SKOLIAD No. 102

Robert Bilinski

Please send solutions to the problems in this edition by ***November 1, 2007***. A copy of **MATHEMATICAL MAYHEM Vol. 4** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

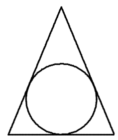
Nos questions proviennent ce mois-ci du Concours 2006 de Mathématique du secondaire de Colombie Britannique. Nous remercions Clint Lee, Okanagan College, Vernon, BC, qui s'occupe de ces concours.

Concours 2006 de Mathématique du secondaire de Colombie Britannique
Ronde Finale Sénior partie B, vendredi 5 mai 2006

1. Déterminer le nombre de séquences d'entiers consécutifs dont la somme est 100.

2. Un réservoir vitré cubique de côté un mètre est placé sur une table horizontale et est rempli à moitié d'eau. Ainsi, la profondeur de l'eau dans le réservoir (la distance entre la surface de l'eau et la surface de la table) est un demi-mètre. Le réservoir est tourné autour d'une des arêtes sur la table afin qu'une des faces du réservoir ait un angle de 30^o avec la table. Trouver la profondeur de l'eau après la rotation.

3. Les longueurs des côtés d'un triangle sont 13, 13 et 10. Le *cercle inscrit* de ce triangle est un cercle ayant son centre à l'intrieur du triangle qui est tangent à chacun des côtés du triangle. (Voir le diagramme.) Trouver le rayon du cercle inscrit.

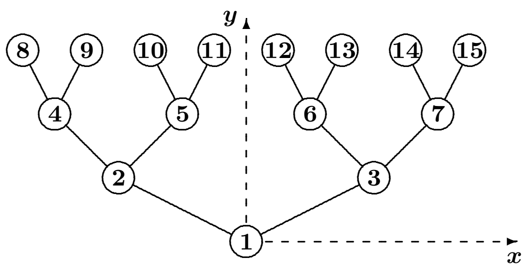


4. Cinq entiers positifs a, b, c, d et e supérieurs à un remplissent les conditions suivantes :

$$\begin{aligned} a(b+c+d+e) &= 128\,, \\ b(a+c+d+e) &= 155\,, \\ c(a+b+d+e) &= 203\,, \\ d(a+b+c+e) &= 243\,, \\ e(a+b+c+d) &= 275\,. \end{aligned}$$

Trouver ces cinq entiers.

5. Un arbre binaire entier consiste en un **noeud** racine qui a deux **enfants**, un noeud droit et un noeud gauche, et chacun des noeuds enfants a deux enfants, jusqu'à ce que le haut de l'arbre soit atteint, où chaque noeud n'a pas d'enfants. Dans un certain arbre binaire entier, chaque noeud est numéroté, en commençant par 1 à la racine et en numérotant de la gauche vers la droite à travers chaque niveau. Le diagramme montre les quatre premiers niveaux d'un tel arbre. La racine d'un tel arbre est placé à l'origine d'un système de coordonnées xy, l'axe x étant à l'horizontal et l'axe y étant à la verticale, comme illustré. Si l'espace entre les niveaux de l'arbre est de 2 unités dans la direction y et l'espacement entre les noeuds du niveau supérieur qui contient le noeud numéroté 2006 est de 2 unités dans la direction x, trouver les coordonnées du noeud numéroté 2006.



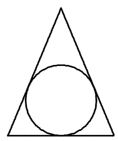
British Columbia Secondary School Mathematics Contest 2006

Senior Final Round, Part B, Friday, May 5, 2006

1. Determine the number of sequences of consecutive integers whose sum is 100.

2. A cubical glass tank with sides of length one metre is placed on a horizontal table and half filled with water. Thus, the depth of the water in the tank (the distance of the surface of the water from the surface of the table) is one half metre. The tank is rotated about one of the edges that is on the table so that one face of the tank makes a 30° angle with the table. Find the depth of the water in the tank after the rotation.

3. The lengths of the sides of a triangle are 13, 13, and 10. The *inscribed circle* of this triangle is the circle with centre inside the circle that is tangent to each of the three sides of the triangle. (See the diagram.) Find the radius of the inscribed circle.

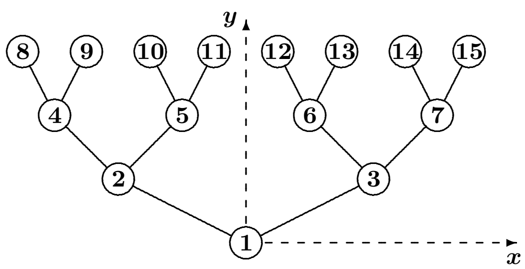


4. Five positive integers a, b, c, d, and e greater than one make the following conditions true:

$$\begin{aligned} a(b+c+d+e) &= 128\,, \\ b(a+c+d+e) &= 155\,, \\ c(a+b+d+e) &= 203\,, \\ d(a+b+c+e) &= 243\,, \\ e(a+b+c+d) &= 275\,. \end{aligned}$$

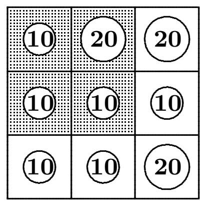
Find the five integers.

5. A full binary tree consists of a root **node** which has two **children**, a right child node and a left child node, and each child node has two children, until the top of the tree is reached, where each node has no children. In a certain full binary tree each node is numbered, starting with 1 at the root, numbering from left to right across each level. The diagram shows the first four levels of such a tree. The root of the tree is placed at the origin of an xy–coordinate system, with the x–axis horizontal and the y–axis vertical, as shown. If the spacing between the levels of the tree is 2 units in the y–direction and the spacing between the nodes at the top level that contains the node numbered 2006 is 2 units in the x–direction, find the coordinates of the node numbered 2006.



Next we give the solutions to the National Bank of New Zealand Competition 2000 [2006 : 417–421].

1. **Grade 9 only**. In this problem we'll be placing various arrangements of 10¢ and 20¢ coins on the nine squares of a 3×3 grid. Exactly one coin will be placed in each of the nine squares. The grid has four 2×2 subsquares each containing a corner, the centre, and the two squares adjacent to these. One example is shown in the diagram.



(a) Find an arrangement where the totals of the four 2×2 subsquares are 40¢, 60¢, 60¢, and 70¢ in any order. (Draw a diagram showing your arrangement.)

(b) Find an arrangement where the totals of the four 2×2 subsquares are 50¢, 60¢, 70¢, and 80¢ in any order. (Draw a diagram showing your arrangement.)

For each part of the problem below, illustrate your answer with a suitable arrangement and an explanation of why no other suitable arrangement contains a larger (part (c)) or a smaller (part (d)) amount of money.

(c) What is the maximum amount of money which can be placed on the grid so that each of the 2×2 subsquares contains exactly 50¢?

(d) What is the minimum amount of money which can be placed on the grid so that the average of the amount of money in each of the 2×2 subsquares is exactly 60¢?

Solution by the editor.

(a) Since one 2×2 subsquare contains only 40¢, all of its entries must be 10¢ coins. In particular, the centre square must contain a 10¢ coin. Since another of the 2×2 subsquares contains 70¢ and it must also contain the 10¢ coin in the centre square, its remaining entries must be 20¢. It is then easy to complete the square as in Figure 1 below. Other squares are possible by rotation.

(b) Since one 2×2 subsquare contains 80¢, all of its entries must be 20¢ coins. In particular, the centre square must contain a 20¢ coin. Since another of the 2×2 subsquares contains only 50¢ and it must also contain the 20¢ coin in the centre square, its remaining entries must be 10¢. It is then easy to complete the square as in Figure 2 below. Other squares are possible by rotation.

(c) The amount of money will be maximized when the number of 20¢ coins on the grid is maximized. If we place a 20¢ coin in the centre square, then all the remaining coins have to be 10¢ coins. If we place a 20¢ coin along one side of the grid, in the middle square of that side, then at most two more 20¢ coins may be placed on the grid (in the corners on the opposite side). But we can use four 20¢ coins if they are placed in the four corners of the grid as in Figure 3 below. Thus, the maximum number of 20¢ coins that can be used is four, giving a total of \$1.30.

(d) The distribution in Figure 4 below satisfies the conditions and uses \$1.30, which includes only three 20¢ coins. If we are to lower that total at all, we must use at most two 20¢ coins. But then none of the 2×2 subsquares could contain more than two 20¢ coins, and therefore none could contain more than 60¢. Since the average across all four 2×2 subsquares must be 60¢, each of the four would then have to contain exactly 60¢, which means that each would have to contain two 20¢ coins. The same two 20¢ coins would then have to be in all four 2×2 subsquares, which is impossible. Thus, the minimum is \$1.30.

10	10	20
10	10	20
20	20	20

Figure 1

10	10	10
10	20	20
20	20	20

Figure 2

20	10	20
10	10	10
20	10	20

Figure 3

10	20	10
10	20	10
10	20	10

Figure 4

2. Humankind was recently contacted by three alien races: the Kweens, the Ozdaks, and the Merkuns. Little is known about these races except that Kweens always speak the truth while Ozdaks always lie. In any group of aliens a Merkun will never speak first. When it does speak, it tells the truth if the previous statement was a lie, and lies if the previous statement was truthful. Although the aliens can readily tell one another apart, of course to humans all aliens look the same.

A high-level delegation of three aliens has been sent to Earth to negotiate our fate. Among them is at least one Kween. On arrival they make the following statements (in order):

First Alien: The second alien is a Merkun.

Second Alien: The third alien is not a Merkun.

Third Alien: The first alien is a Merkun.

Which alien or aliens can you be certain are Kweens?

Official solution, expanded by the editor.

The first alien cannot be a Merkun, since a Merkun never speaks first. Thus, the third statement must be a lie, which means that the third alien cannot be a Kween.

Suppose that the third alien is a Merkun (who is lying). Since Merkuns only lie when the previous statement is true, the second statement must be true. But then we have a contradiction. Hence, the third alien is an Ozdak.

Since there is a Kween among the three aliens, it must be one of the first two aliens. Suppose that the first is a Kween. Then the second is a Merkun and its statement is a lie. But this implies that the third alien is a Merkun, which we have already ruled out. Therefore, the first alien is not a Kween, which means that the first alien is an Ozdak and the second is a Kween.

Thus, only the second alien is a Kween (and the others are both Ozdaks).

3. (Note: In this question an "equal division" is one where the total weight of the two parts is the same.)

(a) Belinda and Charles are burglars. Among the loot from their latest caper is a set of 12 gold weights of 1g, 2g, 3g, and so on, through to 12g. Can they divide the weights equally between them? If so, explain how they can do it, and if not, why not?

(b) When Belinda and Charles take the remainder of the loot to Freddy the fence, he demands the 12g weight as his payment. Can Belinda and Charles divide the remaining 11 weights equally between them? If so, explain how they can do it, and if not, why not?

(c) Belinda and Charles also have a set of 150 silver weights of 1g, 2g, 3g, and so on, through to 150g. Can they divide these weights equally between them? If so, explain how they can do it, and if not, why not?

Official solution.

(a) Yes, it can be done. One way is to pair them from the "extremes": $1 + 12$, $2 + 11$, $\ldots$, $6 + 7$. Each person then takes three of the pairs. There are other posibilities.

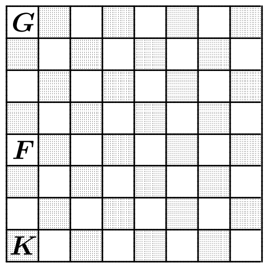
(b) Yes, it can still be done. Belinda gets $6 + 1$ and Charles gets 7. The rest can be paired from the "extremes" ($2 + 11$, $3 + 10$, $4 + 9$, $5 + 8$), with each person taking two of the pairs.

(c) No, it cannot be done. There are 75 even and 75 odd weights, which implies that the total is odd and cannot be split equally.

4. A chessboard is an 8×8 grid of squares. One of the chess pieces, the king, moves one square at a time in any direction, including diagonally.

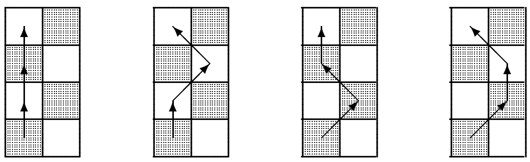
(a) A king (denoted by K in the diagram) stands on the lower left corner of a chessboard. It has to reach the square marked F in exactly three moves. Show that the king can do this in exactly four different ways.

(b) Assume that the king is placed back on the bottom left corner. In how many ways can it reach the upper left corner (marked G) in exactly seven moves?

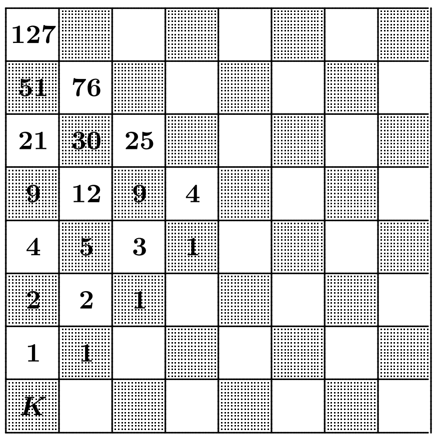


Official solution.

(a) Since each move must find the king one row higher, the following four diagrams illustrate all possible routes.

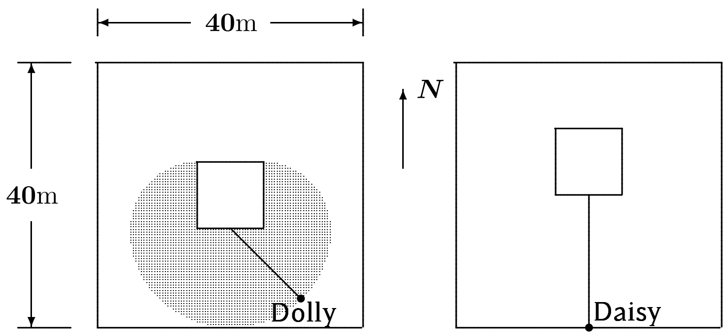


(b) The king must move one row higher on each move. In each square that could be part of the king's seven-move path from bottom left to upper left corner, we place the number of ways the king can reach that square. (Each number may be obtained by adding the numbers in the squares below it and having an edge or corner in common with it.) We find that there are 127 ways the king can reach the upper left corner in seven moves.



5. (Note: For this question answers containing expressions such as $\frac{4\pi}{13}$ are acceptable. If you have a calculator you may use the button for π if you like.)

(a) The Jones family lives in a perfectly square house, 10m by 10m, which is placed exactly in the middle of a 40m by 40m lot, entirely covered (except for the house) in grass. They keep the family pet, Dolly the sheep, tethered to the middle of one side of the house on a 15m rope. What is the area of the part of the lawn (in m^2) in which Dolly is able to graze? (See shaded area.)



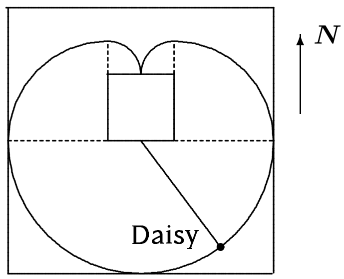
The Jones The Smiths

(b) The Jones' neighbours, the Smiths, have an identical lot to the Jones but their house is located five metres to the North of the centre. Their pet sheep, Daisy, is tethered to the middle of the southern side of the house on a 20m rope. What is the area of the part of the lawn (in m^2) in which Daisy is able to graze?

Official solution.

(a) The part of the lawn in which Dolly can graze consists of a semicircle of radius 15 and two quarter circles of radius 10. Its area is therefore $\frac{1}{2}\pi 15^2 + 2 \cdot \frac{1}{4}\pi 10^2 = \frac{325}{2}\pi$.

(b) The part of the lawn in which Daisy can graze is made up from one semicircle and four other quarter circles, as shown. Its area is therefore $\frac{1}{2}\pi 20^2 + 2 \cdot \frac{1}{4}\pi 15^2 + 2 \cdot \frac{1}{4}\pi 5^2 = 325\pi$.



The Smiths

That brings us to the end of another issue. Please send in solutions!

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Jeff Hooper (Acadia University). The Assistant Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are John Grant McLoughlin (University of New Brunswick), Monika Khbeis (Father Michael Goetz Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

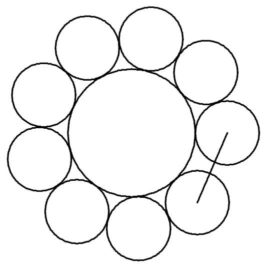
Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier septembre 2007**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

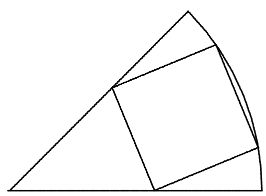
M294. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*



On dessine neuf cercles contigus de rayon $1/2$ et tangents à un cercle de rayon 1.

Trouver la distance entre les centres du premier et du dernier de ces cercles.

M295. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*



Un carré $ABCD$ est inscrit dans un secteur circulaire couvrant le huitième d'un disque de rayon 1. Deux des sommets du carré sont sur les rayons du bord et les deux autres sont sur le cercle.

Trouver l'aire du carré. On exige une réponse exacte de la forme $\frac{a+b\sqrt{c}}{d}$, où a, b, c et d sont des entiers.

M296. *Proposé par Daniel Tsai, étudiant, Taipei American School, Taipei, Taiwan.*

Pour $k = 1, 2, \ldots, n$, soit les points $a_k = (k, n)$ et $b_k = (k, 0)$ et supposons que chaque couple a_k, b_k est relié par un segment de droite vertical. Supposons qu'on dessine un nombre arbitraire fini de segments horizontaux reliant deux verticales adjacentes de sorte qu'aucun point d'une verticale quelconque soit une extrémité de deux segments horizontaux.

On définit une application de l'ensemble $A = \{a_1, a_2, \ldots, a_n\}$ dans l'ensemble $B = \{b_1, b_2, \ldots, b_n\}$ de la manière suivante : partant de a_i, on descend en suivant le segment jusqu'à ce qu'on rencontre l'extrémité d'un segment horizontal, qu'on suit alors jusqu'à son autre extrémité, on descend ensuite le long du nouveau segment vertical et ainsi de suite, pour arriver finalement à b_j pour un certain j. Montrer que deux points d'A n'arrivent pas au même point de B.

M297. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Les nombres 34543 et 713317 sont des nombres palindromes puisque leurs chiffres apparaissent dans le même ordre, quel que soit le sens de lecture. Montrer que tous les nombres palimdromes de quatre chiffres sont des multiples de 11.

M298. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

(a) Etant donné un nombre palindrome de quatre chiffres, quelle est la probabilité qu'il soit un multiple de 99 ?

(b) Etant donné un multiple de 99, quelle est la probabilité qu'il soit un nombre palindrome ?

M299. *Proposé par Titu Zvonaru, Comăneşti, Roumanie.*

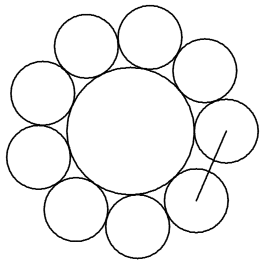
Soit a, b et c trois nombres réels positifs tels que $ab + bc + ca = 3$. Montrer que

$$\frac{ab}{c^2+1} + \frac{bc}{a^2+1} + \frac{ca}{b^2+1} \geq \frac{3}{2}.$$

M300. *Proposé par Geoffrey A. Kandall, Hamden, CT, É-U.*

Dans un triangle quelconque ABC, soit D un point sur le côté AC, E un point sur AB, et soit P le point d'intersection de BD et CE. Si $AE : EB = r$ et $AD : DC = s$, trouver le rapport des aires $[ABC] : [PBC]$ en fonction de r et de s.

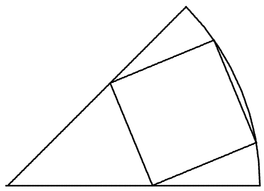
M294. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*



Nine circles of radius $1/2$ are externally tangent to a circle of radius 1 and are tangent to one another, as shown.

Determine the distance between the centres of the first and last of the circles of radius $1/2$.

M295. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*



Square $ABCD$ is inscribed in one-eighth of a circle of radius 1 so that there is one vertex on each radius and two vertices on the arc.

Determine the exact area of the square in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are integers.

M296. *Proposed by Daniel Tsai, student, Taipei American School, Taipei, Taiwan.*

Let n be a positive integer. In the Cartesian plane, consider the points $a_k = (k, n)$ and $b_k = (k, 0)$ for $k = 1, 2, \ldots, n$. We connect each pair a_k, b_k by a straight (vertical) line segment. Then we draw an arbitrary finite number of horizontal line segments, each connecting two adjacent vertical line segments, such that no one point on any vertical segment is the end-point of two horizontal segments.

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Define a map from A to B as follows: starting from a_i, travel down the segment until you meet the end-point of a horizontal segment, go to the other end-point of that segment, and continue on down the new vertical line, repeating this until there are no more horizontal segments to meet, finally ending at b_j for some j. Show that no two points of A map to the same point of B.

M297. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

Numbers such as 34543 and 713317 whose digits can be reversed without changing the number are called *palindromes*. Show that all four-digit palindromes are multiples of 11.

M298. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

(a) Given that a number is a four-digit palindrome, what is the probability that the number is a multiple of 99?

(b) Given that a four-digit number is a multiple of 99, what is the probability that the number is a palindrome?

M299. *Proposed by Titu Zvonaru, Comăneşti, Romania.*

Let a, b, and c be positive real numbers with $ab + bc + ca = 3$. Prove that
$$\frac{ab}{c^2+1} + \frac{bc}{a^2+1} + \frac{ca}{b^2+1} \geq \frac{3}{2}\,.$$

M300. *Proposed by Geoffrey A. Kandall, Hamden, CT, USA.*

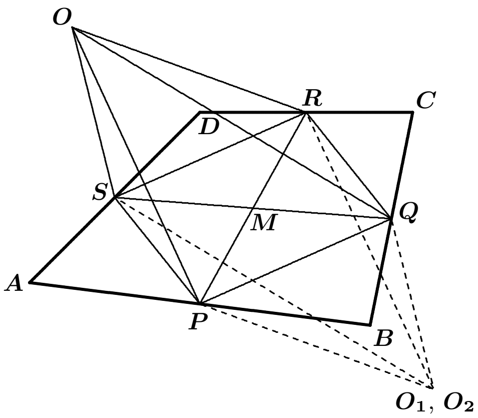
Let ABC be an arbitrary triangle. Let D and E be points on the sides AC and AB, respectively, and let P be the point of intersection of BD and CE. If $AE : EB = r$ and $AD : DC = s$, determine the ratio of areas $[ABC] : [PBC]$ in terms of r and s.

Mayhem Solutions

M244. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $ABCD$ be a convex quadrilateral, and let P, Q, R, S be the mid-points of AB, BC, CD, DA, respectively. Suppose that four distinct lines each passing through one of P, Q, R, S concur at a point O. Draw lines parallel to these four lines but passing through the mid-points of the opposite sides. Prove that these four lines are also concurrent.

A combination of solutions by Hasan Denker, Istanbul, Turkey; and Titu Zvonaru, Comăneşti, Romania.



The points P, Q, R, S form a parallelogram, since they are the mid-points of the sides of a quadrilateral. Therefore, the diagonals of $PQRS$ bisect each other at a point M.

Let O_1 be the point of intersection of the parallel to OR through P and the parallel to OP through R. Since OR is parallel to PO_1 and OP is parallel to RO_1, quadrilateral ORO_1P is a parallelogram. Similarly, if O_2 is the point of intersection of the parallel to OS through Q and the parallel to OQ through S, then OSO_2Q is a parallelogram.

Now, the diagonals of parallelogram ORO_1P bisect at point M and $|OM| = |O_1M|$, whereas the diagonals of parallelogram OSO_2Q bisect at point M resulting in $|OM| = |O_2M|$.

Consequently, $O_1 = O_2$, and the four lines are concurrent.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; ANDREA MUNARO, student, Liceo Scientifico "N. Tron", Schio, Italy; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan.

M245. *Proposed by Ray Killgrove, Vista, CA, USA.*

Given isosceles triangle ABC with $AB = AC$, let the points D and E trisect the third side BC; that is, $BD = DE = EC$. For small angles A, it appears as if $\angle A$ is trisected by the segments AD and AE. Prove that, to the contrary, $\angle BAC$ is never trisected by the segments AD and AE.

Solution by Hasan Denker, Istanbul, Turkey.

Let $AB = AC = b$ and let $AD = AE = y$. Clearly, $\triangle ABD$ is congruent to $\triangle ACE$. Let $\angle BAD = \angle EAC = \alpha$ and let $\angle DAE = \beta$. Now triangles ABD, ADE and ACE all have the same area, since they each have the same altitude and base. Therefore,

$$\tfrac{1}{2}by\sin\alpha \;=\; \tfrac{1}{2}y^2\sin\beta\,,$$

which simplifies to $\sin\alpha = (y/b)\sin\beta$. Since $y/b < 1$, we conclude that $\alpha < \beta$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; ANDREA MUNARO, student, Liceo Scientifico "N. Tron", Schio, Italy; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comăneşti, Romania (two solutions).

Both Kandall and Zvonaru showed that the triangle need not be isosceles to make their arguments work.

M246. *Proposed by the Mayhem Staff.*

Ten points are arranged in a plane so that no three are collinear. What is the maximum number of segments that can be drawn joining two of the points such that no three of these points are all joined to form a triangle?

Solution by Gabriel Krimker, student, and Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

We use the terminology of graph theory. Let G be a graph on n vertices. The sets V and E represent the set of vertices and the set of edges of G, respectively. Let $d(v)$ be the degree of the vertex v in G; that is, $d(v)$ is the number of edges at the vertex v.

Let v_1 be any of the n vertices of G. Suppose that $d(v_1) = m$. Since the maximum degree of any vertex is $n-1$, we have $m < n$. If we consider the m vertices connected to v_1 by an edge, we know that there can be no edge connecting any pair of them (otherwise, we would have a triangle among the two of them and v_1). Then the number of edges will be a maximum if each of these vertices is connected to all the other vertices of G. Since there are $n-m$ other vertices, the maximum degree possible for each of the m vertices joined to v_1 is $n-m$. This also means that those other $n-m$ vertices are only joined to the m vertices which are joined to v_1. Therefore, if we have the maximum number of edges with no triangles, then

$$\sum_{v\in V} d(v) \;=\; (n-m)m + m(n-m) \;=\; 2m(n-m)\,.$$

Since we also have the well-known identity $2|E| = \sum_{v\in V} d(v)$ (where $|E|$ is the number of edges), we see that $|E| = m(n-m)$. This value is a maximum at the integer value of m which is nearest to $\frac{1}{2}n$. Therefore, if n is even, then $m = \frac{1}{2}n$; if n is odd, then $m = \frac{1}{2}(n-1)$. This yields

$$|E| \;=\; \begin{cases} \dfrac{n^2}{4} & \text{if } n \text{ is even,} \\ \dfrac{n^2-1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

For ten points, the maximum number of segments is 25.

[*Ed.:* To see that this maximum can be achieved, we simply take the n vertices and split them into two sets of $\frac{1}{2}n$ vertices if n is even, or a set of $\frac{1}{2}(n-1)$ vertices and a set of $\frac{1}{2}(n+1)$ vertices if n is odd. Then construct all possible edges between the vertices of different sets and no edges between vertices from the same set. In graph theory, these are called complete bipartite graphs; the graph for ten vertices which gives the maximum number of edges is denoted $K_{5,5}$.]

Also solved by HASAN DENKER, Istanbul, Turkey; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan. There was one incomplete solution.

M247. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let a, b, c be positive real numbers with $a+b+c=1$. Given that $ab+bc+ca=\frac{1}{3}$, find the values of:

(a) $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}$, (b) $\dfrac{a}{b+1}+\dfrac{b}{c+1}+\dfrac{c}{a+1}$.

I. *Solution by Titu Zvonaru, Comănești, Romania.*

The given conditions $a+b+c=1$ and $ab+bc+ca=\frac{1}{3}$ imply that $a=b=c=\frac{1}{3}$, because

$$\begin{aligned}(a-b)^2+(b-c)^2+(c-a)^2 &= 2(a^2+b^2+c^2-ab-bc-ca) \\ &= 2[(a+b+c)^2-3(ab+bc+ca)] \\ &= 2\left(1-3\cdot\tfrac{1}{3}\right) \;=\; 0\,.\end{aligned}$$

Therefore, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=3$ and $\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}=\frac{3}{4}$.

II. *Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.*

Substituting $a=1-b-c$ into $ab+bc+ca=\frac{1}{3}$, we get

$$\begin{aligned} (1-b-c)b+bc+c(1-b-c) &= \tfrac{1}{3}\,, \\ \text{or} \quad c^2+(b-1)c+\left(\tfrac{1}{3}-b(1-b)\right) &= 0\,. \end{aligned}$$

We compute the discriminant for this quadratic in c:

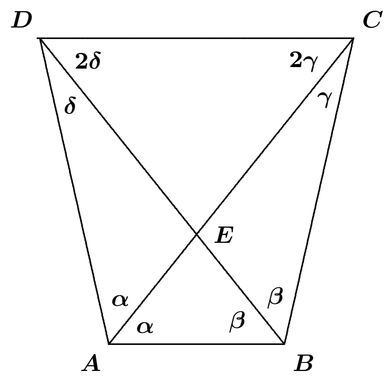
$$(b-1)^2-4\left(\tfrac{1}{3}-b(1-b)\right) \;=\; -3\left(b^2+\tfrac{2}{3}b-\tfrac{1}{9}\right) \;=\; -3\left(b-\tfrac{1}{3}\right)^2.$$

Since c is real, we require the discriminant to be non-negative. Therefore, $b=\frac{1}{3}$. Similarly, we can show that $c=\frac{1}{3}$ and $a=\frac{1}{3}$. Then we have $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=3$ and $\frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}=\frac{3}{4}$.

Also solved by HASAN DENKER, Istanbul, Turkey; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

M248. *Proposed by K.R.S. Sastry, Bangalore, India.*

In the convex quadrilateral $ABCD$, the diagonals AC and BD bisect and trisect the opposite angles as shown.



(a) Find the (acute) angle between AC and BD.

(b) Show that $\frac{\pi}{7}<\alpha<\frac{3\pi}{7}$.

Solution by Hasan Denker, Istanbul, Turkey.

(a) The sum of the angles in quadrilateral $ABCD$ is

$$2(\alpha+\beta)+3(\gamma+\delta) \;=\; 2\pi\,.$$

Summing the angles in $\triangle EAB$ and $\triangle ECD$ yields $\alpha+\beta=2(\delta+\gamma)$. From the above equations, we get

$$\alpha+\beta \;=\; \frac{4\pi}{7}\,. \qquad (1)$$

Hence, the acute angle between AC and BD is $\pi-(\alpha+\beta)=\frac{3\pi}{7}$.

(b) Summing the angles in $\triangle DAB$ gives the equation $2\alpha+\beta+\delta=\pi$. Using (1), we get $\alpha+\delta=\frac{3\pi}{7}$. Since $\delta>0$, we see that $\alpha<\frac{3\pi}{7}$.

On the other hand, summing the angles of $\triangle CAB$ yields the equation $\alpha+2\beta+\gamma=\pi$. Using (1), we get $\beta+\gamma=\frac{3\pi}{7}$; hence, $\left(\frac{4\pi}{7}-\alpha\right)+\gamma=\frac{3\pi}{7}$, or $\gamma=\alpha-\frac{\pi}{7}$. Since $\gamma>0$, we see that $\alpha>\frac{\pi}{7}$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VEDULA N. MURTY, Dover, PA, USA; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comăneşti, Romania.

M249. *Proposed by K.R.S. Sastry, Bangalore, India.*

Determine the real numbers a, b, c, d, given that the roots of the equation $x^2+ax-b=0$ are a and c, and the roots of the equation $x^2+cx+d=0$ are b and d.

Essentially the same solution by Andrea Munaro, student, Liceo Scientifico "N. Tron", Schio, Italy; and Titu Zvonaru, Comăneşti, Romania.

Since the roots of the equation $x^2+ax-b=0$ are a and c, we must have $x^2+ax-b=(x-a)(x-c)=x^2-(a+c)x+ac$; therefore, $a+c=-a$ and $ac=-b$. Similarly, since the roots of the equation $x^2+cx+d=0$ are b and d, we find that $b+d=-c$ and $bd=d$. Thus, we obtain the following system of equations:

$$\begin{aligned} a+c &= -a\,, \\ ac &= -b\,, \\ b+d &= -c\,, \\ bd &= d\,. \end{aligned}$$

If $d\neq 0$, then $b=1$, and consequently, $2a=-c$, $ac=-1$, and $c+d=-1$, which yields $a=\pm\frac{\sqrt{2}}{2}$, $c=\mp\sqrt{2}$ and $d=\pm\sqrt{2}-1$.

If $d=0$, then

$$\begin{aligned} 2a &= -c\,, \\ ac &= -b\,, \\ b &= -c\,, \end{aligned}$$

from which we can conclude that $ac=c$. Hence, if $c\neq 0$, then $a=1$, $b=2$, and $c=-2$. However, if $c=0$, then $a=0$ and $b=0$.

Therefore, the solution is

$$(a,b,c,d)\in\left\{(0,0,0,0),\,(1,2,-2,0),\,\left(\pm\frac{\sqrt{2}}{2},1,\mp\sqrt{2},\pm\sqrt{2}-1\right)\right\}\,.$$

There were two incomplete solutions submitted.

M250. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers satisfying $\sum_{i=1}^{n} x_i = n$. Let $x_{n+1} = x_1$. Show that $\sum_{i=1}^{n} x_i x_{i+1} \le n$ if $n \in \{1, 2, 3, 4\}$, but not necessarily if $n \ge 5$.

Solution by Titu Zvonaru, Comăneşti, Romania.

When $n = 1$, we have $x_1 = 1$. Then $x_1^2 \le 1$.

Let $n = 2$. Then $x_1 + x_2 = 2$. By the AM–GM Inequality, we obtain $\sqrt{x_1 x_2} \le \frac{x_1 + x_2}{2} = 1$. Hence,

$$\sum_{i=1}^{2} x_i x_{i+1} \;=\; x_1 + x_2 \;\le\; 2\,.$$

Equality holds if and only if $x_1 = x_2 = 1$.

Now let $n = 3$. Then $x_1 + x_2 + x_3 = 3$. By the well-known (and easy to prove) inequality

$$3(x_1x_2 + x_2x_3 + x_3x_1) \;\le\; (x_1 + x_2 + x_3)^2\,,$$

we see that

$$\sum_{i=1}^{3} x_i x_{i+1} \;=\; x_1x_2 + x_2x_3 + x_3x_1 \;\le\; \frac{(x_1 + x_2 + x_3)2}{3} \;=\; 3\,.$$

Equality holds if and only if $x_1 = x_2 = x_3 = 1$.

Next consider $n = 4$. Then $x_1 + x_2 + x_3 + x_4 = 4$. By the AM–GM Inequality, we deduce that

$$\begin{aligned}\sqrt{x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1} \;&=\; \sqrt{(x_1 + x_3)(x_2 + x_4)} \\ &\le\; \frac{(x_1 + x_3) + (x_2 + x_4)}{2} \;=\; 2\,.\end{aligned}$$

Hence,

$$\sum_{i=1}^{4} x_i x_{i+1} \;=\; x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 \;\le\; 2^2 \;=\; 4\,.$$

Equality holds if and only if $x_1 = x_2 = x_3 = x_4 = 1$.

Finally, let $n \ge 5$. We set $x_1 = x_2 = \frac{1}{2}n$ and $x_3 = \cdots = x_n = 0$. Then $x_1 + x_2 + \cdots + x_n = n$. Since $n > 4$,

$$\sum_{i=1}^{n} x_i x_{i+1} \;=\; \frac{n}{2} \cdot \frac{n}{2} \;=\; \frac{n}{4} \cdot n \;>\; \frac{4}{4} \cdot n = n\,.$$

Also solved by HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan.

Problem of the Month

Ian VanderBurgh

Here is a problem that requires only some careful reasoning (albeit pretty tricky careful reasoning) and the ability to add.

Problem (2006 Grade 8 Gauss Contest)

In the diagram, the numbers from 1 to 25 are to be arranged in the 5×5 grid so that each number, except 1 and 2, is the sum of two of its neighbours. (Numbers in the grid are *neighbours* if their squares touch along a side or at a corner. For example, the "1" has 8 neighbours.) Some of the numbers have already been filled in. Which number must replace the "?" when the grid is completed?

			20	21
	6	5	4	
23	7	1	3	?
	9	8	2	
25	24			22

This is not another Sudoku—honest! It looks a bit like one, though. That is part of the reason why this problem was included on the Contest—it is nice to have problems that look familiar but, upon closer examination, are a bit different.

Solution: We could just fiddle around by trial and error until we get some numbers that work. But we will walk through the solution in a logical way.

It's tough to know exactly where to start. First, it makes sense to check which numbers are missing. The grid already includes the numbers 1 to 9 and 20 to 25; so those missing are 10 to 19.

Next, we could figure out which numbers in the grid are already the sum of two neighbours. For example, 9 has neighbours 1 and 8 (and $9 = 1 + 8$); 8 has neighbours 1 and 7 (and $8 = 1 + 7$), and so on. Let's italicize every number which is already the sum of two of its neighbours, as well as the entries 1 and 2.

			20	21
	6	*5*	*4*	
23	*7*	*1*	*3*	?
	9	*8*	*2*	
25	24			22

Now what? It's probably time for that tried and true problem-solving technique—panic. After we get that out of our system, we might try looking at some of the numbers that have almost all of their neighbours already filled in. Also, we might as well focus on the part of the grid near the "?".

For example, consider 21. Since 21 already has neighbours 20 and 4, we must write 21 as either $20 + 1$ or $4 + 17$. But the number 1 already appears elsewhere in the grid; thus, the empty space below 21 must be 17.

			20	*21*
	6	*5*	*4*	17
23	*7*	*1*	*3*	?
	9	*8*	*2*	
25	24			22

Looking at 17 as we did with 21, we see that 17 must be $3 + 14$ or $4 + 13$; thus, the "?" must represent either 13 or 14. But we can't say for sure yet which one it is.

How about 22? It cannot be $2 + 20$, as 20 is already accounted for. What two numbers add to 22 and are not yet in the grid? The only possibility is 10 and 12, in some order. But can we tell which of 10 and 12 is placed where? If 10 was above 22, we could not get 10 as the sum of two neighbours, since $2 + 8$ and $3 + 7$ are not possible. If 12 is above 22, then $12 = 10 + 2$ and $10 = 8 + 2$, which can work.

			20	*21*
	6	*5*	*4*	17
23	*7*	*1*	*3*	?
	9	*8*	*2*	*12*
25	24		*10*	*22*

We know that the "?" is either 13 or 14. Could it be 13? Are there two neighbours of "?" that add to 13? No. So the "?" must be 14, which solves the problem.

But wait! We can't stop now! Let's carry on a bit further.

Looking at 25, we see that 25 must be $24 + 1$ (not a possibility) or $9 + 16$. Hence, the number in the space above 25 must be 16. This now allows us to italicize 23, 24, 25, and 16. (Why?)

			20	*21*
	6	*5*	*4*	*17*
23	*7*	*1*	*3*	*14*
16	*9*	*8*	*2*	*12*
25	*24*		*10*	*22*

Try completing the rest of the grid on your own!

Cyclical Diversions from Kirkman's Schoolgirl Problem

Amar Sodhi

A famous example of a recreational mathematics problem is Kirkman's Schoolgirl Problem [3]. This problem, which dates from the mid-nineteenth century, can be phrased as follows:

> A school-teacher wishes to provide a walking schedule for 15 girls so that, over a period of a week, each girl walks daily with two companions but no two girls appear together in a threesome twice. Can you help the teacher find such a schedule?

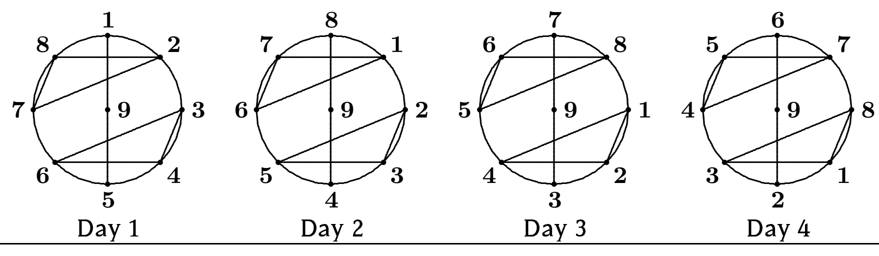
Any such schedule will allow each girl to walk with 14 different companions over the course of a week. In other words, each girl will walk with every other girl exactly once.

It does not take long to discover that finding such a schedule is not a trivial matter. By trial and error, with perseverance, one may possibly be successful. Another approach is to find an appropriate generalization of the problem and discover insights by solving simpler cases. This method sometimes leads to distractions.

Let us look at the case where we vary the number of girls and adjust the number of days to ensure that each girl walks with every other girl exactly once.

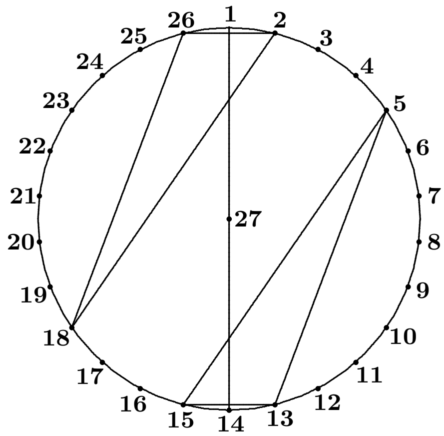
> A school-teacher wishes to provide a walking schedule for v girls so that, over a period of $(v-1)/2$ days, each girl walks daily with two companions but no two girls walk together in a threesome twice. For what values of v can this be done?

For $v = 9$, there is a geometric way of visualizing a solution. Denote the girls by the numbers 1 through 9. Spread the numbers 1 to 8 uniformly along the circumference of a circle and label the centre of the circle as 9. In the circle, draw the triangles $(2, 7, 8)$ and $(3, 4, 6)$ and the diagonal $(1, 9, 5)$ (the diagonal can be viewed as a degenerate triangle). This gives a walking



arrangement for the first day. By rotating the circle, 45 degrees at a time, but keeping the triangles fixed, we obtain a walking schedule where each girl walks with two different companions for each of 4 successive days.

Buoyed by this success, one spreads the numbers 1 to 14 along the circumference of a circle, labels the centre of the circle 15 and draws the diagonal $(1, 15, 8)$. The next hour is cheerfully spent placing four triangles within the circle and applying the rotational technique which worked well for $v = 9$. Frustration sinks in as no solution is forthcoming. Finally, one looks at the case $v = 9$ to see what "went right" and analyzes one's recent scribbling to see what "went wrong" for 15. The problem is left alone for a day or two, then a sudden idea leads to a solution for $v = 27$.



The diagram shows part of the first day's arrangement; namely the diagonal $(1, 27, 14)$ and a congruent pair of triangles $(5, 13, 15)$ and $(2, 26, 18)$. By adding three pairs of congruent 'twin' triangles $(3, 4, 9)$ and $(16, 17, 22)$, $(6, 10, 25)$ and $(12, 19, 23)$, as well as $(7, 21, 24)$ and $(8, 11, 20)$, one completes the arrangement for the first day. Rotating the numbers on the circle, one step at a time, gives the arrangements for the next 12 days. The significant feature of this arrangement is that each pair of congruent triangles can be associated to a triplet $\{a,\ b,\ c\ =\ a + b\}$, where a, b, and c are the lengths of the shortest arcs between the vertices of the triangle. The triangle $(2, 26, 18)$ corresponds to $\{2,\ 8,\ 10\}$; $(3, 4, 9)$ corresponds to $\{1,\ 5,\ 6\}$; $(10, 6, 25)$ corresponds to $\{4,\ 7,\ 11\}$; and $(21, 24, 7)$ corresponds to $\{3, 9, 12\}$. These four triplets form a partition of the set $\{1, 2, \ldots, 12\}$. It is not possible to find such a partition of $\{1, 2, 3, 4, 5, 6\}$; thus, this method cannot be used to solve Kirkman's original problem.

What have we gained? More problems than answers, but we do have a strategy to solve Kirkman's Schoolgirl Problem for certain cases. Here are some new challenges that you may like to tackle yourself.

Challenge 1. Show that if the set of integers $A = \{1,\ 2,\ \ldots,\ 3n\}$ can be partitioned into n disjoint subsets $A_i = \{a_i,\ b_i,\ c_i\}$ (for $i = 1,\ 2,\ \ldots,\ n$) such that $a_i + b_i = c_i$ for each i, then either $n = 4k$ or $n = 4k + 1$ for some integer k.

Skolem [5] has shown that if $n = 4k$ or $n = 4k + 1$ for some integer k, then such a partition of $A = \{1, 2, \ldots, 3n\}$ exists. Your next challenge is more modest than this.

Challenge 2. Find all partitions of the set of integers $A = \{1, 2, \ldots, 3n\}$ into n disjoint subsets $A_i = \{a_i, b_i, c_i\}$ (for $i = 1, 2, \ldots, n$), such that $a_i + b_i = c_i$ for each i. Do this for $n = 4, 5, 8$, and 9.

Challenge 3. Find a partition from Challenge 2 to solve Kirkman's Schoolgirl Problem. Do this for $v = 27, 33, 51$, and 57.

Challenge 3 can be quite frustrating, since not all partitions lead to a solution to the Schoolgirl Problem. Hence, if one partition is troublesome, then try another. There will be at least one partition that works!

Finally, we should not forget the motivation for this article:

Challenge 4. Solve Kirkman's original problem.

Hopefully you can find some revolutionary idea which can be used to solve other cases! Now, how about arranging those girls in groups of four?

Comment. An expository article, *Kirkman's Schoolgirl and Related Problems*, can be found in the May 1980 edition of Scientific American (see also [2]). For a more detailed discussion of cyclical solutions, see [1]. Ray-Chaudhuri and Wilson [4] solved the generalized Kirkman's Schoolgirl Problem by showing that a schedule exists for every odd multiple of 3.

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[2] M. Gardner, Dinner Guests, Schoolgirls and Handcuffed Prisoners, *The Last Recreations*, Springer-Verlag New York, 1997.

[3] T.P. Kirkman, Query VI, *Lady's and Gentleman's Diary* (1850), p. 48.

[4] D.K. Ray-Chaudhuri and R.M. Wilson, Solutions of Kirkman's School-Girl Problem, *Amer. Math. Soc. Symp. Pure Math.* **19** (1971), 187–204.

[5] Th. Skolem, On Certain Distributions of Integers in Pairs with Given Differences, *Math. Scand.* **5** (1957), 57–68.

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THE OLYMPIAD CORNER

No. 262

R.E. Woodrow

We begin this number with problems from the two days of the Final Round of the 18th Korean Mathematical Olympiad written in April 2004. My thanks go to Christopher Small, Canadian Team Leader to the IMO in Athens, for collecting them for our use.

18th KOREAN MATHEMATICAL OLYMPIAD
Final Round
April 10-11, 2004

1. The incircle O of an isosceles triangle ABC with $AB = AC$ meets BC, CA, and AB at K, L, and M, respectively. Let N be the intersection of the lines OL and KM, and let Q be the intersection of the lines BN and CA. Let P be the foot of the perpendicular from A to BQ. If we assume that $BP = AP + 2PQ$, what are the possible values of AB/BC?

2. Show that no pair of positive integers x and y satisfies $3y^2 = x^4 + x$.

3. A computer network is formed by connecting 2004 computers by cables. A set S of these 2004 computers is said to be *independent* if no pair of computers of S is connected by a cable. Suppose that the number of cables used is the minimum under the condition that the size of any independent set does not exceed 50.

(a) Denote by $c(L)$ the number of cables used to connect the computer L with other computers. Show that, for any pair of computers A and B, $c(A) = c(B)$ if they are connected by a cable and $|c(A) - c(B)| \leq 1$ otherwise.

(b) Find the number of cables used for the network.

4. There are n points on a circle, numbered from 1 to n. Let S be the set of these points. Let $\mathcal{G}$ be the family of all k–element subsets A of S which have the property that, between any two distinct points i and j in A, there are at least 3 points of S which are not in A.

For n, $k \geq 2$, find the number of elements of $\mathcal{G}$.

5. Let R and r be the circumradius and the inradius of $\triangle ABC$, respectively. Suppose that $\angle A$ is the largest of the three angles of $\triangle ABC$. Let M be the mid-point of BC and X be the intersection of the tangents to the circumcircle of $\triangle ABC$ at B and C. Show that

$$\frac{r}{R} \geq \frac{AM}{AX}\,.$$

6. Let p be a prime, and define $f_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$.

(a) Show that, for any positive integer m divisible by p, there exists a prime q that divides $f_p(m)$ and is relatively prime to $m(m-1)$.

(b) Show that there are infinitely many positive integers n such that $pn+1$ is a prime.

Next we give the problems of the 21st Balkan Mathematical Olympiad 2004. Thanks again go to Christopher Small for collecting them.

21st BALKAN MATHEMATICAL OLYMPIAD
Pleven 2004

1. The sequence a_0, a_1, a_2, ... satisfies the relation

$$a_{m+n} + a_{m-n} - m + n - 1 = \tfrac{1}{2}(a_{2m} + a_{2n})$$

for all non-negative integers m and n with $m \geq n$. If $a_1 = 3$, find a_{2004}.

2. Find all prime number solutions to the equation $x^y - y^x = xy^2 - 19$.

3. Let O be the circumcentre of the acute triangle ABC. The circles centred at the mid-points of the triangle's sides and passing through O intersect one another at the points K, L, and M. Prove that O is the incentre of triangle KLM.

4. The plane is divided into parts by a finite number of lines. Two parts are called "neighbouring" if they have a common segment, half-line, or line. Each part is to be assigned a real number such that:

(i) the product of the assigned numbers in each pair of neighbouring parts is less than the sum of those two numbers;

(ii) the sum of all numbers assigned to parts lying on the same side of any of the given lines is equal to 0.

Prove that this is possible if and only if the lines are not all parallel.

The remaining set for your problem pleasures are from the 14th Japanese Mathematical Olympiad, Final round, written February 2004. Thanks again go to Christopher Small for obtaining them for our use.

14th JAPANESE MATHEMATICAL OLYMPIAD
Final Round, February 11, 2004 (Time: 4 hours)

1. Prove that there is no positive integer n for which $2n^2 + 1$, $3n^2 + 1$, and $6n^2 + 1$ are all perfect squares.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y, $f\big(xf(x) + f(y)\big) = \big(f(x)\big)^2 + y$.

3. Let ABC be a triangle. Let S be the circle that is perpendicular to the plane ABC and has diameter AB. Let D and E be points on S, and suppose that DE meets AB at P. Show that if CP bisects $\angle ACB$, then CP also bisects $\angle DCE$.

4. For positive real numbers a, b, and c with $a + b + c = 1$, show that

$$\frac{1+a}{1-a} + \frac{1+b}{1-b} + \frac{1+c}{1-c} \;\le\; 2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) .$$

You need not state when equality holds.

5. On the island which Mika toured last year, each village is linked by a road to exactly three other villages. So she set out from a village, visited every village exactly once, and returned to the first village.

This year, she is planning to go on a round trip again, starting from the same village, visiting every village exactly once and returning to the first village. But she does not want to follow the identical course as last year, nor just the reverse route. Show that there is an itinerary that makes her happy.

In December 2006, when we gave readers' solutions to problems of the October 2005 *Corner*, we missed citing the solution of Michel Bataille to problem 1 of the Second Round of the 2002 Yugoslav Qualification for IMO 2002 given in [2005 : 373–374, 2006 : 507]. We also overlooked Bataille's solution to problem 4 of the Vingt-Septième Olympiade Mathématique Belge (Midi Finale) which appeared in [2005 : 374–375; 2006 : 509]. My apologies to Michel Bataille.

Next we present a solution to problem #2 of the 8th Macedonian Mathematical Olympiad. This solution is shorter than the one that was given in [2006 : 218–219].

2. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $n \ge 2$,

$$f\big(f(n-1)\big) \;=\; f(n+1) - f(n)\,?$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Suppose that f is such a function. Then, for all $n \ge 2$,

$$f(n+1) - f(n) \;=\; f\big(f(n-1)\big) \;\ge\; 1\,.$$

Thus, f is increasing for $n \geq 2$. By induction, we have $f(n) \geq n - 1$ for all $n \geq 2$. If $f(8) \leq 9$, then

$$3 = 9 - 6 \geq f(8) - f(7) = f\big(f(6)\big) \geq f(5) \geq 4,$$

which is a contradiction. Hence, $f(8) \geq 10$. Then

$$f(10) > f(10) - f(9) = f\big(f(8)\big) \geq f(10),$$

which is again a contradiction.

Next we turn to solutions to problems of the British Mathematical Olympiad 2002/2003, Round 1, given at [2006 : 215].

1. Given that $34! = 295\,232\,799\,cd9\,603\,140\,847\,618\,609\,643\,5ab\,000\,000$, determine the digits a, b, c, d.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

For any positive integer n and any prime p, it is well known that the exponent of p in the prime decomposition of $n!$ is

$$\nu_p(n!) = \sum_{k \geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor,$$

where $\lfloor . \rfloor$ denotes the integer part. Hence, $\nu_2(34!) = 32$ and $\nu_5(34!) = 7$. It follows that $34!$ is divisible by 10^7, which forces $b = 0$.

Since $\nu_2(34!) = 32$, it follows that $34!/10^7$ is divisible by 8. Then the 3-digit number $35a$ whose digits are the 3 right-most digits of $34!/10^7$ is divisible by 8. Thus, $300 + 50 + a$ is divisible by 8. Then $a + 6 \equiv 0 \pmod 8$, with $6 \leq a + 6 \leq 15$. Therefore, $a + 6 = 8$ and $a = 2$.

For any positive integer n, let $n = \sum\limits_{i=0}^{k} a_i 10^i$ be the decimal expansion of n, and let $S_e(n) = \sum\limits_{i \geq 0} a_{2i}$ and $S_o(n) = \sum\limits_{i \geq 0} a_{2i+1}$. It is well known that $n \equiv S_e(n) + S_o(n) \pmod 9$ and $n \equiv S_e(n) - S_o(n) \pmod{11}$. From the given decimal expansion for $34!$, we find that $S_e(34!) + S_o(34!) = c + d + 140$ and $S_e(34!) - S_o(34!) = c - d - 18$. On the other hand, since 9 and 11 are both factors of $34!$, we have $34! \equiv 0 \pmod 9$ and $34! \equiv 0 \pmod{11}$. Thus, $c + d + 140 \equiv 0 \pmod 9$ and $c - d - 18 \equiv 0 \pmod{11}$. This leads to $c + d \equiv 4 \pmod 9$ and $c - d \equiv 7 \pmod{11}$. Since $0 \leq c + d \leq 18$ and $-9 \leq c - d \leq 9$, it follows that

$$c + d = 4 \quad \text{or} \quad c + d = 13,$$

and

$$c - d = -4 \quad \text{or} \quad c - d = 7.$$

Now, since $c - d$ and $c + d$ have the same parity and $c \in \{0, 1, \ldots, 9\}$, we easily deduce that $c = 0$ and $d = 4$.

Then $(a, b, c, d) = (2, 0, 0, 4)$.

Remark. The problem statement is erroneous. Maple gives

$$34! \;=\; 295\,232\,799\,cd9\,604\,140\,847\,618\,609\,643\,5ab\,000\,000\,.$$

The reasoning above can be adapted word for word to prove that, in that case, $(a,b,c,d) = (2,0,0,3)$.

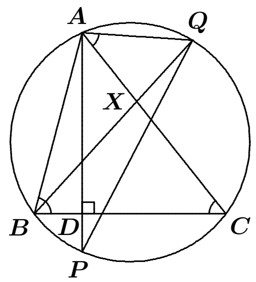
2. The triangle ABC, where $AB < AC$, has circumcircle S. The perpendicular from A to BC meets S again at P. The point X lies on the line segment AC, and BX meets S again at Q.

Show that $BX = CX$ if and only if PQ is a diameter of S.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Kandall's solution.

Suppose that $BX = CX$. Then

$$\begin{aligned}\angle XCB &= \angle XBC\\ &= \angle QBC \;=\; \angle QAC\,;\end{aligned}$$



whence, $AQ \parallel BC$. It follows that $\angle PAQ = 90^\circ$, which means that PQ is a diameter of S.

The argument is reversible.

3. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

$$x^2yz + xy^2z + xyz^2 \;\le\; \tfrac{1}{3}\,.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Geoffrey A. Kandall, Hamden, CT, USA; Vedula N. Murty, Dover, PA, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We first give the solution of Bataille.

From the Cauchy–Schwarz Inequality and the hypothesis, we have

$$\begin{aligned}x^2yz + xy^2z + xyz^2 &= xyz(x+y+z)\\ &\le xyz(1^2+1^2+1^2)^{\frac{1}{2}}(x^2+y^2+z^2)^{\frac{1}{2}} \;=\; \sqrt{3}\,xyz\,.\end{aligned}$$

On the other hand, $\frac{1}{3} = \frac{1}{3}(x^2+y^2+z^2) \ge \sqrt[3]{x^2y^2z^2}$, using the AM–GM Inequality; hence, $xyz \le \frac{1}{3\sqrt{3}}$. The desired inequality follows immediately by combining the two results.

Next we give the solution of Díaz-Barrero.

From the identity $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$ and the well-known inequality $a^2+b^2+c^2 \ge ab+bc+ca$, it follows that

$$3(ab+bc+ca) \;\le\; (a+b+c)^2\,. \qquad (1)$$

Setting $a = xy$, $b = yz$, and $c = zx$ in (1) yields

$$3(x^2yz + xy^2z + xyz^2) \leq (xy + yz + zx)^2 \leq (x^2 + y^2 + z^2)^2 = 1,$$

from which we get $x^2yz + xy^2z + xyz^2 \leq \frac{1}{3}$.

Notice that equality holds when $x = y = z = \frac{1}{\sqrt{3}}$.

4. Let m and n be integers greater than 1. Consider an $m \times n$ rectangular grid of points in the plane. Some k of these points are coloured red in such a way that no three red points are the vertices of a right-angled triangle two of whose sides are parallel to the sides of the grid. Determine the greatest possible value of k.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove, by induction on $p = m + n$, that the greatest possible value of k is $m + n - 2$.

Case 1. $p = 4$.

The grid is 2×2, and the maximum number of red points is clearly $2 = 2 + 2 - 2$. Thus, the result holds for $p = 2$.

Case 2. $p \geq 4$.

Assume that the result holds for all m, $n > 1$ such that $p = m + n$. Let us consider a grid with $p + 1 = m + n$.

With no loss of generality, we may assume that the grid is formed by m rows and n columns with $m \geq n$. Then $m \geq 3$. Consider a colouring with no red right-angled triangle two of whose sides are parallel to the sides of the grid (abbreviated no red RAT). Since $m \geq n$, if each line contains at least two red points, then at least two of them would have a red point on the same column, which would give a red RAT, a contradiction.

Thus, there exists a line with no more than one red point. Deleting this line, we obtain a coloured $(m - 1) \times n$ rectangular grid with no red RAT, and with $p = (m - 1) + n$. The induction hypothesis ensures that there are at most $(m - 1) + n - 2$ red points in this grid. Thus, there are at most $(m - 1) + n - 2 + 1 = m + n - 2$ red points in the initial grid.

Conversely, by colouring red all the points in the first row and the first column except the point which is in both the first row and the first column, we get a colouring with exactly $m + n - 2$ red points and no red RAT.

Therefore the maximum number of red points is $m + n - 2$, which ends the induction and the proof.

5. Find all solutions in positive integers a, b, c to the equation

$$a!b! = a! + b! + c!$$

Solved by Mohammed Aassila, Strasbourg, France; Houda Anoun, Bordeaux, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give the solution of Aassila.

If $a \neq b$, then without loss of generality, $a > b$. Then $a!$ does not divide $b!$. Hence, $c! = a!b! - a! - b!$ is not divisible by $a!$. Then $a > c$, and

$a!b! = a! + b! + c! < 2a! + b!$, which implies that $(a! - 1)(b! - 2) < 2$. It is easy to check that this gives no solutions.

Therefore, we must have $a = b$. Our equation becomes $a!^2 = 2a! + c!$; hence $a! \mid c!$ and thus $a < c$. Write $c = a + k$ where $k > 0$ is an integer. Division by $a!$ now yields $a! = 2 + (a+1)(a+2)\cdots(a+k)$. Then $a! > 2$, which implies that $a > 2$ and $3 \mid a!$. Hence, we must have

$$(a+1)(a+2)\cdots(a+k) \equiv 1 \pmod 3 .$$

Then $k < 3$ (otherwise $(a+1)(a+2)\cdots(a+k)$ is divisible by 3). If $k = 1$, we get $a! = a + 3$, which yields $a = 3$; then $(a,b,c) = (3,3,4)$. If $k = 2$, we get $a! = a^2 + 3a + 4$, which implies that $a \mid 4$; hence, $a = 1$, $a = 2$, or $a = 4$. It is easy to check that these cases fail.

Thus, the only solution is $(a,b,c) = (3,3,4)$.

Now we look at solutions from our readers to Round 2 of the British Mathematical Olympiad 2002–2003 given at [2006 : 215–216].

1. For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n. Determine all triples x, y, z of distinct positive integers satisfying

(i) x, y, z are in arithmetic progression, and

(ii) $p(xyz) \le 3$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France.

Let x, y, and z be distinct positive integers satisfying (i) and (ii). We deduce from (ii) that $x = 2^a \times 3^b$, $y = 2^c \times 3^d$, and $z = 2^e \times 3^f$ where a, b, c, d, e, and f are non-negative integers. Without loss of generality, we may assume that $x < y < z$. Then (i) is equivalent to

$$x + z = 2y . \tag{1}$$

Let $\delta = \gcd(x,y,z)$. Note that $\left(\frac{x}{\delta}, \frac{y}{\delta}, \frac{z}{\delta}\right)$ satisfies (1) and

$$p\left(\frac{x}{\delta} \times \frac{y}{\delta} \times \frac{z}{\delta}\right) \le p(xyz) \le 3 .$$

Thus, $\left(\frac{x}{\delta}, \frac{y}{\delta}, \frac{z}{\delta}\right)$ is a solution. Moreover, (mx, my, mz) is a solution for each $m = 2^p \times 3^q$, where p and q are non-negative integers. Therefore, we may assume that $\delta = 1$. Then $ace = 0 = bdf$.

From (1), we deduce that x and z have the same parity.

Case 1. x and z are odd.

Then (1) reduces to $3^b + 3^f = 2^{c+1} \times 3^d$, with $f > b$ (since $x < z$). If $b > 0$, then the left side of this equation is divisible by 3, which forces $d > 0$.

Then $\delta \geq 3$, which contradicts our assumption that $\delta = 1$. Thus $b = 0$, from which we deduce that $d = 0$. It follows that (1) may be rewritten as $2^{c+1} - 3^f = 1$. But it is well-known (see [1]) that the only integer solution of $2^m - 3^n = 1$ is $(m,n) = (2,1)$. This leads to $(x,y,z) = (1,2,3)$.

Case 2. x and z are even.

Then y must be odd and $d \geq 1$ (otherwise $1 = y > x$). From (1), we deduce that $b = f = 0$. Thus, (1) reduces to $2^a + 2^e = 2 \times 3^d$. Note that $z > y \geq 3$, so that $e \geq 2$. Since the right side of (1) is divisible by 2 but not by 4, this forces $a = 1$. Then (1) is $3^d - 2^{e-1} = 1$. But it is well known (see [1]) that the only integer solutions of $3^m - 2^n = 1$ are $(m,n) = (1,1)$ or $(m,n) = (2,3)$. This leads to $(x,y,z) = (2,3,4)$ or $(x,y,z) = (2,9,16)$.

According to the initial remark, the desired triples are those which have one of the following forms (or their permutations):

$$(2^p \times 3^q, 2^{p+1} \times 3^q, 2^p \times 3^{q+1}),$$
$$(2^{p+1} \times 3^q, 2^p \times 3^{q+2}, 2^{p+2} \times 3^q),$$
$$(2^{p+1} \times 3^q, 2^p \times 3^{q+2}, 2^{p+4} \times 3^1),$$

where p and q are non-negative integers.

References

[1] W. Sierpinski, "250 problèmes de théorie élémentaire des nombres", pb. #5–154 et pb. #5–155.

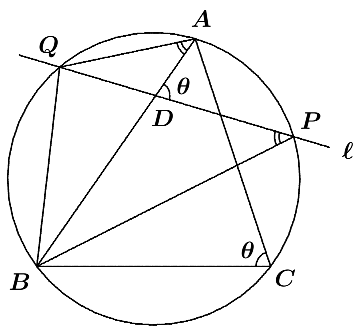
2. Let ABC be a triangle, and let D be a point on AB such that $4AD = AB$. The half-line ℓ is drawn on the same side of AB as C, starting from D and making an angle of θ with DA, where $\theta = \angle ACB$. If the circumcircle of ABC meets the half-line ℓ at P, show that $PB = 2PD$.

Solved by Michel Bataille, Rouen, France; and Geoffrey A. Kandall, Hamden, CT, USA. We give Bataille's solution.

Let Q be the second point of intersection of the line DP with the circumcircle (see figure). Then

$$\angle ADQ = \angle BDP = \pi - \angle ADP = \pi - \angle ACB = \angle AQB$$

and $\angle QAD = \angle BAQ$. Therefore, $\triangle ADQ$ and $\triangle AQB$ are similar. Then $\dfrac{AD}{AQ} = \dfrac{AQ}{AB}$. Since $AB = 4AD$, it follows that $AB = 2AQ$. But $\triangle PDB$ and $\triangle AQB$ are also similar, since $\angle BDP = \angle AQB$ (from above) and $\angle DPB = \angle QPB = \angle QAB$. From this similarity and the relation $AB = 2AQ$, we conclude that $PB = 2PD$.



3. Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation of the set $\mathbb{N}$ of all positive integers.

(a) Show that there is an arithmetic progression a, $a+d$, $a+2d$, where $d > 0$, such that $f(a) < f(a+d) < f(a+2d)$.

(b) Must there be an arithmetic progression a, $a+d$, $\dots$, $a+2003d$, where $d > 0$, such that $f(a) < f(a+d) < \cdots < f(a+2003d)$?

[A permutation of $\mathbb{N}$ is a one-to-one function whose image is the whole of $\mathbb{N}$; that is, a function from $\mathbb{N}$ to $\mathbb{N}$ such that for all $m \in \mathbb{N}$ there is a unique $n \in \mathbb{N}$ such that $f(n) = m$.]

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We give Zhou's write-up.

(a) Let $a = f^{-1}(1)$ and $m = f(a+1)$. Then $m \geq 2$. By the pigeonhole principle, the sequence $\{f(a+2^i)\}_{i=0}^{m-1}$ cannot be monotonically decreasing. Hence, there exists k with $0 \leq k \leq m-2$ such that $f(a+2^k) < f(a+2^{k+1})$. Let $d = 2^k$. Then $f(a) < f(a+d) < f(a+2d)$.

(b) No. In fact, there does not necessarily exist such an arithmetic progression of length 4. Define $f(n) = 4(3^i) - n - 1$ if $3^i \leq n < 3^{i+1}$. Then f is decreasing on the interval $[3^i, 3^{i+1})$ for any integer $i \geq 0$. Suppose that $f(a) < f(a+d) < f(a+2d) < f(a+3d)$ for some a, $d \in \mathbb{N}$. Hence, $3^j \leq a < 3^{j+1}$, $3^k \leq a+d < 3^{k+1}$, $3^l \leq a+2d < 3^{l+1}$, and $3^m \leq a+3d < 3^{m+1}$, with $0 \leq j < k < l < m$. Therefore,

$$2d \;=\; (a+3d)-(a+d) \;>\; 3^m - 3^{k+1} \;\geq\; 3^{k+2} - 3^{k+1} \;=\; 2(3^{k+1})\,.$$

Thus, $d > 3^{k+1}$, which contradicts the fact that $d < a+d < 3^{k+1}$.

4. Let f be a function from the set of non-negative integers into itself such that, for all $n \geq 0$,

(i) $\big(f(2n+1)\big)^2 - \big(f(2n)\big)^2 = 6f(n)+1$, and

(ii) $f(2n) \geq f(n)$.

How many numbers less than 2003 are there in the image of f?

Solved by Mohammed Aassila, Strasbourg, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We give the solution of Zhou.

Since $6f(n)+1$ is odd, both $f(2n+1)-f(2n)$ and $f(2n+1)+f(2n)$ must be odd, by (i). If $f(2n+1)-f(2n) \geq 3$, then

$$f(2n+1)+f(2n) \;\leq\; \tfrac{1}{3}\big(6f(n)+1\big) \;=\; 2f(n)+\tfrac{1}{3}$$

and

$$2f(2n) \;=\; \big(f(2n+1)+f(2n)\big)-\big(f(2n+1)-f(2n)\big) \;\leq\; 2f(2n)+\tfrac{1}{3}-3\,,$$

which is a contradiction. Therefore, $f(2n+1)-f(2n) = 1$, and (i) reduces to $f(2n+1)+f(2n) = 6f(n)+1$. Solving the system, we get $f(2n) = 3f(n)$ and $f(2n+1) = 3f(n)+1$ for all $n \geq 0$.

Now we show, by induction on n, that if the base–2 representation of n is $a_k 2^k + \cdots + a_1 2 + a_0$, then $f(n) = a_k 3^k + \cdots + a_1 3 + a_0$. Since $f(0) = 0$, the claim is true for $n = 0$. Assume that $n \geq 1$ and the claim is true up to $n-1$. Write $n = a_k 2^k + \cdots + a_1 2 + a_0$. Then

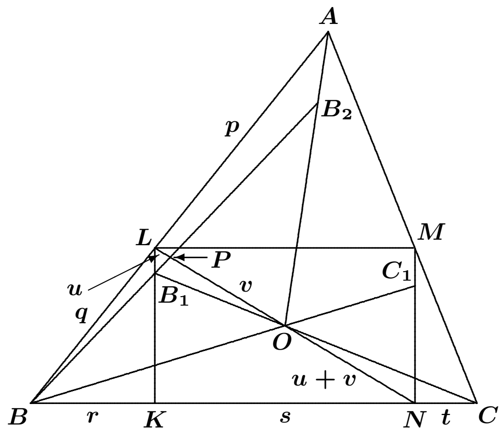
$$\begin{aligned} f(n) &= f(2(a_k 2^{k-1} + \cdots + a_1) + a_0) \\ &= 3f(a_k 2^{k-1} + \cdots + a_1) + a_0 \;=\; a_k 3^k + \cdots + a_1 3 + a_0\,, \end{aligned}$$

where the last step is by the induction hypothesis. Finally, since $3^7 > 2003$, it follows that $3^6+\cdots+3+1 = 1093$ is the largest number less than 2003 in the image of f. So there are $2^7 = 128$ numbers less than 2003 in the image of f.

Next we turn to solutions from our readers to problems of the Kazakh National Mathematical Olympiad 2002-2003 given in [2006 : 216–217].

2. (*S. Mukhanbetkaliev*) Angles B and C of triangle ABC are acute. Side KN of rectangle $KLMN$ belongs to segment BC, points L and M belong to segments AB and AC, respectively. Let O be the intersection point of the diagonals of $KLMN$. Let C_1 be the intersection point of lines BO and MN, and let B_1 be the intersection point of lines CO and LK. Prove that lines AO, BB_1, and CC_1 are concurrent.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.



Let line BB_1 meet LN and OA at P and B_2, respectively. Let C_2 (not shown in the diagram) be the point of intersection of line CC_1 and OA. We use the following notation: $AL = p$, $LB = q$; $BK = r$, $KN = LM = s$, $NC = t$; $LP = u$, $PO = v$.

Since $\triangle ALM \sim \triangle ABC$, we have $\frac{p}{s} = \frac{p+q}{r+s+t} = \frac{q}{r+t}$; that is, $\frac{p}{q} = \frac{s}{r+t}$.

Now we apply Menelaus' Theorem to the triangle-transversal pairs $(\triangle LKN; B_1C)$ and $(\triangle LKN; PB)$ to get

$$\frac{LO}{ON} \cdot \frac{t}{s+t} \;=\; \frac{LB_1}{B_1K} \;=\; \frac{u}{u+2v} \cdot \frac{r+s}{r}\,.$$

Consequently, $\frac{u+2v}{u} = \frac{r+s}{r} \cdot \frac{s+t}{t}$; that is,

$$\frac{v}{u} \;=\; \frac{1}{2}\left(\frac{(r+s)(s+t)}{rt} - 1\right) \;=\; \frac{s(r+s+t)}{2rt}\,.$$

Applying Menelaus' Theorem to the pair $(\triangle OAL; B_2B)$, we obtain

$$\frac{OB_2}{B_2A} \;=\; \frac{v}{u} \cdot \frac{q}{p+q} \;=\; \frac{s(r+s+t)}{2rt} \cdot \frac{r+t}{r+s+t} \;=\; \frac{s(r+t)}{2rt}\,.$$

By symmetry, C_2 divides OA in the same ratio; that is, the points B_2 and C_2 coincide. Thus, AO, BB_1, and CC_1 concur at B_2 $(= C_2)$.

3. (*U. Mukashev*) Find the maximal and minimal values of the sum $a+b+c$ if $a^2 + b^2 \leq c \leq 1$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's solution.

The maximal value is $1 + \sqrt{2}$ and the minimal value is $-\frac{1}{2}$.

Suppose that $a^2+b^2 \leq c \leq 1$, and let $r = \sqrt{a^2+b^2}$. Then $0 \leq r \leq \sqrt{c}$ and $a = r\cos\theta$, $b = r\sin\theta$, for some real number θ. It follows that

$$a + b \;=\; r(\cos\theta + \sin\theta) \;=\; r\sqrt{2}\sin\left(\frac{\pi}{4} + \theta\right)\,.$$

Since $-1 \leq \sin(\frac{\pi}{4} + \theta) \leq 1$ and $r \geq 0$, we have $-r\sqrt{2} \leq a + b \leq r\sqrt{2}$. Then

$$a+b+c \;\leq\; c + r\sqrt{2} \;\leq\; 1 + \sqrt{2}\,,$$

since $c \leq 1$ and $r \leq \sqrt{c} \leq 1$. Also,

$$a+b+c \;\geq\; c - r\sqrt{2} \;\geq\; c - \sqrt{2}\sqrt{c} \;=\; \left(\sqrt{c} - \frac{1}{\sqrt{2}}\right)^2 - \frac{1}{2} \;\geq\; -\frac{1}{2}\,.$$

Thus, $-\frac{1}{2} \leq a+b+c \leq 1+\sqrt{2}$.

Moreover, if we take $a = b = \frac{1}{\sqrt{2}}$ and $c = 1$, then $a^2 + b^2 \leq c \leq 1$ and $a + b + c = 1 + \sqrt{2}$, while with $a = b = -\frac{1}{2}$ and $c = \frac{1}{2}$, we obtain $a^2 + b^2 \leq c \leq 1$ and $a + b + c = -\frac{1}{2}$. The proof is complete.

4. (*U. Mukashev*) Let two sequences of real numbers $\{a_n\}$ and $\{b_n\}$ be such that $a_0 = b_0 = 0$ and for each positive integer n,

$$a_n \;=\; a_{n-1}^2 + 3 \quad \text{and} \quad b_n \;=\; b_{n-1}^2 + 2^n\,.$$

Compare the numbers a_{2003} and b_{2003}.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

For $n \geq 1$, we have $a_n^2 > 2^{2^n}$. This follows by induction on n, since $a_1 = 3$ and $a_n > a_{n-1}^2$ for $n \geq 1$. Another easy induction shows that $2^n > n+3$ for $n \geq 3$. Thus, for $n \geq 3$, we have

$$a_n^2 \;>\; 2^{2^n} \;>\; 2^{n+3}\,. \tag{1}$$

We claim that $2b_n < a_n$ for all $n \geq 3$. The claim is true for $n = 3$, since $a_3 = 147$ and $b_3 = 72$. Proceeding by induction, we assume that the claim holds for some given $n \geq 3$. Using the induction hypothesis and (1), we obtain

$$\begin{aligned} 2b_{n+1} &= 2\left(b_n^2 + 2^{n+1}\right) \;=\; \tfrac{1}{2}\Big((2b_n)^2 + 2^{n+3}\Big) \\ &< \tfrac{1}{2}\left(a_n^2 + a_n^2\right) \;<\; a_n^2 + 3 \;=\; a_{n+1}\,, \end{aligned}$$

which ends the induction.

By using the claim and checking the first few cases, we deduce that $b_n \leq a_n$ for all $n \geq 0$, with equality if and only if $n = 0$. In particular, $a_{2003} > b_{2003}$.

To complete this number, we look at readers' solutions to problems of the Ukrainian Mathematical Olympiad 11$^{\text{th}}$ Form given in [2006 : 217–218].

1. Find all real k such that the following system of equations has a unique solution:

$$\begin{aligned} x^2 + y^2 &= 2k^2\,, \\ kx - y &= 2k\,. \end{aligned}$$

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's write-up.

The system has a unique solution if and only if the line ℓ with equation $kx - y = 2k$ is tangent to the circle γ with equation $x^2 + y^2 = 2k^2$. This condition can be stated equivalently as follows: the radius $\sqrt{2}|k|$ of γ is equal to the distance from ℓ to the origin, which is $\dfrac{|k\cdot 0 - 0 - 2k|}{\sqrt{1+k^2}}$. Thus, we obtain the following condition on k:

$$2k^2 \;=\; \frac{4k^2}{1+k^2}\,.$$

Solving for k, we find that $k \in \{0,\,1,\,-1\}$.

2. Prove that for any triangle, if S denotes its area and r denotes the radius of its inscribed circle, then

$$\frac{S}{r^2} \geq 3\sqrt{3}\,.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Vedula N. Murty, Dover, PA, USA. We give Murty's solution, modified by the editor.

Denote the side lengths of the triangle by a, b, and c and the semi-perimeter by s. Applying the AM–GM Inequality to the positive numbers $s-a$, $s-b$, and $s-c$, we obtain $\sqrt[3]{(s-a)(s-b)(s-c)} \leq \frac{1}{3}s$; that is,

$$(s-a)(s-b)(s-c) \;\leq\; \frac{s^3}{27}\,. \tag{1}$$

Using Heron's Formula along with (1) and the known formula $S = rs$, we obtain

$$S^2 \;=\; s(s-a)(s-b)(s-c) \;\leq\; \frac{s^4}{27} \;\leq\; \frac{S^4}{27r^4}\,.$$

Then, $\frac{S^2}{r^4} \geq 27$, and hence, $\frac{S}{r^2} \geq 3\sqrt{3}$.

4. Let α be a real number such that five consecutive terms of the infinite sequence $\sin\alpha$, $\sin 2\alpha$, $\sin 3\alpha$, $\dots$, $\sin n\alpha$, $\dots$ are rational. Prove that *all* the terms of the sequence are rational.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Lemma. Let p be an integer satisfying any of the following four conditions:

(i) $\sin(p\alpha) = \sin\big((p+1)\alpha\big) = 0$;

(ii) $\sin(p\alpha) = \sin\big((p+2)\alpha\big) = 0$;

(iii) $\cos(p\alpha) = \cos\big((p+1)\alpha\big) = 0$;

(iv) $\cos(p\alpha) = \cos\big((p+2)\alpha\big) = 0$.

Then $\sin\alpha \in \{-1, 0, 1\}$.

Proof. (i) Suppose that $\sin(p\alpha) = \sin\big((p+1)\alpha\big) = 0$. Then $\cos(p\alpha) = \pm 1$ and

$$0 \;=\; \sin\big((p+1)\alpha\big) \;=\; \sin(p\alpha)\cos\alpha + \cos(p\alpha)\sin\alpha \;=\; \pm\sin\alpha\,.$$

Thus, $\sin\alpha = 0$.

(ii) Suppose that $\sin(p\alpha) = \sin\big((p+2)\alpha\big) = 0$. Then $\cos(p\alpha) = \pm 1$ and

$$0 \;=\; \sin\big((p+2)\alpha\big) \;=\; \sin(p\alpha)\cos(2\alpha) + \cos(p\alpha)\sin(2\alpha) \;=\; \pm\sin(2\alpha)\,.$$

Thus, $\sin(2\alpha) = 0$, which implies that $\sin\alpha \in \{-1, 0, 1\}$.

(iii) and (iv) are considered in a similar way. ∎

Suppose $\sin(p\alpha)$ is rational for $p \in \{n, n+1, n+2, n+3, n+4\}$, for some integer $n > 0$.

Case 1. There exists $p \in \{n+1, n+2, n+3\}$ such that $\sin(p\alpha) \neq 0$ and $\cos\big((p-1)\alpha\big) \neq 0$.

Then

$$\cos\alpha = \frac{\sin\big((p-1)\alpha\big) + \sin\big((p+1)\alpha\big)}{2\sin(p\alpha)}$$

is rational. Since $\cos\big((p-1)\alpha\big)$ can be expressed as a polynomial in $\cos\alpha$ with rational coefficients, it follows that $\cos\big((p-1)\alpha\big)$ is rational (and is non-zero, by hypothesis). Therefore,

$$\sin\alpha = \frac{\sin(p\alpha) - \sin\big((p-1)\alpha\big)\cos\alpha}{\cos\big((p-1)\alpha\big)}$$

is rational.

Let k be a positive integer. Since $\sin(k\alpha)$ can be expressed as a polynomial in $\sin\alpha$ with rational coefficients, it follows that $\sin(k\alpha)$ is rational.

Case 2. For each $p \in \{n+1, n+2, n+3\}$, we have $\sin(p\alpha)\cos\big((p-1)\alpha\big) = 0$.

Then, by the pigeonhole principle, either $\sin(p\alpha) = 0$ for two values of $p \in \{n+1, n+2, n+3\}$ or $\cos\big((p-1)\alpha\big) = 0$ for two such values of p. In either case, we may use the lemma to deduce that $\sin\alpha \in \{-1, 0, 1\}$. In particular, $\sin\alpha$ is rational, and the conclusion follows as in Case 1.

5. Does there exist a number $q \in \mathbb{N}$ and a prime number $p \in \mathbb{N}$ such that

$$3^p + 7^p = 2 \cdot 5^q\,?$$

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's version.

There is no such pair (p, q).

Assume such numbers p and q exist. Then $p \neq 2$, since $3^2 + 7^2 = 58$ is not of the form $2 \cdot 5^q$. Thus, p is odd. Then

$$2 \cdot 5^q = 3^p + 7^p = (3+7) \cdot A = 2 \cdot 5 \cdot A\,,$$

where

$$A = 3^{p-1} - 3^{p-2} \cdot 7 + 3^{p-3} \cdot 7^2 - \cdots - 3 \cdot 7^{p-2} + 7^{p-1}\,.$$

It follows that $A = 5^{q-1}$ with $q > 1$ (since $A > 1$). Furthermore, since $3 \equiv -2 \pmod 5$ and $7 \equiv 2 \pmod 5$, we have

$$A \equiv 2^{p-1} + 2^{p-1} + 2^{p-1} + \cdots + 2^{p-1} \equiv p \cdot 2^{p-1} \pmod 5\,.$$

As a result, $p \cdot 2^{p-1} \equiv 0 \pmod 5$, and the only possibility is $p = 5$. Since $3^5 + 7^5 = 2 \cdot 5^2 \cdot 341$ is not of the form $2 \cdot 5^q$, the proof is complete.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(xf(x) + f(y)\big) \;=\; x^2 + y$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We give Zhou's solution.

It is easy to verify that both $f(x) = x$ and $f(x) = -x$ satisfy the functional equation. We will prove that these are the only solutions.

Let f be a solution. Setting $x = 0$ in the functional equation gives $f\big(f(y)\big) = y$ for all $y \in \mathbb{R}$. Hence, for all x, $y \in \mathbb{R}$,

$$x^2 + y \;=\; f\big(xf(x) + f(y)\big) \;=\; f\Big(f(x)f\big(f(x)\big) + f(y)\Big) \;=\; f(x)^2 + y\,.$$

Thus, $f(x)^2 = x^2$ for all $x \in \mathbb{R}$. If $f(1) = 1$, then

$$\begin{aligned} 1 + 2x + x^2 \;&=\; f(1+x)^2 \;=\; f\Big(1f(1) + f\big(f(x)\big)\Big)^2 \\ &=\; \big(1^2 + f(x)\big)^2 \;=\; 1 + 2f(x) + x^2\,, \end{aligned}$$

and thus $f(x) = x$ for all $x \in \mathbb{R}$. Similarly, if $f(1) = -1$, then

$$\begin{aligned} 1 - 2x + x^2 \;&=\; f(-1+x)^2 \;=\; f\Big(1f(1) + f\big(f(x)\big)\Big)^2 \\ &=\; \big(1^2 + f(x)\big)^2 \;=\; 1 + 2f(x) + x^2\,, \end{aligned}$$

and thus $f(x) = -x$ for all $x \in \mathbb{R}$.

That completes this number of the *Corner*. I will be working on the September *Corner* beginning in May, so send me your nice solutions and generalizations soon!

BOOK REVIEWS

John Grant McLoughlin

International Mathematics Tournament of Towns 1997–2002 Book 5
By A.M. Storozhev, AMT Publishing, 2006
ISBN 978-1-876420-19-2, paperbound, 214 pages, AUS$40.00.
Reviewed by **Clint Lee**, *Okanagan College, Vernon, BC*

The International Tournament of Towns is a mathematical problem-solving competition for high school students from towns throughout the world. The first Tournament of Towns took place in Russia in 1979–1980, and it has grown to the point where towns from all regions of the world participate. The Tournament takes place each year and consists of two stages: Autumn and Spring. Each stage has two papers: an "O" level, less difficult but less points; and an "A" level, more difficult and more points. There are two versions of each paper, Junior and Senior. The Senior paper is intended for students from the last two years of high school (in Canada, grades 11 and 12), and the Junior paper is intended for students from lower grades. Individual scores are based on the best of the four papers submitted using the points from the top three problems on each paper, and are scaled based on the student's grade level. A town's score is based on the scores from the best papers from the town, where the number of papers used to determine the score is based on the town's population. Students who exceed a certain minimum score are awarded a diploma by the Russian Academy of Sciences.

This book consists of 5 sections, each containing the eight papers from one Tournament, for the Tournaments 19 through 23, covering the years 1997 to 2002. Each section contains solutions to all of the problems from that Tournament. There is a small amount of duplication of problems, since a few problems appear on both the Junior and Senior versions of a paper. There are 212 different problems altogether in the book. A list of 62 general references is given at the end of the book; however, nowhere in the book is this material cited.

The problems in this book are not for the mathematically faint at heart. Though elementary (in that they do not require knowledge of calculus, linear algebra, or other areas of advanced mathematics), they all require a certain degree of mathematical sophistication. Even the easiest problems require some ingenuity. The more difficult problems require a high level of insight and imagination to solve. Almost every paper contains at least one geometry problem, none of which includes a diagram. Other problems cover areas such as number theory, algebra, counting (but not standard combinatorics), and logic. Some problems are difficult to classify, as they combine one or more standard problem types. Many require the proof of a result rather than a simple calculation. Several problems assume knowledge of standard games such as chess, checkers, and cards.

The following three problems should give some feel for the style of problems encountered in this book:

Tournament 21, Junior, Autumn 1999 (O Level) #1

A right-angled triangle made of paper is folded along a straight line so that the vertex at the right angle coincides with one of the other vertices of the triangle and a quadrilateral is obtained.

(a) What is the ratio into which the diagonals of this quadrilateral divide each other? (2 points)

(b) This quadrilateral is cut along its longest diagonal. Find the area of the smallest piece of paper that is obtained if the area of the original triangle is 1. (2 points)

Tournament 22, Senior, Autumn 2000 (O Level) #4

Among a set of $2N$ coins, all identical in appearance, $2N - 2$ are real and 2 are fake. Any two real coins have the same weight. The fake coins have the same weight, which is different from the weight of a real coin. How can one divide the coins into two groups of equal total weight by using a balance at most 4 times, if

(a) $N = 16$; (3 points)

(b) $N = 11$? (2 points)

Tournament 20, Senior, Spring 1999 (A Level) #6

A rook is allowed to move one cell either horizontally or vertically. After 64 moves the rook visited all cells of the 8×8 chessboard and returned back to the initial cell. Prove that the number of moves in the vertical direction and the number of moves in the horizontal direction cannot be equal. (8 points)

The solutions in this book are elegant and well crafted. The majority were prepared by Andy Liu of the University of Alberta. All of the solutions are terse, with extraneous or elementary justifications left to the reader. Many solutions refer to areas of mathematics that high school students would not normally be familiar with, or to results from more familiar areas that high school students would not have encountered. For example, some concepts and results from graph theory are used in several solutions. This aspect of the solutions could certainly stimulate the interested reader to delve into an unfamiliar topic, but might discourage the more casual reader.

This book would be a valuable resource for anyone interested in mathematical problem-solving. Mathematicians involved in preparing mathematics competitions would find in it inspiration for creating their own problems. High school or university mathematics students would find it useful as preparation for any mathematics competition. The Australian Mathematics Trust should be congratulated for publication of problems of this quality and level.

Tribute to a Mathemagician
Edited by Barry Cipra, Erik D. Demaine, Martin L. Demaine, & Tom Rodgers, published by AK Peters, Wellesley, MA, 2005
ISBN 1-56881-204-3, hardcover, xli+262 pages, US$38.00.
Reviewed by ***John Grant McLoughlin****, University of New Brunswick, Fredericton, NB.*

The *Mathemagician* is Martin Gardner, for whom the Gathering for Gardner (G4G) is named. The fifth gathering (G4G5) in 2004 led to this book, an edited collection of thirty articles contributed by participants. The articles are preceded by *In Memoriam*, four pieces dedicated to the lives and contributions of Edward Hordern and Nobuyuki Yoshigahara. The title of the book is fitting, as "Tribute" sets the tone for the whole book, which not only provides intellectual amusement but also displays the human side of the mathematical community.

Articles from popular authors such as Raymond Smullyan, Peter Winkler, Jerry Slocum, and Dennis Shasha appear amidst the diversity, which is challenging to summarize here. A few examples are offered. First, from Chris Manlanka's "bouquet of brainteasers":

> A bouquet contains red roses, white roses, and blue roses. According to the florist, the number of red roses and white roses comes to 100; the number of white roses and blue roses comes to 53. The number of blue roses and red roses comes to less than that. How many roses of each color are there?

Sliding-coin puzzles, a cryptic crossword, tiling problems, and various other challenges appear. Among the challenges are polyomino activities, including an unusual article entitled Polyomino Number Theory (III) by Uldis Barbans, Andris Cibulis, Gilbert Lee, Andy Liu, and Robert Wainwright, in which the focus is the compatibility of pentominoes (as in pentominoes that share common multiples). Principles of number theory are integrated through definitions such as, "A polyomino A is said to divide another polyomino B if a copy of B may be assembled from copies of A." The article provides a fine example of how mathematical thinking, language, and playfulness meet in this field of recreational mathematics.

Norman L. Sandfield writes on "Chinese Ceramic Puzzle Vessels". Ross Eckler shifts the focus to wordplay with "A History of the Ten-Square", as he examines the development of those alphabetical arrangements in which all columns and rows correspond to English words. Bill Cutler's "Designing Puzzles with a Computer" offers another view on the history and cultural place of puzzles.

The unusual blend of contributions is high calibre, as one expects when keen mathematicians step forth to honour someone they respect. Readers will enjoy the material, as will casual browsers. Indeed the book offers something new on each viewing!

The Locker Problem

Bruce Torrence and Stan Wagon

The locker problem appears frequently in both the secondary and university curriculum [2, 3]. In November 2005, it appeared on National Public Radio's *Car Talk* as a "Puzzler" and so attained even wider circulation [1]. The problem goes like this: A school corridor is lined with 1000 lockers, all closed. There are 1000 students who are sent marching down the hall in turn according to the following rules. The first student opens every locker. The second student closes every second locker, beginning with the second. The third student changes the state of every third locker, beginning with the third, closing it if it is open and opening it if it is closed. This continues, with the n^{th} student changing the state of every n^{th} locker, until all the students have walked the hallway. The problem is: Which lockers remain open after all the students have marched?

The answer is well known: The lockers whose numbers are perfect squares remain open, as only the squares have an odd number of divisors [4]. We note that this is true whether the corridor contains 1000 or any other number of lockers.

In this note we present some simple techniques for dealing with an extension of this problem. At the outset, we wish to extend our thanks to Joe Buhler for lending his attention and sharing his insights.

If we agree that the students are numbered and that, when sent marching, student n will change the state of every n^{th} locker beginning with locker n, then it is known that we can leave *any* collection of lockers open by dispatching precisely the right subset of students [4]. This leads to some interesting problems. For instance, we know that sending all the students leaves the square lockers open. Which subset of students must be sent to leave precisely the cube lockers open? How about the fourth powers? We will show shortly that there is a simple solution to these questions. In the meantime, we state in full generality the *extended locker problem*: Given a subset of the lockers, which students should be dispatched to keep those lockers open? Conversely, given a subset of the students, which lockers will be left open after they march?

We note that there are several problems interspersed throughout the remainder of this discourse. The impatient reader may wish to attempt these without reading the more general results that fill the space between them. While success is certainly possible, the general results provide a means for tackling most of the specific questions with greater efficiency.

Problem 1. Show that there can be no two distinct sets of students who will leave open the same set of lockers. Hint: Given two sets of students, consider the locker whose number is the smallest where the sets differ.

Returning to our main topic, it is shown in [4] that either version of the extended problem amounts to solving an $m \times m$ non-singular system of linear equations modulo 2, where m is the total number of lockers in the corridor. While the system has a unique solution for any subset of the lockers (or any subset of the students), finding it this way is tedious, to say the least.

Henceforth, we will assume that both the number of lockers and the number of students are infinite. The results in Theorems 1, 2, and 3 hold for the case of any finite number m of lockers and students; simply ignore any numbers exceeding m in any of the sets we discuss. We adhere to the convention that the set $\mathbb{N}$ of natural numbers does *not* include 0, so that it corresponds precisely to the complete student and locker sets.

For certain special subsets of the lockers, there is a much more elegant way to approach the problem than by solving a large linear system. Toward this end, define the *signature* of a natural number to be the set of all positive exponents appearing in that number's prime factorization. For our purposes it will be convenient to write members of the signature of a number in ascending order. For example, $12 = 2^2 3^1$ and has signature $\{1, 2\}$; $15 = 3^1 5^1$ and has signature $\{1\}$. Note that the signature of 1 is the empty set, $\emptyset$.

It is evident that the squarefree numbers are precisely the numbers whose signature is contained in the set $\{1\}$. The squares are those numbers whose signature is contained in the set $\{2, 4, 6, \dots\}$, and the cubes are those numbers whose signature is contained in the set $\{3, 6, 9, \dots\}$. In general, given a subset A of $\mathbb{N}$, we let $\sigma(A)$ denote the set of all numbers whose signature is contained in A. The set of squarefree numbers is $\sigma(\{1\})$, the set of squares is $\sigma(\{2, 4, 6, \dots\})$, the set of cubes is $\sigma(\{3, 6, 9, \dots\})$, and so on. Of course, many subsets of $\mathbb{N}$ are not σ of anything. For example, the powers of 2 (or of any specific prime) are not of the form $\sigma(A)$ for any A.

Suppose the set of students is of the form $\sigma(A)$ for some subset A of $\mathbb{N}$. Is there an elegant way to characterize the set of lockers that will be left open by these students? Indeed there is! We need one more definition in order to state the result. Given a set A of natural numbers, let $e(A)$ be all natural numbers that are greater than or equal to an *even* number (including 0) of the elements of A. For example, if $A = \{3, 6, 9, \dots\}$, then

$$\begin{aligned} e(A) &= \{1, 2, 6, 7, 8, 12, 13, 14, 18, 19, 20, \dots\} \\ &= \{n \in \mathbb{N} \mid n \equiv 0, 1, 2 \pmod 6\}. \end{aligned}$$

Theorem 1. Let $A \subseteq \mathbb{N}$. If students $\sigma(A)$ are dispatched, then lockers $\sigma(e(A))$ remain open.

Proof: Locker m will remain open if and only if an odd number of students touch it; that is, if m has an odd number of divisors among the numbers in the student set. Suppose that the students $\sigma(A)$ are sent marching, and suppose that locker m has prime factorization $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Note that the divisors of $p_i^{\alpha_i}$ in $\sigma(A)$ are all numbers of the form p_i^{γ}, where $\gamma \in A$

and $1 \leq \gamma \leq \alpha_i$, together with the number $1 = p^0$ (since $1 \in \sigma(A)$ for all sets A). Now $m \in \sigma(e(A))$ if and only if each $\alpha_i \in e(A)$; that is, if each α_i is greater than or equal to $2a_i$ members of A, for some $a_i \geq 0$. If this is the case, then the number of divisors of m among the students who will march is $(2a_1+1)(2a_2+1)\cdots(2a_n+1)$, an odd number. Hence locker m remains open.

To complete the proof, suppose that some $\alpha_i \notin e(A)$, say α_i is greater than or equal to $2a_i - 1$ members of A. Then the number of divisors of m among the students who march contains the factor $2a_i - 1 + 1 = 2a_i$, making it even. Thus, locker m will be closed. ∎

We note that the proof of this result is a straightforward generalization of the standard formal proof of the solution to the original locker problem (see, for example, [4]). This result makes easy work of several interesting problems (among them the original).

Problem 2. If all the students are sent down the hall, which lockers remain open?

Problem 3. If the squares are sent down the hall, which lockers remain open?

Problem 4. If the n^{th} powers are sent down the hall, which lockers remain open?

Problem 5. If one wishes to keep only locker number 1 open, which students should be sent marching?

The last problem leads to a natural question: Is there an inverse operation to the e function? It is not difficult to see that there is. For a subset A of the natural numbers, let $A + 1$ be the set $\{1\} \cup \{n + 1 \mid n \in A\}$, and define $f(A)$ to be the symmetric difference of A and $A+1$; that is, members of the union $A \cup (A + 1)$ that are not members of both.

Problem 6. Show that e and f are inverse operations. That is, show that $e(f(A)) = f(e(A)) = A$ for any set A of natural numbers.

Problem 6 together with Theorem 1 establish the following (which makes Problem 5 a snap):

Theorem 2. Let $A \subseteq \mathbb{N}$. If lockers $\sigma(A)$ are to remain open, students $\sigma(f(A))$ must be dispatched.

Corollary. The set of marching students is in the image of σ if and only if the set of lockers left open is in the image of σ.

Theorem 2 makes light work of these problems:

Problem 7. If one wishes to keep only the cube lockers open, which students should be sent marching?

Problem 8. If one wishes to keep only the n^{th} powers open, which students should be sent marching?

We see now that the extended locker problem is easily solved for any student or locker set that is determined completely by a signature-containing set A; that is, for sets of students or lockers of the form $\sigma(A)$ for some A. However, this is absolutely no help in cases where the student or locker set is not of this form. A different technique can often be brought to bear for such cases.

Problems 5 and 8 provide us with a necessary insight. Note that the answer to Problem 5 can be stated: To keep open only locker number 1, send only the squarefree students. Likewise, the answer to Problem 8 can be stated: To keep open only the lockers that are n^{th} powers, send all students whose number is the product of a squarefree number with an n^{th} power. The squarefree numbers clearly play a crucial role. We let S denote the set of squarefree numbers; $S = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, \dots\}$. And for any natural number m, we let $mS = \{ms \mid s \in S\}$.

Problem 9. If one wishes to keep only locker m open, show that one should dispatch students mS. Hint: See Problem 5.

Theorem 3. Let $L \subseteq \mathbb{N}$ be the collection of lockers to be kept open. Then student n should be included in the set of marching students if and only if $n \in lS$ for an *odd* number of members $l \in L$.

Proof: Let $L = \{l_1, l_2, \dots\}$ with $l_1 < l_2 < \cdots$. Note that L may be finite or infinite. One way to keep exactly the lockers in L open is as follows: first send students l_1S. After they have marched, only locker l_1 is open. Then send students l_2S. Note that none of the students in l_2S will touch any of the first $l_2 - 1$ lockers. Since $l_1 < l_2$, after this second cadre of students has marched, only lockers l_1 and l_2 will be open. If one were to continue in the fashion, precisely the lockers in L would be open. Now let $n \in \mathbb{N}$. Then $n \in lS$ for only finitely many $l \in L$. Suppose n is an element of precisely k of the sets lS, for $l \in L$. If k is even, then in the above scenario student n will have marched an even number of times. This has the same effect as student n not marching at all. If k is odd, then student n will have marched an odd number of times, and this has the same effect as student n marching just once. ∎

As several people pointed out to us, one can formulate Theorem 3 quite naturally in terms of Möbius inversion. But we have chosen here to present a totally elementary approach. With Theorem 3 in hand, several other cases of the extended locker problem are within reach.

Problem 10. Let p be prime. If one wishes to keep open only those lockers whose numbers are powers of p, which students must be dispatched? Try this both with $1 = p^0$ included in the locker set, and with it not included.

Problem 11. Which students must be dispatched to keep only the prime lockers open?

As a final observation, we note that there can be no non-empty subset A of the students that keeps precisely the lockers A open when the hallway is infinite.

Theorem 4. Let $A \subseteq \mathbb{N}$ be any non-empty subset of students. Then the set of lockers left open by these students is not A.

Proof: Let $A = \{n_1, n_2, \dots\}$ with $n_1 < n_2 < \cdots$. Consider locker $2n_1$. The only proper divisor of $2n_1$ in A is n_1. If $2n_1 \in A$ and the students in A march, then only students n_1 and $2n_1$ will touch locker $2n_1$, and so it will be closed. Conversely, if $2n_1 \notin A$, then only student n_1 will touch locker $2n_1$, and so it will remain open. In either case, the student set does not match the locker set. ∎

Problem 12. The above proof fails in a corridor with a finite number of lockers. Show that there is a set A in any finite corridor where students A will leave open lockers A. If there are more than two lockers in the hallway, there will be many such sets.

We leave the reader with one final problem. For any subset A of the natural numbers, denote by A^2 the set $\{n^2 \mid n \in A\}$.

Problem 13. The solution to the original locker problem shows there is a set A with the property that when students A march, lockers A^2 remain open (just take $A = \mathbb{N}$). Find a non-empty set A so that when students A^2 march, lockers A remain open.

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er novembre 2007**. *Une étoile (⋆) après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

3239. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit n un entier positif. Si $\alpha = 1 + \frac{1}{12(n+1)}$, montrer que

$$e \;<\; \left(\frac{(n+1)^{2n+1}}{(n!)^2}\right)^{\frac{1}{2n}} \;<\; e^{\alpha}\,,$$

3240. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit n un nombre entier positif. Montrer que

$$\left\lfloor \sqrt{n} + \sqrt{n + 2\sqrt[3]{n} + 1} \right\rfloor \;=\; \left\lfloor \sqrt{4n + 4\sqrt[3]{n} + 2} \right\rfloor\,,$$

où $\lfloor x \rfloor$ désigne la partie entière de x.

3241. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit a, b et c trois nombres réels arbitraires tels que $a^2 + b^2 + c^2 = 9$. Montrer que

$$3 \cdot \min\{a,\, b,\, c\} \;\le\; 1 + abc\,.$$

3242. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit $\mathcal{A} = \left\{z \in \mathbb{C}^* : \left|z + \frac{1}{z}\right| \le 2\right\}$. Soit $n \ge 2$ un entier. Montrer que si $\alpha^n \in \mathcal{A}$, alors $\alpha \in \mathcal{A}$.

3243. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Soit P un point intérieur d'un triangle isocèle ABC avec $AB = AC$. Soit D et E les points d'intersection respectifs des droites BP et CP avec les côtés opposés. Trouver le lieu des points P si

$$PD + DC \;=\; PE + EB\,.$$

3244. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Soit P un point intérieur d'un triangle isocèle ABC avec $AB = AC$. Soit D et E les points d'intersection respectifs des droites BP et CP avec les côtés opposés. Trouver le lieu des points P si

$$BD + DC \;=\; BE + EC\,.$$

3245. *Proposé par Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

On suppose que le centre du cercle des neuf points d'un triangle se trouve sur le cercle inscrit du triangle. Montrer que son antipode est le point de Feuerbach, c'est-à-dire le point de tangence des deux cercles.

3246. *Proposé par Marian Tetiva, Bîrlad, Roumanie.*

Soit a, b, c et d des nombres réels positifs arbitraires tels que $d = \min\{a, b, c, d\}$. Montrer que

$$\begin{aligned} & a^4 + b^4 + c^4 + d^4 - 4abcd \\ & \qquad \geq\; 4d\big[(a-d)^3 + (b-d)^3 + (c-d)^3 - 3(a-d)(b-d)(c-d)\big]\,. \end{aligned}$$

3247. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit $a_1, a_2, \ldots, a_n$ des nombres réels, tous plus grands que 1. Montrer que

$$\sum_{k=1}^{n}\big(1 + \log_{a_k}(a_{k+1})\big)^2 \;\geq\; 4n\,,$$

où $a_{n+1} = a_1$.

3248. *Proposé par Titu Zvonaru, Comănești, Roumanie, et Bogdan Ioniţă, Bucarest, Roumanie.*

Si a, b et c sont des nombres réels positifs, montrer que

$$\frac{a^2(b+c-a)}{b+c} + \frac{b^2(c+a-b)}{c+a} + \frac{c^2(a+b-c)}{a+b} \;\leq\; \frac{ab+bc+ca}{2}\,.$$

3249. *Proposé par Titu Zvonaru, Comănești, Roumanie, et Bogdan Ioniţă, Bucarest, Roumanie.*

Soit a, b et c les longueurs des côtés d'un triangle. Montrer que

$$\frac{(b+c)^2}{a^2+bc} + \frac{(c+a)^2}{b^2+ca} + \frac{(a+b)^2}{c^2+ab} \;\geq\; 6\,.$$

3250. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Soit ABC un triangle isocèle avec $AB = AC$ et tel que l'angle $BAC = 100°$. Soit D le point sur le prolongement de AB au-delà de A et tel que $AD = BC$. Trouver l'angle ADC.

. . . .

3239. *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer. If $\alpha = 1 + \frac{1}{12(n+1)}$, prove that

$$e < \left(\frac{(n+1)^{2n+1}}{(n!)^2}\right)^{\frac{1}{2n}} < e^{\alpha},$$

3240. *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer. Prove that

$$\left\lfloor \sqrt{n} + \sqrt{n + 2\sqrt[3]{n} + 1} \right\rfloor = \left\lfloor \sqrt{4n + 4\sqrt[3]{n} + 2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of x.

3241. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let a, b, c be any real numbers such that $a^2 + b^2 + c^2 = 9$. Prove that

$$3 \cdot \min\{a, b, c\} \leq 1 + abc.$$

3242. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let $\mathcal{A} = \left\{ z \in \mathbb{C}^* : \left| z + \frac{1}{z} \right| \leq 2 \right\}$. Let $n \geq 2$ be an integer. Prove that, if $\alpha^n \in \mathcal{A}$, then $\alpha \in \mathcal{A}$.

3243. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let ABC be an isosceles triangle with $AB = AC$, and let P be an interior point. Let the lines BP and CP intersect the opposite sides at the points D and E, respectively. Find the locus of P if

$$PD + DC = PE + EB.$$

3244. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let ABC be an isosceles triangle with $AB = AC$, and let P be an interior point. Let the lines BP and CP intersect the opposite sides at the points D and E, respectively. Find the locus of P if

$$BD + DC = BE + EC.$$

3245. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Suppose that the centre of the nine-point circle of a triangle lies on the incircle of the triangle. Show that its antipodal point is the Feuerbach Point; that is, the point where the nine-point circle and the incircle are tangent to each other.

3246. *Proposed by Marian Tetiva, Bîrlad, Romania.*

Let a, b, c, d be any positive real numbers with $d = \min\{a, b, c, d\}$. Prove that

$$\begin{aligned}&a^4+b^4+c^4+d^4-4abcd\\&\qquad\ge\ 4d\big[(a-d)^3+(b-d)^3+(c-d)^3-3(a-d)(b-d)(c-d)\big]\,.\end{aligned}$$

3247. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $a_1, a_2, \ldots, a_n$ be real numbers, each greater than 1. Prove that

$$\sum_{k=1}^{n}\big(1+\log_{a_k}(a_{k+1})\big)^2\ \ge\ 4n\,,$$

where $a_{n+1} = a_1$.

3248. *Proposed by Titu Zvonaru, Comăneşti, Romania, and Bogdan Ioniţă, Bucharest, Romania.*

If a, b, and c are positive real numbers, prove that

$$\frac{a^2(b+c-a)}{b+c}+\frac{b^2(c+a-b)}{c+a}+\frac{c^2(a+b-c)}{a+b}\ \le\ \frac{ab+bc+ca}{2}\,.$$

3249. *Proposed by Titu Zvonaru, Comăneşti, Romania, and Bogdan Ioniţă, Bucharest, Romania.*

Let a, b, and c be the lengths of the sides of a triangle. Prove that

$$\frac{(b+c)^2}{a^2+bc}+\frac{(c+a)^2}{b^2+ca}+\frac{(a+b)^2}{c^2+ab}\ \ge\ 6\,.$$

3250. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let ABC be an isosceles triangle with $AB = AC$ and $\angle BAC = 100^\circ$. Let D be the point on the production of AB such that $AD = BC$. Find $\angle ADC$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3138. [2006 : 173, 176] *Proposed by Paul Bracken, University of Texas, Edinburg, TX, USA.*

Let a_1 be a non-zero real number, and define the sequence $\{a_n\}_{n=1}^{\infty}$ by $a_{n+1} = n^2/a_n$ for $n \geq 1$. Prove that

$$\sum_{n=1}^{N} \frac{1}{a_n} = \left(\frac{1}{\pi a_1} + \frac{\pi a_1}{4}\right) \ln(N) + O(1)\,.$$

Solution by Joel Schlosberg, Bayside, NY, USA, modified by the editor.

It is straightforward to prove by induction that for $n \geq 0$,

$$a_{2n+1} = \left(\frac{2^{2n} n!^2}{(2n)!}\right)^2 a_1\,.$$

According to Stirling's Formula, $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left[1 + O\left(\frac{1}{n}\right)\right]$ for $n \to \infty$. Therefore, for $n \geq 1$,

$$\frac{2^{2n} n!^2}{(2n)!} = \frac{2^{2n} 2\pi n \left(\frac{n}{e}\right)^{2n}}{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}} \left[1 + O\left(\frac{1}{n}\right)\right] = \sqrt{\pi n}\left[1 + O\left(\frac{1}{n}\right)\right]\,.$$

Then, for $n \geq 1$,

$$\frac{1}{a_{2n+1}} = \frac{1}{\pi n a_1}\left[1 + O\left(\frac{1}{n}\right)\right] = \frac{1}{\pi a_1}\left[\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right]$$

and

$$\begin{aligned} \frac{1}{a_{2n+2}} &= \frac{a_{2n+1}}{(2n+1)^2} = \frac{\pi n a_1}{4n^2\left(1 + \frac{1}{2n}\right)^2}\left[1 + O\left(\frac{1}{n}\right)\right] \\ &= \frac{\pi a_1}{4n}\left[1 + O\left(\frac{1}{n}\right)\right] = \frac{\pi a_1}{4}\left[\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right]\,. \end{aligned}$$

Therefore, for $N \geq 4$,

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{a_n} &= \frac{1}{a_1} + \frac{1}{a_2} + \sum_{n=1}^{\lfloor\frac{N-1}{2}\rfloor} \frac{1}{a_{2n+1}} + \sum_{n=1}^{\lfloor\frac{N-2}{2}\rfloor} \frac{1}{a_{2n+2}} \\ &= \frac{1}{\pi a_1} \sum_{n=1}^{\lfloor\frac{N-1}{2}\rfloor} \left[\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right] + \frac{\pi a_1}{4} \sum_{n=1}^{\lfloor\frac{N-2}{2}\rfloor} \left[\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right] + O(1) \\ &= \left(\frac{1}{\pi a_1} + \frac{\pi a_1}{4}\right) \ln\frac{N}{2} + O(1)\,.\end{aligned}$$

Noting that $\ln(N/2) = \ln N - \ln 2 = \ln N + O(1)$, we arrive at the desired result.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3139. [2006 : 238, 240] *Proposed by Michel Bataille, Rouen, France.*

Let ε be the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$. Two parallel tangents to ε intersect a third tangent at $M_1(x_1,y_1)$ and $M_2(x_2,y_2)$. Show that the value of

$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}-1\right)$$

is independent of the chosen tangents.

A combination of similar solutions by Apostolis K. Demis, Varvakeio High School, Athens, Greece; and Peter Y. Woo, Biola University, La Mirada, CA, USA.

We shall see that for all choices of the three tangents, the product in question is always 1. We denote by O the centre of the given ellipse, and by C_1, C_2, and M the points of tangency of the three tangents C_1M_1, C_2M_2, and M_1MM_2.

The affine transformation $(x,y) \to \left(x, \frac{a}{b}y\right)$ transforms the given ellipse to the circle with centre O and radius a. It also transforms the points $M_1(x_1,y_1)$, $M_2(x_2,y_2)$, C_1, C_2, and M to the points $M_1'\left(x_1, \frac{a}{b}y_1\right)$, $M_2'\left(x_2, \frac{a}{b}y_2\right)$, C_1', C_2', and M', respectively, and it transforms the tangents C_1M_1, C_2M_2, and M_1M_2 of the ellipse to the tangents $C_1'M_1'$, $C_2'M_2'$, and $M_1'M_2'$, respectively, of the circle.

Consequently, $M_1'O \perp M_2'O$ (bisectors of $\angle C_1'OM'$ and $\angle M'OC_2'$, where $\angle C_1'OM' + \angle M'OC_2' = \pi$), and $OM' \perp M_1'M_2'$ (since $M_1'M_2'$ is tangent to the circle and OM' is a radius). Because $\triangle M_1'M'O \sim \triangle OM'M_2'$, it follows that $M'M_1'^2 \cdot M'M_2'^2 = M'O^4$. Then

$$\begin{aligned}
(M_1'O^2 - M'O^2)\cdot(M_2'O^2 - M'O^2) &= a^4\,,\\
\left(\left(x_1^2 + \frac{a^2}{b^2}y_1^2\right) - a^2\right)\cdot\left(\left(x_2^2 + \frac{a^2}{b^2}y_2^2\right) - a^2\right) &= a^4\,,\\
\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)\cdot\left(\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} - 1\right) &= 1\,.
\end{aligned}$$

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

3140. [2006 : 238, 240] *Proposed by Michel Bataille, Rouen, France.*

Let $a_1, a_2, \ldots, a_n$ be n distinct positive real numbers, where $n \geq 2$. For $i = 1, 2, \ldots, n$, let $p_i = \prod\limits_{j\neq i}(a_j - a_i)$. Show that $\prod\limits_{i=1}^{n} a_i^{\frac{1}{p_i}} < 1$.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, modified by the editor.

We will apply the following theorem on convex functions of higher order (see [1, pp. 4–5]).

Theorem. Let I be an open interval, and let $f : I \to \mathbb{R}$ be a function which is n-times differentiable. The following statements are equivalent:

(i) If $x_0, x_1, \ldots, x_n$ are any $n+1$ distinct points in I, then $\sum\limits_{i=0}^{n} \frac{f(x_i)}{w'(x_i)} > 0$, where $w(x) = \prod\limits_{j=0}^{n}(x - x_j)$.

(ii) $f^{(n)}(x) > 0$ for all $x \in I$.

A function f satisfying the above conditions is said to be *strictly n-convex*.

For our problem, we take $I = (0, \infty)$ and $f(x) = (-1)^n \ln x$. Then $f^{(n-1)}(x) = (n-2)!/x^{n-1} > 0$ for all $x > 0$. Therefore, f is strictly $(n-1)$-convex.

Using the n distinct numbers $a_1, a_2, \ldots, a_n$ given in the problem, we let $w(x) = \prod\limits_{j=1}^{n}(x - a_j)$. Then for each $i = 1, 2, \ldots, n$, we have

$$w'(a_i) \;=\; \prod_{j\neq i}(a_i - a_j) \;=\; (-1)^{n-1}p_i\,.$$

According to condition (i) in the theorem, $\sum_{i=1}^{n} \frac{f(a_i)}{w'(a_i)} > 0$. Then

$$\sum_{i=1}^{n} \frac{\ln a_i}{p_i} \;=\; -\sum_{i=1}^{n} \frac{(-1)^n \ln a_i}{(-1)^{n-1} p_i} \;=\; -\sum_{i=1}^{n} \frac{f(a_i)}{w'(a_i)} \;<\; 0\,.$$

Taking exponentials, we obtain $\prod_{i=1}^{n} a_i^{\frac{1}{p_i}} < 1$.

References

[1] Mitrinovic, D.S., *Classical and New Inequalities in Analysis*, Kluwer Acad. Publ., Dordrecht, 1993.

Also solved by JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

3141. [2006 : 238, 240] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be the sides of a scalene triangle ABC. Prove that

$$\sum_{\text{cyclic}} \frac{(a+1)bc}{(\sqrt{a}-\sqrt{b})(\sqrt{a}-\sqrt{c})} \;<\; \frac{a^4+b^4+c^4}{abc}\,.$$

Solution by Michel Bataille, Rouen, France, modified by the editor.

Write the left side as $S_1 + S_2$, where

$$S_1 \;=\; \sum_{\text{cyclic}} \frac{abc}{\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}-\sqrt{c}\right)}\,, \quad S_2 \;=\; \sum_{\text{cyclic}} \frac{bc}{\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}-\sqrt{c}\right)}\,.$$

Observe that

$$S_1 \;=\; abc \cdot \frac{\left(\sqrt{c}-\sqrt{b}\right)+\left(\sqrt{a}-\sqrt{c}\right)+\left(\sqrt{b}-\sqrt{a}\right)}{\left(\sqrt{c}-\sqrt{b}\right)\left(\sqrt{a}-\sqrt{c}\right)\left(\sqrt{b}-\sqrt{a}\right)} \;=\; 0\,.$$

We can simplify S_2 by making use of the identity

$$x^2y^2(y-x)+y^2z^2(z-y)+z^2x^2(x-z) \;=\; (x-y)(y-z)(z-x)(xy+yz+zx).$$

Dividing both sides by $(x-y)(y-z)(z-x)$ and setting $x=\sqrt{a}$, $y=\sqrt{b}$, and $z=\sqrt{c}$, we obtain

$$S_2 \;=\; \sqrt{ab}+\sqrt{bc}+\sqrt{ca}\,.$$

Thus, the given inequality turns out to be equivalent to

$$abc\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right) \;<\; a^4+b^4+c^4\,. \tag{1}$$

Now, from the Weighted AM–GM Inequality, we have

$$a^{\frac{3}{2}}b^{\frac{3}{2}}c \;=\; (a^4)^{\frac{3}{8}}(b^4)^{\frac{3}{8}}(c^4)^{\frac{1}{4}} \;<\; \tfrac{3}{8}a^4+\tfrac{3}{8}b^4+\tfrac{1}{4}c^4$$

(where the inequality is strict because a, b, and c are distinct). Similarly, we have $a^{\frac{3}{2}}bc^{\frac{3}{2}} < \frac{3}{8}a^4+\frac{1}{4}b^4+\frac{3}{8}c^4$ and $ab^{\frac{3}{2}}c^{\frac{3}{2}} < \frac{1}{4}a^4+\frac{3}{8}b^4+\frac{3}{8}c^4$. If we add all three inequalities, we obtain (1).

Note that the result actually holds whenever a, b, and c are distinct positive real numbers.

Also solved by ARKADY ALT, San Jose, CA, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Several solvers mentioned that the result is true for any three distinct positive real numbers. Most solvers began, like Bataille, by showing that the given inequality is equivalent to $abc(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) < a^4+b^4+c^4$. At that point, there were several ways to complete the solution. Janous provided several such ways himself.

3142. [2006 : 238, 241] *Proposed by Mihály Bencze, Brasov, Romania.*

If $x_k > 0$ for $k = 1, 2, \ldots, n$, prove that

(a) $$\cos\left(\frac{n}{\sum\limits_{k=1}^{n} x_k}\right) - \sin\left(\frac{n}{\sum\limits_{k=1}^{n} x_k}\right) \;\geq\; \frac{1}{n}\sum_{k=1}^{n}\left(\cos\frac{1}{x_k}-\sin\frac{1}{x_k}\right);$$

(b) $$\frac{\sum\limits_{k=1}^{n}\sin\frac{1}{x_k}}{\sum\limits_{k=1}^{n}\cos\frac{1}{x_k}} \;\geq\; \tan\left(\frac{n}{\sum\limits_{k=1}^{n} x_k}\right).$$

Editor's comment.

Unfortunately, the inequalities are incorrect as stated. Both Walther Janous, Ursulinengymnasium, Innsbruck, Austria and Peter Y. Woo, Biola University, La Mirada, CA, USA gave counterexamples and then attempted to impose additional restrictions on the variables to make the inequalities correct. Janous succeeded in repairing part (a). Woo's counterexample for part (a) is $n = 2$, $x_1 = 0.4$, and $x_2 = 0.3$, and Janous' counterexample for part (b) is $n = 2$, $x_1 = 1/\pi$, and $x_2 = 3/\pi$.

Solution to adjusted part (a) *by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $f(x) = \cos\frac{1}{x} - \sin\frac{1}{x}$. Inequality (a) can now be written as

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) \;\leq\; f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right).$$

This is Jensen's Inequality characterizing the concavity of the function f.

Using a computer algebra system, we can show that

$$f''(x) = \frac{(1-2x)\sin\frac{1}{x} - (2x+1)\cos\frac{1}{x}}{x^4} < 0$$

for $x \in (0.601451551, \infty)$. (The editor verified Janous' result using Maple.) Thus, inequality (a) is true if $x_k \in (0.601451551, \infty)$ for $k = 1, 2, \ldots, n$.

Solution to adjusted part (b) *by the editor using ideas of the proposer.*

Let $g(x) = \sin\frac{1}{x}$ and $h(x) = \cos\frac{1}{x}$. Then

$$g''(x) = \frac{2}{x^3}\cos\frac{1}{x} - \frac{1}{x^4}\sin\frac{1}{x} \quad\text{and}\quad h''(x) = -\left(\frac{2}{x^3}\sin\frac{1}{x} + \frac{1}{x^4}\cos\frac{1}{x}\right).$$

For $x \in (0.928613759, \infty)$, we have $g''(x) > 0$ and $h''(x) < 0$. In fact, $h''(x) < 0$ if $x \in (0.436885409, \infty)$. Since $0.928613759 > 2/\pi$, we have

$$0 < \frac{n}{\sum\limits_{k=1}^{n} x_k} < \frac{\pi}{2} \qquad\text{and}\qquad 0 < \frac{1}{x_k} < \frac{\pi}{2}.$$

Now, Jensen's Inequality applied to the functions $g(x)$ and $h(x)$ yields

$$\frac{1}{n}\sum_{k=1}^{n}\sin\frac{1}{x_k} \ge \sin\left(\frac{n}{\sum\limits_{k=1}^{n} x_k}\right) > 0$$

$$\text{and}\quad 0 < \frac{1}{n}\sum_{k=1}^{n}\cos\frac{1}{x_k} \le \cos\left(\frac{n}{\sum\limits_{k=1}^{n} x_k}\right),$$

from which the inequality in part (b) follows immediately.

3143. [2006 : 239, 241] *Proposed by Mihály Bencze, Brasov, Romania.*

For $n \ge 1$ let $a_n = 1 + \sqrt{2} + \sqrt[3]{3} + \cdots + \sqrt[n]{n}$. Prove that

$$\sum_{k=1}^{n}\frac{\sqrt[k]{k}}{a_k^2} < \frac{2n+1+(\ln n)^2}{n+1+\frac{1}{2}(\ln n)^2}.$$

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA, modified by the editor.

Define

$$L_n = \sum_{k=1}^{n}\frac{\sqrt[k]{k}}{a_k^2} \qquad\text{and}\qquad R_n = \frac{2n+1+(\ln n)^2}{n+1+\frac{1}{2}(\ln n)^2}.$$

By direct computation, we see that $L_1 = 1$, $L_2 \approx 1.2426$, $L_3 \approx 1.3396$, $L_4 \approx 1.3905$, $R_1 = 1.5$, $R_2 \approx 1.6914$, $R_3 \approx 1.7828$, and $R_4 \approx 1.8322$. Hence, $L_n < R_n$ for $n = 1, 2, 3, 4$.

Now assume that $n \geq 5$. Since the function $f(x) = x^{1/x}$ is decreasing on (e, ∞), we have, for each $k \geq 5$, $1 < k^{1/k} \leq 5^{1/5} \approx 1.3797$. Note also that $a_k \geq k$ for all $k \geq 1$. Hence,

$$\begin{aligned} L_n &= L_4 + \sum_{k=5}^{n} \frac{\sqrt[k]{k}}{a_k^2} \;<\; L_4 + (1.3797) \sum_{k=5}^{n} \frac{1}{k^2} \;<\; L_4 + (1.4) \int_4^n \frac{dx}{x^2} \\ &= L_4 + (1.4)\left(\frac{1}{4} - \frac{1}{n}\right) \;<\; (1.4) + (1.4)\left(\frac{1}{4}\right) \;=\; 1.75\,. \end{aligned}$$

Since $R_n = 2 - \dfrac{1}{n+1+\frac{1}{2}(\ln n)^2} > 2 - \dfrac{1}{n+1} \geq 2 - \dfrac{1}{6} > 1.8$, it follows that $L_n < R_n$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.

Janous' solution, which is based on computer verifications, actually establishes the stronger inequality $\sum\limits_{k=1}^{n} \frac{\sqrt[k]{k}}{k^2} < R_n$. *Both Janous and Hess believe that the minimum of* $R_n - L_n$ *is attained when* $n = 8$*; thus,* $L_n \leq R_n - 0.4409$ *where* $0.4409 \approx R_8 - L_8$*, but they gave no proof. The solution by the proposer applied simple telescoping together with the inequality* $a_n < n + 1 + \frac{1}{2}(\ln n)^2$*, which he claimed can be shown by induction but did not supply any proof.*

3144. [2006 : 239, 241] *Proposed by Mihály Bencze, Brasov, Romania.*

Let A, $B \in M_n(\mathbb{C})$, and let $\omega = e^{2\pi i/n}$. Prove that

$$\sum_{k=1}^{n} \det(A + \omega^{k-1}B) + \sum_{k=1}^{n} \det(B + \omega^{k-1}A) \;=\; 2n(\det A + \det B)\,.$$

[*Ed.* The problem has been corrected to state that $\omega = e^{2\pi i/n}$, as the proposer intended, rather than $\omega = e^{2\pi/n}$. This correction was made by the solvers.]

Solution by Michel Bataille, Rouen, France.

For $M \in M_n(\mathbb{C})$, we denote by $M^{(1)}$, $M^{(2)}$, …, $M^{(n)}$ the columns of M. Let x be an indeterminate. Since an $n \times n$ determinant is an n–linear function of its columns, we can expand $\det(A + xB)$ as follows:

$$\begin{aligned} \det(A + xB) &= \det\left(A^{(1)} + xB^{(1)}, A^{(2)} + xB^{(2)}, \ldots, A^{(n)} + xB^{(n)}\right) \\ &= \det\left(A^{(1)}, A^{(2)}, \ldots, A^{(n)}\right) + \sum_{j=1}^{n-1} \alpha_j x^j \\ &\qquad + x^n \det\left(B^{(1)}, B^{(2)}, \ldots, B^{(n)}\right) \\ &= \det A + \sum_{j=1}^{n-1} \alpha_j x^j + x^n \det B\,, \end{aligned}$$

where α_1, α_2, $\dots$, α_{n-1} are complex numbers (independent of x). Taking $x = 1$, ω, ω^2, $\dots$, ω^{n-1} in succession and adding the results, we obtain

$$\begin{aligned}\sum_{k=0}^{n-1} \det(A + \omega^k B) &= n \det A + \sum_{k=0}^{n-1}\sum_{j=1}^{n-1} \alpha_j \omega^{kj} + \sum_{k=0}^{n-1} \omega^{kn} \det B \\ &= n \det A + \sum_{j=1}^{n-1} \alpha_j \sum_{k=0}^{n-1} \omega^{kj} + \sum_{k=0}^{n-1} \omega^{kn} \det B\,.\end{aligned}$$

Now, since $\omega^n = 1$ and $\sum\limits_{k=0}^{n-1} \omega^{kj} = \dfrac{1-\omega^{nj}}{1-\omega^j} = 0$ for $j = 1, 2, \dots, n-1$, we have

$$\sum_{k=0}^{n-1} \det(A + \omega^k B) \;=\; n(\det A + \det B)\,.$$

Then $\sum\limits_{k=0}^{n-1} \det(B + \omega^k A) = n(\det B + \det A)$, and the result follows.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3145★. [2006 : 239, 241] *Proposed by Yuming Chen, Wilfrid Laurier University, Waterloo, ON.*

Let $f(x) = x - c^2 \tanh x$, where $c > 1$ is an arbitrary constant. It is not hard to show that $f(x)$ is decreasing on the interval $[-x_0, x_0]$, where $x_0 = \ln\left(c + \sqrt{c^2-1}\right)$ is the positive root of the equation $\cosh x = c$. For each $x \in (-x_0, x_0)$, the horizontal line passing through $\big(x, f(x)\big)$ intersects the graph of f at two other points with abscissas $x_1(x)$ and $x_2(x)$. Define a function $g : (-x_0, x_0) \to \mathbb{R}$ as follows:

$$g(x) \;=\; x + c^2 \tanh\big(x_1(x)\big) + c^2 \tanh\big(x_2(x)\big)\,.$$

Prove or disprove that $g(x) > 0$ for all $x \in (0, x_0)$.

Editor's note: No solutions were received for this problem; hence, it remains open. The proposer believes that the conjecture is true, since there is ample empirical evidence.

3146. [2006 : 239, 242] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $p > 1$, and let $a, b, c, d \in \left[1/\sqrt{p}, \sqrt{p}\right]$. Prove that

(a) $\dfrac{p}{1+p} + \dfrac{2}{1+\sqrt{p}} \;\le\; \dfrac{a}{a+b} + \dfrac{b}{b+c} + \dfrac{c}{c+a} \;\le\; \dfrac{1}{1+p} + \dfrac{2\sqrt{p}}{1+\sqrt{p}}$;

(b) $\dfrac{p}{1+p} + \dfrac{3}{1+\sqrt[3]{p}} \;\le\; \dfrac{a}{a+b} + \dfrac{b}{b+c} + \dfrac{c}{c+d} + \dfrac{d}{d+a} \;\le\; \dfrac{1}{1+p} + \dfrac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}$.

Solution by Arkady Alt, San Jose, CA, USA, modified by the editor.

(a) The transposition $(a,b,c) \mapsto (b,a,c)$ in the inequality

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \;\le\; \frac{1}{1+p}+\frac{2\sqrt{p}}{1+\sqrt{p}} \tag{1}$$

gives the equivalent inequality

$$\frac{b}{b+a}+\frac{a}{a+c}+\frac{c}{c+b} \;\le\; \frac{1}{1+p}+\frac{2\sqrt{p}}{1+\sqrt{p}}\,.$$

Since

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \;=\; 3-\left(\frac{b}{b+a}+\frac{a}{a+c}+\frac{c}{c+b}\right)\,,$$

we see that (1) is satisfied if and only if

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \;\ge\; 3-\left(\frac{1}{1+p}+\frac{2\sqrt{p}}{1+\sqrt{p}}\right) \;=\; \frac{p}{1+p}+\frac{2\sqrt{p}}{1+\sqrt{p}}\,.$$

Thus, to prove (a), it is sufficient to prove (1).

Let $x=b/a$, $y=c/b$, and $z=a/c$. Then (1) becomes

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z} \;\le\; \frac{1}{1+p}+\frac{2\sqrt{p}}{1+\sqrt{p}}\,. \tag{2}$$

Note that $xyz=1$ and x, y, $z \in [1/p,p]$. To prove (1), it is sufficient to prove (2) for all such x, y, and z.

By the symmetry in (2), we may assume that $z=\max\{x,\,y,\,z\}$. Then, since $xyz=1$ and $z\le p$, we must have $1\le z\le p$ and $1/p\le xy\le 1$. Let $t=\sqrt{xy}$. Then $t^2z=1$ and $1/\sqrt{p}\le t\le 1$. Since $x+y\ge 2\sqrt{xy}=2t$, we have

$$\begin{aligned}\frac{1}{1+x}+\frac{1}{1+y} \;&=\; \frac{2+x+y}{1+x+y+xy} \;=\; 1+\frac{1-t^2}{1+x+y+t^2}\\ &\le\; 1+\frac{1-t^2}{1+2t+t^2} \;=\; 1+\frac{1-t}{1+t} \;=\; \frac{2}{1+t}\,.\end{aligned} \tag{3}$$

Hence,

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z} \;\le\; \frac{2}{1+t}+\frac{1}{1+z} \;=\; \frac{2}{1+t}+\frac{t^2}{1+t^2}\,.$$

Let $h(t)=\dfrac{2}{1+t}+\dfrac{t^2}{1+t^2}$. Since $h'(t)=\dfrac{-2(1-t)(1-t^3)}{(1+t)^2(1+t^2)^2}$, it follows that h is decreasing on $(0,1]$. Consequently, for $1/\sqrt{p}\le t\le 1$,

$$h(t) \;\le\; h(1/\sqrt{p}) \;=\; \frac{1}{1+p}+\frac{2\sqrt{p}}{1+\sqrt{p}}\,.$$

This proves inequality (2) and completes the proof of (a).

(b) This is treated similarly. The transposition $(a, b, c, d) \mapsto (b, a, d, c)$ in the inequality

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} \;\leq\; \frac{1}{1+p} + \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}} \tag{4}$$

yields the equivalent inequality

$$\frac{b}{b+a} + \frac{a}{a+d} + \frac{d}{d+c} + \frac{c}{c+b} \;\leq\; \frac{1}{1+p} + \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}\,.$$

Since

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} \;=\; 4 - \left(\frac{b}{b+a} + \frac{a}{a+d} + \frac{d}{d+c} + \frac{c}{c+b}\right),$$

we see that (4) is satisfied if and only if

$$\begin{aligned} \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} \;&\geq\; 4 - \left(\frac{p}{p+1} + \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}\right) \\ &=\; \frac{p}{p+1} + \frac{3}{1+\sqrt[3]{p}}\,. \end{aligned}$$

Thus, to prove (b), it is sufficient to prove (4).

Let $x = b/a$, $y = c/b$, $u = c/d$, and $v = d/a$. Then (4) becomes

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+u} + \frac{1}{1+v} \;\leq\; \frac{1}{1+p} + \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}\,. \tag{5}$$

Note that $xyuv = 1$ and $x, y, u, v \in [1/p, p]$. To prove (4), it is sufficient to prove (5) for all such x, y, u, and v.

Let $t = \sqrt{xy}$ and $s = \sqrt{uv}$. By the symmetry in (5), we may assume that $t \leq s$. Then, since $ts = 1$, we see that $t \leq 1 \leq s$. Furthermore, since $s^2/u = v \leq p$, we have $s^2/p \leq u$, and thus, $s^2/p \leq u \leq p$.

Now, for fixed s,

$$\begin{aligned} &\max\left\{u+v \;\middle|\; uv = s^2, \frac{s^2}{p} \leq u \leq p\right\} \\ &\qquad\qquad = \;\max\left\{u + \frac{s^2}{u} \;\middle|\; \frac{s^2}{p} \leq u \leq p\right\} \;=\; p + \frac{s^2}{p}\,. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{1}{1+u} + \frac{1}{1+v} \;&=\; \frac{2+u+v}{1+u+v+uv} \;=\; 1 - \frac{s^2-1}{1+u+v+s^2} \\ &\leq\; 1 - \frac{s^2-1}{1+p+\dfrac{s^2}{p}+s^2} \;=\; \frac{2+p+\dfrac{s^2}{p}}{1+p+\dfrac{s^2}{p}+s^2} \\ &=\; \frac{p}{s^2+p} + \frac{1}{1+p}\,. \end{aligned} \tag{6}$$

Using inequalities (6) and (3), we get

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+u}+\frac{1}{1+v} \;\le\; \frac{2s}{1+s}+\frac{p}{s^2+p}+\frac{1}{1+p}\,.$$

Let $g(s) = \frac{2s}{1+s} + \frac{p}{s^2+p}$. Since $g'(s) = \frac{2(s-p)(s^3-p)}{(s+1)^2(s^2+p)^2}$, this function has a local maximum at $s = \sqrt[3]{p}$, which is in the interval $(1,p)$. We have $g(1) = -1 + \frac{p}{1+p} < 0$ and $g(p) = -\frac{2}{p+1} + \frac{1}{p+1} < 0$; whence, $\max\limits_{s\in[1,p]} g(s) = g\big(\sqrt[3]{p}\big)$, and therefore,

$$\begin{aligned}\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+u}+\frac{1}{1+v} &\le \frac{2\sqrt[3]{p}}{1+\sqrt[3]{p}}+\frac{p}{\sqrt[3]{p^2}+p}+\frac{1}{1+p}\\ &= \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}+\frac{1}{1+p}\,.\end{aligned}$$

This proves (4) and completes the proof of (b).

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (a)); PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3147. [2006 : 239, 242] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania; and Gabriel Dospinescu, Paris, France.*

Let $n \ge 3$, and let $x_1, x_2, \dots, x_n$ be positive real numbers such that $x_1x_2\cdots x_n = 1$. For $n = 3$ and $n = 4$, prove that

$$\frac{1}{x_1^2+x_1x_2}+\frac{1}{x_2^2+x_2x_3}+\cdots+\frac{1}{x_n^2+x_nx_1} \;\ge\; \frac{n}{2}\,.$$

Solution by the proposer.

Using the substitutions $x_1 = \sqrt{\frac{a_2}{a_1}}$, $x_2 = \sqrt{\frac{a_3}{a_2}}$, $\dots$, $x_n = \sqrt{\frac{a_1}{a_n}}$, the given inequality becomes

$$\frac{a_1}{a_2+\sqrt{a_1a_3}}+\frac{a_2}{a_3+\sqrt{a_2a_4}}+\cdots+\frac{a_n}{a_2+\sqrt{a_na_2}} \;\ge\; \frac{n}{2}\,.$$

Since $\sqrt{a_1a_3} \le \frac{a_1+a_3}{2}$, $\dots$, $\sqrt{a_na_2} \le \frac{a_n+a_2}{2}$, it suffices to show that

$$\frac{a_1}{a_1+2a_2+a_3}+\frac{a_2}{a_2+2a_3+a_4}+\cdots+\frac{a_n}{a_n+2a_1+a_2} \;\ge\; \frac{n}{4}\,.$$

By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}(a_1+\cdots+a_n)^2 \;&\le\; [a_1(a_1+2a_2+a_3)+\cdots+a_n(a_n+2a_1+a_2)]\\ &\qquad\cdot\left(\frac{a_1}{a_1+2a_2+a_3}+\cdots+\frac{a_n}{a_n+2a_1+a_2}\right)\,.\end{aligned}$$

Thus, it suffices to show that

$$4(a_1+\cdots+a_n)^2 \;\ge\; n[a_1(a_1+2a_2+a_3)+\cdots+a_n(a_n+2a_1+a_2)]\,.$$

This inequality is an identity for $n=4$. For $n=3$, it is

$$a_1^2+a_2^2+a_3^2 \;\ge\; a_1a_2+a_2a_3+a_3a_1\,,$$

which is true, because

$$\begin{aligned}2(a_1^2+a_2^2+a_3^2-a_1a_2-a_2a_3-a_3a_1)&\\ =\;(a_1-a_2)^2+(a_2-a_3)^2+(a_3-a_1)^2 \;\ge\; 0\,.&\end{aligned}$$

This completes the proof. Equality holds when $x_i=1$ for all i.

The case $n=3$ was also solved by MOHAMMED AASSILA, Strasbourg, France. There was one incorrect submission.

3148. [2006 : 240, 242] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $0<m<1$, and let a, b, $c\in\left[\sqrt{m},1/\sqrt{m}\right]$. Prove that

$$\frac{a^3+b^3+c^3+3(1+m)abc}{ab(a+b)+bc(b+c)+ca(c+a)} \;\ge\; 1+\frac{m}{2}\,.$$

Solution by Joel Schlosberg, Bayside, NY, USA.

Let $f(x)=-2x^2+\left(m^2+3m\right)x+\left(m^3-3m^2-2m+2\right)$. For $x\ge m$,

$$f'(x) \;=\; -4x+m^2+3m \;\le\; -4m+m^2+3m \;=\; m(m-1) \;<\; 0\,,$$

which implies that $f(x)$ is decreasing.

Suppose that $a=m$, $b\in[m,1]$, and $c=1$. Then $m=a\le b\le c=1$ and

$$\begin{aligned}&2[a^3+b^3+c^3+3(1+m)abc]\\ &\qquad -(m+2)[a^2b+ab^2+b^2c+bc^2+c^2a+ca^2]\\ &=\; 2[m^3+b^3+1+3(1+m)mb]\\ &\qquad -(m+2)[m^2b+mb^2+b^2+b+m+m^2]\\ &=\; 2b^3-(m^2+3m+2)b^2+(-m^3+4m^2+5m-2)b\\ &\qquad +(m^3-3m^2-2m+2)\\ &=\; (1-b)[-2b^2+(m^2+3m)b+(m^3-3m^2-2m+2)]\\ &=\; (1-b)f(b) \;\ge\; (1-b)f(1) \;=\; (1-b)m(m-1)^2 \;\ge\; 0\,,\end{aligned}$$

which yields

$$\frac{a^3+b^3+c^3+3(1+m)abc}{a^2b+ab^2+b^2c+bc^2+c^2a+ca^2} \;\ge\; 1+\frac{m}{2}\,.$$

Suppose that a, b, $c \in [\sqrt{m}, 1/\sqrt{m}]$. Without loss of generality, we can assume that $a \leq b \leq c$. Then

$$m \;\leq\; \frac{a}{c} \;\leq\; \frac{b}{c} \;\leq\; 1\,.$$

Hence, for $(m', a', b', c') = \left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, 1\right)$, we have

$$m \;\leq\; m' \;=\; a' \;\leq\; b' \;\leq\; c' \;=\; 1\,,$$

and therefore,

$$\frac{a'^3 + b'^3 + c'^3 + 3(1+m')a'b'c'}{a'^2b' + a'b'^2 + b'^2c' + b'c'^2 + c'^2a' + c'a'^2} \;\geq\; 1 + \frac{m'}{2}\,,$$

which can be written as

$$\begin{aligned}&\frac{a'^3 + b'^3 + c'^3 + 3(1+m')a'b'c'}{a'^2b' + a'b'^2 + b'^2c' + b'c'^2 + c'^2a' + c'a'^2} - 1\\ &\qquad\geq\; m'\left(\frac{1}{2} - \frac{3a'b'c'}{a'^2b' + a'b'^2 + b'^2c' + b'c'^2 + c'^2a' + c'a'^2}\right).\end{aligned}$$

By the AM–GM Inequality,

$$\frac{a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2}{6} \;\geq\; \sqrt[6]{a^6b^6c^6} \;=\; abc\,.$$

Hence,

$$\frac{1}{2} - \frac{3abc}{a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2} \;\geq\; 0\,.$$

Since $a : b : c = a' : b' : c'$, we have

$$\begin{aligned}&\frac{a^3 + b^3 + c^3 + 3abc}{a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2} - 1\\ &\qquad=\; \frac{a'^3 + b'^3 + c'^3 + 3a'b'c'}{a'^2b' + a'b'^2 + b'^2c' + b'c'^2 + c'^2a' + c'a'^2} - 1\\ &\qquad\geq\; m'\left(\frac{1}{2} - \frac{3a'b'c'}{a'^2b' + a'b'^2 + b'^2c' + b'c'^2 + c'^2a' + c'a'^2}\right)\\ &\qquad=\; m'\left(\frac{1}{2} - \frac{3abc}{a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2}\right)\\ &\qquad\geq\; m\left(\frac{1}{2} - \frac{3abc}{a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2}\right),\end{aligned}$$

and therefore,

$$\frac{a^3 + b^3 + c^3 + 3(1+m)abc}{a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2} \;\geq\; 1 + \frac{m}{2}\,.$$

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. There was also one incorrect solution submitted.

3149. Replacement. [2006 : 303, 306] *Proposed by David Martinez Ramirez, student, Universidad Nacional Autonoma de Mexico, Mexico.*

Let $P(z)$ be any non-constant complex monic polynomial. Show that there is a complex number w such that $|w| \leq 1$ and $|P(w)| \geq 1$.

I. *Solution by Michel Bataille, Rouen, France.*

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. Suppose, to the contrary, that $|P(z)| < 1$ for all complex numbers z such that $|z| \leq 1$. Consider $Q(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + 1$. Note that $Q(z) = z^n P\left(\frac{1}{z}\right)$ for $z \neq 0$. Thus, if $|z| = 1$, then $|Q(z)| = \left|P\left(\frac{1}{z}\right)\right| < 1$.

It follows that $\int\limits_0^{2\pi} |Q(e^{it})|\, dt < 2\pi$. However,

$$\int_0^{2\pi} Q(e^{it})\, dt \;=\; \int_0^{2\pi} \left(a_0 e^{int} + a_1 e^{i(n-1)t} + \cdots + a_{n-1}e^{it} + 1\right) dt \;=\; 2\pi\,,$$

since $\int\limits_0^{2\pi} e^{ikt}\, dt = \frac{1}{ik}\left(e^{2k\pi i} - 1\right) = 0$ for all $k = 1, 2, \ldots, n$.

Hence, $2\pi = \left|\int\limits_0^{2\pi} Q(e^{it})\, dt\right| \leq \int\limits_0^{2\pi} |Q(e^{it})|\, dt < 2\pi$, a contradiction.

II. *Solution by Bin Zhao, student, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China, modified by the editor.*

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. Let $\zeta = e^{i\pi/n}$. Then $\zeta^n = -1$, $\zeta^{2n} = 1$, and $|\zeta^k| = 1$ for $k = 0, 1, 2, \ldots, 2n-1$. Now,

$$\begin{aligned} \sum_{k=0}^{2n-1} |P(\zeta^k)| \;&\geq\; \big|\left(P(1) + P(\zeta^2) + \cdots + P(\zeta^{2n-2})\right) \\ &\qquad - \left(P(\zeta) + P(\zeta^3) + \cdots + P(\zeta^{2n-1})\right)\big| \\ &=\; \left|\sum_{k=0}^{n-1} \left(P(\zeta^{2k}) - P(\zeta^{2k+1})\right)\right| \end{aligned} \qquad (1)$$

and

$$\begin{aligned} \sum_{k=0}^{n-1} \left(P(\zeta^{2k}) - P(\zeta^{2k+1})\right) \;&=\; \sum_{k=0}^{n-1} \left((\zeta^{2k})^n - (\zeta^{2k+1})^n\right) \\ &\qquad + \sum_{j=1}^{n-1} a_j \sum_{k=0}^{n-1} \left((\zeta^{2k})^j - (\zeta^{2k+1})^j\right). \end{aligned} \qquad (2)$$

Note that $(\zeta^{2k})^n - (\zeta^{2k+1})^n = (\zeta^n)^{2k} - (\zeta^n)^{2k+1} = 1 - (-1) = 2$, and hence,

$$\sum_{k=0}^{n-1} \left((\zeta^{2k})^n - (\zeta^{2k+1})^n\right) \;=\; 2n\,. \qquad (3)$$

On the other hand, for each $j = 1, 2, \ldots, n-1$, we have

$$\sum_{k=0}^{n-1}\left((\zeta^{2k})^j - (\zeta^{2k+1})^j\right) = (1-\zeta^j)\sum_{k=0}^{n-1}(\zeta^{2j})^k = (1-\zeta^j)\frac{1-(\zeta^{2j})^n}{1-\zeta^{2j}} = 0\,, \qquad (4)$$

since $(\zeta^{2j})^n = (\zeta^{2n})^j = 1$.

Substituting (3) and (4) into (2) and using (1), we then have

$$\sum_{k=0}^{2n-1}\left|P(\zeta^k)\right| \ge 2n\,,$$

from which we deduce that there must be some k, with $0 \le k \le 2n-1$, for which $|P(\zeta^k)| \ge 1$, completing the proof.

Also solved by KEE-WAI LAU, Hong Kong, China; and the proposer. There was also one incorrect solution.

The proof given by Lau used Rouché's Theorem from complex analysis. Both solutions featured above show that w can be chosen so that $|w| = 1$ and $|P(w)| \ge 1$. But this is not surprising, in view of the well-known Maximum Modulus Principle.

3150. [2006 : 240, 242; corrected 2006 : 303, 306] *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let a, b, c be the three sides of a triangle, and let h_a, h_b, h_c be the altitudes to the sides a, b, c, respectively. Prove that

$$\frac{h_a^2}{b^2+c^2}\cdot\frac{h_b^2}{c^2+a^2}\cdot\frac{h_c^2}{a^2+b^2} \le \left(\frac{3}{8}\right)^3 .$$

Essentially the same solution by D. Kipp Johnson, Beaverton, OR, USA; Joel Schlosberg, Bayside, NY, USA; and D.J. Smeenk, Zaltbommel, the Netherlands.

Using the formula $h_a = b\sin C$, the Law of Sines, and the AM–GM Inequality, we obtain

$$\frac{h_a^2}{b^2+c^2} = \frac{b^2\sin^2 C}{b^2+c^2} = \frac{\sin^2 B\sin^2 C}{\sin^2 B+\sin^2 C} \le \frac{\sin^2 B\sin^2 C}{2\sin B\sin C} = \frac{1}{2}\sin B\sin C\,,$$

and similarly,

$$\frac{h_b^2}{c^2+a^2} \le \frac{1}{2}\sin C\sin A \quad\text{and}\quad \frac{h_c^2}{a^2+b^2} \le \frac{1}{2}\sin A\sin B\,.$$

Multiplying these and using the well-known inequality

$$\sin A \sin B \sin C \;\le\; \frac{3\sqrt{3}}{8}$$

(see [1, p. 18]), we obtain

$$\begin{aligned}\frac{h_a^2}{b^2+c^2}\cdot\frac{h_b^2}{c^2+a^2}\cdot\frac{h_c^2}{a^2+b^2} &\le \frac{1}{8}(\sin A \sin B \sin C)^2 \\ &\le \frac{1}{8}\left(\frac{3\sqrt{3}}{8}\right)^2 \;=\; \left(\frac{3}{8}\right)^3 .\end{aligned}$$

Equality holds if and only if the triangle is equilateral.

References

[1] O. Bottema et al., *Geometric Inequalities*, Groningen, 1969.

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); MICHEL BATAILLE, Rouen, France; QUANG CAO MINH, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; KEE-WAI LAU, Hong Kong, China; D.M. MILOŠEVIĆ, Pranjani, Yugoslavia; VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; BIN ZHAO, student, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer.