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# Sets of convergent power series given by differential conditions

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**Abstract**

Given an ideal $I \subset \mathbb{R}\{x_1,\dots,x_m,y\}$ and $f \in \mathbb{R}\{x_1,\dots,x_m,y\}$, we study the subset $\{P\cdot f \mid \partial_y(P) \in I\}$. This idea comes from the classification of singularities of illuminated surfaces. We prove a (Hironaka-type) division theorem for subsets of this form and give a Buchberger-like reduction algorithm. © 1997 Elsevier Science B.V.

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## 1. Introduction

The classification of singularities of illuminated surfaces (see [3–5, 7]) needs the search of real analytic functions $g(x,y)$ such that the set of series which can be written like

$$\frac{\partial g}{\partial x}\cdot P(x,y) + \frac{\partial g}{\partial y}\cdot \phi(y,g) \tag{1}$$

with the condition

$$\frac{\partial P}{\partial y} \in \left(\frac{\partial g}{\partial y}\right)\mathbb{R}\{x,y\}$$

is of finite codimension in the ring $\mathbb{R}\{x,y\}$ of convergent power series ($\phi(y,g)$ is any convergent power series in two variables $y$ and $z$ computed in $z=g(x,y)$).

In the present paper we study subsets of the ring $\mathbb{R}\{x_1,\dots,x_m,y\}$ of the form

$$\mathscr{K} = \{f\cdot P \mid P \in \mathbb{R}\{x_1,\dots,x_m,y\}, \partial_y(P) \in I\},$$

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where $I$ is any ideal of $\mathbb{R}\{x_1,\ldots,x_m,y\}$ and $f$ is any convergent power series in the same ring. This is a partial generalization of the original problem: with $m=1$, $f=\partial g/\partial x$ and $I=(\partial g/\partial y)$ we get the first part of Eq. (1). In [3] we used the division by the set $\mathscr{K}$ as a first step to study the whole of Eq. (1). Notice that $I$ do not need to be of finite codimension (in the case $I=(\partial g/\partial y)$ it is not), so we will have infinite-dimensional vector spaces.

Note that the membership problem is easily solved via Gröbner basis and the tangent cone algorithm [2, 8, 9]. Our purpose is to prove a division theorem in the sense of Weierstrass for ideals of analytic functions. Notice that $\mathscr{K}$ is not an ideal in our case; if we call

$$\mathscr{B}=\{P \mid \partial_y(P)\in I\},$$

then $\mathscr{B}$ is a subring of $\mathbb{R}\{x_1,\ldots,x_m,y\}$ so $\mathscr{K}$ is a $\mathscr{B}$-module generated by $f$. More interesting, $\mathscr{K}$ is a $\mathbb{R}\{x_1,\ldots,x_m\}$-module: although it is not in general finitely generated, this structure will be useful for the division theorem.

We will show the existence of an infinite-dimensional vector space $\mathscr{H}$ subset of $\mathbb{R}\{x_1,\ldots,x_m,y\}$ such that

$$\mathbb{R}\{x_1,\ldots,x_m,y\}/\mathscr{K}\sim\mathscr{H}.$$

This will be the subject of Section 2, whereas in Section 3 we show that the projection $\mathbb{R}\{x_1,\ldots,x_m,y\}\to\mathscr{H}$ can be effectively computed via a kind of tangent cone algorithm. In Section 4, as a conclusion, we review some possible generalizations of these ideas to broader contexts.

**Remark.** (1) The base field $\mathbb{R}$ of the ring may be replaced by $\mathbb{C}$ or any other complete-valued field without any modification of the results.

(2) The key point of the following theory is *convergence*: the same results in the ring of *formal* power series are much more easy to obtain, whereas a special care is needed to deal with convergent power series because of the derivative $\partial_y$ in the definition of $\mathscr{K}$.

## 2. Division theorem

From now on, we will use the following notations: call $\mathbb{R}\{X\}:=\mathbb{R}\{x_1,\ldots,x_m\}$ and $\mathbb{R}\{Z\}:=\mathbb{R}\{x_1,\ldots,x_m,y\}=\mathbb{R}\{X,y\}$. So let $I$ be an ideal of the ring $\mathbb{R}\{Z\}$, $f\in\mathbb{R}\{Z\}$ and

$$\mathscr{K}=\{f\cdot P \mid \partial_y(P)\in I\}.$$

Let $<$ be any monomial ordering on $\mathbb{R}\{Z\}$ such that for all $i=1,\ldots,m$, $x_i\ll y$ (this condition is not strictly necessary and can be replaced by a weaker one, see the proof of Lemma 3), and let $(g_1,\ldots,g_k)$ be a Gröbner basis of $I$ with respect to this ordering.

We will moreover suppose that $f$ and all $g_i$, $i=1,\dots,k$ have a leading coefficient equal to 1. Call $\mathrm{in}(g)$ the *initial monomial* of $g$ with respect to the term ordering $<$.

For $i=1,\dots,k$, call

$$Z^{E_i} = \mathrm{in}(g_i),$$

$$D_i = E_i + \mathbb{N}^{m+1} \Big\backslash \left( \bigcup_{j=1}^{i-1} D_j \right),$$

and

$$D = \bigcup_{i=1}^{k} D_i.$$

Then it is well known (see, for instance, [2]) that the set of initial exponent of the elements of $I$, $\mathrm{in}(I)$, is equal to $D$.

We prove a similar result for $\mathscr{K}$.

**Proposition 1.** *Let* $Z^F = \mathrm{in}(f)$; *then*

$$\mathrm{in}(\mathscr{K}) = (F + (0,\dots,0,1) + D) \cup (F + \mathbb{N}^m + \{0\}).$$

**Proof.** Call $\Delta := F + (0,\dots,0,1) + D$, $\Delta_0 := F + \mathbb{N}^m + \{0\}$ and $\Delta^K$ their union.

Let $h \in \mathscr{K}$ and $Z^A = \mathrm{in}(h)$; since $f$ divides $h$, we can write $A = F + B$ with $B \in \mathbb{N}^{m+1} = \mathrm{in}(h/f)$. Now either $\partial_y(h/f)$ is zero or belongs to $I$. In the first case $h/f \in \mathbb{R}\{X\}$ thus $B \in \mathbb{N}^m + \{0\}$, so $A \in \Delta_0$. In the second case $\mathrm{in}(\partial_y(h/f)) = \mathrm{in}(h/f) - (0,\dots,0,1) \in D$, thus $A \in \Delta$.

Conversely, let $A \in \Delta^K$. If $A \in \Delta$, then $B = A - F - (0,\dots,0,1) \in D$; let $i$ such that $B \in D_i$ and write

$$g = f \int g_i \cdot Z^{B-E_i}\, \mathrm{d}y.$$

Then $g \in \mathscr{K}$ and $\mathrm{in}(g) = Z^A$. If $A \in \Delta_0$, then

$$g = f \cdot X^{A-F}$$

has the good properties. □

We can now state the main division theorem:

**Theorem 2.** *Given any* $g \in \mathbb{R}\{Z\}$, *there exists a unique pair* $(q,r)$ *of series of* $\mathbb{R}\{Z\}$ *such that*

$$g = f \cdot q + r, \qquad \partial_y q \in I, \qquad \mathrm{Supp}(r) \subset \mathbb{N}^{m+1} \backslash \Delta^K.$$

The technique of the proof is inspired by a paper of Briançon ("Weierstrass préparé à la Hironaka" [1], see also [6]) from a course of Houzel: we use a perturbation of linear isomorphisms of Banach spaces.

**Proof.** Call $\mathscr{H}=\{r\in\mathbb{R}\{Z\}\mid \operatorname{Supp}(r)\subset\mathbb{N}^{m+1}\backslash\Delta^K\}$, and, for $i=1,\dots,k$, $\Delta_i=D_i+F+(0,\dots,0,1)$ and $E_i'=E_i+F+(0,\dots,0,1)$. Remark that the subsets $\Delta_0,\Delta_1,\dots,\Delta_k$ and $\mathbb{N}^{m+1}\backslash\Delta^K$ are a partition of $\mathbb{N}^{m+1}$.

Consider the linear map $\varphi_1$:

$$\varphi_1:\mathbb{R}\{X\}\times(\mathbb{R}\{X,y\}^m)\times\mathscr{H}\longrightarrow\mathbb{R}\{X,y\},$$

$$(h_0,h_1,\dots,h_k;r)\longmapsto\sum_{i=1}^{k}\left(Z^F\int h_iZ^{E_i}\,\mathrm{d}y\right)+Z^F\cdot h_0+r.$$

Then $\varphi_1$ is injective because the initial exponents of the terms on the left belong to distinct sectors of $\mathbb{N}^{m+1}$. This map is also surjective because for any series $u\in\mathbb{R}\{Z\}$, we can write

$$u=\sum_{A\in\Delta_0}u_AZ^A+\sum_{i=1}^{k}\left(\sum_{A\in\Delta_i}u_AZ^A\right)+\sum_{A\in\mathbb{N}^{m+1}\backslash\Delta^K}u_AZ^A.$$

Define

$$r=\sum_{A\in\mathbb{N}^{m+1}\backslash\Delta^K}u_AZ^A, \tag{2}$$

$$h_0=\sum_{A\in\Delta_0}u_AZ^{A-F}; \tag{3}$$

then $r\in\mathscr{H}$ and $h_0\in\mathbb{R}\{X\}$. On the other hand, for $i=1,\dots,k$,

$$\begin{aligned}\sum_{A\in\Delta_i}u_AZ^A&=Z^F\sum_{A\in\Delta_i}u_AZ^{A-F}\\&=Z^F\int\sum_{A\in\Delta_i}u_A(a_{m+1}-e^i_{m+1})\frac{Z^{A-F}}{y}\,\mathrm{d}y\\&=Z^F\int Z^{E_i}\sum_{A\in\Delta_i}u_A(a_{m+1}-e^i_{m+1})Z^{A-E_i'}\,\mathrm{d}y,\end{aligned}$$

where $a_{m+1}$ and $e^i_{m+1}$ are the $(m+1)$th element of $A$ and $E_i$, respectively (those corresponding to the $y$ variable). Call

$$h_i=\sum_{A\in\Delta_i}u_A(a_{m+1}-e^i_{m+1})Z^{A-E_i'}, \tag{4}$$

then $h_i\in\mathbb{R}\{X,y\}$ and $\varphi_1(h_0,h_1,\dots,h_k;r)=u$.

We have thus proved Theorem 2 when $f$ and the $g_i$ are monomials. □

To prove Theorem 2 in the general case we have to make a deformation of $\varphi_1$. We will use Banach spaces techniques to show that such a deformation exists.

Let $\rho=(\rho_1,\dots,\rho_{m+1})\in(\mathbb{R}_+^*)^{m+1}$ be a poly-radius and $\mathbb{R}\{Z\}_\rho$ be the Banach algebra of power series converging for radius $\rho$ with the following norm:

$$\left\| \sum_{A\in\mathbb{N}^{m+1}} u_A Z^A \right\|_\rho = \sum_{A\in\mathbb{N}^{m+1}} |u_A|\rho^A.$$

The restriction of $\varphi_1$ to $\varphi_1^{-1}(\mathbb{R}\{Z\}_\rho)$ is obviously a bijection onto $\mathbb{R}\{Z\}_\rho$. The subspace $\varphi_1^{-1}(\mathbb{R}\{Z\}_\rho)$ is made of $(h_0,h_1,\dots,h_k;r)$ such that

(i) $h_0\in\mathbb{R}\{X\}$ and $X^F\cdot h_0\in\mathbb{R}\{X,y\}_\rho$, which means $h_0\in\mathbb{R}\{X\}_\rho$;

(ii) for $i=1,\dots,k$, $h_i\in\mathbb{R}\{Z\}$ and $Z^F\int h_iZ^{E_i}\,\mathrm{d}y\in\mathbb{R}\{Z\}_\rho$, i.e.,

$$\left\| Z^F\int h_iZ^{E_i}\,\mathrm{d}y\right\|_\rho<\infty,$$

call

$$\mathbb{R}\{Z\}_\rho^* := \left\{ u\in\mathbb{R}\{Z\} \mid \left\| Z^F\int uZ^{E_i}\,\mathrm{d}y\right\|_\rho<\infty,\ i=1,\dots,k\right\},$$

we have easily that $\mathbb{R}\{Z\}_\rho\subset\mathbb{R}\{Z\}_\rho^*$ and if $\rho'$ is a multi-index such that $\rho'_{m+1}<\rho_{m+1}$ and $\rho'_j\le\rho_j$ for $j=1,\dots,m$, we have $\mathbb{R}\{Z\}_\rho^*\subset\mathbb{R}\{Z\}_{\rho'}$;

(iii) $r\in\mathscr{H}$ and $r\in\mathbb{R}\{Z\}_\rho$ so that $r\in\mathscr{H}\cap\mathbb{R}\{Z\}_\rho=:\mathscr{H}_\rho$.

So if $H^*(\rho)=\mathbb{R}\{X\}_\rho\times(\mathbb{R}\{X,y\}_\rho^*)^m\times\mathscr{H}_\rho$ then $\varphi_1$ is a linear bijection from $H^*(\rho)$ onto $\mathbb{R}\{X,y\}_\rho$. Moreover, if we give $H^*(\rho)$ the norm induced by $\varphi_1$, that is

$$\|(h_0,h_1,\dots,h_k;r)\|_{H_\rho^*} := \rho^F\|h_0\|_\rho+\sum_{i=1}^k\left\|Z^F\int h_iZ^{E_i}\,\mathrm{d}y\right\|_\rho+\|r\|_\rho,$$

then $H^*(\rho)$ becomes a Banach space and $\|\varphi_1\|=\|\varphi_1^{-1}\|=1$.

To demonstrate Theorem 2 we have to prove that the map

$$\varphi:\mathbb{R}\{X\}\times(\mathbb{R}\{X,y\})^m\times\mathscr{H}\longrightarrow\mathbb{R}\{X,y\},$$

$$(h_0,h_1,\dots,h_k;r)\longmapsto\sum_{i=k}^k\left(f\int h_i\cdot g_i\,\mathrm{d}y\right)+f\cdot h_0+r$$

is also a bijection. Note that for the same reasons as for $\varphi_1$, this map is injective.

If we write $\varphi=\varphi_1+\varphi_2$, $f=X^F+u$ and, for $i=1,\dots,k$, $g_i=X^{E_i}+v_i$, we have

$$\varphi_2:\mathbb{R}\{X\}\times(\mathbb{R}\{X,y\})^m\times\mathscr{H}\longrightarrow\mathbb{R}\{X,y\},$$

$$(h_0,h_1,\dots,h_k;r)\longmapsto\sum_{i=1}^k\left(u\int h_iZ^{E_i}\,\mathrm{d}y\right)+\sum_{i=1}^k\left(Z^F\int h_i\cdot v_i\,\mathrm{d}y\right)$$

$$+\sum_{i=1}^k\left(u\int h_i\cdot v_i\,\mathrm{d}y\right)+u\cdot h_0$$

Suppose $u, v_i \in \mathbb{R}\{Z\}_\rho$, then $\varphi_2 \in (H^*(\rho)) \subset \mathbb{R}\{Z\}_\rho$ because we have

$$\|\varphi_2\|_\rho \leq \frac{\|\varphi_2(h_0, h_1, \ldots, h_k; r)\|_\rho}{\|(h_0, h_1, \ldots, h_k; r)\|_{H_\rho^*}}$$

$$\leq \frac{\sum_{i=1}^k \|u \int h_i Z^{E_i}\,\mathrm{d}y\|_\rho + \sum \|Z^F \int h_i v_i\,\mathrm{d}y\|_\rho + \sum \|u \int h_i v_i\,\mathrm{d}y\|_\rho + \|u \cdot h_0\|_\rho}{\sum_{i=1}^k \|Z^F \int h_i Z^{E_i}\,\mathrm{d}y\|_\rho + \rho^F \cdot \|h_0\|_\rho + \|r\|_\rho}$$

and

**Lemma 3.** *For $i = 1, \ldots, k$, the following inequalities hold*:

$$\frac{\|u \int h_i Z^{E_i}\,\mathrm{d}y\|_\rho}{\|Z^F \int h_i Z^{E_i}\,\mathrm{d}y\|_\rho} \leq \rho^{-F} \|u\|_\rho, \tag{5}$$

$$\frac{\|Z^F \int h_i v_i\,\mathrm{d}y\|_\rho}{\|Z^F \int h_i Z^{E_i}\,\mathrm{d}y\|_\rho} \leq \rho^{-E_i} \|v_i\|_\rho, \tag{6}$$

$$\frac{\|u \int h_i v_i\,\mathrm{d}y\|_\rho}{\|Z^F \int h_i Z^{E_i}\,\mathrm{d}y\|_\rho} \leq \rho^{-E_i - F} \|u\|_\rho \cdot \|v_i\|_\rho, \tag{7}$$

so that

$$\|\varphi_2\| \leq \frac{\|u\|_\rho}{\rho^F} + \sum_{i=1}^k \frac{\rho^{E_i}\|u\|_\rho + \rho^F \|v_i\|_\rho + \|u\|_\rho \cdot \|v_i\|_\rho}{\rho^{E_i + F}} \tag{8}$$

$$\leq \rho^{-F} \|u\|_\rho \left( k + 1 + \sum_{i=1}^k \rho^{-E_i} \|v_i\|_\rho \right) + \sum_{i=1}^k \rho^{-E_i} \|v_i\|_\rho. \tag{9}$$

**Proof of Lemma 3.** Inequality (5) is straightforward since

$$\left\| u \int h_i Z^{E_i}\,\mathrm{d}y \right\|_\rho \leq \|u\|_\rho \cdot \left\| \int h_i Z^{E_i}\,\mathrm{d}y \right\|_\rho$$

and

$$\left\| Z^F \int h_i Z^{E_i}\,\mathrm{d}y \right\|_\rho = \rho^F \cdot \left\| \int h_i Z^{E_i}\,\mathrm{d}y \right\|_\rho .$$

To prove (6), let us write

$$v_i = \sum_{A > F} v_A^i Z^A, \qquad h_i = \sum_{B \in \mathbb{N}^{m+1}} h_B^i Z^B,$$

then

$$\left\| \int h_i v_i \, \mathrm{d}y \right\|_\rho = \left\| \int h_i \cdot \sum_{A>F} v_A^i Z^A \, \mathrm{d}y \right\|_\rho = \left\| \int \sum_{A>F} h_i \cdot v_A^i Z^A \, \mathrm{d}y \right\|_\rho$$

$$= \left\| \sum_{A>F} v_A^i \int h_i Z^A \, \mathrm{d}y \right\|_\rho \leq \sum_{A>F} |v_A^i| \left\| \int h_i Z^A \, \mathrm{d}y \right\|_\rho$$

We have

$$\left\| \int h_i Z^A \, \mathrm{d}y \right\|_\rho = \rho^A \sum_B \frac{|h_B^i|}{a_{m+1} + b_{m+1}^i + 1} \rho^{B+(0,\dots,0,1)}$$

and, since $h_i \in \mathbb{R}\{X, y\}_\rho^*$, we get

$$\left\| \int h_i Z^{E_i} \, \mathrm{d}y \right\|_\rho = \rho^{E_i} \sum_B \frac{|h_B^i|}{h_{m+1}^i + f_{m+1} + 1} \rho^{B+(0,\dots,0,1)} < \infty.$$

The monomial order we choose on $\mathbb{N}^{m+1}$ is such that $A > F \Rightarrow a_{m+1} \geq f_{m+1}$ so

$$\left\| \int h_i Z^A \, \mathrm{d}y \right\|_\rho \leq \left\| \int h_i Z^F \mathrm{d}y \right\|_\rho < \infty;$$

thus

$$\frac{\|Z^F \int h_i v_i \, \mathrm{d}y\|_\rho}{\|Z^F \int h_i Z^{E_i} \, \mathrm{d}y\|_\rho} = \frac{\| \int h_i v_i \, \mathrm{d}y\|_\rho}{\| \int h_i Z^{E_i} \, \mathrm{d}y\|_\rho} \leq \frac{\sum_A |v_A^i| \rho^A}{\rho^{E_i}},$$

and inequality (6) is proved.

Combining the results of the first two inequalities, one easily gets the proof of (7). □

Now suppose $u, v_i$ are in $\mathbb{R}\{Z\}_\nu$, with $\nu = (\nu, \dots, \nu) \in \mathbb{N}^{m+1}$. Let $L = (l_1, l_2, \dots, l_{m+1})$ be a positive linear form such that $\forall A > F$, $L(A) > L(F)$ and for $i = 1, \dots, k$, $\forall A > E_i$, $L(A) > L(E_i)$, there exist positive real numbers $\alpha_0, \alpha_1, \dots, \alpha_k$ such that $A > F \Rightarrow L(A - F) > \alpha_0$ and for $i = 1, \dots, k$, $A > E_i \Rightarrow L(A - E_i) > \alpha_i$.

For $0 < \eta < 1$, let us take

$$\rho = (\nu\eta^{l_1}, \nu\eta^{l_2}, \dots, \nu\eta^{l_{m+1}}). \tag{10}$$

Then $u, v_i \in \mathbb{R}\{Z\}_\rho$ and, for $i = 1, \dots, k$,

$$\rho^{-E_i} \|v_i\|_\rho = \sum_{L(A-E_i)>0} |v_A^i| \rho^{A-E_i} = \sum_{L(A-E_i)>0} |v_A^i| \nu^{A-E_i} \eta^{L(A-E_i)}$$

$$= \sum_{L(A-E_i)>0} |v_A^i| \nu^A \nu^{-E_i} \eta^{L(A-E_i)} \leq \sum_{L(A-E_i)>0} |v_A^i| \nu^A \nu^{-E_i} \eta^{\alpha_i} \leq \nu^{-E_i} \eta^{\alpha_i} \|v_i\|_\nu$$

and in the same way

$$\rho^{-F}\|u\|_\rho \le v^{-F}\eta^{\alpha_0}\|u\|_v.$$

If we choose $\eta > 0$ such that, for $i = 1,\dots,k$, the following inequalities hold:

$$\eta^{\alpha_0} < \frac{1}{k+1}\frac{v^F}{\|u\|_v}, \tag{11}$$

$$\eta^{\alpha_i} < \frac{1}{2k}\frac{v^{E_i}}{\|v_i\|_v}, \tag{12}$$

then from Eq. (9) with $\rho$ given by (10), we get

$$\|\varphi_2\| < 1.$$

The map $\varphi_1$ is a bijection between $H^*(\rho)$ and $\mathbb{R}\{Z\}_\rho$ ($\rho$ given by (10) depending on $\eta$ such that (11) and (12) hold) so

$$(\varphi_1 + \varphi_2)\circ\varphi_1^{-1} = I + \varphi_2\circ\varphi_1^{-1}.$$

Since $\|\varphi_2\circ\varphi_1^{-1}\| \le \|\varphi_2\| < 1$, the map $I + \varphi_2\circ\varphi_1^{-1}$ is invertible; then

$$(\varphi_1 + \varphi_2)\circ\varphi_1^{-1}\circ(I + \varphi_2\circ\varphi_1^{-1})^{-1} = I,$$

so that that $\varphi = \varphi_1 + \varphi_2$ is a bijection between $H^*(\rho) = (\mathbb{R}\{X,y\}^*_\rho)^m \times \mathbb{R}\{X\}_\rho \times \mathscr{H}_\rho$ and $\mathbb{R}\{X,y\}_\rho$.

More precisely, when $f$ and $g_1,\dots,g_k$ belong to $\mathbb{R}\{X,y\}_v$, if we take $\eta > 0$ such that Eqs. (11) and (12) hold then for

$$\rho = (v\eta^{l_1}, v\eta^{l_2},\dots, v\eta^{l_{m+1}}),$$

any $\varphi \in \mathbb{R}\{X,y\}_\rho$ can be written in a unique way as

$$\varphi = f\cdot h_0 + f\sum_{i=1}^{k}\left(\int g_i\cdot h_i\,\mathrm{d}y\right) + r,$$

where $h_0 \in \mathbb{R}\{X\}_\rho$, $h_i \in \mathbb{R}\{X,y\}^*_\rho$ for $i = 1,\dots,k$ and $r \in \mathscr{H}_\rho$. Notice that $h_i \in \mathbb{R}\{X,y\}^*_\rho$, implies that for any $\rho' < \rho$, $h_i \in \mathbb{R}\{X,y\}_{\rho'}$.

For any $\bar v < v$ we have $\|u\|_{\bar v} \le \|u\|_v$ and $\|v\|_{\bar v} \le \|v\|_v$; then we can choose $\bar\eta < \eta$ and thus the above decomposition holds for the corresponding $\bar\rho$ and $\mathbb{R}\{X,y\}_\rho \subset \mathbb{R}\{X,y\}_{\bar\rho}$. At the limit we have the decomposition in the ring $\mathbb{R}\{Z\}$.

## 3. Division algorithm

From Theorem 2 we obtain the isomorphism

$$\mathbb{R}\{Z\}/\mathscr{K} \sim \mathscr{H}.$$

Call $r$ the projection map from $\mathbb{R}\{Z\}$ to $\mathscr{H}$; in the present section we give an algorithm to compute $r(h)$ for any $h \in \mathbb{R}\{Z\}$. More precisely, we suppose that the ideal $I$ is given by a polynomial Gröbner basis $g_i$ and that $f \in \mathbb{R}[x_1, \dots, x_m, y]$. In this hypothesis, $r(h)$ is not a polynomial even if $h$ is: for this reason our algorithm only gives the initial term of the remainder $r(h)$.

For $A \in \Delta_0$, call

$$\phi_A = f \cdot X^B \quad \text{where } A = F + B \times \{0\},$$

and for $i = 1, \dots, k$, for $A \in \Delta_k$

$$\phi_A = f \int Z^B g_i \, \mathrm{d}y \quad \text{where } A = F + E_i + (0, \dots, 0, 1) + B.$$

Then, for each $A \in \Delta^K$, we have a polynomial $\phi_A$ such that

$$\phi_A \in \mathscr{K}, \qquad \mathrm{in}(\phi_A) = Z^A.$$

Thus, a straightforward method to reduce a polynomial $h$ consists in repeatedly subtracting real multiples of $\phi_A$ as long as the leading term of $h$ is in $\Delta^K$. This method is not convergent because even if the leading term increases strictly (with respect to the term ordering $<$) there is no reason for the leading term of $h$ to leave $\Delta^K$ after a finite number of iterations. The situation is similar to classical Buchberger reduction algorithm [2] in the local case for which tangent cone algorithm [8, 9] gives the solution.

We propose the following algorithm, adapting the tangent cone algorithm in our situation, and using the fact that $\mathscr{K}$ is a $\mathbb{R}\{X\}$-module. Consider $g$ in $\mathbb{R}\{Z\}$ as an element of $\mathbb{R}\{X\}\{y\}$, i.e., as a power series in $y$ with coefficients in $\mathbb{R}\{X\}$; since our term ordering is such that $x_i \ll y$ for $i = 1, \dots, m$ we have that the leading term of $g$ is the leading term (as an element of $\mathbb{R}\{X\}$ with induced ordering) of the coefficients of the lowest degree (in $y$) term of $g$.

With the following notations: let $\mathtt{in}_y(g)$ be the lowest $y$-degree term of $g$ as an element of $\mathbb{R}\{X\}\{y\}$, $\mathtt{deg}_y(g)$ be its degree and $\mathtt{coeff}_y(g)$ be its coefficient (an element of $\mathbb{R}\{X\}$); thus: $\mathtt{in}_y(g) = \mathtt{coeff}_y(g) \cdot y^{\mathtt{deg}_y(g)}$; call $\mathtt{in}_X(g)$ the leading term of $\mathtt{coeff}_y(g)$, with the induced ordering, such that $\mathrm{in}(g) = \mathtt{in}_X(g) \cdot y^{\mathtt{deg}_y(g)}$; define $\mathtt{ecart}_X(g)$ as the *écart* of $\mathtt{coeff}_y(g) \in \mathbb{R}\{X\}$, i.e., the difference between its greatest and lowest exponent:

The algorithm is the following:

`Input:` $h \in \mathbb{R}\{Z\}$

`Output:` $g$ `such that` $g - h \in \mathscr{K}$ `and` $\mathrm{in}(g) = \mathrm{in}(r(h))$.

$h_0 := h$; $i := 0$; `LIST := ` $\emptyset$;

`while` $\mathrm{in}(h_i) \in \Delta^K$ `do`

  $A := \mathrm{in}(h_i)$;

```
04          put phi_A in LIST;
05          find q in LIST such that:
06               deg_y(q) = deg_y(Z^A)
07               deg_X(q) <= deg_X(Z^A)
08               ecart_X(q) is minimal
09          h_{i+1} := simplify(h_i,q);
10          if deg_y(h_{i+1}) > deg_y(h_i) then
11               LIST := ∅;
12          else
13               put h_i in LIST;
14          fi
15          i := i + 1;
16    od
17    return g := h_i.
```

*Comments.* (i) The `simplify` step (line `09`) corresponds to

$$h_{i+1} := h_i - c \cdot X^B \cdot q$$

where $c \in \mathbb{R}$ and $B \in \mathbb{N}^m$ are chosen in order to simplify $\mathrm{in}(h_i)$.

(ii) At any step we have

$$h_{i+1} \equiv h_i \pmod{\mathscr{K}}.$$

(iii) After a finite number of iterations of the `while` loop either the degree in $y$ of $h_i$ increases strictly or $\mathrm{in}(h_i)$ leaves the sector $\Delta^K$. Suppose the degree in $y$ is stationary then the écart decreases strictly and that can happen only a finite number of times.

Thus, if we know *a priori* that $h$ does not belong to $\mathscr{K}$ (this can easily be checked, see Section 1), then $r(h)$ exists and is different from zero. Let $d$ be the degree in $y$ of its leading term; then the degree in $y$ of $h_i$ cannot exceed $d$ (otherwise we would have $h_i - r(h) \in \mathscr{K}$ with $\mathrm{in}(h_i - r(h)) = \mathrm{in}(r(h)) \in \mathbb{N}^{m+1} \setminus \Delta^K$ which is impossible from Proposition 1). Thus, from Comment (iii), there exists $i$ such that $\mathrm{in}(h_i) \notin \Delta^K$ thus

$$\mathrm{in}(h_i) = \mathrm{in}(r(h)).$$

## 4. Generalizations and conclusion

The problem that generalizes this situation is the following: for a given differential operator $D$ and an ideal $I \in \mathbb{R}\{Z\}$, consider $D^{-1}(I)$ when this makes sense, i.e., when for every $g \in I$ the equation

$$D(f) = g$$

has a solution in $\mathbb{R}\{Z\}$.

After $D = \partial_y$, the next simpler operator is $D = a \cdot \partial_y$ where $a$ is any element of $\mathbb{R}\{Z\}$. This problem can be reduced to the first one, with the ideal $J := (I : a)$. Next, if $D = \partial_y + b$ with $b \in \mathbb{R}\{Z\}$, the problem can still be solved with the same techniques. For general operators, even regular, we think that the problem is far from being that simple.

In any case we have shown that classical techniques of computational commutative algebra can be applied with appropriate changes in some situation involving differential relations.

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