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Sequence domains and integer-valued polynomials

Alan Loper

Department of Mathematics, The Ohio State University–Newark, 1179 University Drive, Newark, OH 43055-1797, USA

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Abstract

A problem of recent interest has been to characterize all commutative integral domains D such that Int(D) (the integer-valued polynomial ring on D) is Prufer. It is known that if D is Noetherian, then Int(D) is Prufer if and only if D is Dedekind with all residue fields finite. Moreover, it is known that if Int(D) is Prufer (D Noetherian or not), then D is almost Dedekind with all residue fields finite. The case where D is non-Noetherian has been attacked by Chabert [2,3], Glimer [5], and Loper [10], but is far from settled.

This paper considers a special class of non-Noetherian almost Dedekind domains with finite residue fields which can be constructed by intersecting a sequence of Noetherian valuation domains which has a particular convergence property. These domains are called sequence domains. The especially simple ideal structure of sequence domains allows us to draw conclusions about the ideal structure of the integer-valued polynomial rings. For example, we show that a two-part boundedness condition proposed by Chabert in [3] completely characterizes the sequence domains D for which Int(D) is Prufer. Also, in [3] Chabert posed a condition he called "behaving well under localization" which he proved to be a sufficient condition for Int(D) to be Prufer, but left unsettled the question of its necessity. We characterize the sequence domains D for which Int(D) behaves well under localization and show by means of an example that this condition is not necessary. We construct many other examples as well, all of which are overrings of $Z[x]$.
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1. Introduction

Let D be a commutative integral domain with quotient field K and let Int(D) be the ring of integer-valued polynomials on D. Thus, $\mathrm{Int}(D) = \{f(x) \in K[x] \mid f(D) \subseteq D\}$. A problem of recent interest has been to characterize all domains D such that Int(D) is Prüfer. Chabert has shown [1, Proposition 6.3] that if Int(D) is Prüfer, then D is almost Dedekind with all residue fields finite. (Recall that a domain D is almost Dedekind

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provided D_P is a Noetherian valuation domain for each maximal ideal P of D [6, Ch. 36].) Chabert [1, Corollaire 6.5] and McQuillan [13, Corollary 2.5 and Theorem 5.3] have shown, independently, that if D is Noetherian, then Int(D) is Prüfer if and only if D is Dedekind with all residue fields finite. The non-Noetherian case appears to be less tractable. We call a non-Noetherian almost Dedekind domain with all residue fields finite an NaDf domain. Gilmer [5], Chabert [2,3] and Loper [10] have constructed the only examples of NaDf domains in the literature. Int(D) is Prüfer for many of the NaDf domains which have been constructed. However, Gilmer [5, Example 14] and Chabert [3, Example 6.2] have each given (very different) examples of NaDf domains D such that Int(D) is not Prüfer. Gilmer and Chabert each constructed NaDf domains as infinite degree algebraic extensions of Dedekind domains and made extensive use of the extensions in obtaining their results. In this paper, we try to gain insight into the question of when Int(D) is Prüfer by considering NaDf domains as intersections of Noetherian valuation domains and examining the interrelationships of the valuation domains.

In Section 2 we present Gilmer's and Chabert's negative (Int(D) not Prüfer) examples and discuss in intuitive terms the properties D possesses which cause Int(D) to not be Prüfer.

In Section 3 we use the intuition gained in Section 2 to define a class of NaDf domains which we call sequence domains. We then explore some of the ideal-theoretic properties of sequence domains which will be utilized later.

Section 4 deals with the question of when Int(D) is Prüfer for D a sequence domain. In [3] Chabert, as noted above, dealt primarily with NaDf domains constructed using infinite degree algebraic extensions of Dedekind domains. In this context, he gave a 2-part condition (Theorem 1.2) (essentially avoiding the two negative examples cited above) which he proved to be necessary for Int(D) to be Prüfer. He also asked (Q6) if this condition is sufficient. We call this condition double boundedness. In Section 4 we show that a translation of double-boundedness into the setting of sequence domains is both necessary and sufficient.

In Section 5 we employ ultrafilters and P-adic completions of D to describe the maximal ideals of Int(D) in the case that D is a doubly bounded sequence domain. In all cases in the literature where Int(D) is known to be Prüfer (D Dedekind or not – and D not necessarily a sequence domain) the maximal ideals of Int(D) which intersect nontrivially with D correspond precisely to the maximal ideals of Int(D_P), where P runs over the maximal ideals of D. Chabert calls this condition "behaving well under localization". He proves [3, Theorem 2.1] that behaving well under localization is a sufficient condition for Int(D) to be Prüfer and asks (Q7) if it is also necessary. In Section 5 we apply our knowledge of the maximal ideals of Int(D) to give a complete characterization of those doubly bounded sequence domains D for which Int(D) behaves well under localization.

In Section 6 we construct explicit examples of sequences domains of various types. We construct negative examples which parallel those of Gilmer and Chabert. We also construct several examples of doubly bounded sequence domains for which the results of Section 5 show that Int(D) does not behave well under localization, giving a negative answer to Chabert's (Q7). We preface the examples with a short summary of a 1935 paper of Saunders MacLane which provides the principal tools for building the examples.

Finally, in Section 7 we indicate how the special case of sequence domains relates to the general problem of characterizing all NaDf domains such that Int(D) is Prüfer.

2. The negative examples of Gilmer and Chabert

Gilmer's Example 14 [5] and Chabert's Example 6.2 [3] are the only examples in the literature of NaDf domains D for which Int(D) is (known to be) not Prüfer. In this section we analyze these two examples in an effort to gain intuition for approaching the general problem. Our analysis has a strong heuristic component which is not present in [5] or [3]. This approach seems worthwhile because it provides motivation for the definition of sequence domains and the results concerning them.

In both Gilmer's and Chabert's examples, an NaDf domain D was constructed and an exceptional maximal ideal M was found such that $\mathrm{Int}(D) \subset D_M[x]$. This essentially is the proof that Int(D) is not Prüfer, since $D_M[x]$ is an overring of Int(D) which is not Prüfer, whereas every overring of a Prüfer domain is Prüfer. The choice of the maximal ideal M is not random. In particular, it seems worthwhile in both examples to view M as being, in some sense, the "limit" of a sequence $\{M_i \mid i \in Z^+\}$ of maximal ideals of D such that the sequence has a particular "negative" property. Or, equivalently, we can associate a valuation v_i on the quotient field of D to each maximal ideal M_i and view v (the valuation associated with M) as being the "limit" of the sequence $\{v_i \mid i \in Z^+\}$ of valuations.

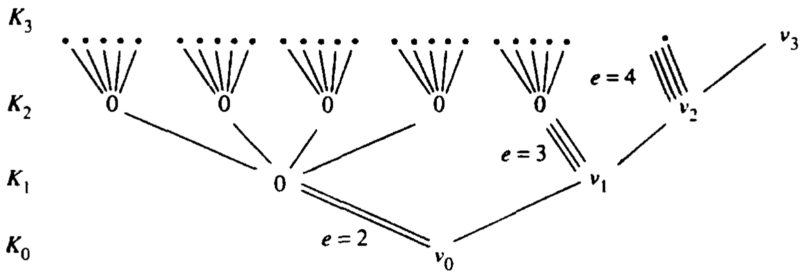
We now explain what the negative property is for each example and give some intuitive discussion as to why each property is negative. Then we offer some discussion as to why the notion of "limit" seems appropriate in each case.

Gilmer's Example. D is an NaDf domain with a maximal ideal M such that M is the "limit" of a sequence $\{M_i \mid i \in Z^+\}$ of maximal ideals such that $\lim_{i\to\infty} |D/M_i| = \infty$. Intuitively, if such a sequence were viewed as "converging" to M, then M should have an infinite residue field. M has a finite residue field, but Int(D) behaves as if it were infinite. (Recall that all residue fields of a domain D are finite if Int(D) is Prüfer.) In particular, $\mathrm{Int}(D) \subseteq D_M[x]$.

Chabert's Example. D is an NaDf domain with quotient field K. Again, D has a maximal ideal M such that $\mathrm{Int}(D) \subseteq D_M[x]$ and we view M as being the "limit" of a sequence $\{M_i \mid i \in Z^+\}$ of maximal ideals. However, for this example we focus on the sequence $\{v_i^{(\mathrm{N})} \mid i \in Z^+\}$ of normed valuations on K associated to the M_i's, and view this sequence as converging to $v^{(\mathrm{N})}$, the normed valuation associated with M. The negative property consists of the existence of an element $d \in D\backslash\{0\}$ such that $\lim_{i\to\infty} v_i^{(\mathrm{N})}(d) = \infty$. Intuitively, if the sequence $\{v_i^{(\mathrm{N})}\}$ were viewed as converging to

$v^{(N)}$, then we should have $v^{(N)}(d) = \infty$. This would imply that M has height larger than one. M is a height one prime and $v^{(N)}(d) < \infty$, but Int(D) behaves otherwise. (Recall that Int(D) Prüfer implies D is almost Dedekind, which in turn implies that all maximal ideals of D have height one.)

Both Gilmer and Chabert begin with a Noetherian valuation domain D_0 with quotient field K_0. They then build an infinite tower $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ of finite degree field extensions of K_0. In particular $[K_n : K_{n-1}] = n+2$. The NaDf domain D is the integral closure of D_0 in $K = \bigcup_{i=0}^{\infty} K_i$. The splitting and ramification of maximal ideals in each of the finite-degree extensions K_i can be viewed using a tree. The diagram below represents Chabert's example. Gilmer's example can be obtained by replacing e with f. At each stage the ramification index e and the relative degree f is equal to one except where it is specified otherwise in the diagram. The first four stages of the tree are given below. The pattern should be clear from this picture.



Each branch of the (infinite) tree corresponds to a maximal ideal of D. The right hand branch $v_0 - v_1 - v_2 - v_3 \cdots$ corresponds to M. To construct M_i for any $i \in Z^+$, choose any branch which begins $v_0 - v_1 - v_2 - \cdots - v_i$ and then goes to the left from v_i. It seems clear then that M_{i+1} should have, in some sense, a "larger base of agreement" with M than does M_i. The goal of the next section is to make the vague notions and assertions of this section more explicit. The reader who is interested in more precise details about the examples of Gilmer and Chabert is referred to their papers.

3. Sequence domains

In this section we use the (so far undefined) notion of a "convergent" sequence of maximal ideals/valuations to define a class of NaDf domains. Recall that Chabert's example was phrased in the language of valuations while Gilmer's example was discussed using only the language of maximal ideals. For our purposes, it seems preferable to follow Chabert's lead and define sequence domains using valuations.

Note: The rank-one discrete valuations we use will not necessarily be normed. If ω is such a valuation on a field K, we will use $\omega^{(N)}$ to denote the corresponding normed valuation.

3.1. Definition. Suppose that D is an NaDf domain with quotient field K. We call D a *sequence domain* provided it satisfies the following conditions:

(1) There exists a collection of maximal ideals $S = \{P_i | i \in Z^+\}$ of D such that
  (a) $D = \bigcap_{i=1}^{\infty} D_{P_i}$,
  (b) each residue field D/P_i has the same characteristic p,
  (c) the collection $\{P_i\}$ does not constitute all of the maximal ideals of D.

(2) There exists a collection $\{v_i | i \in Z^+\}$ of valuations on K such that
  (a) $v_i^{(N)}$ is the normed valuation on K corresponding to P_i for each i,
  (b) the sequence $\{v_i(d) | i \in Z^+\}$ is eventually constant for each $d \in D \backslash \{0\}$,
  (c) $v^*(d) = \lim_{i\to\infty} v_i(d) \in Z^+ \cup \{0\}$ for each $d \in D\backslash\{0\}$,
  (d) there exists an element $\pi \in D$ such that $v_i(\pi) = 1$ for each $i \in Z^+$.

Several remarks are in order regarding Definition 3.1. First, we deal with notation.

3.2. Notation. With D and v^* as in Definition 3.1 we say that $P^* = \{d \in D | v^*(d) > 0\} \cup \{0\}$. Further, we note that through the end of Section 5 we will use the notation P^* and the notations introduced in Definition 3.1 with those symbols retaining the hypotheses described in 3.1 and 3.2.

It also seems worthwhile to discuss the motivation of Definition 3.1. It is easy to see that P^* is a maximal ideal of D and that v^* is the corresponding normed valuation on K. Further, $M, \{M_i\}, v$ and $\{v_i\}$ of Section 2 correspond to $P^*, \{P_i\}, v^*$ and $\{v_i^{(N)}\}$ of Definition 3.1. In this regard, part 2c of Definition 3.1 formalizes the notion of convergence which was alluded to earlier. Also note that for any fixed $d \in D\backslash\{0\}, v_i(d)$ is an integer for all sufficiently large values of i, but for any fixed $i \in Z^+$ it is possible that $v_i(d)$ is not an integer for some $d \in D\backslash\{0\}$. If we recall that the valuations considered in Chabert's example were normed, the preceding remark explains how we can speak of convergence in a "Chabert type" example. Assuming Definition 3.1 notation, we can have $v_i(\pi) = 1$ for all $i \in Z^+$ while the sequence $\{v_i^{(N)}(\pi) | i \in Z^+\}$ goes to ∞. We make a similar clarifying observation concerning Gilmer's example following Proposition 3.3.

3.3. Proposition. *D/P^* is isomorphic to a subfield of D/P_i for all but finitely many values of $i \in Z^+$.*

Proof. Let $\{u_0, u_1, u_2, \ldots, u_{q-1}\}$ be a complete residue system for P^*. Then $u_j - u_i \notin P^*$ whenever $i \neq j$. Hence, if $i \neq j$, then $u_j - u_i \notin P_k$ for all but finitely many $k \in Z^+$. The result follows immediately. □

Proposition 3.3 will prove to be useful later in proving further results. However, we can make use of it now to clarify "Gilmer type" examples. Suppose $d \in D$. Then $d - u_i \in P^*$ for some i. Hence, $d - u_i \in P_k$ for all but finitely many values of k. Hence, any d in D considered modulo (successively) $P_1, P_2, P_3, P_4, \ldots$ will at some point "converge" into the equivalence classes corresponding to some u_i. However, it may be that for any fixed value of k there can exist an element $d \in D$ such that $d - u_i \notin P_k$ for all $0 \le i \le q-1$. This explains how the residue field D/P^* can be finite while the sequence $\{|D/P_i| \mid i \in Z^+\}$ goes to ∞.

We now give a number of structure theorems concerning sequence domains. We begin by giving the definition of a non-D-ring on which many of the subsequent results hinge. Details concerning non-D-rings can be found in [8, 9].

3.4. Definition. A domain T is called a *non-D-ring* provided there exists a nonconstant polynomial $f(x) \in T[x]$ such that $f(a)$ is a unit in T for every $a \in T$. The polynomial $f(x)$ is called a uv (unit-valued) polynomial.

3.5. Proposition. *Suppose that the set $\{|D/P_i| \mid i \in Z^+\}$ is bounded. Then D is a non-D-ring. Moreover, two monic, irreducible uv-polynomials exist for D with relatively prime degrees.*

Proof. Any monic polynomial of degree greater than one with coefficients in the set $\{0, 1, 2, \dots, p-1\}$ which is irreducible over each residue field D/P_i is a uv-polynomial for D. The result follows easily. □

3.6. Corollary. *Suppose the set $\{|D/P_i| \mid i \in Z^+\}$ is bounded. Then D is Bezout and each P_i is principal.*

Proof. Since D is a Prüfer non-D-ring with monic uv-polynomials of relatively prime degrees, [9, Corollary 2.7] implies that D is Bezout.

Now choose a maximal ideal P_i. Set $d \in P_i \backslash P^*$. The definition of P^* implies that $v_j(d) \ne 0$ for at most finitely many $j \in Z^+$. Let $P_{j_1}, P_{j_2}, \dots, P_{j_t}, P_i$ be the maximal ideals in the collection $\{P_k | k \in Z^+\}$ which contain d. For $1 \le k \le t$, choose an element $d_k \in D_i \backslash D_{j_k}$. Let $f(x)$ be a monic, irreducible uv-polynomial for D whose existence is guaranteed by Proposition 3.5. Let $\deg(f(x)) = n_f$. Now let $b_0 = d$ and for $1 \le k \le t$, let $b_k = b_{k-1}^{n_f}(f(d_{j_k}/b_{k-1}))$. It follows from [9, Proposition 2.2] that P_i is the only maximal ideal of D in $\{P_i | i \in Z^+\}$ which contains b_t. It follows that the intersection $D = \bigcap_{j=1}^{\infty} D_{P_j}$ is irredundant in the sense that $\bigcap_{j \in Z^+ \backslash \{k\}} D_{P_j} \ne D$ for each $k \in Z^+$. Then [7, Corollary 1.11] implies that P_i is the only maximal ideal of D which contains b_t. Since P_i extends to a principal ideal in D_{P_i}, [6, Lemma 37.3] implies that P_i is finitely generated. Hence, P_i is principal. □

3.7. Corollary. *Suppose that the set $\{|D/P_i| \mid i \in Z^+\}$ is bounded. Then the set $\{P_i | i \in Z^+\} \cup \{P^*\}$ comprises all of the maximal ideals of D.*

Proof. Suppose that $d \in D$ is a nonunit and that $d \notin P^*$. The definition of P^* implies that d is contained in only finitely many maximal ideals of D in the set $\{P_i | i \in Z^+\}$. Then [7, Corollary 1.11] implies that these are the only maximal ideals of D which contain d. Any maximal idea of D other than P^* must contain an element which does not lie in P^*. Hence, we have shown that $\{P_i | i \in Z^+\}$ are the only maximal ideals of D other than P^*. $\square$

We now give the definition of double-boundedness in the context of sequence domains. As noted in the introduction, the condition essentially amounts to avoiding the negative features of both Gilmer's and Chabert's examples.

3.8. Definition. We say that the sequence domain D is *doubly bounded* provided the following two conditions hold:

(1) The set $\{|D/P_i| \mid i \in Z^+\}$ is bounded.

(2) For each $d \in D\backslash\{0\}$, the set $\{v_i^{(\mathrm{N})}(d) | i \in Z^+\}$ is bounded.

Note that results 3.5–3.7 assume condition 1 of Definition 3.8 and so the properties ascribed to D by those results hold when D is assumed to be doubly bounded.

3.9. Proposition. *Suppose that the set $\{|D/P_i| \mid i \in Z^+\}$ is bounded. Then $v_i^{(\mathrm{N})}(d) = v_i^{(\mathrm{N})}(\pi)v_i(d)$ for all $d \in D\backslash\{0\}$ and for all $i \in Z^+$.*

Proof. For each $i \in Z^+$ there exists $e_i \in Z^+$ such that $v_i^{(\mathrm{N})}(d) = e_i v_i(d)$ for each $d \in D\backslash\{0\}$. Hence, $v_i^{(\mathrm{N})}(\pi) = e_i v_i(\pi) = e_i$. $\square$

3.10. Corollary. *Suppose D is doubly bounded. There exists a positive integer b such that for all $i \in Z^+$ and for all $d \in D\backslash\{0\}$, $bv_i(d) \in Z^+ \cup \{0\}$.*

Proof. The set $\{v_i^{(\mathrm{N})}(\pi) | i \in Z^+\}$ is bounded. Let b_1 be the largest element in this set. It follows easily from Proposition 3.9 that $(b_1)! v_i(d) \in Z^+ \cup \{0\}$ for all $i \in Z^+$ and for all $d \in D\backslash\{0\}$. $\square$

3.11. Corollary. *Suppose that the set $\{|D/P_i| \mid i \in Z^+\}$ is bounded and the set $\{v_i^{(\mathrm{N})}(d) | i \in Z^+\}$ is unbounded for some $d \in D\backslash\{0\}$. Then the set $\{v_i^{(\mathrm{N})}(\pi) | i \in Z^+\}$ is unbounded.*

Proof. The result follows immediately from Proposition 3.9 and from the fact that $\{v_i(d) | i \in Z^+\}$ is bounded for each $d \in D\backslash\{0\}$. $\square$

We close this section by noting that the definition of sequence domain involves some rather strong conditions. An interesting collection of domains could perhaps be obtained by weakening this definition. We will not pursue this line here. The strong conditions have been imposed to suit our goal of characterizing the domains D for which Int(D) is Prüfer.

4. Int(D) classification theorems

The objective of this section is to prove that when D is a sequence domain, Int(D) is Prüfer if and only if D is doubly bounded. As noted previously, Chabert also posed a double-bounded condition for a different class of domains [3, Theorem 1.2]. In that context he proved double-boundedness to be necessary and asked (Q6) if it was also sufficient. Hence, in the context of sequence domains (Q6) has an affirmative answer.

We begin by proving the necessity of the double-boundedness condition. The proof given here is essentially a translation of the corresponding proof given by Chabert [3, Theorem 1.2] to the context of sequence domains.

Recall that the notation introduced in Definition 3.1 and Notation 3.2 will be used (with the same hypotheses as in 3.1 and 3.2) throughout Section 4.

4.1. Theorem. *Suppose that* $\mathrm{Int}(D)$ *is Prüfer. Then* D *is doubly bounded.*

Proof. We prove the contrapositive.

Suppose first that the set $\{|D/P_i| \mid i \in Z^+\}$ is not bounded. For each $i \in Z^+$ let $q_i = |D/P_i|$. Choose $h(x) \in \mathrm{Int}(D)\backslash\{0\}$. We wish to show that $h(x) \in D_{P^*}[x]$. Let $h(x) = a_m x^m + \cdots + a_1 x + a_0$. For each $i \in Z^+$ define $v_i^{(\mathrm{N})}(h(x)) = \inf\{v_i^{(\mathrm{N})}(a_j)|0 \leq j \leq m, a_j \neq 0\}$. Then [3, 1.3 and 1.4] imply that $v_i^{(\mathrm{N})}(h(x)) > -m/(q_i - 1)$ for each $i \in Z^+$. Hence, if $q_i > m+1$, it follows that $v_i^{(\mathrm{N})}(h(x)) > -1$. Since $q_i > m+1$ for infinitely many values of i and $v_i^{(\mathrm{N})}(h(x)) \in Z$ for all i, the definition of sequence domain implies that $v_i^{(\mathrm{N})}(h(x)) \geq 0$ for all but finitely many $i \in Z^+$. Hence, $v^*(a_j) \geq 0$ for each nonzero a_j. This implies that $h(x) \in D_{P^*}[x]$ and so $\mathrm{Int}(D)$ is not Prüfer since $\mathrm{Int}(D) \subseteq D_{P^*}[x]$.

Now suppose that each set $\{|D/P_i| \mid i \in Z^+\}$ is bounded and that for some $d \in D\backslash\{0\}$ the set $\{v_i^{(\mathrm{N})}(d)|i \in Z^+\}$ is not bounded. Corollary 3.11 implies that the set $\{v_i^{(\mathrm{N})}(\pi)|i \in Z^+\}$ is also unbounded. Also recall that Proposition 3.9 implies that $v_i^{(\mathrm{N})}(d) = v_i^{(\mathrm{N})}(\pi)v_i(d)$ for all $i \in Z^+$ and for all $d \in D\backslash\{0\}$. Choose $h(x) \in \mathrm{Int}(D)\backslash\{0\}$ and define $v_i^{(\mathrm{N})}(h(x))$ and q_i as in the preceding paragraph. Then we have $v_i^{(\mathrm{N})}(h(x)) > -m/(q_i-1) > -m$ for all $i \in Z^+$. However, for all $i \in Z^+$ we have

$$v_i^{(\mathrm{N})}(h(x)) = v_i^{(\mathrm{N})}(\pi)\left[\inf_j\{v_i(a_j)|0 \leq j \leq m, a_j \neq 0\}\right].$$

The definition of sequence domain implies that for sufficiently large values of $i, \inf_j \{v_i(a_j)|0 \leq j \leq m, a_j \neq 0\} \in Z$. Also, $\{v_i^{(\mathrm{N})}(\pi)|i \in Z^+\}$ is an unbounded set of positive integers. Hence, $v_i^{(\mathrm{N})}(h(x)) > -m$ for all $i \in Z^+$ forces $v_i^{(\mathrm{N})}(h(x)) \geq 0$ for sufficiently large i. It follows that $v^*(a_j) \geq 0$ for each nonzero a_j and so again $h(x) \in D_{P^*}[x]$. Hence, $\mathrm{Int}(D) \subseteq D_{P^*}[x]$ and so $\mathrm{Int}(D)$ is not Prüfer. $\square$

Before we turn to the question of the sufficiency of double-boundedness we need to introduce a miscellany of notations, terminologies and assumptions.

4.2. Note. All of the following will be assumed to hold through Sections 4 and 5.

(1) D is a doubly bounded sequence domain.

(2) $P_i = \rho_i D$ for each $i \in Z^+$. (Recall that Corollary 3.6 implies that P_i is principal.)

(3) $q_i = |D/P_i|$ for each $i \in Z^+$.

(4) If M is a maximal ideal of $\mathrm{Int}(D)$ such that $M \cap D \neq \{0\}$ then M will be called unitary. Otherwise M will be called nonunitary.

(5) If M is a unitary maximal ideal of $\mathrm{Int}(D)$ which is the restriction to $\mathrm{Int}(D)$ of a unitary maximal ideal of $\mathrm{Int}(D_P)$ for some maximal ideal P of D (either $P = P_i$ for some i or $P = P^*$) then M will be called a well-behaved ideal. If M cannot be represented as such a restriction, then M will be called unruly.

Now we turn to the sufficiency proof. This is accomplished by showing that $\mathrm{Int}(D)_M$ is a valuation domain for each maximal ideal M of $\mathrm{Int}(D)$. We begin with an easy result concerning nonunitary maximal ideals.

4.3. Lemma. *Let M be a nonunitary maximal ideal of* $\mathrm{Int}(D)$. *Then* $\mathrm{Int}(D)_M$ *is a valuation domain.*

Proof. The definition of $\mathrm{Int}(D)$ implies that $D[x] \subseteq \mathrm{Int}(D) \subseteq K[x]$. Since $M \cap D = \{0\}$, then $\mathrm{Int}(D)_M \supseteq K$. Hence, $\mathrm{Int}(D)_M \supseteq K[x]$. Since $K[x]$ is Prüfer, the result follows. □

Now we turn to the unitary maximal ideals of $\mathrm{Int}(D)$. We begin with an approach which will be unsuccessful in showing that $\mathrm{Int}(D)$ is Prüfer. We use this approach because it gives us information (Corollary 4.5 and subsequent remarks) concerning well-behaved and unruly ideals.

4.4. Proposition. *Let M be a unitary maximal ideal of* $\mathrm{Int}(D)$ *such that $M \cap D = P_i$ for some $i \in Z^+$. Then* $\mathrm{Int}(D)_M$ *is a valuation domain.*

Proof. Let $Y_i = \{y_0, y_1, y_2, \ldots, y_{q_i-1}\}$ be a complete system of residues for P_i in D. Also, let A_i be the ring generated by Y_i and ρ_i. Gilmer indicates [5, Proposition 9] that a D_{P_i} module basis for $\mathrm{Int}(D_{P_i})$ can be constructed of the form $F_{P_i} = \{f_0(x), f_1(x), f_2(x), \ldots\}$ with $f_j(x) = (a_0 + a_1x + \cdots + a_jx^j)/\rho_i^{k_j}$ such that $k_j \in Z^+ \cup \{0\}$ and $\{a_0, a_1, \ldots, a_j\} \subseteq A \subseteq D$. Note that if $j \neq i$ then ρ_i is a unit in D_{P_j}. Hence, $F_{P_i} \subseteq \mathrm{Int}(D_{P_j})$ for each $j \in Z^+$. It follows that $F_{P_i} \subseteq \bigcap_{j=1}^{\infty} \mathrm{Int}(D_{P_j}) = \mathrm{Int}(D) \subseteq \mathrm{Int}(D)_M$. Since $D_{P_i} \subseteq \mathrm{Int}(D)_M$ as well and F_{P_i} is a D_{P_i} module basis for $\mathrm{Int}(D_{P_i})$, it follows that $\mathrm{Int}(D_{P_i}) \subseteq \mathrm{Int}(D)_M$. Since $\mathrm{Int}(D_{P_i})$ is well known to be Prüfer, it follows that $\mathrm{Int}(D)_M$ is a valuation domain. □

4.5. Corollary. *The unitary maximal ideals of* $\mathrm{Int}(D)$ *which lie above the ideals $\{P_i | i \in Z^+\}$ of D are all well behaved.*

Proof. This follows immediately from the proof of Proposition 4.4. □

The proof of Proposition 4.4 does not go through if P_i is replaced by P^*. We will demonstrate in Section 6 that unruly unitary maximal ideals can, in fact, exist in $\mathrm{Int}(D)$, lying over P^*. We now take a different approach which will show that $\mathrm{Int}(D)_M$ is a valuation domain for any unitary maximal ideal M. We begin with some elementary results.

4.6. Proposition. *There exist positive integers n and r such that $((f(x))^r - f(x))^n/\pi \in \mathrm{Int}(D)$ for any $f(x) \in \mathrm{Int}(D)$.*

Proof. Since the set $\{|D/P_i| \mid i \in Z^+\} = \{q_i\}$ is bounded, it follows easily that there exists a positive integer r such that for any $d \in D$, for any $f(x) \in \mathrm{Int}(D)$, and for any $i \in Z^+$ we have either $(f(d)^r - f(d)) = 0$ or $v_i((f(d))^r - f(d)) > 0$. Then Corollary 3.10 implies that there exists a positive integer n such that for any $d \in D$, for any $f(x) \in \mathrm{Int}(D)$, and for any $i \in Z^+$ we have either $(f(d)^r - f(d)) = 0$ or $v_i[(f(d)^r - f(d))^n] \in Z^+$. Since $v_i(\pi) = 1$ for every $i \in Z^+$, the result follows. □

Note. We assume n and r to be as in Proposition 4.6 for the duration of Section 4 and throughout Section 5.

4.7. Corollary. *If $f(x) \in \mathrm{Int}(D)$, then $((f(x))^r - f(x))$ is contained in every unitary maximal ideal of* $\mathrm{Int}(D)$.

Proof. If $(f(x))^r - f(x) = 0$, the result is clear. If $(f(x))^r - f(x) \neq 0$, then Proposition 4.6 implies that $1/\pi \in \mathrm{Int}(D)[1/((f(x))^r - f(x))]$. Since π is contained in every maximal ideal of D, it follows that $K[x] \subseteq \mathrm{Int}(D)[1/((f(x))^r - f(x))]$. The result follows immediately. □

4.8. Corollary. *Every unitary prime ideal of* $\mathrm{Int}(D)$ *is maximal.*

Proof. Suppose not. Choose a nonmaximal unitary prime ideal M_1 and choose a maximal ideal M such that $M_1 \subseteq M$. Choose $f(x) \in M \backslash M_1$. Proposition 4.6 implies that $[(f(x))^r - f(x)]^n/\pi \in \mathrm{Int}(D)$. Since π is contained in every maximal ideal of $D, \pi \in M_1$. Hence $(f(x))^n[(f(x))^{r-1} - 1]^n = [(f(x))^r - f(x)]^n = \pi[((f(x))^r - f(x))^n/\pi] \in M_1$. $(f(x))^n \notin M_1$ so $(f(x))^{r-1} - 1 \in M_1$. Hence, $f(x) \in M$ and $(f(x))^{r-1} - 1 \in M_1 \subseteq M$ and so $1 \in M$, a contradiction. □

4.9. Lemma. *Suppose that A is an integrally closed domain with quotient field F. Then* $\mathrm{Int}(A)$ *is integrally closed.*

Proof. Suppose not. Then there exists $f(x) \in F(x) \backslash \mathrm{Int}(A)$ and a monic polynomial $g(y) \in \mathrm{Int}(A)[y]$ such that $g(f(x)) = 0$. The fact that $F[x]$ is integrally closed forces $f(x) \in F[x]$. Then $f(x) \notin \mathrm{Int}(A)$ implies that $f(a) \notin A$ for some $a \in A$. Now evaluate the coefficients of $g(y)$ at $x = a$. The result is a monic polynomial $g_1(y) \in A[y]$ such that $g_1(f(a)) = 0$. This contradicts the integral closure of A. □

Now we focus explicitly on proving that $\mathrm{Int}(D)_M$ is a valuation domain for any unitary maximal ideal M. The proof will be broken into a sequence of propositions and will be accomplished by showing that if $f(x), g(x) \in \mathrm{Int}(D) \backslash \{0\}$, then either $f(x)/g(x) \in \mathrm{Int}(D)_M$ or $g(x)/f(x) \in \mathrm{Int}(D)_M$.

4.10. Proposition. *Suppose that M is a unitary maximal ideal of* $\mathrm{Int}(D)$, *and that* $f(x) \in M$. *Then either* $(f(x))^n/\pi^j \in M\,\mathrm{Int}(D)_M$ *for all* $j \in Z^+$ *or* $(f(x))^n/\pi^j$ *is a unit in* $\mathrm{Int}(D)_M$ *for some* $j \in Z^+$.

Proof. Let $f_0(x) = f(x)$, let $f_1(x) = [(f_0(x))^r - f_0(x)]^n/\pi = (f_0(x))^n[(f_0(x))^{r-1} - 1]^n/\pi$ and for $j > 1$, let $f_j(x) = f_{j-1}(x)[(f_{j-1}(x))^{r-1} - 1]^n/\pi$. Proposition 4.6 implies that $f_1(x) \in \mathrm{Int}(D)$. Note that for $j > 1, f_{j-1}(x)$ is not raised to the exponent n in the definition of $f_j(x)$. Nevertheless, $f_j(x) \in \mathrm{Int}(D)$ for each $j \in Z^+$. To justify this, we first say that a polynomial $g(x) \in K[x]$ satisfies property $*$ provided for every $i \in Z^+$ and for every $d \in D$, either $g(d) = 0$ or $v_i(g(d)) \in Z^+ \cup \{0\}$. The choice of n (see the proof of Proposition 4.6) insures that $f_1(x)$ satisfies $*$. We claim that $f_j(x)$ satisfies $*$ for each $j \in Z^+$. It will then follow easily that $f_j(x) \in \mathrm{Int}(D)$ for all $j \in Z^+$. Suppose that $f_t(x)$ satisfies $*$ for some $t \geq 1$. We want to show that $f_{t+1}(x)$ satisfies $*$. Let $g_{t+1}(x) = [f_t(x)]^n[(f_t(x))^{r-1} - 1]^n/\pi$. The choice of n insures that $g_{t+1}(x)$ satisfies $*$. Choose $i \in Z^+$ and $d \in D$ and suppose that $g_{t+1}(d) \neq 0$. Suppose that $v_i(f_t(d)) = 0$. Then $v_i(f_{t+1}(d)) = v_i(g_{t+1}(d)) \in Z^+ \cup \{0\}$. Suppose that $v_i(f_t(d)) > 0$. Then $v_i(f_{t+1}(d)) = v_i(f_t(d)/\pi) = v_i(f_t(d)) - 1 \in Z^+ \cup \{0\}$. Finally, note that $f_{t+1}(d) = 0$ if and only if $g_{t+1}(d) = 0$. Hence, $f_{t+1}(x)$ satisfies $*$ and it follows that $f_j(x) \in \mathrm{Int}(D)$ for each $j \in Z^+$. Suppose that $k \in Z^+$ such that $f_j(x) \in M$ for $0 \leq j \leq k$. Then $[(f_j(x))^{r-1} - 1]^n \notin M$ for each $0 \leq j \leq k$. Hence, $f_{k+1}(x) = (f(x))^n h_{k+1}(x)/\pi^{k+1}$ where $h_{k+1}(x)$ is a product of elements of $\mathrm{Int}(D)$ which do not lie in M. Hence, $(f(x))^n/\pi^{k+1} \in \mathrm{Int}(D)_M$. Suppose that $f_{k+1}(x) \notin M$. Then $(f(x))^n/\pi^{k+1}$ is a unit in $\mathrm{Int}(D)_M$. However, if $f_j(x) \in M$ for all $j \in Z^+$, then $(f(x))^n/\pi^j \in M\mathrm{Int}(D)_M$ for all $j \in Z^+$. $\square$

4.11. Proposition. *Suppose that M is a unitary maximal ideal of* $\mathrm{Int}(D)$ *and that* $f(x), g(x) \in \mathrm{Int}(D)\backslash\{0\}$ *such that* $(f(x))^n/\pi^j$ *and* $(g(x))^n/\pi^j \in \mathrm{Int}(D)_M$ *for all* $j \in Z^+$. *Then $f(x)$ and $g(x)$ have a common factor over* $K[x]$.

Proof. Suppose not. Then there exist $h(x), k(x) \in \mathrm{Int}(D)$ such that $h(x)(f(x))^n + k(x)(g(x))^n = d$ for some $d \in D\backslash\{0\}$. It follows that $d/\pi^j \in \mathrm{Int}(D)_M$ for all $j \in Z^+$, which is impossible. $\square$

4.12. Proposition. *Let M be a unitary maximal ideal of* $\mathrm{Int}(D)$ *and let* $f(x), g(x) \in \mathrm{Int}(D)\backslash\{0\}$. *Then, either* $f(x)/g(x) \in \mathrm{Int}(D)_M$ *or* $g(x)/f(x) \in \mathrm{Int}(D)_M$.

Proof. If either $f(x) \notin M$ or $g(x) \notin M$, the result is trivial. So, suppose that $f(x), g(x) \in M\backslash\{0\}$. There are three cases to consider.

(1) Suppose there exists $j_f, j_g \in Z^+$ such that $(f(x))^n/\pi^{j_f}$ and $(g(x))^n/\pi^{j_g}$ are units in $\mathrm{Int}(D)_M$. Suppose without loss of generality that $j_g \geq j_f$. Then $(g(x))^n/\pi^{j_f} \in \mathrm{Int}(D)_M$ and hence $(g(x)/f(x))^n = (\pi^{j_f}/(f(x))^n)((g(x))^n/\pi^{j_f}) \in \mathrm{Int}(D)_M$. Then Lemma 4.9 implies that $g(x)/f(x) \in \mathrm{Int}(D)_M$.

(2) Suppose there exists $j_f \in Z^+$ such that $(f(x))^n/\pi^{j_f}$ is a unit in $\mathrm{Int}(D)_M$ and suppose that $(g(x))^n/\pi^j \in \mathrm{Int}(D)_M$ for all $j \in Z^+$. In this case $(g(x))^n/\pi^{j_f} \in \mathrm{Int}(D)_M$ and so $(g(x)/f(x))^n = (\pi^{j_f}/(f(x))^n)((g(x))^n/\pi^{j_f}) \in \mathrm{Int}(D)$. Then Lemma 4.9 again implies that $g(x)/f(x) \in \mathrm{Int}(D)_M$.

(3) Suppose that $(g(x))^n/\pi^j$ and $(f(x))^n/\pi^j \in \mathrm{Int}(D)_M$ for all $j \in Z^+$. Proposition 4.11 implies that $f(x)$ and $g(x)$ have a common factor over $K[x]$. Write $g(x)/f(x) = h(x)/k(x)$ where $h(x), k(x) \in \mathrm{Int}(D)$, but $h(x)$ and $k(x)$ have no common factors over $K[x]$. Then we need to show that either $h(x)/k(x) \in \mathrm{Int}(D)_M$ or $k(x)/h(x) \in \mathrm{Int}(D)_M$. Since $h(x)$ and $k(x)$ have no common factors, this is settled by one of the previous cases.

Proposition 4.10 indicates that one of the cases considered must hold for $f(x)$ and $g(x)$. Hence, the result is proven. □

4.13. Corollary. *If M is a unitary maximal ideal of* Int(D), *then* Int$(D)_M$ *is a valuation domain.*

Proof. The result follows immediately from Proposition 4.12. □

We now have all of the necessary components to state the main theorem of this section.

4.14. Theorem. *If D is a doubly bounded sequence domain then* Int(D) *is Prüfer.*

Proof. Lemma 4.3 and Corollary 4.13, taken together, show that if M is any maximal ideal of Int(D) then Int$(D)_M$ is a valuation domain. This is sufficient to show that Int(D) is Prüfer. □

5. Maximal ideals of Int(D)

In this section we analyze the structure of the unitary maximal ideals of Int(D) when D is a doubly bounded sequence domain. First, however, we make two brief digressions.

5.1. Definition. Let B be an infinite set and let U be a collection of nonempty subsets of B. We say that U is an *ultrafilter* on B provided it satisfies the following three properties:

(1) If $C, E \in U$, then $C \cap E \in U$.

(2) If $C \in U$ and $C \subseteq E \subseteq B$ then $E \in U$.

(3) If $B = C \cup E$ and $C \cap E = \emptyset$, then either $C \in U$ or $E \in U$.

Several relevant facts concerning ultrafilters are given in the following lemma. The results follow easily from Definition 5.1 and the proof will be omitted.

5.2. Lemma. *The following statements are valid for any ultrafilter U on any infinite set B:*

(1) *If $C, E \in U$ then $C \cap E$ is nonempty.*

(2) *If $B = C \cup E$, $C \cap E = \emptyset$ and $C \in U$, then $E \notin U$.*

(3) *If $C \in U, C = E \cup G$, and $E \cap G = \emptyset$, then either $E \in U$ and $G \notin U$ or $G \in U$ and $E \notin U$.* □

Given an infinite set B, it is easy to produce examples of ultrafilters on B. In particular if $\alpha \in B$, then the collection $U_\alpha = \{C \subseteq B | \alpha \in C\}$ is an ultrafilter on B. Ultrafilters of this type will be called principal ultrafilters. The next result gives us a means of producing nonprincipal ultrafilters.

5.3. Proposition. *Suppose that B is an infinite set and that U_1 is a collection of nonempty subsets of B which satisfies condition 1 of Definition 5.1. Then U_1 can be extended to an ultrafilter U on B.*

Proof. Suppose that $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ is a tower of collections of subsets of B, each of which satisfies condition 1. Then $U_\infty = \bigcup_{i=1}^{\infty} U_i$ also satisfies condition 1. Hence, Zorn's Lemma can be applied to obtain a collection U of subsets of B which contains U_1 and is maximal with respect to satisfying condition 1. Conditions 2 and 3 follow easily from the maximality of U. □

5.4. Corollary. *Let B be an infinite set. A nonprincipal ultrafilter U on B exists.*

Proof. Let $U_1 = \{C \subseteq B | B \setminus C$ is finite or empty$\}$. It is apparent that U_1 satisfies condition 1 of Definition 5.1 and that no single element of B is contained in all of the members of U_1. Hence, Proposition 5.3 can be applied to extend U_1 to a nonprincipal ultrafilter on B. □

Before we proceed to our discussion of the maximal ideals of Int(D), we consider the ideal structure of Int(V) where V is a Noetherian valuation domain with maximal ideal Q and with finite residue field. It is well known that Int(V) is Prüfer. The following result characterizes the unitary maximal ideals of Int(V). The assertions of Proposition 5.5 and of Corollary 5.6 are all either contained in or are easy consequences of [1, Proposition 2.2] and will not be proven here.

5.5. Proposition. *Let V and Q be as in the preceding paragraph and let $\hat{V}_Q$ be the Q-adic completion of V. The unitary maximal ideals of* Int(V) *can be naturally indexed by the elements of $\hat{V}_Q$. In particular, if $\alpha \in \hat{V}_Q$, the ideal $M_\alpha = \{f(x) \in \text{Int}(V) | f(\alpha) \in Q\hat{V}_Q\}$ is a unitary maximal ideal of* Int(V) *and, conversely, each unitary maximal ideal of* Int(V) *is equal to M_α for some $\alpha \in \hat{V}_Q$.*

5.6. Corollary. *Let V and Q be as in Proposition 5.5 and let $\beta \in V$ be a generator of the ideal Q. Choose $\alpha \in \hat{V}_Q$. Then the following statements hold concerning the* *ideal* M_α:

(1) $|\mathrm{Int}(V)/M_\alpha| = |V/Q|$.

(2) $M_\alpha \mathrm{Int}(V)_{M_\alpha}$ *is a principal ideal of* $\mathrm{Int}(V)_{M_\alpha}$ *generated by* β.

Now we turn to the subject of unitary maximal ideals of $\mathrm{Int}(D)$. We give two different constructions of such ideals, each involving ultrafilters and P_i-adic completions of D. The first construction is simpler and, perhaps, more intuitive than the latter, but it is unclear as to whether or not it yields all unitary maximal ideals of $\mathrm{Int}(V)$. The second construction is, perhaps, less intuitive, but does yield all unitary maximal ideals.

We begin by giving some motivation for our first construction. Note that Corollary 4.5 together with Proposition 5.5 have already given a complete characterization of the unitary maximal ideals of $\mathrm{Int}(D)$ which lie over P_i for any $i \in Z^+$. In particular, for any $i \in Z^+$ these ideals consist of the restrictions to $\mathrm{Int}(D)$ of the unitary maximal ideals of $\mathrm{Int}(D_{P_i})$ (i.e., maximal ideals of $\mathrm{Int}(D)$ lying above P_i are all well behaved) and the unitary maximal ideals of $\mathrm{Int}(D_{P_i})$ are characterized as in Proposition 5.5 using elements of $\hat{D}_{P_i}$ (the completion of D with respect to P_i). As noted in Section 4, the maximal ideals of $\mathrm{Int}(D)$ lying above P^* are problematic. The restrictions to $\mathrm{Int}(D)$ of the unitary maximal ideals of $\mathrm{Int}(D_{P^*})$ are well-behaved maximal ideals lying above P^*. However, there may also be unruly maximal ideals of $\mathrm{Int}(D)$ lying above P^*. To further analyze this issue, recall that the original intuitive view of P^* was that it was the "limit" of the sequence $\{P_i | i \in Z^+\}$. Hence, it seems plausible that a unitary maximal ideal of $\mathrm{Int}(D)$ lying over P^* could be viewed as a "limit" of a sequence $\{M_i | i \in Z^+\}$ with each M_i being a unitary maximal ideal of $\mathrm{Int}(D)$ lying over P_i. Or, in view of Proposition 5.5, the sequence of M_i's might be replaced by a sequence $\{\alpha_i | i \in Z^+\}$ with each $\alpha_i \in \hat{D}_{P_i}$.

5.7. Construction.

(1) Choose a sequence $B = \{\alpha_i | i \in Z^+\}$ such that for each i, $\alpha_i \in \hat{D}_{P_i}$.

(2) Let U be an ultrafilter on B.

(3) Given $f(x) \in \mathrm{Int}(D)$ and $C \in U$, we say that $f(x)$ satisfies C provided $f(\alpha_i) \in P_i\hat{D}_{P_i}$ for each $\alpha_i \in C$.

(4) Let $M_{B(U)} = \{f(x) \in \mathrm{Int}(D) | f(x)$ satisfies some set $C \in U\}$.

5.8. Theorem. *Assume the terminology of Construction* 5.7. *Then* $M_{B(U)}$ *is a unitary maximal ideal of* $\mathrm{Int}(D)$.

Proof. First we show that $M_{B(U)}$ is a unitary ideal of $\mathrm{Int}(D)$. Note that $\pi \in M_{B(U)}$ and so $M_{B(U)}$ is unitary. Choose $f(x), g(x) \in M_{B(U)}$. Then choose $C_f, C_g \in U$ which are satisfied by $f(x)$ and $g(x)$ respectively. Then $C_f \cap C_g$ lies in U and is satisfied by $f(x) + g(x)$. Hence, $f(x) + g(x) \in M_{B(U)}$. Now suppose that $f(x) \in M_{B(U)}$ and $g(x) \in \mathrm{Int}(D)$ and choose $C_f \in U$ which is satisfied by $f(x)$. Then $f(x)g(x)$ satisfies C_f as well and so $f(x)g(x) \in M_{B(U)}$. Hence, $M_{B(U)}$ is an ideal of $\mathrm{Int}(D)$.

Now we show that $M_{B(U)}$ is prime. Choose $f(x), g(x) \in \mathrm{Int}(D)$ such that $f(x)g(x) \in M_{B(U)}$ and choose $C_{fg} \in U$ which is satisfied by $f(x)g(x)$. Let $C' = \{\alpha_i \in C_{fg} | f(\alpha_i) \in P_i\hat{D}_{P_i}\}$. If $C' = C_{fg}$, then $f(x)$ satisfies $C' \in U$ and so $f(x) \in M_{B(U)}$. Suppose that $C' \neq C_{fg}$ and let $C'' = C_{fg} \backslash C'$. If $C'' = C_{fg}$, then $g(x)$ satisfies $C'' \in U$ and so $g(x) \in M_{B(U)}$. Suppose that C' and C'' are both nonempty. Then $f(x)$ satisfies C' and $g(x)$ satisfies C''. Moreover, Lemma 5.2 (part 3) implies that either $C' \in U$ or $C'' \in U$. Hence, either $f(x) \in M_{B(U)}$ or $g(x) \in M_{B(U)}$.

Finally, Corollary 4.8 implies that $M_{B(U)}$ is maximal since it is both prime and unitary. □

There is no indication in either Construction 5.7 or Theorem 5.8 that every unitary maximal ideal of $\mathrm{Int}(D)$ has the form $M_{B(U)}$. However, we can exploit Construction 5.7 to determine the existence or nonexistence of unruly ideals of $\mathrm{Int}(D)$. Recall (Note 4.2 (part 5)) that a unitary maximal ideal of $\mathrm{Int}(D)$ is well behaved if it is the restriction to $\mathrm{Int}(D)$ of a unitary maximal ideal of $\mathrm{Int}(D_P)$ for some maximal ideal P of D and is called unruly otherwise. These notions of well-behaved and unruly ideals carry over naturally to the setting of $\mathrm{Int}(A)$ where A is an almost Dedekind domain with finite residue fields. In this regard, we make the following definition.

5.9. Definition. Suppose that A is an almost Dedekind domain with all residue fields finite. We say that $\mathrm{Int}(A)$ is *well behaved under localization* provided every unitary maximal ideal of $\mathrm{Int}(A)$ is well behaved.

For every domain A in the literature for which it is known that $\mathrm{Int}(A)$ is Prüfer, it is also known that $\mathrm{Int}(A)$ behaves well under localization. Also, Chabert has proven the following related result.

5.10. Theorem [3, Theorem 2.1]. *Suppose that A is an almost Dedekind domain with finite residue fields and that* $\mathrm{Int}(A)$ *behaves well under localization. Then* $\mathrm{Int}(A)$ *is Prüfer.* □

Chabert also asks the following question.

5.11 (Q7). For $\mathrm{Int}(A)$ to be Prüfer (A almost Dedekind with finite residue fields), is it necessary that it behaves well under localization?

We utilize Construction 5.7 to give a complete characterization of doubly bounded sequence domains D such that $\mathrm{Int}(D)$ behaves well under localization. In Section 6 we utilize the results obtained here to give a negative answer to Chabert's question Q7.

We begin by recalling that if M is a unitary maximal ideal of $\mathrm{Int}(D)$ which lies over P_i for some $i \in Z^+$, then M is well-behaved (see Corollary 4.5). Also note that such a maximal ideal can be constructed by Construction 5.7. In particular, suppose that $M \cap D = P_i$ and that $M = \{f(x) \in \mathrm{Int}(D) | f(\alpha_i) \in P_i\hat{D}_{P_i}\}$ for some $\alpha_i \in \hat{D}_{P_i}$. For each $j \neq i$ choose $\alpha_j \in \hat{D}_{P_j}$ and let $B = \{\alpha_j | j \in Z^+\}$. Then let U_{α_i} be the principal ultrafilter on B consisting of all subsets of B which contain α_i. Then $M = M_{B(U_{\alpha_i})}$. Hence, we look to nonprincipal ultrafilters to find unruly ideals of Int(D).

Next we recall a result from [10, Corollary 12]. The statement given here is different from that given in [10]. In particular, the object in [10] was to prove that Int(D) was Prüfer, while the object here is to prove that Int(D) behaves well under localization. The proof given in [10] proves that Int(D) is Prüfer by first showing that it behaves well under localization. Hence, the result given here is actually proven in [10].

5.12. Proposition. *Suppose the set C of all maximal ideals of D can be partitioned into subsets $\{C_i | i \in I\}$ such that for each $i \in I$, the following hold.*

(1) *There exists $t_i \in D$ such that $t_i D_P = PD_P$ for each maximal ideal $P \in C_i$. Also, t_i is a unit in D_P for each maximal ideal P of D such that $P \notin C_i$.*

(2) *There exists a set $W_i = \{\omega_0, \omega_1, \ldots, \omega_{m_i}\} \subseteq D$ which constitutes a complete residue system for D modulo P for each maximal ideal P in C_i.*

Then Int(D) *behaves well under localization.*

Now we are prepared to give our classification theorem.

5.13. Theorem. Int(D) *behaves well under localization if and only if both of the following conditions hold:*

(1) $q_i = |D/P_i| = |D/P^*|$ *for all but finitely many $i \in Z^+$.*

(2) $v_i = v_i^{(\mathrm{N})}$ *for all but finitely many $i \in Z^+$.*

Proof. Suppose first that there exists $m \in Z^+$ such that $q_i = |D/P^*|$ and $v_i = v_i^{(\mathrm{N})}$ for all $i \geq m$. Let $W^* = \{\omega_0, \omega_1, \ldots, \omega_{q^*-1}\}$ be a complete residue system for P^* in D. It follows easily from the proof of Proposition 3.3 that we can assume, without loss of generality, that W^* also serves as a complete system of residues for all P_i in D such that $i \geq m$. Now partition the maximal ideals of D as follows: For $1 \leq i \leq m-1$, let $C_i = \{P_i\}$ and let $C_m = \{P_i | i \geq m\} \cup \{P^*\}$. Since each P_i is principal, we can choose $\pi_m \in D$ such that $v_i^{(\mathrm{N})}(\pi_m) = 1$ if $i \geq m$ and $v_i^{(\mathrm{N})}(\pi_m) = 0$ if $i < m$. Then π_m and W^* can play the roles of t_m and W_m in Proposition 5.12. For $i < m$, ρ_i and any complete residue system for P_i can play the roles of t_i and W_i in Proposition 5.12. Hence, Proposition 5.12 implies that Int(D) behaves well under localization.

Now suppose that there exists an infinite set of positive integers $I = \{n_1, n_2, n_3, \ldots\}$ such that $q_{n_i} > |D/P^*|$ for each n_i. Let $q^* = |D/P^*|$. For each $i \in Z^+$ choose $\alpha_i \in D$ such that $\alpha_i + P_i$ has order $q_i - 1$ in the multiplicative group $(D/P_i)\backslash\{0 + P_i\}$. Let $B = \{\alpha_i | i \in Z^+\}$ and let $B_N = \{\alpha_{n_i} | n_i \in I\}$. We want to build a nonprincipal ultrafilter U on B such that $B_N \in U$. To do this, recall that the set $U_1 = \{E \subseteq B | B\backslash E$ is finite or empty$\}$ satisfies condition 1 of the definition (5.1) of ultrafilter. Let $U_2 = \{E \cap B_N | E \in U_1\} \cup U_1$. U_2 also satisfies condition 1 of Definition 5.1 and, hence, we can apply Proposition 5.3 to extend U_2 to an ultrafilter U on B. Note that $B_N \in U_2$ and $U_2 \subseteq U$ so that $B_N \in U$. Also U is nonprincipal since it extends U_1 (see proof of Corollary 5.4).

We claim that $M_{B(U)}$ is unruly. Note first that $\rho_i \notin M_{B(U)}$ for each i and so $M_{B(U)}$ lies over P^*. Corollary 5.6 implies that a well-behaved maximal ideal which lies over P^* should have residue field of order q^*. We show that $|\mathrm{Int}(D)/M_{B(U)}| > q^*$ by showing that $f(x) = x^{q^*} - x \notin M_{B(U)}$. Suppose that $f(x) \in M_{B(U)}$. Then $f(x)$ satisfies some set $E \in U$, and hence also satisfies $B_N \cap E \subset U$. However, $f(\alpha_{n_i}) \notin P_{n_i}$ for each $\alpha_{n_i} \in B_N$ and so $f(x)$ does not satisfy $B_N \cap E$, a contradiction. It follows that $M_{B(U)}$ is unruly.

Now suppose that $I = \{n_1, n_2, n_3, \ldots\}$ is an infinite set of positive integers such that v_{n_i} is not a normed valuation for each $n_i \in I$. In particular, $v_{n_i}^{(\mathrm{N})}(\pi) > 1$ for each n_i. Let $B = \{\rho_i | i \in Z^+\}$ and let $B_N = \{\rho_{n_i} | n_i \in I\}$. As in the preceding paragraph, we build a nonprincipal ultrafilter U on B such that $B_N \in U$. Again, we claim that $M_{B(U)}$ is unruly. $\rho_i \notin M_{B(U)}$ for each $i \in Z^+$ and so $M_{B(U)}$ lies over P^*. Note that $P^* D_{P^*} = \pi D_{P^*}$. Hence, if $M_{B(U)}$ is well-behaved, Corollary 5.6 implies that $M_{B(U)} \mathrm{Int}(D)_{M_{B(U)}} = \pi\, \mathrm{Int}(D)_{M_{B(U)}}$. Suppose that $M_{B(U)}$ is well-behaved. Now consider the polynomial $g(x) = x$. Clearly, $g(x)$ satisfies B_N and so $g(x) \in M_{B(U)}$. Hence, $g(x) \in \pi\ \mathrm{Int}(D)_{M_{B(U)}}$ and so $g(x)/\pi \in \mathrm{Int}(D)_{M_{B(U)}}$. Then we can write $g(x)/\pi = h(x)/k(x)$ such that $h(x)$, $k(x) \in \mathrm{Int}(D)$ and $k(x) \notin M_{B(U)}$. Then $g(x) = \pi h(x)/k(x)$. By our choice of B_N, $v_{n_i}(\pi) > v_{n_i}(g(\rho_{n_i}))$ for each $n_i \in I$. Since, $h(x) \in \mathrm{Int}(D)$, we must then have $v_{n_i}(k(\rho_{n_i})) > 0$ for each $n_i \in I$. Thus, $k(x)$ satisfies B_N and so $k(x) \in M_{B(U)}$, a contradiction. It follows that $M_{B(U)}$ is unruly. This completes the proof of Theorem 5.13. $\square$

The method given in Construction 5.7 for constructing maximal ideals $M_{B(U)}$ does contain some redundancy. For example, choose $d \in D$ and define $B = \{\alpha_i | i \in Z^+\}$ with $\alpha_i = d$ for each i. Now let $f(x) \in \mathrm{Int}(D)$. The definition of sequence domain implies that either $f(d) \in P_i$ for all but finitely many values of i or $f(d) \notin P_i$ for all but finitely many values of i. It follows that applying Construction 5.7 to two different nonprincipal ultrafilters on B would give rise to the same maximal ideal. We do not know the extent of this redundancy. This example hints that perhaps the ultrafilters on sequences could be replaced by some type of convergence condition on sequences. We are currently unaware of what such a condition would be.

Also, as noted previously, it is not clear that Construction 5.7 yields all of the unitary maximal ideals of $\mathrm{Int}(D)$. However, we next give a slightly different construction, also using ultrafilters, which does yield all of the unitary maximal ideals of $\mathrm{Int}(D)$. We begin with several lemmas, in the spirit of Proposition 4.6 and Corollary 4.7, which describe which polynomials lie in various unitary maximal ideals of $\mathrm{Int}(D)$.

5.14. Lemma. *Suppose $f(x) \in \mathrm{Int}(D)$ such that $f(d)$ is a unit in D for all $d \in D$. Then $f(x)$ is not contained in any unitary maximal ideal of* $\mathrm{Int}(D)$.

Proof. Suppose that M is a unitary maximal ideal of $\mathrm{Int}(D)$ such that $f(x) \in M$. Proposition 4.6 implies that $((f(x))^r - f(x))^n/\pi \in \mathrm{Int}(D)$. Since $f(x)$ is a unit for all $d \in D$, this implies that $((f(x))^{r-1} - 1)/\pi \in \mathrm{Int}(D)$. Then since π lies in every maximal ideal of D, $\pi \in M$ and so $((f(x))^{r-1} - 1)^n = \pi[((f(x))^{r-1} - 1)^n/\pi] \in M$. However, $((f(x))^{r-1} - 1)^n \in M$ and $f(x) \in M$ implies $1 \in M$, a contradiction. $\square$

5.15. Lemma. *There exists a monic, irreducible polynomial $h(y) \in \mathrm{Int}(D)[y]$ of degree $m \geq 2$ with coefficients in the set $\{0, 1, 2, \ldots, p-1\}$ which is a uv-polynomial for $\mathrm{Int}(D)_M$ for each maximal unitary ideal M.*

Proof. Corollary 4.7 implies that the set $\{|\mathrm{Int}(D)/M| \mid M$ a unitary maximal ideal of $\mathrm{Int}(D)\}$ is bounded. Hence, as in the proof of Proposition 3.5, we can choose $h(y)$ to be any monic polynomial of degree $m \geq 2$, with coefficients in the set $\{0, 1, \ldots, p-1\}$ which is irreducible over each residue field $\mathrm{Int}(D)/M$ (M a unitary maximal ideal). □

5.16. Lemma. *Let $h(y)$ be as in Lemma 5.15. Let $f(x), g(x) \in \mathrm{Int}(D)$ and let*

$$k(x) = \begin{cases} (g(x))^m h(f(x)/g(x)) & \text{if } g(x) \neq 0, \\ (f(x))^m & \text{if } g(x) = 0. \end{cases}$$

Then $k(x)$ is contained in exactly the set of unitary maximal ideals which contain both $f(x)$ and $g(x)$.

Proof. [8, Proposition 1.12] implies that for each unitary maximal ideal M of $\mathrm{Int}(D)$, $k(x) \in M$ if and only if $f(x), g(x) \in M$. The result follows immediately. □

Now we turn our attention to our second construction. Our method is very similar to that of Construction 5.7, with the principal difference being that the set on which we define the ultrafilters is larger. The set B used in Construction 5.7 essentially amounted to a collection of well-behaved ideals of $\mathrm{Int}(D)$ lying over the ideals P_i of D such that B contained exactly one such ideal for each $i \in Z^+$. (See the discussion preceding Construction 5.7.) In Construction 5.17 we expand to the set of all well-behaved ideals of $\mathrm{Int}(D)$ lying over each P_i for $i \in Z^+$.

5.17. Construction.

(1) Let H be the collection of ordered pairs

$$H = \{(\alpha, P_i) | i \in Z^+, \alpha \in \hat{D}_{P_i}\}.$$

(Note that it is important to specify the maximal ideal P_i as well as the P_i-adic element α. In particular, if $\alpha \in D$ and $i \neq j$, then $(\alpha, P_i) \neq (\alpha, P_j)$.)

(2) Let U be an ultrafilter on H.

(3) Given $f(x) \in \mathrm{Int}(D)$ and $C \in U$, we say that $f(x)$ satisfies C provided $f(\alpha) \in P_i\hat{D}_{P_i}$ for each $(\alpha, P_i) \in C$.

(4) Let $M[H, U] = \{f(x) \in \mathrm{Int}(D) | f(x)$ satisfies some set $C \in U\}$. (We call an ideal of the form $M[H, U]$ an mul (maximal-ultrafilter) ideal.)

5.18. Proposition. *Assume the terminology of Construction 5.17. Then $M[H, U]$ is a unitary maximal ideal of $\mathrm{Int}(D)$.*

Proof. The proof is virtually identical to that of Theorem 5.8 and will be omitted. □

Now we prove that each unitary maximal ideal of Int(D) is an mul-ideal. We accomplish this by showing that each unitary ideal of Int(D) is contained in an mul-ideal. For this we need one additional piece of notation and a preliminary lemma.

5.19. Note. For $f(x) \in \text{Int}(D)$, let $E(f) = \{(\alpha, P_i) \in H \mid f(\alpha) \in P_i\hat{D}_{P_i}\}$.

5.20. Lemma. *Let I be a unitary ideal of* Int(D) *and let $f(x), g(x) \in I$. Then there exists a polynomial $k(x) \in I$ such that $E(f) \cap E(g) = E(k)$ and $E(k)$ is not empty.*

Proof. For each polynomial $k_1(x) \in \text{Int}(D)$ and each ordered pair $(\alpha, P_i) \in H$ the statement $(\alpha, P_i) \in E(k_1)$ is equivalent to saying that $k_1(x)$ lies in the well-behaved ideal $M = \{h(x) \in \text{Int}(D) \mid h(\alpha) \in P_i\hat{D}_{P_i}\}$. Viewed from this perspective, Lemma 5.16 immediately implies the existence of a polynomial $k(x)$ such that $k(x) \in I$ and $E(k) = E(f) \cap E(g)$. Then since $k(x) \in I$ and I is unitary, Lemma 5.14 implies that $E(k)$ is not empty. □

5.21. Theorem. *Every unitary maximal ideal of* Int(D) *is an mul-ideal.*

Proof. Let I be a unitary ideal of Int(D). Let $U_1 = \{E(f) \mid f(x) \in I\}$. Lemma 5.20 implies that U_1 satisfies condition 1 of the definition of ultrafilter. Hence, Proposition 5.3 implies that U_1 can be extended to an ultrafilter on H. Let U be such an extension. Define $M[H, U]$ as in Construction 5.17. Since $U_1 \subseteq U$, it follows immediately that $I \subseteq M[H, U]$. Hence, every unitary ideal of Int(D) is contained in an mul-ideal. It follows that every unitary maximal ideal is an mul-ideal. □

We close this section with some comments regarding Construction 5.17 and Theorem 5.21. As noted in the discussion preceding Construction 5.17, the ordered pairs in the set H used in that construction correspond in a natural way to well-behaved ideals of Int(D) lying over the ideals P_i of D. Just as we have viewed P^* as being the "limit" of the sequence $\{P_i \mid i \in Z^+\}$, it seems plausible that the unitary maximal ideals of Int(D), lying over P^*, could be viewed as "limits" of sequences of unitary maximal ideals lying over P_i's. Hence, the ultrafilter description of maximal ideals given in this note is probably not a final answer to describing the unitary maximal ideals of Int(D). P-adic completions of D are perfectly adequate to handle the well-behaved ideals. So, perhaps the focus should be on expanding the P-adic completions to some larger topological spaces which can accommodate the unruly ideals as well.

6. Examples

In this section we construct examples of sequence domain overrings of $Z[x]$ with various different properties. The construction relies heavily on the work of MacLane in [11]. We will review the necessary results from [11].

Our method consists of constructing a sequence $\{W_i | i \in Z^+\}$ of Noetherian valuation domains such that $D = \bigcap_{i=1}^{\infty} W_i$ is a sequence domain with the desired properties. In particular, $W_i = D_{P_i}$ where P_i is as in Sections 3–5.

We need to specify some of the notations and assumptions we will be using.

6.1. Preliminaries. (1) All valuation domains will be constructed by extending p-adic valuations on Q to valuations on $Q(x)$ which correspond to valuation overrings of $Z[x]$. MacLane deals with the more general problem of extending valuations of a field K to the field $K(x)$. For our purposes, little would be gained from utilizing this extra generality.

(2) For any valuation v_i on $Q(x)$ constructed, we assume that Γ_i, the value group of v_i, is a cyclic subgroup of the additive group of rational numbers and is generated by $1/e_i$ for some $e_i \in Z^+$. This is also more restrictive than MacLane's work, but the restriction is essential for our purposes.

(3) For any valuation v considered, we assume $v(0) = \infty$. Occasionally (it will be made explicit when it occurs), we will allow $v(f(x)) = \infty$ for a nonzero polynomial $f(x)$. We define $a + \infty = \infty$ for any real number a.

We now give a 4-step outline of the construction of a single Noetherian valuation domain W to be used in constructing a sequence domain D.

6.2. Construction.

Step 1. [11, Section 3] Begin with a prime $p \in Z^+$ and the p-adic valuation ω_p on Q. Extend ω_p to a valuation v_1 on $Q(x)$ as follows:

(a) Choose a nonnegative rational number μ_1.

(b) For $a \in Q$ set $v_1(a) = \omega_p(a)$ and set $v_1(x) = \mu_1$.

(c) For $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ set

$$v_1(f(x)) = \min\{v_1(a_i x^i) | 0 \le i \le m\} = \min\{\omega_p(a_i) + i\mu_i | 0 \le i \le m\}.$$

(d) Extend v_1 multiplicatively to $Q(x)$.

Step 2. [11, Section 4] For a given $n \ge 2$, assume that v_{n-1} has been constructed. Then construct v_n as follows:

(a) Select a particular nonconstant, irreducible polynomial $\varphi_n(x)$, which is called a key polynomial. (Many, but not all, nonconstant irreducible polynomials are eligible to be selected as a key polynomial. We elaborate on some, but not all, of the criteria used in the sequel.)

(b) Choose a rational number μ_n such that $\mu_n > v_{n-1}(\varphi_n(x))$.

(c) Given $f(x) \in Q[x]$, write $f(x) = f_m(x)(\varphi_n(x))^m + f_{m-1}(x)(\varphi_n(x))^{m-1} + \cdots + f_1(x)(\varphi_n(x)) + f_0(x)$ such that each $f_i(x)$ is either zero or has degree smaller than $\varphi_n(x)$. (Simply divide by successive powers of $\varphi(x)$ to obtain the expansion.)

(d) Given $f(x) \in Q[x]$ as in (c), set $v_n(f(x)) = \min\{v_{n-1}(f_i(x)) + i\mu_n | 0 \le i \le m\}$. In particular, $v_n(\varphi_n(x)) = \mu_n$.

Step 3. [11, Section 6] For $f(x) \in Q[x]$, define $v_\infty(f(x)) = \lim_{i\to\infty} v_i(f(x))$. Part 1 of Proposition 6.3 implies that v_∞ is well defined provided we allow ∞ as a value for a nonzero polynomial. We call v_∞ a limit-valuation.

Step 4. Suppose that $v_\infty(f(x)) < \infty$ for all $f(x) \in Q[x]\backslash\{0\}$. We say in this case that v_∞ is finite. If v_∞ is finite, we extend it multiplicatively to $Q(x)$. Let W be the valuation domain corresponding to v_∞.

Each sequence domain constructed will be given as $D = \bigcap_{i=1}^{\infty} W_i$ with each W_i being given by Step 4 of Construction 6.2. Now we give several results concerning Construction 6.2 which will fill in details to show how Construction 6.2 can yield the specific valuation domains we require.

6.3. Proposition. *Suppose that the sequence $\{v_i | i \in Z^+\}$, and hence v_∞ as well, has been constructed using Construction 6.2. Then the following statements hold*:

(1) [11, Theorem 6.5] *For each $n \geq 2$ and for each $f(x) \in Q[x], v_n(f(x)) \geq v_{n-1}(f(x))$.*

(2) *If $f(x), g(x) \in Q[x]\backslash\{0\}$ are irreducible polynomials such that $v_\infty(f(x)) = v_\infty(g(x)) = \infty$, then $f(x) = cg(x)$ for some rational number c.*

(3) [11, Theorem 6.5] *If $f(x) \in Q[x]$ and $v_\infty(f(x)) < \infty$, then there exists $k_f \in Z^+$ such that $v_j(f(x)) = v_{k_f}(f(x))$ for all $j \geq k_f$.*

(3′)(Special case of 3) [11, Theorem 6.4] *For $n \geq 2, v_j(\varphi_n(x)) = \mu_n$ for all $j \geq n$.*

(4) [11, Theorem 13.1] *Let F_0 be the residue field of the valuation ω_p on Q. Then for $n \geq 1$, the residue field of v_n is a field $F_n(y)$ of rational functions in the variable y over the field F_n, which is a finite-degree integral extension of F_{n-1}. (Note that $F_1 = F_0$.)*

(5) [11, Theorem 12.1] *For each $n \geq 1$, $\deg(\varphi_n(x)) | \deg(\varphi_{n+1}(x))$ and $e_n | e_{n+1}$. In particular,* $\deg(\varphi_{n+1}(x)) = m_{n+1}\tau_n \deg(\varphi_n(x))$ *with* $m_{n+1} = \deg[F_{n+1} : F_n]$ *and* $\tau_{n+1} = e_{n+1}/e_n$.

(6) [11, Theorem 14.1] *If the sequence $\{\deg(\varphi_i(x)) | i \in Z^+\}$ is eventually constant and v_∞ is finite, then v_∞ is a rank one discrete valuation on $Q(x)$ with a finite residue field. (Say $F_\infty =$ the residue field of v_∞.) In particular, if* $\deg(\varphi_i) = \deg(\varphi_{i+1}) = \deg(\varphi_{i+2}) = \cdots$, *then $F_\infty = F_i$.*

(7) [11, Theorems 5.1 and 12.1] *If $f(x) \in Q[x]\backslash\{0\}$ and* $\deg(f(x)) < \deg(\varphi_n(x))$ *for some $n \in Z^+$, then $v_i(f(x)) = v_n(f(x))$ for all $i \geq n$.*

Proof. Proofs of 1, 3′, and 4–6 are given in [11] and 7 follows immediately from the two results cited from [11].

We prove 2 by contradiction. Suppose $f(x), g(x) \subset Q[x]\backslash\{0\}$ are irreducible polynomials such that $v_\infty(f(x)) = v_\infty(g(x)) = \infty$ and $f(x)/g(x) \notin Q$. Then there exists integers a and b such that $c = af(x) + bg(x) \in Q\backslash\{0\}$. Hence, $v_\infty(c) = \infty$ for some $c \in Q\backslash\{0\}$. However, statement 7 above implies that $v_\infty(c) = \omega_p(c) < \infty$ for all $c \in Q\backslash\{0\}$.

The statement of 3 is slightly stronger than that of [11, Theorem 6.5]. If the sequence $\{\deg(\varphi_i(x))|i \in Z^+\}$ is unbounded, then the result follows by also considering statement 7 above. Suppose that the sequence $\{\deg(\varphi_i(x))|i \in Z^+\}$ is eventually constant. Then statement 5 above implies that the sequence $\{\Gamma_i|i \in Z^+\}$ is eventually constant. The result then follows from [11, Theorem 6.5] since a monotone increasing sequence within a single value group Γ_i must go to ∞. □

Consideration of Construction 6.2 and Proposition 6.3 indicates that an important component in building the specific type of valuation domains we require is making careful choices of the key polynomials $\varphi_n(x)$. Our next result gives us the freedom to make the necessary choices.

6.4. Proposition [11, Theorem 13.2]. *Suppose that $v_1, v_2, \ldots, v_{n-1}$ have been constructed as in Construction 6.2 and that v_n has not been constructed. Given any predetermined positive integer m, a key polynomial $\varphi_n(x)$ can be chosen so that $m_n = \deg[F_n : F_{n-1}] = m$.*

Proof. The result is an immediate corollary of [11, Theorem 13.1] and the fact that, given a finite field F, there exist irreducible polynomials in $F[y]$ of all positive integer degrees. □

One obstacle remains before we can construct sequence domains with specified properties. In order to use Construction 6.2 to construct Noetherian valuation domains, we need v_∞ to be finite. However, we have given no indication that finite limit-valuations exist. The next two results will combine to resolve this issue.

6.5. Lemma. *Suppose that $v_1, v_2, \ldots, v_{n-1}$ have been constructed as in Construction 6.2 and that v_n has not been constructed. Suppose also that a key polynomial $\varphi_n(x)$ has been chosen. Then the following statements hold:*

(1) Any rational number μ_n such that $\mu_n > v_{n-1}(\varphi_n(x))$ is a valid choice for $v_n(\varphi_n(x))$.

(2) If v_n and v_n' are constructed such that $v_n(\varphi_n(x)) = \mu_n \neq \mu_n' = v_n'(\varphi_n(x))$ and the two sequences $v_1, v_2, \ldots, v_{n-1}, v_n$ and $v_1, v_2, \ldots, v_{n-1}, v_n'$ are extended to two limit-valuations v_∞ and v_∞', respectively, then v_∞ and v_∞' will be distinct valuations on $Q(x)$.

Proof. The freedom to choose μ_n arbitrarily as stated in 1 was built into MacLane's construction by assumption. 2 follows immediately from part 3′ of Proposition 6.3. □

Since Proposition 6.4 guarantees the existence of key polynomials of any desired degree (subject to part 5 of Proposition 6.3), Lemma 6.5 leads to the existence of uncountably many limit-valuations.

6.6. Proposition. *There exist only countably many limit-valuations which are not finite.*

Proof. [12, Theorem 2.1] and [12, Theorem 10.1] together show that for a given prime $p \in Z$ and a given nonconstant polynomial $f(x) \in Q[x]$, there exist only finitely many limit-valuations v_∞ on $Q(x)$ which extend ω_p and which satisfy $v_\infty(f(x)) = \infty$. Since Z and $Z[x]$ are both countable, the result follows. □

Now we have all the necessary tools to prove a result which will enable us to construct the specific valuation domains used in our examples.

6.7. Theorem. *Suppose that $v_1, v_2, \ldots, v_{n-1}$ have been constructed as in Construction 6.2 and that v_n has not been constructed. Recall that Γ_{n-1}, the value group of v_{n-1}, is generated by $1/e_{n-1}$ for some $e_{n-1} \in Z^+$ and that $F_{n-1}(y)$ is the residue field of v_{n-1} with F_{n-1} being a finite field. Let e be a positive integer such that $e_{n-1}|e$ and let F be a finite field which is an extension field of F_{n-1}. Then the sequence $v_1, v_2, \ldots, v_{n-1}$ can be extended to uncountably many distinct finite limit-valuations which have $\Gamma = \langle 1/e \rangle$ as value group and have F as residue field.*

Proof. Let $t = \deg[F : F_{n-1}]$. Then choose a key polynomial $\varphi_n(x)$ of degree $\tau_{n-1} \cdot t \cdot \deg(\varphi_{n-1}(x))$ and set $\mu_n = m + (1/e)$ for some positive integer m such that $m > v_{n-1}(\varphi_n(x))$. This gives $\Gamma = \Gamma_n$ and $F = F_n$. Then we can force $\Gamma_{n+i} = \Gamma_n$ and $F_{n+i} = F_n$ for all $i \in Z^+$ by choosing $\varphi_{n+i}(x)$ so that $\deg(\varphi_{n+1}(x)) = \tau_n \deg(\varphi_n(x))$ and $\deg(\varphi_{n+i}(x)) = \deg(\varphi_{n+1}(x))$ for $i > 1$ and by choosing $\mu_{n+i} \in Z^+$ for all $i \geq 1$. This gives a limit valuation v_∞ which has value group Γ and residue field F, assuming v_∞ is finite. It follows easily from Lemma 6.5 and Proposition 6.6 that uncountably many distinct finite limit-valuations can be constructed using the above procedure. □

Construction 6.2 gave a general outline for the construction of a single limit-valuation. Now we give a comparable outline for the construction of a sequence domain.

6.8. Construction.

Step 1. Use Construction 6.2 and the subsequent results (especially Theorem 6.7) to build a sequence $\{v_1^*, v_2^*, v_3^*, \ldots\}$ of valuations on $Q(X)$ with associated limit valuation v_∞^* such that the following hold.

(a) v_∞^* is finite.

(b) v_∞^* has a finite residue field, F_∞^*.

(c) The additive group of integers, $Z = \Gamma_\infty^*$, is the value group of v_∞^*.

(d) For each $i \in Z^+, \varphi_{i+1}^*(x)$ is a key polynomial used to extend from v_i^* to v_{i+1}^* with value $v_{i+1}^*(\varphi_{i+1}^*(x)) = \mu_{i+1}^*$.

Step 2. For each $j \in Z^+$, use Construction 6.2 and the subsequent results (especially Theorem 6.7) to build a sequence $\{v_1^{(j)}, v_2^{(j)}, v_3^{(j)}, \ldots\}$ of valuations on $Q(x)$ with associated limit-valuation $v_\infty^{(j)}$ such that the following hold:

(a) Each $v_\infty^{(j)}$ is finite.

(b) Each $v_\infty^{(j)}$ has a finite residue field, $F_\infty^{(j)}$.

(c) For each $j \in Z^+$, the value group $\Gamma_\infty^{(j)}$ of $v_\infty^{(j)}$ is generated by $1/e_\infty^{(j)}$ for some $e_\infty^{(j)} \in Z^+$.

(d) For each $j \in Z^+, v_i^{(j)} = v_i^*$ whenever $i \le j$ but $v_{j+1}^{(j)} \ne v_{j+1}^*$.

Step 3. For each $j \in Z^+$, let W_j be the valuation domain corresponding to $v_\infty^{(j)}$ and let $D = \bigcap_{j=1}^\infty W_j$.

The idea of Construction 6.8 is very similar to the tree illustration of Section 2. Each sequence $\{v_1^{(j)}, v_2^{(j)}, v_3^{(j)}, \ldots\}$ agrees with the sequence $\{v_1^*, v_2^*, v_3^*, \ldots\}$ for j terms before diverging. Hence, it is reasonable that v_∞^* should be the "limit" of the sequence $\{v_\infty^{(1)}, v_\infty^{(2)}, v_\infty^{(3)}, \ldots\}$. This is made more explicit in our next result.

6.9. Theorem. *Assume the notation of Construction* 6.8. *Then D is a sequence domain. In particular, $v_\infty^{(j)}, v_\infty^*$ and W_j of Construction* 6.8 *correspond to v_j, v^* and D_{P_i} of Sections* 3–5.

Proof. Part 3 of Proposition 6.3 implies that if $f(x) \in Q[x]\backslash\{0\}$, then $v_\infty^{(j)}(f(x)) = v_\infty^*(f(x))$ for all sufficiently large $j \in Z^+$. The prime number $p \in Z^+$ serves both as the characteristic of each residue field and as the element π. The other conditions of the definition of sequence domain are easily verified. □

We now have all of the tools necessary to construct a wide variety of examples of sequence domains. Assume the notation/terminology of Sections 3–5. The results of this section then allow us to do the following.

6.10. Generic Example. (1) Choose a prime $p \in Z^+$.

(2) Choose a finite field F^* of order $q^* = p^t$ for some $t \in Z^+$.

(3) Choose two sequences of positive integers $\{b_i | i \in Z^+\}$ and $\{e_i | i \in Z^+\}$.

(4) Use Construction 6.8 to construct a sequence domain D such that

(a) $q^* = |D/P^*| = p^t$,

(b) $q_i = |D/P_i| = p^{tb_i}$ for each $i \in Z^+$,

(c) $v_i^{(\mathrm{N})}(\pi) = e_i = 1/v_i(\rho_i)$ for each $i \in Z^+$.

The key point here is that we can construct sequence domains with the most important features specified essentially arbitrarily. We now use this strength to construct some explicit examples. We begin with several examples of sequence domains that are not doubly bounded.

6.11. Example. Choose a prime p. Construct a sequence domain D such that $q^* = p, q_i = p^i$ and $e_i = 1$ for each $i \in Z^+$. This is comparable to Gilmer's Example 14 in that the sizes of the residue fields of the P_i's go to ∞. D is not doubly bounded, so Int(D) is not Prüfer.

6.12. Example. Choose a prime p. Construct a sequence domain D such that $q^* = q_i = p$ and $e_i = i$ for each $i \in Z^+$. This example corresponds to Chabert's

Example 6.2 in that $\lim_{i\to\infty} v_i^{(N)}(\pi) = \infty$. D is not doubly bounded so Int(D) is not Prüfer.

Next, we give two examples of doubly bounded sequence domains D such that Int(D) is Prüfer, but does not behave well under localization. We will fix a prime p for each of the remaining examples.

6.13. Example. Construct D such that $q^* = p, q_i = p^2$ and $e_i = 1$ for each $i \in Z^+$. Int(D) is Prüfer since D is doubly bounded, but Theorem 5.13 implies that Int(D) does not behave well under localization since $q_i \neq q^*$ for each $i \in Z^+$.

6.14. Example. Construct D such that $q^* = p$, $q_i = p$ and $e_i = 2$ for each $i \in Z^+$. Int(D) is Prüfer since D is doubly bounded, but Theorem 5.13 implies that Int(D) does not behave well under localization since $v_i(p) = 1 \neq 2 = v_i^{(N)}(p)$ for each $i \in Z^+$. (Here p serves the role of π.)

Finally, we note that examples can be constructed so that Int(D) is Prüfer and well-behaved under localization.

6.15. Example. Construct D such that $q^* = q_i = p$ and $e_i = 1$ for each $i \in Z^+$. D is doubly bounded so Int(D) is Prüfer and Theorem 5.13 implies that Int(D) behaves well under localization. In fact D is an overring of Int(Z) and Int(D) was proven to be Prüfer and well-behaved under localization in [10].

Inferences one might make from the constructions and examples of this section are that:

(1) Amongst all sequence domains, doubly bounded sequence domains are rare.

(2) Amongst all doubly bounded sequence domains D, those for which Int(D) is well-behaved under localization are rare.

Hence, it appears that the Noetherian setting where

(1) A Notherian and Int(A) Prüfer is equivalent to A Dedekind with all residue fields finite, and

(2) A Notherian implies Int(A) Prüfer if and only if Int(A) behaves well under localization

does not carry over well to the setting where A is non-Noetherian.

7. Conclusion

The problem of characterizing the set of all NaDf domains D such that Int(D) is Prüfer has not been resolved. Sequence domains represent a very special case of the larger problem. However, we note that Chabert observed in [3] that his version of double-boundedness appeared to work well when considered as a "local" property in a general "nonlocal" setting and did not work well when considered as a more "global" property. Sequence domains represent a very "local" type of setting. Hence a possible approach to the general problem may be to attempt to divide a given NaDf domain into topological components of some type and consider the notion of double-boundedness on the components, perhaps by looking at sequence domain overrings. In any case it appears that a solution to the general problem may well be of a topological nature.

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