Sign Patterns, Nonsingularity, and the Solvability of $Ax = b$

Charles R. Johnson*
Institute for Physical Science and Technology
and Department of Economics
University of Maryland
College Park, Maryland 20742

Frank Uhlig
Institut fur Geometrie und Praktische Mathematik
RWTH Aachen
51 Aachen, West Germany

and

Dan Warner
Mathematical Sciences Department
Clemson University
Clemson, South Carolina 29631

Submitted by George P. Barker

ABSTRACT

We investigate conditions on the sign-pattern class of the $(n-1)$st compound of a real n-by-n matrix A such that the solvability of $Ax = b^{(i)}$ for $i = 1,\dots,k$, $k < n$, with specific $b^{(i)}$, insures the nonsingularity of A. The number and choice of right-hand sides $b^{(i)}$ sufficient for the task depends only on the sign-pattern class of the $(n-1)$st compound of A. The result for $k = 1$ generalizes a known fact about totally nonnegative matrices and an observation about M-matrices, thus providing another unifying result for these two classes of matrices.

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INTRODUCTION

Let $A=(a_{ij})$ be an n-by-n real matrix. We raise the following question. What information about A permits the deduction that A is nonsingular from knowing that

$$Ax=b, \qquad b\neq 0, \tag{1}$$

can be solved for a specified set of $k<n$ right-hand sides b? This question is motivated by the intriguing, but not too well-known, fact that if A is totally nonnegative (the determinant of each square submatrix is nonnegative), then the existence of an x such that the components of $b=Ax$ alternate in sign implies that A is nonsingular [2]. We refer to this as the "totally nonnegative result," but actually somewhat more can be said. In applications, the totally nonnegative hypothesis is often assured by the structure of a given problem, so that the result takes on added significance.

Our interest is a theoretical one. We try to generalize, and in the process re-prove, the totally nonnegative result by relating sign patterns and the solvability of (1) to the nonsingularity of A. [The nonsingularity of A, of course, may, in principle, be determined in the same degree of complexity as a solution to (1).] In this spirit, our focus is on information about the minors, most specifically the signs of the $(n-1)$-by-$(n-1)$ minors of A. It is clear that, if highly specific information is known, then little more may need to be done in order to verify nonsingularity for A. For example, if it is known that any particular $(n-1)$-by-$(n-1)$ submatrix has nonzero determinant, say that achieved by deletion of row i and column j, then A is nonsingular if and only if (1) is solvable for b equal to the ith unit vector (Cramer's rule). However, such specific information may not be available, and our sign-pattern generalizations of the totally nonnegative result, which include the above observation as a special case, do not generally assume that any particular minor is nonzero. They do, however, usually assume that some $(n-1)$-by-$(n-1)$ minor is nonzero, for if all were zero, then A would be singular.

NOTATION AND CONCEPTS

For two index sets $I,J\subseteq\{1,2,\dots,n\}$, we denote by $A(I,J)$ the *submatrix* of A resulting from *deletion* of the rows indicated by I and the columns indicated by J. When $I=\{i\}$ and $J=\{j\}$, each contains exactly one index, and we abbreviate $A(\{i\},\{j\})$ to $A(i,j)$. A k-by-k *minor* of A is a scalar $\det A(I,J)$, where $I=\{i_1,\dots,i_{n-k}\}$ and $J=\{j_1,\dots,j_{n-k}\}$, and a minor is said to be *principal* if it is the determinant of a principal submatrix, i.e. if $I = J$. We let $C_k(A)$ denote the *kth compound matrix* of A: the $\binom{n}{k}$-by-$\binom{n}{k}$ matrix of k-by-k minors of A, ordered lexicographically. If F is the nonsingular diagonal matrix, $F \equiv \mathrm{diag}((-1)^i)$, whose diagonal entries alternate between -1 and $+1$, then the *adjoint matrix* of A is defined by

$$\mathrm{adj}\, A \equiv FC_{n-1}(A)^T F. \tag{2}$$

Of course, $A^{-1} = (\det A)^{-1}\, \mathrm{adj}\, A$ if A is nonsingular, and otherwise, $\mathrm{adj}\, A$ [and $C_{n-1}(A)$] is of rank 1 (if rank $A = n-1$) or rank 0 (if rank $A \leqslant n-2$). We note at this point that, although we state our results in terms of the compound $C_{n-1}(A)$, it will be clear that there are equivalent statements in terms of $\mathrm{adj}\, A$. Finally, we call a real entried vector or matrix *uniformly signed* if all entries are nonnegative or all entries are nonpositive. A particular uniformly signed vector is the vector e, each of whose entries is equal to 1. We define *the* alternating-sign vector to be $f \equiv Fe$.

We utilize two notions of a *sign-pattern matrix* and associated *sign-pattern class*. A *weak-sign-pattern* matrix P may have entries of four possible types: "+" denotes a nonnegative entry, "−" a nonpositive entry, "0" allows only the entry zero, and "*" indicates unrestricted entries. Such a sign-pattern matrix P defines a class of real matrices $\mathcal{P}$ in a natural way, except that we make the additional requirement that for A to be in $\mathcal{P}$, not all entries of A corresponding to +'s and −'s in P may be zero. Note then that $\mathcal{P}$ is empty if P has no entries equal to + or −. For example, if

$$P = \begin{bmatrix} + & 0 \\ * & - \end{bmatrix},$$

then

$$\begin{bmatrix} 1 & 0 \\ -2 & -5 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} \in \mathcal{P},$$

while

$$\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$$

are not. Our results will be based upon sign-pattern information about $C_{n-1}(A)$ and will thus apply to classes of matrices.

Our second notion of a sign pattern is that of a *strong-sign-pattern* matrix, which is, in a certain sense, dual to the weak-sign-pattern notion mentioned above. The sign-pattern matrix $Q = P^d$ has "+" entries and "−" entries wherever P does, but these two symbols now require the strong interpretation of positive or negative entries, respectively. The remaining entries of Q are determined by replacing the "*" entries of P with 0's and the "0" entries with *'s. These two symbols are interpreted as before, and, again, Q naturally determines a class of real matrices, which we denote by $\mathcal{Q}$. For example, if

$$P = \begin{bmatrix} * & + \\ 0 & - \end{bmatrix},$$

then

$$Q = P^d = \begin{bmatrix} 0 & + \\ * & - \end{bmatrix}$$

is the strong-sign-pattern matrix associated with P, and

$$\begin{bmatrix} 0 & 3 \\ -1 & -2 \end{bmatrix} \in \mathcal{Q}, \qquad \text{while} \qquad \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \notin \mathcal{Q}.$$

Notationally, if $Q = P^d$, we also write $\mathcal{Q} = \mathcal{P}^d$. By construction, the strong sign-pattern class $\mathcal{Q} = \mathcal{P}^d$ is such that the trace of each AB^T, $A \in \mathcal{P}$, $B \in \mathcal{Q}$, is positive, and this is the sense of duality mentioned above. The requirements placed upon $\mathcal{Q}$ are minimally necessary to insure this.

By the *rank* of a strong-sign-pattern matrix Q, we mean the minimum of the ranks assumed by all matrices in the class $\mathcal{Q}$, and we shall say that the rank of a weak sign-pattern matrix P is k if $k = \operatorname{rank} P^d$. There will always be matrices $B \in \mathcal{Q}$ such that $\operatorname{rank} B = \operatorname{rank} Q$, but the rank of any matrix in $\mathcal{Q}$ serves as an upper bound for rank Q. In particular, the special matrix $C(Q)$, in which +'s in Q are replaced by 1's, −'s by −1's, and 0's or *'s by 0's, provides an upper bound, which makes it clear that

$$\begin{bmatrix} + & \cdots & + \\ \vdots & & \vdots \\ + & \cdots & + \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} + & - & + & - & \cdots \\ - & & & & \\ + & & & & \\ \vdots & & & & \end{bmatrix}$$

are rank-1 sign patterns. It is an interesting question how to determine the rank of a sign pattern. Note that multiplication of any strong sign pattern by a diagonal strong-sign-pattern matrix (a signature matrix) is well defined, and such a transformation (no zero diagonal entries) on the right or left, and left or right multiplication by a permutation matrix, as well as transposition, do not change the rank of a strong sign pattern. Another interesting question is to determine the equivalence classes of strong-sign-pattern matrices under these transformations collectively. The rank of each sign pattern in such an equivalence class is the same.

PRIMARY RESULT

An observation which underlies our results is the following very simple fact.

LEMMA. *If, for an n-by-n matrix A, there exists an x such that* (1) *is satisfied for a given b, then*

$$x_j \det A = \sum_{i=1}^{n} (-1)^{i+j} b_i \det A(i,j) \tag{3}$$

for each $j = 1, \ldots, n$.

Proof. This is essentially Cramer's rule before dividing by $\det A$. In general, $(\operatorname{adj} A)A = (\det A)I$, so that $(\operatorname{adj} A)b = (\operatorname{adj} A)Ax = (\det A)x$, of which (3) is a component-by-component version based upon (2). ∎

Another vectorial version of (3) is

$$(\det A)(Fx)^T = b^T F C_{n-1}(A). \tag{4}$$

The utility of (3) occurs when the right-hand side can be shown to be nonzero for some j; for then the existence of a solution x of (3) implies $\det A \neq 0$. Alternatively stated, if (1) is solvable and A is singular, then b must be orthogonal to every column of $FC_{n-1}(A)$. [Conversely, if $b^T FC_{n-1}(A) = 0$, then A is singular.]

Our principal observation is the

THEOREM. *Let $\mathcal{P}$ be a rank $k \geqslant 1$ weak-sign-pattern class. Then there exist k vectors*

$$b^{(1)}, b^{(2)}, \ldots, b^{(k)},$$

which depend only upon the class $\mathcal{P}$, *such that if* $C_{n-1}(A)\in\mathcal{P}$, *then* A *is nonsingular if and only if* (1) *is solvable for each* $b=b^{(i)}$, $i=1,\dots,k$.

Proof. Choose $B\in\mathcal{Q}=\mathcal{P}^d$, the strong-sign-pattern class, such that rank $B=k$, and let $b^{(1)},\dots,b^{(k)}$ be k linearly independent columns of FBF. We show that these $b^{(i)}$ satisfy the requirements of the theorem. Suppose, as we may by the definition of $\mathcal{P}$, that column j of $C_{n-1}(A)$ contains a nonzero entry in a position corresponding to a $+$ or $-$ in the matrix P which defines $\mathcal{P}$, and consider column j of FBF, which we call $\hat{b}$. If (1) is solvable for each right-hand side $b^{(i)}$, $i=1,\dots,k$, then (1) is solvable for $b=\hat{b}$, since it is in the span of the $b^{(i)}$. By virtue of the construction of $\mathcal{Q}$, if the ith entry of column j of $C_{n-1}(A)$ is positive, the corresponding entry of $\hat{b}$ has the sign of $(-1)^{i+j}$ or is 0, and, if the ith entry of column j of $C_{n-1}(A)$ is negative, the corresponding entry of $\hat{b}$ has the sign of $(-1)^{i+j+1}$ or is 0. Furthermore, there is at least one nonzero entry in column j of $C_{n-1}(A)$ which corresponds to a $+$ or $-$ in P, and any such nonzero entry must correspond to a nonzero entry in $\hat{b}$; therefore, since $\mathcal{Q}$ is a strong-sign-pattern class, the right-hand side of (3) is positive for this j and $b=\hat{b}$, and it follows that $\det A\neq 0$, since (1) is solvable for $b=\hat{b}$. Thus the solvability of (1) for each $b=b^{(i)}$, $i=1,\dots,k$, implies that A is nonsingular. If A is nonsingular, then (1) is, of course, solvable for any right-hand side $b^{(i)}$ and the proof is complete. ■

IMPLICATIONS

The case $k=1$ of the theorem is of note because the fewest right-hand sides are required. Moreover, rank-1 patterns are easily recognized.

REMARK. Let J be the strong sign pattern

$$J\equiv\begin{bmatrix} + & \cdots & + \\ \vdots & & \vdots \\ + & \cdots & + \end{bmatrix}.$$

It is clear that rank $J=1$. Let D_1 and D_2 be two diagonal sign-pattern matrices with the restriction that each has at least a $+$ or $-$ on the diagonal. It is furthermore clear that the sign-pattern product

$$Q=D_1JD_2 \tag{5}$$

has rank 1, and it is easy to check that any $+$, $-$, 0 rank-1 class is determined by a sign-pattern matrix of the form (5). Therefore, a strong-sign-pattern matrix Q has rank 1 if and only if the particular matrix $C(Q)$ is of rank 1. Thus, it is particularly easy to determine if a pattern Q has rank 1. The rank of $C(Q)$ may be calculated directly, or, since 0-entries of rank-1 patterns can occur only in 0-rows or 0-columns, first any 0-rows or columns must be sorted out of Q, and then, when the remaining rows and columns are sign scaled so that their initial entries are $+$, Q is of rank 1 if and only if the result is uniformly signed $(+)$.

EXAMPLE. Although $C(Q)$ suffices to identify rank-1 sign patterns Q, it does not in general determine the rank of a sign pattern. For example, if

$$Q=\begin{bmatrix} - & - & + \\ + & + & + \\ - & + & - \end{bmatrix},$$

then rank $C(Q)=3$, but

$$\operatorname{rank} Q=\operatorname{rank}\begin{bmatrix} -1 & -4 & 2 \\ 4 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}=2.$$

In fact, the rank of any 3-by-3 $+,-$ sign pattern is at most 2.

We may record the case $k=1$ of the theorem as

COROLLARY 1. *Let $\mathcal{P}$ be a rank-1 weak-sign-pattern class, and let b be the first nonzero column of $FC(P^d)$. If $C_{n-1}(A)\in\mathcal{P}$, then A is nonsingular if and only if* (1) *is solvable for b.*

Two special cases of Corollary 1 are the totally nonnegative matrices of rank at least $n-1$ ($C_{n-1}(A)\in\mathcal{P}$ for $P=J$) and the (possibly singular) M-matrices of rank at least $n-1$ ($C_{n-1}(A)\in\mathcal{P}$ for $P=FJF$). Thus, Corollary 1 may be viewed as a unifying result for totally nonnegative matrices and M-matrices.

COROLLARY 2. *Suppose that A is an n-by-n real matrix such that $C_{n-1}(A)$ is uniformly signed and $C_{n-1}(A)\neq 0$. If* (1) *is solvable for $b=f$, then A is nonsingular.*

COROLLARY 3. *Suppose that A is an n-by-n real matrix such that* $FC_{n-1}(A)F$ *is uniformly signed and* $C_{n-1}(A) \neq 0$. *If* (1) *is solvable for* $b = e$, *then A is nonsingular.*

Actually, when Corollaries 2 and 3 are specialized further to the cases of totally nonnegative and (possibly singular) M-matrices respectively, the assumption $C_{n-1}(A) \neq 0$ (rank at least $n-1$) may be omitted. This is because the defining property of each class is inherited under the extraction of principal submatrices and because the vectors e and f retain their form when components are deleted serially, which permits an argument on a principal submatrix in case the rank is smaller than $n-1$. Recall that an M-matrix (possibly singular) is one in the weak-sign-pattern class determined by

$$Z = \begin{bmatrix} + & & & \\ & + & & - \\ & & \ddots & \\ & - & & + \end{bmatrix},$$

all of whose eigenvalues have nonnegative real parts. An M-matrix is called a nonsingular M-matrix if the minimum of these real parts is positive.

COROLLARY 4. *If a real n-by-n matrix A is totally nonnegative, then A is nonsingular if and only if* (1) *is solvable for* $b = f$.

Proof. We proceed by induction. The assertion is clear for $n = 1$. If $Ax = f$ and A is singular, then there is a $y \neq 0$ such that $A(x + \alpha y) = f$ for all real α. Choose α so that $x_i + \alpha y_i = 0$ for some i, and define $\tilde{x}$ to be the $(n-1)$-vector equal to $x + \alpha y$ with the ith component deleted. It follows that $A(n,i)\tilde{x} = f \in R^{n-1}$. By the induction hypothesis, $A(n,i)$ is nonsingular, $\det A(n,i) \neq 0$, and Corollary 2 applies to A. This means that A was actually nonsingular. ∎

COROLLARY 5. *An n-by-n M-matrix A is nonsingular if and only if* (1) *is solvable for* $b = e$.

Proof. This is similar to Corollary 4 except that $A(i,i)$ is used, along with the fact that any component deleted from e leaves a vector of the same form. ∎

REMARK. It is clear that all the results mentioned here actually depend only on the sign patterns of the specified right-hand-side vectors (and not on the magnitudes of their components).

REFERENCES

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