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SERIAL RINGS AND SUBDIRECT PRODUCTS

Joel K. HAACK
Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA

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A basic Artinian serial ring can be realized as the subdirect product of factor rings of (S,M)-upper triangular matrix rings with S a local Artinian ring and M the maximal ideal of S. As an application the serial subdirect product of (S,M)-rings is shown to have self-duality.

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A ring R is said to be *serial* if R, both as a left and as a right module over itself, is a direct sum of modules that have linearly ordered submodule lattices. The model for Artinian serial rings is the ring of upper triangular matrices over a division ring. The structure of serial rings has been analyzed by Nakayama [17], Goldie [6], Kupisch [10], Murase [14,15,16], Michler [11], Eisenbud and Griffith [3,4], Fuller [5], Warfield [18], and Ivanov [9]. Michler and Warfield employ (S,M)-upper triangular matrix rings to describe the structure of Noetherian non-Artinian serial rings, where an (S,M)-upper triangular matrix ring is a matrix ring over a local serial ring S with the entries below the main diagonal restricted to the unique maximal ideal M of S, thus generalizing the class of upper triangular matrix rings over division rings. Here we show that (S,M)-upper triangular matrix rings underlie the structure of all Artinian serial rings in that every Artinian serial ring is Morita equivalent to a finite subdirect product of factor rings of (S,M)-upper triangular matrix rings. We then use this characterization to show that members of a broad class of rings, namely those that are factor rings of serial finite subdirect products of (S,M)-upper triangular matrix rings, are *self-dual*; that is, they admit a functorial duality between their categories of left and right finitely generated modules.

First, let us fix notation and recall some facts about the structure of Artinian serial rings. For a module M, let $c(M)$ denote the composition length of M and $\operatorname{soc}(M)$ the socle of M. The right annihilator of X in Y is given by

$$r_Y(X)=\{y\in Y \mid xy=0 \text{ for all } x\in X\}.$$

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Let R be an indecomposable Artinian serial ring with (Jacobson) radical $J=J(R)$. Let

$$\{e_1=e_{11},\ldots,e_{1m_1},\ldots,e_n=e_{n1},\ldots,e_{nm_n}\}$$

be a complete set of primitive orthogonal idempotents of R, indexed so that $Re_{ij}\cong Re_{kl}$ iff $i=k$ and so that $Re_1,\ldots,Re_n$ forms a Kupisch series for R; that is, Re_i is a projective cover of Je_{i+1} for $1\le i\le n-1$ and either $Je_1=0$ or Re_n is a projective cover of Je_1 [10].

Let $[k]$ denote the least strictly positive residue of k modulo n. Knowledge of the Kupisch series of a serial ring allows one to identify the composition factors of each Re_j: If $J^ke_j\ne 0$, then

$$J^ke_j/J^{k+1}e_j\cong Re_{[j-k]}/Je_{[j-k]} \quad [5].$$

The sequence $c(Re_1), c(Re_2),\ldots,c(Re_n)$ of composition lengths of the Re_j is called an *admissible sequence* of R; it is unique up to cyclic permutation, so that we may assume Re_1 is of minimal length among the Re_j. An admissible sequence satisfies the inequality

$$c(Re_{[j+1]})\le c(Re_j)+1 \qquad (j=1,\ldots,n).$$

A member Re_k of the Kupisch series is called a *chain end* if $c(Re_k)\ge c(Re_{[k+1]})$ [16]. A consequence of our assumption that Re_1 is of minimal length is that Re_n is a chain end.

Of course, any ring is a subdirect product of subdirectly irreducible factor rings, and a ring is subdirectly irreducible iff it has a unique minimal non-zero ideal. Let $e_j'=e_j+e_{j2}+\cdots+e_{jn}$. Murase [15, Theorem 11] has characterized the ideals of an Artinian serial ring as being of the form $\sum J^{b_j}e_j'$, where the b_j satisfy $b_{[j+1]}\le b_j+1$. It follows that the subdirectly irreducible Artinian serial rings are those with exactly one chain end. The conditions of having exactly one chain end, having homogeneous socle, and having a strictly increasing admissible sequence are easily seen to be equivalent for a serial ring.

1. Proposition. *The minimal non-zero ideals of an Artinian serial ring R are precisely those ideals of the form $J^{b_k}e_k'$, where Re_k is a chain end of R and $b_k=c(Re_k)-1$. Hence an Artinian serial ring R is subdirectly irreducible iff R has a strictly increasing admissible sequence.*

Proof. Let $c_i=c(Re_i)$. Re_k is a chain end of R iff $c_{[k+1]}\le c_k$; hence

$$\sum_{i\ne k} J^{c_i}e_i'+J^{c_k-1}e_k'=J^{b_k}e_k'$$

is an ideal that is clearly minimal. Any non-zero ideal I of R must contain $J^{c_i-1}e_i'$ for some i. If Re_i is a chain end, then I minimal implies that $I=J^{b_i}e_i'$. If Re_i is not a chain end, let Re_k be the first chain end in the Kupisch series occuring after Re_i; then also $J^{c_j-1}e_j'$ is contained in I for $i\le j\le k$. In particular, the ideal $J^{b_k}e_k'\subseteq I$ so that in this case, I is not minimal.

2. Corollary. *If R is an Artinian serial ring, then R is a subdirect product of serial rings having strictly increasing admissible sequences.*

More specifically, let Re_k be a chain end of an indecomposable Artinian serial ring R. Let $c_k = c(Re_k)$ and let

$$I_k = J^{c_k - 1}e'_{[k-1]} + \cdots + J^{c_k - n + 1}e'_{[k-n+1]}$$
$$= \sum_{i=1}^{n} J^{c_k - i + 1}e'_{[k-i+1]}.$$

Then I_k is an ideal of R and R/I_k is a serial ring with a unique chain end $(R/I_k)(e_k + I_k)$. Moreover,

$$0 = \bigcap \{I_k \mid Re_k \text{ is a chain end of } R\}.$$

To see this, let $0 \neq r \in R$. Choose i with $re'_i \neq 0$. Let $k = i$ if Re_k is chain end; otherwise define k by letting Re_k be the first chain end appearing after Re_i in the Kupisch series of R. Then the terms from c_i to c_k in the corresponding admissible sequence must be $c_i, c_i + 1, c_i + 2, \ldots, c_k = c_i + (k - i)$. Hence $c_i = c_k - (k - i)$, so that

$$I_k \cap Re'_i = J^{c_k - (k-i)}e'_i = J^{c_i}e'_i = 0.$$

Hence neither re'_i nor r is in I_k and the claim is established. As a consequence, we have

3. Proposition. *Let R be an indecomposable Artinian serial ring. For each chain end Re_k, let I_k be defined as above. Then R is a subdirect product of the subdirectly irreducible serial rings R/I_k.*

The basic ring R_0 of the serial ring R is $R_0 = (e_1 + \cdots + e_n)R(e_1 + \cdots + e_n)$; R is Morita equivalent to R_0 and if $R_0 = R$, then R is said to be a basic ring [1, Section 27]. The following proposition, communicated by Warfield in a private correspondence, has a proof similar to that of [18, Theorem 5.14].

4. Proposition [Warfield]. *Let R be a basic indecomposable Artinian serial ring with homogeneous socle and Kupisch series $Re_1, \ldots, Re_n$. Let $S = e_nRe_n$, $M = e_nJe_n$, and T the $(n \times n)$-(S, M)-upper triangular matrix ring. Then T is serial and R is isomorphic to a factor ring of T. Moreover, R is isomorphic to T iff n divides $c(Re_n)$.*

Proof. Let X be the direct sum of n copies of Re_n. For $1 \leq i \leq n-1$,

$$J^{n-i}e_n / J^{n-i+1}e_n \cong Re_i / Je_i.$$

Since the admissible sequence of R is strictly increasing, we see that

$$c(Re_i) = c(Re_n) - (n - i) = c(J^{n-i}e_n).$$

Consequently, $J^{n-i}e_n \cong Re_i$. Thus the submodule

$$P = J^{n-1}e_n \oplus J^{n-2}e_n \oplus \cdots \oplus Re_n$$

of X is isomorphic to ${}_RR$. Consequently, $R \cong \mathrm{End}({}_RP)$.

Let $T = \{f \in \mathrm{End}({}_RX) \mid f(P) \subseteq P\}$. Since X is injective (because the chain end Re_n is [5, Theorem 2.5]), any endomorphism of P extends to one of X, so $\mathrm{End}({}_RP)$ is a factor ring of T. Let $S = e_nRe_n$, $M = e_nJe_n = J(S)$ and identify $\mathrm{End}({}_RX)$ with the $(n \times n)$-matrix ring $M_n(S)$ over S. Let U be the (M,S)-upper triangular subring of $M_n(S)$. We must show that, under this identification, the (i,j)-entry f_{ij} of $f \in T$ is contained in M if $i > j$. But $f_{ij} : J^{n-i}e_n \to J^{n-j}e_n$ must have non-zero kernel since $c(J^{n-i}e_n) = c(Re_i) > c(Re_j) = c(J^{n-j}e_n)$, so also $f_{ij} : Re_n \to Re_n$ has non-zero kernel and must therefore be in M. Finally, let f be in U and let f_{ij} be the (i,j)-entry. If $i \le j$, then

$$J^{n-i}e_nf_{ij} \subseteq J^{n-i}e_nRe_n = J^{n-i}e_n \subseteq J^{n-j}e_n;$$

if $i > j$, then $f_{ij} \in M = e_nJe_n = e_nJ^ne_n$, and

$$J^{n-i}e_nf_{ij} \subseteq J^{n-i}e_nJ^ne_n = J^{n-(i-n)}e_n \subseteq J^{n-j}e_n.$$

Hence $f \in T$ and thus $T \cong U$. For the last statement, notice that the last column of T has composition length $nc(S)$. For the converse, suppose $c(Re_n) = mn$. By Proposition 1, $T \cong R$ if any homomorphism $f : J^{n-1}e_n \to Re_n$ has a unique extension to $g : Re_n \to Re_n$, since the unique minimal ideal of T is non-zero only in the $(1,n)$-position of $M_n(S)$. Now the kernel of a map h from Re_n to Re_n must be one of the submodules $Re_n, J^ne_n, J^{2n}e_n, \ldots, J^{mn}e_n = 0$ since $Re_n/\ker h = 0$ or $\mathrm{soc}(Re_n/\ker h) \cong \mathrm{soc}(Re_n) \cong Re_1/Je_1$. Hence if g and $\bar{g}$ are two extensions of a map $f : J^{n-1}e_n \to Re_n$, $J^{n-1}e_n \subseteq \ker(g - \bar{g})$ and thus $\ker(g - \bar{g}) = Re_n$.

5. Theorem. *If R is an Artinian serial ring, then the basic ring of R is a subdirect product of factor rings of (S,M)-upper triangular matrix rings where each S is a local Artinian serial ring and $M = J(S)$.*

Proof. Corollary 2 and Proposition 4.

As an application of this characterization of serial rings, we shall show that if the representation in Proposition 3 of an Artinian serial ring R is as a subdirect product of (S,M)-upper triangular matrix rings (rather than merely as factors of such rings), then R is self-dual. Results of Morita [12] and Azumaya [2] show that an Artinian ring R is self-dual in that there is a functorial duality between the categories of finitely generated left and right R-modules if $R \cong \mathrm{End}({}_RE)$, where E is a left injective cogenerator. We shall use the theorems and techniques of [7] and [8].

It is now sufficient to restrict our attention to a basic indecomposable Artinian serial ring R. The indecomposable injective R-modules are factors of the chain ends of R [5, Theorem 2.5]; we say that the simple R-module Re_i/Je_i *belongs to* the chain end Re_k (or simply that *i belongs to k*) if the injective envelope E_i of Re_i/Je_i is a factor module of Re_k. We shall need the following calculations.

6. Lemma. *Let R be an (S,M)-upper triangular matrix ring with Kupisch series* $Re_1, \ldots, Re_n$ *and indecomposable injective modules* $E_i = E(Re_i/Je_i)$. *For each* $i = 1, \ldots, n$,

$$c(E_i) = c(Re_n) - i + 1,$$

so that

$$c(Re_i) + c(E_i) = 2c(Re_n) - n + 1.$$

Proof. Let $c({}_SS) = m$. Then $c({}_RRe_n) = nc({}_SS) = nm$. [5, Theorem 2.5] shows that $E_i \cong Re_n/J^{b_i}e_n$, where $b_i = c(e_iR_R)$. But

$$\begin{aligned} c(e_iR) &= (i-1)c({}_SM) + (n-i+1)c({}_SS) \\ &= (i-1)(m-1) + (n-i+1)m \\ &= nm - i + 1 = c(Re_n) - i + 1. \end{aligned}$$

Similarly,

$$c(Re_i) = ic({}_SS) + (n-i)c({}_SM) = c(Re_n) - n + i.$$

The lemma follows.

The *trace* of a module M in another module N is the submodule

$$\operatorname{tr}_N(M) = \sum \{\operatorname{im} f \mid f : M \to N\} \quad \text{of } N.$$

7. Lemma. *Let R be a basic indecomposable Artinian ring with Kupisch series* $Re_1, \ldots, Re_n$. *Let* $c_i = c(Re_i)$. *For each chain end* Re_k, *let*

$$I_k = J^{c_k - 1}e_{[k-1]} + \cdots + J^{c_k - n + 1}e_{[k-n+1]}.$$

Assume that $J^n \neq 0$ *and that* R/I_k *is an (S, M)-upper triangular matrix ring for each chain end* Re_k. *Let* Re_k *and* Re_l *be chain ends of R and let i belong to k. Then*

(a) *Every chain end of R has the same composition length.*

(b) $\operatorname{tr}_{Re_k}(Re_i) \subsetneq \operatorname{tr}_{Re_k}(Re_l)$.

(c) *If* $\psi : Re_l \to Re_k$ *is an R-homomorphism with* $\psi(\operatorname{tr}_{Re_l}(Re_i)) = 0$, *then* $\psi = 0$.

Proof. (a) If $J^n \neq 0$, then for some e_i, $J^n e_i \neq 0$; so also for some chain end Re_k, $J^n e_k \neq 0$ and $c_k > n$. Hence $c_k - n + 1 > 1$, so that each $(R/I_k)(e_i + I_k)$ is non-zero and $\{e_1 + I_k, \ldots, e_n + I_k\}$ is a basic set of primitive orthogonal idempotents of R/I_k. Thus by Lemma 6, $c({}_RRe_k) = c({}_{(R/I_k)}Re_k/Ie_k)$ is a multiple of n and is at least $2n$. The condition $c_{[i+1]} \leq c_i + 1$ implies that the largest possible difference among the c_i is $n-1$; hence every chain end Re_l has length greater than n, so satisfies $J^n e_l \neq 0$, and must have length the same multiple of n.

b) Now let Re_k and Re_l be chain ends of length mn. The composition factors of Re_k are

$$Re_k/Je_k, Je_k/J^2e_k \cong Re_{[k-1]}/Je_{[k-1]}, \dots,$$
$$J^{mn-2}e_k/J^{mn-1}e_k \cong Re_{[k+2]}/Je_{[k+2]},$$
$$J^{mn-1}e_k \cong Re_{[k+1]}/Je_{[k+1]}.$$

Similarly, the last composition factors of Re_l are

$$\dots, J^{mn-2}e_l/J^{mn-1}e_l \cong Re_{[l+2]}/Je_{[l+2]},$$
$$J^{mn-1}e_l \cong Re_{[l+1]}/Je_{[l+1]}.$$

Since the injective envelope of Re_j/Je_j is the maximal essential extension of Re_j/Je_j, we are guaranteed that the simples corresponding to $k, \dots, [l+2], [l+1]$ do not belong to Re_k. Thus if Re_i/Je_i does belong to Re_k, either $Re_i = Re_l$ or Re_l/Je_l occurs before Re_i/Je_i as a composition factor of Re_k. Hence $\mathrm{tr}_{Re_k}(Re_i) \subseteq \mathrm{tr}_{Re_k}(Re_l)$.

c) Referring to the composition factors of Re_k and Re_l, again either $Re_i = Re_l$ or Re_i/Je_i occurs *after* Re_l/Je_l as a composition factor of Re_k. Hence if $\psi : Re_l \to Re_k$ is non-zero, then $\psi(\mathrm{tr}_{Re_l}(Re_i)) \neq 0$.

A ring R has a *weakly symmetric self-duality* if there is an isomorphism $\phi : R \to \mathrm{End}({}_RE)$ such that $E\phi(e)$ is the injective envelope of Re/Je for each idempotent e in a basic set for R [7, Poposition 3.1]. Homomorphisms of left R-modules are written on the right in the following proof.

8. Theorem. *Let R be a basic indecomposable Artinian serial ring with Kupisch series $Re_1, \dots, Re_n$. For each chain end Re_k, let*

$$I_k = J^{c_k-1}e_{[k-1]} + \dots + J^{c_k-n+1}e_{[k-n+1]}.$$

If R/I_k is an (S,M)-upper triangular matrix ring for each chain end Re_k, then R has a weakly symmetric self-duality.

Proof. By [7, Corollary 4.5], if $J^n = 0$, then R has a weakly symmetric self-duality. Assume $J^n \neq 0$. Let E_i be the injective envelope of Re_i/Je_i and let $E = \bigoplus E_i$ be the minimal injective cogenerator. Let $S = \mathrm{End}({}_RE)$. Let $E_i^k = r_{E_i}(I_k)$ and let $E^k = r_E(I_k)$. Then E_i^k is the injective envelope of $(R/I_k)(e_i + I_k)/(J/I_k)(e_i + I_k)$ and E^k is a minimal injective cogenerator over R/I_k for Re_k a chain end of R. Also, let $S^k = \mathrm{End}({}_{R/I_k}E^k)$; $S^k \cong S/r_S(E^k)$ [13]. By [7, Theorem 2.4] there is a ring isomorphism $\phi_k : R/I_k \to S^k$ yielding a weakly symmetric self-duality. Moreover, S is a subdirect product of the rings S^k where the coordinate map from S to S^k is the restriction of $s \in S$ to E^k, for $r_s(r_E(\ ))$ provides an isomorphism between the lattices of ideals of R and S [1, Section 24]. Thus, the product ϕ of the ϕ_k provides an isomorphism from $\prod R/I_k$ to $\prod S^k$. Regard R as the subdirect product of the R/I_k; if the maps ϕ_k can be chosen so that $\phi(R) = S$, then it will follow that R is self-dual. Hence we must use some care in defining the ϕ_k from the proof of [7, Theorem

2.4]. Our hypothesis on R guarantees that only case i of that proof need be considered.

To this end, fix $i \in \{1,\dots,n\}$. Order the chain ends $Re_{k_0},\dots,Re_{k_q}$ so that $c(Re_i/I_{k_p}e_i)<c(Re_i/I_{k_{p+1}}e_i)$ for $p=0,\dots,q-1$. Then i belongs to k_0, for by Lemma 7, a longer $E_i^{k_p}$ corresponds to a shorter $Re_i/I_{k_p}e_i$. Choose a monomorphism $\alpha_i^{k_0}: Re_i/I_{k_0}e_i \to Re_{k_0}$ and an epimorphism $\beta_i^{k_0}: Re_{k_0} \to E_i^{k_0}$. Since i belongs to k_0, $E_i^{k_0}=E_i$. Assume that $\alpha_i^{k_{p-1}}$ and $\beta_i^{k_{p-1}}$ have been defined for some $p \geq 1$. Since $Re_i/I_{k_{p-1}}e_i$ is shorter than $Re_i/I_{k_p}e_i$, $E_i^{k_p}$ is a submodule of $E_i^{k_{p-1}}$ (Lemma 7); denote this inclusion map by $\iota_i^{k_p}$. Let $\eta_i^{k_p}$ be the natural epimorphism $\eta_i^{k_p}: Re_i/I_{k_p}e_i \to Re_i/I_{k_{p-1}}e_i$. Since $Re_{k_{p-1}}$ has a linearly ordered submodule lattice, there exists $\theta_i^{k_p}: Re_{k_p} \to Re_{k_{p-1}}$ with $\operatorname{im}\theta_i^{k_p}=\operatorname{tr}_{Re_{k_{p-1}}}(Re_{k_p})$. Because

$$\operatorname{im}\eta_i^{k_p}\alpha_i^{k_{p-1}} \subseteq \operatorname{tr}_{Re_{k_{p-1}}}(Re_i) \subseteq \operatorname{tr}_{Re_{k_{p-1}}}(Re_{k_p}) = \operatorname{im}\theta_i^{k_p}$$

(Lemma 7 applied to $R/(I_{k_p} \cap I_{k_{p+1}} \cap \cdots \cap I_{k_q})$) and since ${}_{R/I_{k_p}}(Re_i/I_{k_p}e_i)$ is projective, there exists a map $\alpha_i^{k_p}: Re_i/I_{k_p}e_i \to Re_{k_p}$ such that

$$\alpha_i^{k_p}\theta_i^{k_p} = \eta_i^{k_p}\alpha_i^{k_{p-1}}.$$

In fact $\alpha_i^{k_p}$ is monic, for $\operatorname{soc}(Re_i/I_{k_p}e_i) \cong \operatorname{soc}(Re_{k_p})$, so that $c(\ker\alpha_i^{k_p})$ is a multiple of n. But

$$\ker\alpha_i^{k_p} \subseteq \ker\alpha_i^{k_p}\theta_i^{k_p} = \ker\eta_i^{k_p}\alpha_i^{k_{p-1}} = \ker\eta_i^{k_p} = I_{k_{p-1}}e_i/I_{k_p}e_i,$$

which has composition length less than n. Finally $\operatorname{im}\theta_i^{k_p}\beta_i^{k_{p-1}} \subseteq \operatorname{im}\iota_i^{k_p}$ since $\operatorname{im}\iota_i^{k_p}=E_i^{k_p}$ is the maximal submodule of $E_i^{k_{p-1}}$ with $E_i^{k_p}/JE_i^{k_p} \cong Re_{k_p}/Je_{k_p}$; and $\operatorname{im}\iota_i^{k_p} \subseteq \operatorname{im}\theta_i^{k_p}\beta_i^{k_{p-1}}$ since $\beta_i^{k_p}$ is epic, Re_{k_p} is projective, and $\operatorname{im}\theta_i^{k_p}=\operatorname{tr}_{Re_{k_{p-1}}}(Re_{k_p})$. Thus there exists $\beta_i^{k_p}$ with $\beta_i^{k_p}\iota_i^{k_p}=\theta_i^{k_p}\beta_i^{k_{p-1}}$; $\beta_i^{k_p}$ necessarily epic. Now for chain ends $Re_l=Re_{k_p}$ and $Re_m=Re_{k_t}$ with $p<t$, define $\theta_i^{ml}=1\cdot\theta_i^{k_t}\theta_i^{k_{t-1}}\cdots\theta_i^{k_{p+1}}: Re_m \to Re_l$, define η_i^{ml} to be the natural epimorphism $\eta_i^{ml}: Re_i/I_me_i \to Re_i/I_le_i$, and define ι_i^{ml} to be the natural inclusion $\iota_i^{ml}: E_i^m \to E_i^l$. Then for chain ends $Re_l=Re_{k_p}$ and $Re_m=Re_{k_t}$ with $p<t$, the following diagram is commutative:

$$\begin{array}{ccccc}
Re_i/I_le_i & \xrightarrow{\ \alpha_i^l\ } & Re_l & \xrightarrow{\ \beta_i^l\ } & E_i^l \\
\uparrow{\scriptstyle \eta_i^{ml}} & & \uparrow{\scriptstyle \theta_i^{ml}} & & \uparrow{\scriptstyle \iota_i^{ml}} \\
Re_i/I_me_i & \xrightarrow[\ \alpha_i^m\ ]{} & Re_m & \xrightarrow[\ \beta_i^m\ ]{} & E_i^m
\end{array}$$

The commutativity of these diagrams will produce the desired result that $\phi(R)=S$.

For each chain end Re_l, define $\phi_l: R/I_l \to S^l$ as in [7, Theorem 2.4] using the above choices for α_i^l and β_i^l; that is, given $r \in R$, define $\gamma^l, \delta^l, \varepsilon^l$, and $s_{ij}^l=\phi_l(e_ire_j+I_l)$ as the unique maps that make the following diagram commutative:

$$\begin{array}{ccccc}
Re_i/I_le_i & \xrightarrow{\alpha_i^l} & Re_l & \xrightarrow{\beta_i^l} & E_i^l \\
{\scriptstyle e_ire_j+I_l}\Big\downarrow & \searrow^{\gamma^l} & \Big\downarrow{\scriptstyle \delta^l} & \searrow^{\varepsilon^l} & \Big\downarrow{\scriptstyle s_{ij}^l} \\
Re_j/I_le_j & \xrightarrow[\alpha_j^l]{} & Re_l & \xrightarrow[\beta_j^l]{} & E_j^l
\end{array}$$

Then extend the definition of ϕ_l linearly. It remains to be shown that for any $r\in R$ and for any two chain ends Re_l and Re_m, $\phi_l(r)$ and $\phi_m(r)$ are restrictions of one endomorphism $s\in S$. This will be accomplished by showing that for any $r\in R$ any pair of idempotents e_i and e_j in a basic set for R, and any chain end Re_l, $\phi_l(e_ire_j)$ is the restriction of $\phi_k(e_ire_j)$ to E_i^l, where i belongs to k.

Denote by ϱ_{ij}^k (respectively, ϱ_{ij}^l) right multiplication by $e_ire_j+I_k(e_ire_j+I_l)$. Consider the following diagram:

$$\begin{array}{ccccc}
Re_j/I_le_j & \xrightarrow{\alpha_j^l} & Re_l & \xrightarrow{\beta_j^l} & E_j^l \\
{\scriptstyle \varrho_{ij}^l}\Big\uparrow & & \Big\uparrow{\scriptstyle \gamma^l} & & \Big\uparrow{\scriptstyle s_{ij}^l} \\
Re_i/I_le_i & \xrightarrow{\alpha_i^l} & Re_l & \xrightarrow{\beta_i^l} & E_i^l \\
{\scriptstyle \eta_i^{lk}}\Big\downarrow & & \Big\downarrow{\scriptstyle \theta_i^{lk}} & & \Big\downarrow{\scriptstyle \iota_i^{lk}} \\
Re_i/I_ke_i & \xrightarrow{\alpha_i^k} & Re_k & \xrightarrow{\beta_i^k} & E_i^k \\
{\scriptstyle \varrho_{ij}^k}\Big\downarrow & & \Big\downarrow{\scriptstyle \gamma^k} & & \Big\downarrow{\scriptstyle s_{ij}^k} \\
Re_j/I_ke_j & \xrightarrow[\alpha_j^k]{} & Re_k & \xrightarrow[\beta_j^k]{} & E_j^k
\end{array}$$

Case (i). If $c(Re_j/I_ke_j)\geq c(Re_j/I_le_j)$, then we have defined maps such that the following diagram is commutative:

$$\begin{array}{ccccc}
Re_j/I_le_j & \xrightarrow{\alpha_j^l} & Re_l & \xrightarrow{\beta_j^l} & E_j^l \\
{\scriptstyle \eta_i^{kl}}\Big\uparrow & & \Big\uparrow{\scriptstyle \theta_j^{kl}} & & \Big\uparrow{\scriptstyle \iota_j^{kl}} \\
Re_j/I_ke_j & \xrightarrow[\alpha_j^k]{} & Re_k & \xrightarrow[\beta_j^k]{} & E_j^k
\end{array}$$

Hence $\alpha_i^l\gamma^l=\alpha_i^l\theta_i^{lk}\gamma^{kl}\theta_j^{kl}$, so that the map $\psi=\gamma^l-\theta_i^{lk}\theta^k\theta_j^{kl}$ restricted to $\operatorname{im}\alpha_i^l=\operatorname{tr}_{R_i}(Re_l)$ is the 0-map. Because $I_le_l=0$ and i belongs to l in R/I_l, we may apply Lemma 7 to see that $\psi=0$; that is, $\gamma^l=\theta_i^{lk}\gamma^k\theta_j^{kl}$. Therefore

$$\beta_i^ls_{ij}^l=\gamma^l\beta_j^l=\theta_i^{lk}\gamma^k\theta_j^{kl}\beta_j^l=\beta_i^l\iota_i^{lk}s_{ij}^k\iota_j^{lk}.$$

Cancel the epimorphism β_i^l to obtain $s_{ij}^l = \iota_i^{lk} s_{ij}^k \iota_j^{kl}$.

Case (ii) is handled similarly with the conclusion that $\iota_i^{lk} s_{ij}^k = s_{ij}^l \iota_j^{lk}$ if $c(Re_j/I_k e_j) < c(Re_j/I_l e_j)$. Hence every s_{ij}^l is a restriction of s_{ij}^k when i belongs to k. Thus $\phi(R) = S$ and R has a weakly symmetric self-duality.

A consequence of Theorem 8 and [7, Proposition 4.1] is that every (serial) ring that is a factor ring of a serial ring satisfying the hypotheses of Theorem 5 also has self-duality. Unfortunately, not all serial rings are such factors. An example is given in [7, Example 3.4]. This ring R has admissible sequence 3,3 with $e_1Re_1 \cong \mathbb{Z}_4$ and $e_2R_2e_2 \cong \Lambda = \mathbb{Z}_2[x]/(x^2)$ and is a subdirect product of factors of the (2×2)-$(\mathbb{Z}_4, 2\mathbb{Z}_4)$-upper triangular matrix ring and the (2×2)-$(\Lambda, x\Lambda)$-upper triangular matrix ring. But R is *not* a factor of a serial subdirect product T of (S,M)-rings, for such a ring T must have admissible sequence $2m-1, 2m$ or $2m$, $2m$ for some m. It would then follow from [9, Theorem 11] that $\Lambda \cong \mathbb{Z}_4$, a contradiction.

References

[1] F.W. Anderson and K.R. Fuller, Rings and Categories of Modules (Springer-Verlag, Berlin, 1974).

[2] G. Azumaya, A duality theory for injective modules, Amer. J. Math. 81 (1959) 249–278.

[3] D. Eisenbud and P. Griffith, Serial rings, J. Algebra 17 (1971) 389–400.

[4] D. Eisenbud and P. Griffith, The structure of serial rings, Pacific J. Math. 36 (1971) 109–121.

[5] K.R. Fuller, Generalized uniserial rings and their Kupisch series, Math. Z. 106 (1968) 248–260.

[6] A.W. Goldie, Torsion-free modules and rings, J. Algebra 1 (1964) 268–287.

[7] J.K. Haack, Self-duality and serial rings, J. Algebra 59 (1979) 345–363.

[8] J.K. Haack, Finite subdirect products of rings and Morita duality, Comm. Algebra, to appear.

[9] G. Ivanov, Left generalized uniserial rings, J. Algebra 31 (1974) 166–181.

[10] H. Kupisch, Beitrage zur Theorie nichthalbeinfacher Ringe mit Minimalbedingung, J. Reine Angew. Math. 201 (1959) 100–112.

[11] G.O. Michler, Structure of semi-perfect hereditary Noetherian rings, J. Algebra 13 (1969) 327–344.

[12] K. Morita, Duality for modules and its applications to the theory of rings with minimum condition, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 6 (1958) 83–142.

[13] B.J. Müller, On Morita duality, Canad. J. Math. 6 (1969) 1338–1347.

[14] I. Murase, On the structure of generalized uniserial rings, I, Sci. Papers College Gen. Ed. Univ. Tokyo 13 (1963) 1–22.

[15] I. Murase, On the structure of generalized uniserial rings, II, Sci. Papers College Gen. Ed. Univ. Tokyo 13 (1963) 131–158.

[16] I. Murase, On the structure of generalized uniserial rings, III, Sci. Papers College Gen. Ed. Univ. Tokyo 14 (1964) 14–25.

[17] T. Nakayama, Note on uniserial and generalized uniserial rings, Proc. Imp. Acad. Japan 16 (1940) 285–289.

[18] R.B. Warfield, Jr., Serial rings and finitely presented modules, J. Algebra 37 (1975) 187–222.