of the induction.

LEMMA 2.18. *Let $m = 4$ and $n = 3$, and suppose $\mathfrak{A}(R, S)$ has no invariant positions. Then there exists $A \in \mathfrak{A}(R, S)$ with a 3×3 submatrix of the form* (1.3).

Proof. It follows from Theorem 2.9 that $d(R, S) = 2$. Suppose there is no matrix in $\mathfrak{A}(R, S)$ with a 3×3 submatrix of the form (1.3). Then it follows that there exists a matrix $A \in \mathfrak{A}(R, S)$ having the form

$$(2.19) \qquad P\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ c & 1 & 0 \\ d & 0 & 1 \end{bmatrix}Q$$

or

$$(2.20) \qquad P\begin{bmatrix} 1 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \\ 0 & 1 & d \end{bmatrix},$$

where P, Q are permutation matrices. In case of (2.19), it follows easily that $a = d = 1$ and $b = c = 0$. In case of (2.20), it follows easily that $a = b$ and $c = d$. We conclude that $R = (2,2,1,1)$ and $S = (2,2,2)$. But then the matrix B in (2.13),

$$(2.21) \qquad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

is in $\mathfrak{A}(R, S)$, contradicting the assumption that no matrix in $\mathfrak{A}(R, S)$ has a 3×3 submatrix of the form (1.3).

LEMMA 2.22. *Suppose $m = n = 4$ and $\mathfrak{A}(R, S)$ has no invariant positions. If $d(R, S) = 2$, then $R = S = (3,3,3,1)$ or $R = S = (3,1,1,1)$.*

Proof. Consider the special matrix $\tilde{A} \in \mathfrak{A}(R,S)$. Let $\tilde{A}_3 \in \mathfrak{A}(R_3,S_3)$ be the leading 4×3 submatrix of $\tilde{A}$. First suppose $\mathfrak{A}(R_3,S_3)$ has no invariant positions. Then it follows from Theorem 2.6 that $d(R_3,S_3)=2$. By Lemma 2.18 there exists a matrix $B \in \mathfrak{A}(R_3,S_3)$ having a 3×3 submatrix of the form (1.3). Since $d(R,S)=2$, no matrix in $\mathfrak{A}(R,S)$ can have the form (1.3). It now follows in a straightforward manner that there exist permutation matrices P,Q such that $P\tilde{A}Q$ has one of the forms

(2.23)
$$\begin{bmatrix} 1 & 0 & a & 0 \\ b & 1 & 0 & 1 \\ 0 & c & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

(2.24)
$$\begin{bmatrix} 1 & 0 & a & 0 \\ b & 1 & 0 & 1 \\ 0 & c & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

In the case of (2.23) it follows readily that $b=1$, $c=0$, and $a=0$, so that $R=S=(3,1,1,1)$. Now suppose (2.24) holds. Then it is easy to verify that $a=0$, $b=1$, and $c=1$, so that $R=S=(3,3,3,1)$.

Now suppose $\mathfrak{A}(R_3,S_3)$ has invariant positions. Then it follows from Lemma 2.5 that (2.7) or (2.8) holds where $m=n=4$. First suppose (2.7) holds. If $e=3$, then $d(R',S')=2$ and since the last column of $\tilde{A}$ contains a 0, it is easily seen that $d(R,S) \geqslant 3$. Suppose $e=2$. Then $R'=(2,1)$ and $S'=(1,1,1)$. Hence $S=(1,1,1,s_n)$, $s_n \geqslant 2$, which contradicts the monotonicity of S. Now suppose (2.8) holds. If $e=1$, then $d(R',S')=2$ and it follows again that $d(R,S) \geqslant 3$. Suppose $e=2$. Then $R'=(2,1)$ and $S'=(1,1,1)$. Hence $R=(3,3,3,1)=S$. This completes the proof of the lemma. ■

We note that an alternative proof of Lemma 2.22 can be obtained by examining all possibilities of R and S and exhibiting a pair A,B of matrices with $d(A,B)>2$ when R and S do not satisfy the conclusion of the lemma.

THEOREM 2.2.5. *Suppose $\mathfrak{A}(R,S)$ has no invariant positions and $m,n \geqslant 4$. Then $d(R,S)=2$ if and only if*

(2.26) $$R=(n-1,\dots,n-1,1), \qquad S=(m-1,\dots,m-1,1)$$

or

(2.27) $$R=(n-1,1,\dots,1), \qquad S=(m-1,1,\dots,1).$$

Proof. If (2.26) holds, then

$$\left[\begin{array}{ccc|c} & & & 0 \\ & J_{m-1,n-1} & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right]$$

is a central vertex of $G(R,S)$ and it follows that $d(R,S)=2$. If (2.27) holds, switching 0's and 1's in the above matrix and rearranging rows and columns, we obtain a central vertex of $G(R,S)$.

Now suppose $d(R,S)=2$. We prove that (2.26) or (2.27) holds by induction on $m+n$. If $m+n=8$, the conclusion holds by Lemma 2.22. Now suppose $m+n>8$. For definiteness we assume $n>4$. Consider the special matrix $\tilde{A}\in\mathfrak{A}(R,S)$, and let $\tilde{A}_{n-1}$ be the leading $m\times(n-1)$ submatrix of $\tilde{A}$, so that $\tilde{A}_{n-1}\in\mathfrak{A}(R_{n-1},S_{n-1})$, where both R_{n-1} and S_{n-1} are monotone. It follows that $d(R_{n-1},S_{n-1})\leqslant 2$.

First suppose that $\mathfrak{A}(R_{n-1},S_{n-1})$ has no invariant positions. Then it follows from Theorem 2.2 that $d(R_{n-1},S_{n-1})=2$. Thus by induction

$$R_{n-1}=(n-2,\dots,n-2,1),\qquad S_{n-1}=(m-1,\dots,m-1,1)$$

or

$$R_{n-1}=(n-2,1,\dots,1),\qquad S_{n-1}=(m-1,1,\dots,1).$$

It follows from the monotonicity of S that $s_n=1$. In the latter case, it is a consequence of the definition of $\tilde{A}$ that (2.27) holds. In the former case,

$$\tilde{A}=\left[\begin{array}{cccccc|cc} & & & & & & 0 & 1 \\ & & J_{m-2,n-2} & & & & \vdots & \vdots \\ & & & & & & 0 & 0 \\ \hline 1 & 0 & 1 & \cdots & & 1 & 1 & 0 \\ 0 & 1 & 0 & \cdots & & 0 & 0 & 0 \end{array}\right].$$

An interchange within the last two columns of $\tilde{A}$ produces a matrix having a 4×4 submatrix of the form (1.3). But then $d(R,S)>2$, a contradiction.

Now suppose that $\mathfrak{A}(R_{n-1}, S_{n-1})$ has invariant positions. Then by Lemma 2.7 either (2.7) or (2.8) holds. First suppose (2.7) holds. Then

$$\tilde{A} = \left[\begin{array}{c|c} A' & \begin{matrix} * \\ \vdots \\ * \end{matrix} \\ \hline O_{m-e,n-1} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \end{array}\right]$$

where $A' \in \mathfrak{A}(R', S')$ and $\mathfrak{A}(R', S')$ has no invariant positions. Clearly, $d(R', S') \leqslant 2$, and since $\mathfrak{A}(R', S')$ has no invariant positions, $d(R', S') \geqslant 1$. Suppose $d(R', S') = 1$. Then it follows from Theorem 2.2 that $e = 2$ and $R' = (n-2, 1)$, $S' = (1, \ldots, 1)$. From the monotonicity of S we conclude that $m - e = 1$. Hence $m = 3$, which is a contradiction. Now suppose $d(R', S') = 2$. Using the fact that the last column of $\tilde{A}$ contains a 0, it is now easy to see that $d(R, S) > 2$, contradicting our assumption.

Now suppose (2.8) holds. Then

$$\tilde{A} = \left[\begin{array}{c|c} J_{e,n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline A' & \beta \end{array}\right],$$

where $A' \in \mathfrak{A}(R', S')$ and $\mathfrak{A}(R', S')$ has no invariant positions, and where β is not a column of 0's. It is now easy to verify that $d(R, S) \geqslant d(R', S') + 1$. Since $d(R, S) = 2$ and $\mathfrak{A}(R', S')$ has no invariant positions, we conclude that $d(R', S') = 1$. Applying Theorem 2.2, we conclude that $m - e = 2$ and $R' = (n-2, 1)$, $S = (1, \ldots, 1)$. It follows that $R = (n-1, \ldots, n-1, 1)$ and $S = (m-1, \ldots, m-1, 1)$. This completes the induction and the theorem is proved. ■

An examination of the interchange graphs of diameter at most 2 leads one to speculate that interchange graphs are rarely bipartite and indeed usually contain a triangle. In the next section we prove a general theorem.

3. BIPARTITE INTERCHANGE GRAPHS

Recall that a graph is *bipartite* if its vertices can be partitioned into two sets X and Y such that every edge joins a vertex in X with a vertex in Y.

Equivalently, a graph is bipartite if and only if it has no cycles of odd length. A cycle of length 3 is called a *triangle*. In this section we characterize R and S for which $\mathfrak{A}(R,S)$ has no invariant positions and $G(R,S)$ is bipartite. First we prove the following.

LEMMA 3.1. *Suppose $\mathfrak{A}(R,S)$ has no invariant positions, and let $1 \leqslant i < j \leqslant m$. Then there exists $A \in \mathfrak{A}(R,S)$ having a 2×2 submatrix of the form* (1.1) *or* (1.2) *in rows i and j.*

Proof. Since $\mathfrak{A}(R,S)$ has no invariant positions, $m, n \geqslant 2$. If $n = 2$, then $R = (1,\dots,1)$ and $S = (a, m-a)$ for some integer a with $1 \leqslant a \leqslant m-1$, and the lemma follows easily in this case. We now proceed by induction on n. Let $n \geqslant 3$, and let $\tilde{A}_{n-1}$ be the matrix obtained from the special matrix $\tilde{A} \in \mathfrak{A}(R,S)$ by eliminating its last column. Let $\tilde{A}_{n-1} \in \mathfrak{A}(R_{n-1}, S_{n-1})$. Then by Lemma 2.5, either (2.6), (2.7), or (2.8) holds. If (2.6) holds, then the conclusion follows from the inductive assumption. Now suppose (2.7) holds. Then there exists an integer e with $1 \leqslant e \leqslant m-1$ such that

$$\tilde{A} = \left[\begin{array}{c|c} A' & \alpha \\ \hline & 1 \\ O_{m-e,\,n-1} & \vdots \\ & 1 \end{array}\right],$$

where $A' \in \mathfrak{A}(R', S')$ and $\mathfrak{A}(R', S')$ has no invariant positions, and $\alpha = (\alpha_1, \dots, \alpha_e)^t$ is not a column of 0's. If $1 \leqslant i < j \leqslant e$, the conclusion follows from the inductive assumption. Suppose $1 \leqslant i \leqslant e < j \leqslant m$. If $\alpha_i = 0$, then row i of A' contains a 1 and the conclusion holds. Suppose $\alpha_i = 1$. Choose k so that $\alpha_k = 0$. Then rows i and k of A' contain a 2×2 submatrix of the form (1.1) or (1.2). An interchange involving column n of $\tilde{A}$ is now available to produce a matrix in $\mathfrak{A}(R,S)$ for which the conclusion holds. Now suppose $e < i < j \leqslant m$. Choose k such that $\alpha_k = 0$. Since row k of A' contains a 1, there is an interchange which when applied to $\tilde{A}$ gives a matrix for which the conclusion holds.

Finally, suppose (2.8) holds. Then there exists an integer e with $1 \leqslant e \leqslant m-1$ such that

$$\tilde{A} = \left[\begin{array}{c|c} & 0 \\ J_{e,\,n-1} & \vdots \\ & 0 \\ \hline A' & \beta \end{array}\right]$$

where $A' \in \mathfrak{A}(R', S')$ and $\mathfrak{A}(R', S')$ has no invariant positions and $\beta = (\beta_{e+1}, \ldots, \beta_m)^t$. If $e < i < j \leq m$, the conclusion follows from the inductive assumption. Suppose $1 \leq i \leq e < j \leq m$. If $\beta_j = 1$, then since row $j - e$ of A' contains a 0, the conclusion holds. Suppose $\beta_j = 0$. Then an argument similar to the above shows there is a matrix, obtainable from $\tilde{A}$ by an interchange, for which the conclusion holds. Now suppose $1 \leq i < j \leq e$. Again an argument similar to the above establishes the conclusion, and the lemma holds by induction. ■

COROLLARY 3.2. *Suppose $\mathfrak{A}(R, S)$ has no invariant positions. Suppose i and j are such that $r_i > r_j$. Then there exists a matrix in $\mathfrak{A}(R, S)$ having within row i and j a 2×3 submatrix of the form*

$$P \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q \tag{3.3}$$

for some permutation matrices P and Q.

Proof. By Lemma 3.1 there exists a matrix in $\mathfrak{A}(R, S)$ having a 2×2 submatrix of the form (1.1) or (1.2) in rows i and j. Since $r_i > r_j$, the result follows. ■

The proof of the following lemma is easy.

LEMMA 3.4. *If $\mathfrak{A}(R, S)$ contains a matrix having a submatrix of the form* (3.3) *or its transpose, then $G(R, S)$ contains a triangle.*

We now prove the main result of this section.

THEOREM 3.5. *Suppose $\mathfrak{A}(R, S)$ has no invariant positions. Then the following are equivalent:*

$$G(R, S) \text{ is bipartite.} \tag{3.6}$$

$$G(R, S) \text{ has no triangle.} \tag{3.7}$$

$$m = n, \quad \text{and} \quad R = S = (1, \ldots, 1) \text{ or } R = S = (n-1, \ldots, n-1). \tag{3.8}$$

Proof. Clearly (3.6) implies (3.7). Suppose (3.7) holds. Then it follows from Corollary 3.2 and Lemma 3.4 that $r_1 = \cdots = r_n$ and $s_1 = \cdots = s_n$. Hence $R = (c, \ldots, c)$ and $S = (d, \ldots, d)$ for some integers c and d where

$1 \leqslant c \leqslant n-1$, $1 \leqslant d \leqslant m-1$, and $cm = dn$. Let $u =$ g.c.d.$\{m, n\}$. Then $m = m'u$, $n = n'u$, and $cm' = dn'$. It follows that there is an integer k such that $c = kn'$ and $d = km'$. Let $E = [e_{ij}]$ be the $u \times u$ permutation matrix with $e_{12} = \cdots = e_{u-1,u} = e_{u,1} = 1$, and consider a $u \times u$ circulant matrix $A = a_0 I + a_1 E + \cdots + a_{u-1} E^{u-1}$ where $a_0 = \cdots = a_{k-1} = 1$ and $a_k = \cdots = a_{u-1} = 0$. Replacing each 1 of A with $J_{m',n'}$ and each 0 with $O_{m',n'}$, we obtain a matrix B in $\mathfrak{A}(R, S)$. It is now easy to see that if $1 < c < n-1$, then B has a 2×3 submatrix of the form (3.3) in rows m' and $m'+1$. Thus it follows from Lemma 3.4 that $c = 1$ or $c = n-1$. Similarly, $d = 1$ or $d = m-1$. If $c = d = 1$ or $c = n-1$, $d = m-1$, then $m = n$ and (3.8) holds. If $c = n-1$, $d = 1$ or $c = 1$, $d = m-1$, then $m = n = 2$ and (3.8) holds. Thus (3.7) implies (3.8). Finally suppose (3.8) is satisfied. It suffices to assume $R = S = (1, \dots, 1)$. Then $\mathfrak{A}(R, S)$ consists of all $n \times n$ permutation matrices, which themselves correspond to the permutations of $\{1, \dots, n\}$. Consider permutation matrices $P, Q \in \mathfrak{A}(R, S)$. Then P and Q are joined by an edge in $G(R, S)$ if and only if P^tP and P^tQ are. Since $P^tP = I_n$, the latter holds if and only if $P^tQ = F$ (thus $Q = PF$) where F is a permutation matrix corresponding to a transposition. It follows that if P and Q are joined by an edge in $G(R, S)$, then they correspond to permutations of different parity. Hence (3.6) holds, and the proof is complete. ■

In closing we remark that it is not difficult to show that if A, B, and C are the vertices of a triangle in $G(R, S)$, then A, B, and C agree everywhere except on a 2×3 or 3×2 submatrix, which then is of the form (3.3) or its transpose.

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