Separating Cocircuits in Binary Matroids

William H. Cunningham*
Department of Mathematics and Statistics
Carleton University
Ottawa, Canada K1S 5B6

Submitted by Richard A. Brualdi

ABSTRACT

A cocircuit of a matroid is separating if deleting it leaves a separable matroid. We give an efficient algorithm which finds a separating cocircuit or a Fano minor in a binary matroid, thus proving constructively a theorem of Tutte. Using this algorithm and a new recursive characterization of bond matroids, we give a new method for testing binary matroids for graphicness. We also give an efficient algorithm for finding a special kind of separating cocircuit: one whose deletion leaves a matroid having a coloop.

1. INTRODUCTION

A cocircuit Y of a nonseparable matroid M is said to be separating if deleting Y from M leaves a separable matroid. In this case a number of smaller matroids, called Y-components, can be defined, and the possibility of testing M for a property by testing the Y-components suggests itself. This recursive approach to matroid problems was pioneered by Tutte [10, 13], who gave a characterization of polygon matroids, and an algorithm for recognizing them. (See also [3], which refines Tutte's algorithm.) This approach was also successfully applied to recognizing 3-connected matroids [2].

An important ingredient in these recursive algorithms is a method of finding separating cocircuits, and this is the main theme of the present paper. It happens that, in both of the applications cited above, if a very simple-minded search fails to reveal such a cocircuit, then the question at issue can be immediately resolved. In other applications, such as a (proposed) recursive

*Research partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada and by Bell Laboratories.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/82/020069+18$02.75

approach to testing a binary matroid for regularity, finding a separating cocircuit is more difficult. Tutte [12] proved that a nonseparable binary matroid of rank at least 3, having no Fano minor, has a separating cocircuit. A main result of the present paper (Section 3) is an efficient algorithm which finds either a separating cocircuit or a Fano minor. A new sufficient condition for the existence of a Fano minor is used to improve the computation bound for a variant of the algorithm.

An application of the separating cocircuit algorithm is described in Section 4: a new algorithm for determining whether a given binary matroid is a bond matroid. In many ways this algorithm is "dual to" Tutte's algorithm [10]; in particular, a new recursive characterization of bond matroids, having a strong resemblance to Tutte's theorem [13] for polygon matroids, is presented. In Section 5 the separating cocircuit algorithm is used to find cocircuits having single-element bridges, that is, cocircuits whose deletion leaves a matroid having a coloop. This construction is related to some previous results on such cocircuits.

2. PRELIMINARIES

Matroid terminology used but not defined here is defined by Welsh [14]. This section describes some less standard terminology, and some background material on binary matrices. Where r is the rank function of a matroid M on E, a *separator* of M is a set $S \subseteq E$ such that $r(S)+r(E \setminus S)=r(E)$. An *elementary separator* of M is a minimal nonempty separator. A matroid is *nonseparable* if it has at most one elementary separator. Given a basis B of M, a *B-fundamental cocircuit* of M is a cocircuit having just one element from B.

Where $X \subseteq E$, $M \setminus X$ (M/X) denotes the matroid obtained from M by deleting (contracting) X. For $e \in E$, $M \setminus e$ denotes $M \setminus \{e\}$, and similarly for contraction. A *component* of M is obtained by deleting all but one elementary separator of M. If Y is a cocircuit of M, a *bridge* of Y in M is an elementary separator of $M \setminus Y$. A *Y-component* of M is a matroid obtained from M by contracting all but one of the bridges of Y. If M has more than one Y-component, Y is *separating*; otherwise Y is *nonseparating*. It is easy to see that Y is a nonseparating cocircuit of each Y-component. It is not difficult to show that Y-components of nonseparable matroids are nonseparable; a proof is given in [2].

Given a $\{0,1\}$-valued matrix A whose columns are indexed by E, let $M(A)$ denote the (binary) matroid on E which is the linear independence matroid of A over the binary field. If A has an identity submatrix of size r, the number of rows of A, then A is a *standard representative matrix* (SRM) for $M=M(A)$.

In this case, the identity columns correspond to a basis B of M, and the rows of A are the incidence vectors of B-fundamental cocircuits of M. It is convenient to consider the rows to be indexed by the cocircuits whose incidence vectors they are. Another SRM A' is a SRM for M if and only if A' can be obtained from A by elementary row operations. More particularly, in this paper we will consider only certain sequences of row operations. Given a 1 in position (i, j) of a binary matrix A, the sequence of row operations consisting of adding row i to every other row having a 1 in column j is called *pivoting on position* (i, j). The amount of computation required to perform a pivot in an r-by-c binary matrix is $O(rc)$.

Given a SRM A for M, the elementary separators of M can be calculated, using a notion of "paths" in A. (Although the algorithm of Section 3 is motivated by results of Tutte [12], the tools for studying and manipulating SRMs are reminiscent of another paper of Tutte [11].) A *path* in A *from* column j_0 *to* column j_n, for $n \geq 0$, is a sequence $j_0, i_1, j_1, \ldots, i_n, j_n$, where the j_k are column indices, the i_k are row indices, and, for $1 \leq k \leq n$, A has 1's in positions (i_k, j_k), (i_k, j_{k-1}). Similarly, we can define a path from a row to a column, from a column to a row, or from a row to a row. A path is *minimal* if no proper subsequence of it is also a path, having the same first and last term. If P is a minimal path, then the submatrix of A induced by P, that is, the submatrix consisting of the positions (i_k, j_l) for which i_k is a row term of P and j_l is a column term of P, has at most two 1's per row and per column. Moreover, the rows or columns of this submatrix corresponding to the first and the last term of P have exactly one 1.

The elementary separators of M are the equivalence classes under the relation on E of "connectedness"; that is, $a, b \in E$ are connected if and only if there is a path in A from a to b. Thus there exists an $O(rc)$ algorithm to find the elementary separators of M. (In fact, under appropriate assumptions, the computation bound is linear in the number of 1's of A.) One such algorithm, using "breadth-first search," will find minimal paths from a given row or column to all rows and columns connected to it. Given an elementary separator E_1 of M, the submatrix A_1 of A having a column for each element of E_1 and a row for each row Y of A meeting E_1, is a SRM for the component of M corresponding to E_1.

Certain submatrices of a binary SRM A are SRMs for minors of $M(A)$. Any submatrix A' which intersects the identity submatrix of A in an identity submatrix is of this type; $M(A')$ can be obtained from $M(A)$ by contracting (appropriate) elements corresponding to identity columns of A and deleting elements corresponding to nonidentity columns. We will call such a submatrix A' a *standard* submatrix of A.

Given a row Y of a SRM A for the nonseparable binary matroid M, we can use similar ideas to identify the bridges of Y in M, and to find submatrices of A which are SRMs for the Y-components of M. Each of these submatrices will contain a column for each element of Y, and each will contain a row corresponding to row Y of A; each column of A indexed by an element of $E \setminus Y$ will appear in exactly one of them, and each row of A other than Y will appear in exactly one of them. Thus there is an $O(rc)$ algorithm to compute the Y-components of M. (All of the above material is elementary; more details can be found in [3].)

3. SEPARATING COCIRCUITS AND FANO MINORS

In this section we take up the problem of finding separating cocircuits in binary matroids. Of course, not all matroids have such cocircuits. For example, a nonseparable matroid having rank 2 or less cannot have a separating cocircuit. An example of a rank-3 binary matroid having this property is $M(F)$, where

$$F = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

We call F the *Fano matrix* and $M(F)$ the *Fano matroid*. It is easy to see that F is the only SRM for $M(F)$. Therefore, if $M(A)$ has a Fano minor, then A can be transformed by pivots to an SRM having a standard submatrix which is Fano.

It is possible for a nonseparable binary matroid to have both a separating cocircuit and a Fano minor; Tutte showed that it must have one or the other.

THEOREM 3.1[12, 8.63]). *Let M be a nonseparable binary matroid having rank at least* 3. *Then M has a separating cocircuit or M has a Fano minor.*

Tutte's proof of Theorem 3.1 suggested the following algorithm for attempting to find a separating cocircuit in a binary matroid. There are two slightly different versions of the algorithmic problem. The first is to find a separating cocircuit of M or *find* a Fano minor of M; the second is to find a separating cocircuit of M or *conclude* that M *has* a Fano minor. Algorithm 3.2 solves the first problem (and thus the second); later we will describe a modified version which solves the second problem more efficiently.

Tamari [9] has given algorithmic interpretations of a number of Tutte's proofs, including an algorithmic proof of [12, 8.62]. This result is the basis for Tutte's proof of [12, 8.63], called Theorem (3.1) here. Thus Tamari's work essentially contains an algorithmic proof of Theorem 3.1; moreover, his algorithm and Algorithm 3.2, both being based on Tutte's proof, are conceptually the same. The main differences seem to be that here a careful implementation and complexity analysis is given, and that the case in which the algorithm may discover a Fano minor is treated explicitly. I am grateful to the referee for making me aware of this work of Tamari.

ALGORITHM 3.2 (Input is a nonseparable binary matroid M having rank at least 3, represented by a binary SRM A having r rows and c columns).

Step 1. Choose a row Y. If Y is separating, stop.

Step 2. Choose a column $a \in Y$ having more than one 1, and let W be the set of columns identical to a. (*Comment*: Such a column a must exist, because if every column in Y has just one 1, $M(A)$ is separable.) Choose Z_1 to be a row such that $Z_1 \supseteq W$ and $|Z_1 \cap Y|$ is as small as possible.

Step 3. If $Z_1 \cap Y = W$, then pivot on the 1 in column a and row Y, and row $Z_1 + Y$ of the new SRM will be separating; stop. (*Comment*: In this case W will be a bridge of $Z_1 + Y$.)

Step 4. Find a minimal path $Z_1, j_1, Z_2, j_2, \ldots, j_{n-1}, Z_n$ in A such that each j_k is not in Y and $Z_1 \cap Y$ meets both $Y \cap Z_n$ and $Y \setminus Z_n$. (*Comment*: Z_n must exist because $Z_1 \cap Y \neq W$, and such a path exists because Y is nonseparating.)

Step 5. For $k = n, n-1, \ldots, 4, 3$ pivot on the 1 in column j_{k-1} and row Z_k, and replace Z_{k-1} by $Z_k + Z_{k-1}$. (*Comment*: For each k, step 5 leaves a sequence having the same construction as the sequence of step 4, but having $k-1$ row terms. At the end of step 5, Z_1, j_1, Z_2 is such a sequence.)

Step 6. Find columns $b \in Z_1 \cap Y$ and $d \notin Y$ such that exactly one of a, b is in Z_2, and $d \in Z_1 \cap Z_2$; and let x be the element in column a and row Z_2. (*Comment*: We can choose d to be j_1; b exists because $Z_1 \cap Y \neq W$.) If there exists $c \in (Y \setminus Z_1) \cap Z_2$, then we have a standard Fano submatrix

$$\begin{array}{ccccccc} & & & a & b & c & d \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & x & 1-x & 1 & 1 \end{bmatrix};$$

stop.

Step 7. Otherwise, if $x = 1$, then $Z_2 \cap Y \subset Z_1 \cap Y$; replace Z_1 by Z_2 and go to step 3. If $x = 0$, pivot on the 1 in column d and row Z_2; replace Z_1 by $Z_1 + Z_2$ and go to step 3. (*Comment*: In either case $|Z_1 \cap Y|$ is smaller than before.)

EXAMPLE 3.3. Let us apply the algorithm to find a separating cocircuit in $M(A)$, where

$$A=\begin{bmatrix} 1&0&0&0&0&1&1&1&0&0&0&0\\ 0&1&0&0&0&1&1&0&1&1&0&0\\ 0&0&1&0&0&0&0&1&0&1&0&1\\ 0&0&0&1&0&1&0&1&0&0&1&1\\ 0&0&0&0&1&0&1&0&1&0&1&0 \end{bmatrix}.$$

We suppose the columns of A to be indexed, left to right, by $1,2,\ldots,12$. It happens that all of the rows of A provide nonseparating cocircuits. Suppose that we choose the top row to be Y in step 1; then we go to step 2. If we choose $a=7$, then we will choose Z_1 to be the fifth row, and the algorithm will terminate in step 2. But suppose instead that we choose $a=6$. Then $W=\{6\}$, and we choose Z_1 to be the second row. We go to step 3 and find $Z_1\cap Y\neq W$, and go on to step 4. If the algorithm were to find a *shortest* path of the required kind, which the breadth-first search method would do, it would find a sequence Z_1, j_1, Z_2 where $j_1=9$ and Z_2 is the bottom row. However, let us suppose instead that the sequence turns out to be Z_1, j_1, Z_2, j_2, Z_3, where $j_1=10$, $j_2=12$, Z_2 is the third row, and Z_3 is the fourth row. We go on to step 5, where we are instructed to pivot on position (4, 12). The resulting transformed matrix is

$$\begin{bmatrix} 1&0&0&0&0&1&1&1&0&0&0&0\\ 0&1&0&0&0&1&1&0&1&1&0&0\\ 0&0&1&1&0&1&0&0&0&1&1&0\\ 0&0&0&1&0&1&0&1&0&0&1&1\\ 0&0&0&0&1&0&1&0&1&0&1&0 \end{bmatrix}.$$

Now we proceed to step 6, and find $b=7$, $d=10$, $x=1$, and that there is no column c having 1 in rows 1 and 3 and 0 in row 2. We go on to step 7, replace Z_1 by row 3, and then return to step 3. We now have $Z_1\cap Y=W$, so we pivot on position $(1,6)$, and the resulting matrix is

$$\begin{bmatrix} 1&0&0&0&0&1&1&1&0&0&0&0\\ 1&1&0&0&0&0&0&1&1&1&0&0\\ 1&0&1&1&0&0&1&1&0&1&1&0\\ 1&0&0&1&0&0&1&0&0&0&1&1\\ 0&0&0&0&1&0&1&0&1&0&1&0 \end{bmatrix}.$$

The third row of this matrix gives a separating cocircuit of $M(A)$, and the algorithm terminates.

A proof of validity and finiteness for Algorithm 3.2 is easily deduced from the comments. Therefore, we have a constructive proof of Theorem 3.1. The complexity of the algorithm is clearly determined by step 5, which requires at most r pivots per execution, and therefore at most $O(r^2c)$ work per execution. The number of executions of Step 5 is bounded by the number of distinct columns in Y; this is certainly bounded by c, and so we have an $O(r^2c^2)$ computation bound for the algorithm.

For the second version of the algorithmic problem, which does not require actually finding a Fano minor, the computation bound is improved by making an appropriate choice of Y, using the following result (whose discovery was motivated by this application).

THEOREM 3.4. *Let M be a simple binary matroid having rank r and having no Fano minor, and let B be a basis of M. Then there exists a B-fundamental cocircuit of cardinality at most r.*

The proof of Theorem 3.4 is rather difficult and uses some deep results of Seymour [7]. It provides a simple necessary condition for a real matrix to be totally unimodular. This theorem, together with some other interesting work on "short cocircuits," will appear in a separate paper. Using Example 3.3 we can modify the algorithm as follows. Step 1 now reads: "Choose a row Y such that Y has at most r distinct columns. If no such Y exists, stop; M has a Fano minor. Otherwise, if Y is separating, stop." This modified algorithm requires at most r executions of step 5, and thus has a computation bound of $O(r^3c)$.

4. A DUAL ALGORITHM FOR GRAPH REALIZABILITY

Given a graph G, the binary matroid whose circuits are the edge-sets of circuits of G is called the *polygon matroid*, PM(G), of G. The dual of PM(G) is BM(G), the *bond matroid* of G. A fundamental problem is that of determining whether a given binary matroid is graphic (that is, a polygon matroid) or, dually, whether it is cographic (that is, a bond matroid). An algorithm for this problem is the main tool in a method for determining whether a given linear program can be converted to a network flow problem [3]; the problem also generalizes the graph-theoretic problem of planarity testing. There are several algorithms available for graph realizability, some of which ([3], for example) are extremely efficient. In this section we describe a new algorithm which is not computationally competitive with the best algorithms; however, it is interesting because it uses Algorithm 3.2, and because of its relationship to Tutte's algorithm [10], which is used in [3].

We briefly review the recursive approach to determining whether a binary matroid is graphic; the details are in [3]. Given a cocircuit Y of a nonseparable matroid M, one can define a "bridge graph," whose vertices are the bridges of Y in M, and whose edges join "overlapping" pairs of vertices. The recursive approach to recognizing graphicness is based on the following result of Tutte [13].

THEOREM 4.1. *Let M be a nonseparable binary matroid, and let Y be a cocircuit of M. Then M is a graphic if and only if*

(4.1a) *each Y-component of M is graphic;*
(4.1b) *the bridge graph determined by Y and M is bipartite.*

The recursive approach to determining whether a binary matroid is cographic requires an analogue of (4.1). The necessary ideas can be discovered by considering the situation when M *is* the bond matroid of a graph G; then Y is the edge set of a circuit C of G. The bridges of Y are the edge sets of blocks of the graph obtained from G by contracting C. The Y-component M_i of M corresponding to the bridge B_i of Y is the bond matroid of the connected subgraph G_i of G having edge set $B_i \cup Y$. (Notice that for $|E| \neq |Y|+1$, Y will be nonseparating if and only if C is chordless and deleting its vertex set does not disconnect G.) Given these graphs G_i, it is easy to reconstruct G; one simply "joins the G_i at C." The difficulty in a recursive approach is that we may know that each M_i is cographic and have a corresponding graph G_i', but the elements of Y may not occur in the same cyclic order in all of the G_i'. We may need to modify the G_i' to make these cyclic orders coincide, or recognize that it is impossible to do this. Fortunately, only a very simple type of modification of the G_i' will be required, as we shall see.

A *cyclic partition* of a set Y is a partition $\pi=(P_0, P_1,\dots,P_{k-1})$ of Y together with a symmetric adjacency relation stating that P_i is adjacent to P_{i+1} for $0 \leq i \leq k-1$. (Subscripts are modulo k.) Two cyclic partitions are regarded as the same if they are equal as partitions and also have the same adjacency relation. An *interval* of a cyclic partition π is a set of the form $\bigcup(S_j: 1 \leq j \leq m)$ where each S_j is a member of π and S_j is adjacent to S_{j+1} for $1 \leq j \leq m-1$. A *refinement* of a cyclic partition π of Y is a cyclic partition π' of Y such that every interval of π is an interval of π'. A collection of cyclic partitions of a set Y are *compatible* if they have a common refinement. Given a graph G and a circuit C having edge set Y, a cyclic partition of Y is determined, as follows: its members are the edge sets of maximal subpaths of C each of whose interior vertices has degree 2 in G, and two of these are adjacent if and only if the associated paths have a common end. The next result implies that if a Y-component of a matroid is cographic, then this cyclic partition is determined by the Y-component, and does not depend on the choice of representing graph.

LEMMA 4.2. *If Y is a nonseparating cocircuit of a nonseparable bond matroid M, then every graph G having* $\mathrm{BM}(G)=M$ *determines the same cyclic partition of Y.*

Proof. Suppose that the lemma is not true, and choose graphs G_1, G_2 with $\mathrm{BM}(G_1)=\mathrm{BM}(G_2)=M$ but determining different cyclic partitions π_1, π_2 of Y. Let C_i be the circuit of G_i having edge set Y, for $i=1$ and 2. First consider the case in which π_1 and π_2 are not equal as partitions. Then one of the graphs, say G_1, has a 2-element edge cutset $\{e, f\}$ which is not contained in the edge set of a subpath of C_2, each of whose interior vertices has degree 2. It follows that when $\{e, f\}$ is deleted from G_2, each component contains a vertex not in $V(C)$. Thus Y is separating in $\mathrm{BM}(G_2)$, a contradiction.

Therefore, we may suppose that there exist subpaths P_1, Q_1 of C_1 and P_2, Q_2 of C_2 such that $E(P_1)=E(P_2)\in\pi_1$, $E(Q_1)=E(Q_2)\in\pi_1$, $V(P_1)\cap V(Q_1)\neq\varnothing$, $V(P_2)\cap V(Q_2)=\varnothing$. Let p, q be the vertices of degree greater than two in G_1 satisfying $p\in V(P_1)\backslash V(Q_1)$, $q\in V(Q_1)\backslash V(P_1)$. Let p', q' be vertices not in $V(C_1)$ adjacent to p, q, respectively. Since Y is nonseparating, there is a simple path in G_1 from p' to q' which contains no vertex from $V(C_1)$. It follows that M has a cocircuit Z such that $Y\cup Z$ is a minimal union of two cocircuits ("coline"), and $Y\cap Z=E(P_1)\cup E(P_2)$. But clearly $\mathrm{BM}(G_2)$ cannot have these properties, a contradiction. ■

The hypothesis in Lemma 4.2 that Y is nonseparating is easily seen to be necessary. The value of Lemma 4.2 lies in the observation that if Y is a cocircuit of a nonseparable matroid M, then each Y-component is nonseparable, and has Y as a nonseparating cocircuit. We now present the dual version of Theorem 4.1.

THEOREM 4.3. *Let M be a nonseparable binary matroid, and let Y be a circuit of M. Then M is cographic if and only if*

(4.3a) *each Y-component of M is cographic;*

(4.3b) *the cyclic partitions of Y determined by the Y-components of M are compatible.*

Proof. The "only if" part of the theorem is immediate, from Lemma 4.2 and the fact that minors of cographic matroids are cographic. Now suppose that M and Y satisfy (4.3a), (4.3b). Let G_i be a graph such that $\mathrm{BM}(G_i) = M_i$, $1 \leqslant i \leqslant k$, where $M_1, \ldots, M_k$ are the Y-components of M. Then by rearranging the order of edges of G_i on paths whose interior vertices have degree 2, we can obtain a graph G_i' such that $\mathrm{BM}(G_i') = M_i$, $1 \leqslant i \leqslant k$, and such that the edges of Y occur in the same cyclic order in all of the G_i'. Now we combine the G_i' to form a graph G'; let M' be its bond matroid. We want to show that $M' = M$. Clearly (Y is a cocircuit of M' and) the Y-components of M' are $M_1, \ldots, M_k$, so it will be enough to show that the Y-components of any binary matroid N determine N. Choose a basis D of N such that Y is D-fundamental, and let e be the common element of D and Y. Then for each bridge B_i of N, $D_i = (D \cap B_i) \cup \{e\}$ is a basis of the corresponding Y-component N_i; moreover, the D-fundamental cocircuits of N are Y together with the D_i-fundamental cocircuits of the N_i. Since the D-fundamental cocircuits of N determine N, the proof is finished. ∎

The hypothesis in Theorem 4.3 that M is binary was used in the proof. Similarly, Tutte's proof [13] of Theorem 4.1 uses the binary hypothesis. However, Bixby [1] has recently proved that Theorem 4.1 is true even if M is not assumed to be binary. This raises the question of whether a similar strengthening of Theorem 4.3 can be proved. In fact, this is so and it can be proved with the aid of a recent result of Seymour [8].

We are now ready to state the new recursive algorithm for graph realizability. First, we need to make two observations. The first is that the Fano matroid is not cographic; this is well known and easy to prove. The second is that any binary matroid having rank at most two is cographic, and representing graphs are easily constructed. In the case in which the rank is exactly two, the graph consists of two vertices joined by three internally disjoint paths whose edges are $Y_1 \cap Y_2$, $Y_1 \setminus Y_2$, and $Y_2 \setminus Y_1$, where Y_1 and Y_2 are the fundamental cocircuits for some basis. We remark that implementation of the algorithm will require the actual construction of a representing graph, in order to apply the recursion; however, its statement does not mention this explicitly.

Algorithm 4.4 (Input is a nonseparable binary matroid M, represented by an SRM having r rows and c columns).

Step 1. If M has rank $\leqslant 2$, stop; M is cographic.

Step 2. Apply Algorithm 3.2 in modified form, to find a separating cocircuit Y of M. If instead it is determined that M has a Fano minor, stop; M is not cographic.

Step 3. Apply this algorithm recursively to the Y-components of M. If any is not cographic, stop; M is not cographic.

Step 4. If the cyclic partitions of Y determined by the Y-components of M are not compatible, stop; M is not cographic. Otherwise, stop; M is cographic.

The computational effort required for step 2 dominates that for the other steps of Algorithm 4.4. Each application of step 2 results either in termination or in consideration of a collection of $k \geqslant 2$ Y-components whose ranks r_i satisfy $\Sigma(r_i : 1 \leqslant i \leqslant k) = r + k - 1$. Therefore, we can prove by induction that the total number of applications of step 2 is at most $r - 2$. This is certainly true for small r, and otherwise we compute $1 + \Sigma((r_i - 2) : 1 \leqslant i \leqslant k) = 1 + (r + k - 1) - 2k = r - k \leqslant r - 2$, as required. Therefore, the total computational effort is $O(r^4 c)$. The reason that this bound is inferior to that for an algorithm based on Theorem 4.1 is that the analogue of Step 2 does not require application of Algorithm 3.2. If the SRM has at most two 1's per column, M is known to be graphic; if a column having three 1's is not in a row Y which is a separating cocircuit, then M is known not to be graphic.

The problem of determining efficiently whether a given collection of m cyclic partitions of a set Y are compatible deserves a little more attention. The claim above that this work does not affect the time bound for Algorithm 4.4 would be substantiated if we could demonstrate an $O(m^3|Y|)$ algorithm for this problem [because $m \leqslant r$, $|Y| \leqslant c$ in step 4, and step 2 requires $O(r^3 c)$ time]. In fact, there is an $O(m|Y|)$ algorithm for testing compatibility. We will not give such a method in detail, but we describe a few of the important ideas for simplifying the problem. First, it will be enough to show that we can construct the "coarsest" common refinement of *two* cyclic partitions of Y, or conclude that none exists, in time $O(|Y|)$, for we can then use this algorithm $m - 1$ times. Therefore, suppose that we are given cyclic partitions $\pi = (S_0, \ldots, S_{k-1})$ and $\pi' = (S'_0, \ldots, S'_{l-1})$ of Y. Notice that, except for its adjacency relation, the desired partition π'' is easily constructed. Its members are just the nonempty sets of the form $S_i \cap S'_j$. It is now easy to see that there is no loss of generality in assuming that each member of π'' is a singleton. It follows that, if π and π' are compatible, then every member of each of them has cardinality one or two. We leave the remaining details of the algorithm to the reader.

EXAMPLE 4.5. We illustrate Algorithm 4.4 by applying it to $M = M(A)$, where A is the matrix given in Example 3.3.

The first step is to find a separating cocircuit Y, which was done in Example 3.3; $Y = \{1,3,4,7,8,10,11\}$. There are 3 Y-components, having

SRMs

$$A_1=\begin{array}{c} \begin{matrix} 1 & 3 & 4 & 6 & 7 & 8 & 10 & 11 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array},$$

$$A_2=\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 7 & 8 & 9 & 10 & 11 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array},$$

$$A_3=\begin{array}{c} \begin{matrix} 1 & 3 & 4 & 7 & 8 & 10 & 11 & 12 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}.$$

Since A_1 and A_3 each has just 2 rows, when we apply the algorithm recursively to $M(A_1)$ and $M(A_3)$, we immediately obtain the graphs G_1, G_3 in Figure 1. On the other hand, applying the algorithm to $M(A_2)$ requires finding a separating cocircuit and using recursion one more time. We omit the details of these steps, and claim that the graph G_2 of Figure 1 satisfies $\mathrm{BM}(G_2)=M(A_2)$.

Now cyclic partitions π_1, π_2, π_3 of Y arising from G_1, G_2, G_3 are

$$\pi_1=(\{1,7,8\},\ \{3,4,10,11\}),$$

$$\pi_2=(\{1,8,10\},\ \{7,11\},\ \{3,4\}),$$

$$\pi_3=(\{1,4,7,11\},\ \{3,8,10\}).$$

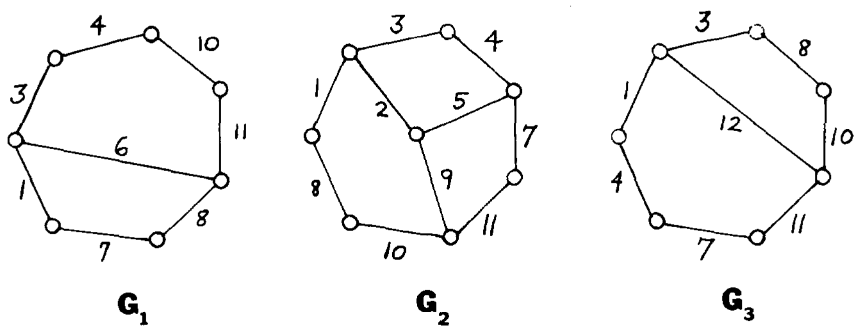

FIG. 1.

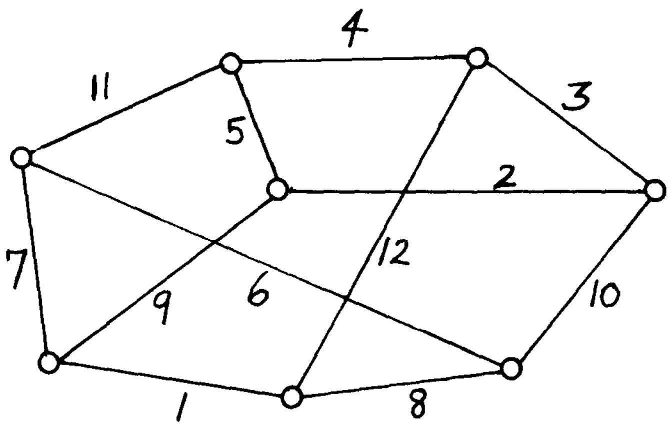

FIG. 2.

A common refinement of π_1, π_2, π_3 is the cyclic partition $\pi = (\{1\}, \{7\}, \{11\}, \{4\}, \{3\}, \{10\}, \{8\})$. Rearranging the order of edges of Y in each of G_1, G_2, G_3 to correspond to π, and combining the resulting graphs, we obtain the graph G of Figure 2. We conclude that M is cographic and that $M = \mathrm{BM}(G)$.

5. SINGLE-ELEMENT BRIDGES

Let Y be a circuit of a matroid M on E. An element $e \in E$ is a *chord* of Y if $e \notin Y$ but e is spanned by Y in M. In [4], conditions for a given element e to be a chord of some circuit were given for a large class of matroids. In particular, the following is a consequence of the main result proved there.

THEOREM 5.1. *Let M be a nonseparable binary matroid on E, $|E| \geq 3$, such that M has no minor which is the dual of the Fano matroid. For any $e \in E$, e is a chord of a circuit of M if and only if $M \backslash e$ is nonseparable.*

The fact that the Fano matroid appears in both Theorem 5.1 and Theorem 3.1 is not a coincidence. If Y is a circuit of M, then Y is a cocircuit of M^*, the dual of M. An element e is a chord of Y if and only e is a loop of M/Y, that is, a coloop of $M^* \backslash Y$. Thus Y has a chord in M if and only if Y has a single-element bridge in M^*. It follows that, provided M^* has rank at least 3, Y is a special kind of separating cocircuit of M^*.

An earlier result on single-element bridges of separating cocircuits has been pointed out to me by Jack Edmonds. Hansen [6] proved that, among the hyperplanes determined by a full-dimensional finite set of points in real n-space, $n \geqslant 2$, there is one for which all but one point is contained in a flat of dimension $n-2$. (The special case in which $n=2$ is the classical Sylvester-Gallai theorem.) A restatement of Hansen's result is the following.

THEOREM 5.2. *A simple real matroid M having rank at least 2 has a cocircuit with a single-element bridge.*

The binary matroids which are also real are precisely the regular ones. Thus Theorem 5.2 can be applied to yield a result on regular matroids. In fact, using Theorem 5.1 or the methods of the current paper, we can obtain a result for a slightly larger class of matroids. (For convenience only, Theorem 5.3 assumes nonseparability; it is easy to see that this hypothesis can be dropped.)

THEOREM 5.3. *Let M be a simple nonseparable binary matroid having rank at least 2 and having no Fano minor. Then M has a cocircuit having a single-element bridge.*

Proof. We will show that the result follows quite easily from (5.1). It will be enough to show that M has an element e such that M/e is nonseparable. (This part has nothing to do with Fano minors.) This is easily seen to be true if M has exactly 3 elements, and M cannot have fewer elements. Suppose that it is not true in general, and choose M to be a counterexample having as few elements as possible. For any element e for which M/e is separable, it follows, by a well-known result, that $M\setminus e$ is nonseparable. Moreover, $M\setminus e$ is simple and has rank at least 2. Thus $M\setminus e$ has an element f such that $(M\setminus e)/f$ is nonseparable. Since M/f is separable, it follows that e is a loop of M/f, so that $\{e, f\}$ is a circuit of M, a contradiction. ∎

It is not difficult to use the above argument to show that such a matroid has at least 3 elements which occur as single-element bridges of cocircuits. However, this proof does not provide an efficient algorithm for finding a cocircuit having a single-element bridge. We now show how Algorithm 3.2 can be used to do just that, in the process giving another proof of Theorem 5.3.

First, we observe that if Algorithm 3.2 is begun with a nonseparating cocircuit Y in step 1, then termination will occur in step 3 (assuming that there is no Fano minor), and then W is a bridge of Z_1+Y having rank 1; since M is simple, the required cocircuit and single-element bridge are at hand. However, we must be able to deal with the possibility that Y of step 1 is separating, but each of its bridges has rank at least 2. In this case, we can apply the algorithm to a Y-component M' of M. This must be done carefully; generally, the bridges in M' of a cocircuit Y' of M' are not related in a simple way to its bridges in M. But this difficulty can be handled by restricting attention to fundamental cocircuits. The following result is implicit in [2] (see the Proposition, p. 97, and its proof). Alternatively, the lemma can be proved, in the case in which M is binary, using path connectivity on SRMs.

LEMMA 5.4. *Let M be a nonseparable matroid and Y, Y' be cocircuits of M which are fundamental with respect to some basis of M. Suppose Y' is a cocircuit of the Y-component M' of M, and let the bridges of Y' in M' be $B_1, B_2, \ldots, B_k$. Then one of the B_i, say B_k, contains $Y \setminus Y'$, and $B_1, \ldots, B_{k-1}$ are bridges of Y' in M.*

The next result extends Lemma 5.4 in a straightforward way.

LEMMA 5.5. *Let M be a nonseparable matroid, and let $Y_1, Y_2, \ldots, Y_n$ be fundamental cocircuits of M with respect to some basis of M. Let $M_0 = M$ and, for $1 \leqslant i \leqslant n$, suppose that Y_i is a cocircuit of M_{i-1}, let B_i be a bridge of Y_i in M_{i-1} such that, for $i \neq 1$, $B_i \cap (Y_{i-1} \setminus Y_i) = \varnothing$, and let M_i be the corresponding Y_i-component of M_{i-1}. Then B_n is a bridge of Y_n in M.*

Proof. The proof is by induction on n; for $n = 1$, the result is trivially true. If $n \geqslant 2$, we can conclude from the induction hypothesis that B_{n-1} is a bridge of Y_{n-1} in M. It is easy to see that the corresponding Y_{n-1}-component of M is M_{n-1}. Now applying Lemma 5.4 with $Y = Y_{n-1}$, $Y' = Y_n$, and $M' = M_{n-1}$, we conclude that B_n is a bridge of Y_n in M, as required. ■

The algorithm for finding a cocircuit having a single-element bridge (Algorithm 5.6 below) will construct sequences such as those described in Lemma 5.5, and will terminate when n such that $|B_n| = 1$ is encountered. As indicated previously, Y_i is generated by applying Algorithm 3.2 to a Y_{i-1}-component. The important observation is that this can be done in such a way that the Y_i are all fundamental cocircuits with respect to some basis of M.

ALGORITHM 5.6 (Input is a simple nonseparable binary matroid M having rank at least 3, represented by an SRM having r rows and c columns).

Step 1. Apply algorithm 3.2 to M. If it terminates with a Fano minor, stop. Otherwise, let Y be the separating cocircuit it produces. If Y has a rank-one bridge, stop.

Step 2. Put $M_0 = M$, put $Y_1 = Y$, let B_1 be a bridge of Y in M, and let M_1 be the corresponding Y-component. Put $i = 1$.

Step 3. Apply Algorithm 3.2 to M_i, choosing Y in step 1 to be Y_i, and choosing a in step 2 so that $a \notin Y_j$ for $j < i$. If Algorithm 3.2 terminates with a Fano minor, stop. Otherwise, let Y_{i+1} be the separating cocircuit of M_i which it produces. If Y_{i+1} has a rank-one bridge B_{i+1} in M_i, such that B_{i+1} is disjoint from $Y_i \backslash Y_{i+1}$, stop.

Step 4. Choose a bridge B_{i+1} of Y_{i+1} in M_i such that B_{i+1} is disjoint from $Y_i \backslash Y_{i+1}$, and let M_{i+1} be the corresponding Y_{i+1}-component. Replace i by $i+1$ and go to step 3.

We now prove the validity of Algorithm 5.6. In general, Algorithm 5.6 requires the application of Algorithm 3.2 to a number of different matroids, so it may appear that a succession of SRMs need to be created and transformed. In fact, we will see that each application of Algorithm 3.2 can be viewed as transforming by pivots the SRM for M itself. Each time M_{i+1} is defined, an SRM for M_{i+1} occurs as a submatrix of the current SRM for M. We must show that any pivots on this submatrix required by the application of Algorithm 3.2 to M_{i+1} are actually pivots on the SRM for M, and moreover that each Y_k, $k \leq i+1$, remains as a row of the SRM for M. (The latter fact will allow us to apply Lemma 5.5.)

Suppose that $Y_1, Y_2, \ldots, Y_i$ are all rows of the current SRM for M at the time Algorithm 3.2 is applied to M_i. Because B_k has been chosen to be disjoint from $Y_{k-1} \backslash Y_k$ for all $k \leq i$, it follows that any element of M_i which is not in Y_i, is not in Y_k for all $k \leq i$. Therefore, any pivot occurring in step 5 or step 7 of Algorithm 3.2 will be on a column j such that $j \notin Y_k$ for $k \leq i$, and so $Y_1, \ldots, Y_i$ will still be rows of the new SRM. Moreover, by the choice of a in step 3 of Algorithm 5.6, any pivot occurring in step 3 of Algorithm 3.2 applied to M_i will be on row Y_i and a column which is not in Y_k for $k < i$, and thus $Y_1, Y_2, \ldots, Y_i$ will still be rows of the SRM after the pivot. Finally, Y_{i+1} is then chosen to be a row created by the pivot, and so it follows inductively that the cocircuits $Y_1, Y_2, \ldots, Y_n$ constructed by Algorithm 5.6 are fundamental cocircuits of M with respect to some basis.

The preceding argument made use of the special choice of a in step 3 of Algorithm 5.6; we must verify that such an a always exists. Suppose, on the contrary, that every column $j \in Y_i$ either has no other 1's or is an element of Y_k for some $k < i$. It is easy to see that any j of the second kind must satisfy $j \in Y_{i-1}$. But then $Y_i \backslash Y_{i-1}$ is a rank-one bridge of Y_{i-1} in M_{i-2}, and it is disjoint from $Y_{i-2} \backslash Y_{i-1}$; this contradicts the termination criterion in step 3 of Algorithm 5.6. Now the rank of M_{i+1} is less than the rank of M_i, so Algorithm 5.6 will require at most r major iterations to find a cocircuit Y_n having a rank-one bridge B_n (or a Fano minor). Since M is simple, clearly B_n is a single-element bridge, so we have a constructive proof of Theorem 5.3. (The case in which M has rank less than 3, excluded by Algorithm 5.6, is trivial.) If the modified form of Algorithm 3.2 is used as a subroutine, the computation bound for Algorithm 5.6 is $O(r^4c)$.

There also exists an efficient algorithm to determine whether a given element e of a binary matroid M having no Fano minor, is a single-element bridge of some cocircuit. This algorithm provides a constructive proof of Theorem 5.1. However, the method is rather roundabout, so we give only a summary. First, e cannot be in a two-element circuit of M, and the case in which M has rank less than 3 can be handled directly. Otherwise, if a nonseparating cocircuit Y containing e is known, we can choose a basis B such that $B \cap Y = \{f\}$, $f \neq e$, and apply Algorithm 3.2 to the SRM corresponding to B, beginning with Y in step 1, and choosing $a = e$ in step 2. The algorithm will terminate with a cocircuit having e as a single-element bridge. The proof of Theorem 3 of [2] contains the essence of an efficient algorithm to find such a cocircuit Y, provided that M is 3-connected. The case in which M is not 3-connected can be handled by applying decomposition techniques [5], together with the above solution for the 3-connected case.

I am grateful to Professor Robert E. Bixby for a number of suggestions and comments which have improved this paper.

REFERENCES

1 R. E. Bixby, Bridges in general matroids, to appear.

2 R. E. Bixby and W. H. Cunningham, Matroids, graphs, and 3-connectivity, in *Graph Theory and Related Topics*, (Bondy and Murty, Eds.) Academic, New York, 1979, pp. 91–103.

3 R. E. Bixby and W. H. Cunningham, Converting Linear Programs to Network Problems, *Math. Oper. Res.* 5:321–357 (1980).

4 W. H. Cunningham, Chords and disjoint paths in matroids, *Discrete Math.* 19:7–15 (1977).

5 W. H. Cunningham and J. Edmonds, A combinatorial decomposition theory, *Canad. J. Math.* 32:734–765 (1980).

6 S. Hansen, A generalization of a theorem of Sylvester on the lines determined by a finite point set, *Math. Scand.* 16:175–180 (1965).

7 P. D. Seymour, Decomposition of regular matroids, *J. Combin. Theory Ser. B* 28:305–359 (1980).

8 P. D. Seymour, Recognizing graphic matroids, *Combinatorica* 1:75–78 (1981).

9 R. Tamari, Combinatorial algorithms in certain classes of binary matroids, Ph.D. Thesis, Cornell Univ., 1977.

10 W. T. Tutte, An algorithm for determining whether a given binary matroid is graphic, *Proc. Amer. Math. Soc.* 11:905–917 (1960).
11 W. T. Tutte, From matrices to graphs, *Canad. J. Math.* 16:108–127 (1964).
12 W. T. Tutte, Lectures on matroids, *J. Res. Nat. Bur. Standards Sect. B* 69:1–47 (1965).
13 W. T. Tutte, On even matroids, *J. Res. Nat. Bur. Standards Sect. B* 71:213–214 (1967).
14 D. J. A. Welsh, *Matroid Theory*, Academic, New York, 1976.

Received 15 February 1981; revised 24 July 1981