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Separable functors in graded rings

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Abstract

Separable functors were introduced by C. Năstăsescu et al. (J. Algebra 123 (1989) 397–413). We characterize separability of left or right adjoint functors defined on a Grothendieck category having a set of projective generators. This general results are particularized to the canonical functors arising from a graded homomorphism of group-graded rings (restriction of scalars, induction and coinduction functors). We relate the separability of these functors with that of their ungraded versions. In particular, we recover the characterizations given in loc. cited for the ungraded restriction of scalars and induction functors. © 1998 Elsevier Science B.V. All rights reserved.

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Introduction

Separable functors were introduced by Năstăsescu et al. [4]. They investigated mainly the separability of the usual induction and restriction of scalar functors associated to a ring homomorphism and those concerning with a group-graded ring. In [7], the separability of adjoint functors between general categories are characterized by mean of splitting properties of the unit and the counit of the adjunction. Consequently, they obtain a criterion for the separability of a right adjoint functor defined on a category \mathcal{C} of modules over a ring R [7, Proposition 2.2]. This idea is also exploited in [2] to give a criterion of separability for a right adjoint functor defined on a category \mathcal{C} of graded modules by a G -set over a G -graded ring R (here, G denotes a group) [2, Proposition 4.2]. In both cases, they assume that the category \mathcal{C} contains nice projective generators.

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In this paper we give criteria for the separability of left or right adjoint functors defined on a Grothendieck category having a set of projective generators (Section 1, Theorems 1.2 and 1.3). These theorems can be applied to recover the characterizations given in [4] of the separability for the induction and restriction of scalars functors (denoted by $\text{Ind}(-)$, resp. φ_*) arising from a ring homomorphism $\varphi: R \rightarrow S$ (Corollary 1.4). Moreover, we give a characterization of the separability of the functor $\text{hom}_R(S, -)$ (Corollary 1.5), which will be improved in Section 4 (Corollary 3.10). In Section 2 we illustrate how to use our general criteria to obtain characterizations of the separability of the graded functors $\varphi_*^{gr}, \text{IND}(-), \text{HOM}(S, -)$ associated to a homomorphism of graded rings $\varphi: R \rightarrow S$. In Section 3 we introduce the notion of strongly separable graded functor. This concept allows to compare the separability of the graded functors $\varphi_*^{gr}, \text{IND}(-), \text{HOM}(S, -)$ and the ungraded functors $\varphi_*, \text{Ind}(-)$ investigated in [4] and $\text{hom}_R(S, -)$ (Theorem 3.9). In particular, it is proved that $\text{Ind}(-)$ is separable if and only if $\text{hom}_R(S, -)$ is separable if and only if $\varphi: R \rightarrow S$ is a splitting monomorphism of R -bimodules.

1. A criterion for separability

Although the notion of separable functor is given for arbitrary categories, this paper concerns mainly to abelian categories. We refer the reader to [6, 3]. If \mathcal{C} is a category, the expression $C \in \mathcal{C}$ means that C is an object of \mathcal{C} . Given $C, C' \in \mathcal{C}$, the set of all morphisms from C to C' will be denoted by $\text{hom}_{\mathcal{C}}(C, C')$. However, in particular cases we will use special notations. If A is a unitary associative ring, the Grothendieck category of all unital left A -modules will be denoted by ${}_A\text{Mod}$. Given $M, N \in {}_A\text{Mod}$, the abelian group consisting of all the morphisms of left A -modules will be denoted by $\text{hom}_A(M, N)$.

The notion of separable functor was introduced in [4] in the following terms.

Definition 1.1. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is said to be separable if there is a family of maps

$$v = \{v_{M,N} : \text{hom}_{\mathcal{D}}(F(M), F(N)) \rightarrow \text{hom}_{\mathcal{C}}(M, N)\},$$

where the pairs M, N runs over all the objects of \mathcal{C} such that the following conditions are fulfilled:

1. $v_{M,N}(F(\alpha)) = \alpha$ for every $\alpha \in \text{hom}_{\mathcal{C}}(M, N)$ and every $M, N \in \mathcal{C}$.
2. Given $M, N, M', N' \in \mathcal{C}$, $\alpha \in \text{hom}_{\mathcal{C}}(M, M')$, $\beta \in \text{hom}_{\mathcal{C}}(N, N')$, $f \in \text{hom}_{\mathcal{D}}(F(M), F(N))$ and $g \in \text{hom}_{\mathcal{D}}(F(M'), F(N'))$ such that the following diagram commutes:

$$\begin{array}{ccc}
 F(M) & \xrightarrow{f} & F(N) \\
 \downarrow F(\alpha) & & \downarrow F(\beta) \\
 F(M') & \xrightarrow{g} & F(N')
 \end{array}$$

then the diagram

$$\begin{array}{ccc}
 M & \xrightarrow{\quad} & N \\
 \downarrow x & \text{\scriptsize } v_{M,N}(f) & \downarrow \beta \\
 M' & \xrightarrow{\quad} & N' \\
 & \text{\scriptsize } v_{M',N'}(f) &
 \end{array}$$

commutes.

The family v will be said to be a *separability mapping* for the separable functor F .

The next two theorems show our criteria to identify separable adjoint functors in Grothendieck categories with projective generators.

Theorem 1.2. *Let \mathcal{D} be a Grothendieck category that contains a full subcategory \mathcal{P} whose objects form a set of projective generators of \mathcal{D} . Consider an abelian category \mathcal{C} and functors*

$$G: \mathcal{F} \rightleftarrows \mathcal{C}: F$$

such that F is left adjoint to G with counit $\varepsilon: FG \rightarrow 1_{\mathcal{C}}$. Let us denote by $\iota_{\mathcal{P}}: \mathcal{P} \rightarrow \mathcal{D}$ the inclusion functor. The functor G is separable if and only if there is a natural transformation $u: \iota_{\mathcal{P}} \rightarrow FG\iota_{\mathcal{P}}$ such that $\varepsilon_P \circ u_P = 1_P$ for every $P \in \mathcal{P}$.

Proof. Let us adopt the following notation: $S = FG$, $T = 1_{\mathcal{D}}$, $S' = FG\iota_{\mathcal{P}}$, $T' = \iota_{\mathcal{P}}$. Assume that there is a natural transformation $u: T' \rightarrow S'$ such that $\varepsilon_P \circ u_P = 1_P$ for every $P \in \mathcal{P}$. By [6, Theorem 6.5, p. 103], u extends uniquely to a natural transformation $\bar{u}: T \rightarrow S$. Let $\{P_i: i \in I\}$ be a set of objects of \mathcal{P} . Consider the canonical morphisms $u_i: P_i \rightarrow \bigoplus P_i$ and $v_i: S(P_i) \rightarrow \bigoplus S(P_i)$ and let $r: \bigoplus S(P_i) \rightarrow S(\bigoplus P_i)$ the morphism uniquely determined by the conditions $r \circ v_i = S(u_i)$ for every $i \in I$. The morphism $\bar{u}_{\bigoplus P_i}$ is given by the composite arrow

$$T(\bigoplus P_i) = \bigoplus T(P_i) \xrightarrow{\bigoplus u_{P_i}} \bigoplus S(P_i) \xrightarrow{r} S(\bigoplus P_i).$$

Consider $\varepsilon_{\bigoplus P_i}: S(\bigoplus P_i) \rightarrow T(\bigoplus P_i)$. We claim that $\varepsilon_{\bigoplus P_i} \circ r = \bigoplus \varepsilon_{P_i}$. In fact, for every $j \in I$, $(\bigoplus \varepsilon_{P_i}) \circ v_j = \varepsilon_j \circ \varepsilon_{P_i} = T(u_j) \circ \varepsilon_{P_i}$. This last arrow is $\varepsilon_{\bigoplus P_i} \circ S(u_j)$ since ε is natural. Therefore, $\bigoplus \varepsilon_{P_i} \circ v_j = \varepsilon_{\bigoplus P_i} \circ S(u_j) = \varepsilon_{\bigoplus P_i} \circ r \circ v_j$ for every $j \in I$, whence $\varepsilon_{\bigoplus P_i} \circ r = \bigoplus \varepsilon_{P_i}$. Now, consider any object D of \mathcal{D} . There is an epimorphism $f: \bigoplus P_i \rightarrow D$ for some set $\{P_i\} \subseteq \mathcal{P}$. We have

$$\begin{aligned}
 \varepsilon_{\bigoplus P_i} \circ \bar{u}_{\bigoplus P_i} &= \varepsilon_{\bigoplus P_i} \circ r \circ (\bigoplus u_{P_i}) = (\bigoplus \varepsilon_{P_i}) \circ (\bigoplus u_{P_i}) \\
 &= \bigoplus (\varepsilon_{P_i} \circ u_{P_i}) = \bigoplus 1_{P_i} = 1_{\bigoplus P_i}.
 \end{aligned}$$

Since $\varepsilon \circ \bar{u}$ is natural, we obtain that

$$\begin{aligned} \varepsilon_D \circ \bar{u}_D \circ f &= f \circ (\varepsilon_{\oplus P_i} \circ \bar{u}_{\oplus P_i}) \\ &= f \circ 1_{\oplus P_i} = f = 1_D \circ f. \end{aligned}$$

The fact that f is an epimorphism provides that $\varepsilon_D \circ \bar{u}_D = 1_D$. Now, we can apply [7, Theorem 1.2] to obtain that G is separable.

The converse can be easily deduced from [7, Theorem 1.2]. \square

Theorem 1.3. *Let \mathcal{C} be a Grothendieck category that contains a full subcategory \mathcal{Q} whose objects form a set of projective generators of \mathcal{C} . Consider an abelian category \mathcal{D} and functors*

$$G: \mathcal{D} \rightleftarrows \mathcal{C}: F$$

such that F is left adjoint to G with unit $\eta: 1_{\mathcal{C}} \rightarrow GF$. Let us denote by $\iota_{\mathcal{Q}}: \mathcal{Q} \rightarrow \mathcal{C}$ the inclusion functor and assume that GF preserves direct limits. The functor F is separable if and only if there is a natural transformation $u: GF\iota_{\mathcal{Q}} \rightarrow \iota_{\mathcal{Q}}$ such that $u_Q \circ \eta_Q = 1_Q$ for every $Q \in \mathcal{Q}$.

Proof. Let us adopt the following notation: $T = GF$, $S = 1_{\mathcal{C}}$, $T' = GF\iota_{\mathcal{Q}}$, $S' = \iota_{\mathcal{Q}}$. Assume that there is a natural transformation $u: T' \rightarrow S'$ such that $u_Q \circ \eta_Q = 1_Q$ for every $Q \in \mathcal{Q}$. By [6, Theorem 6.5, p.103], u extends uniquely to a natural transformation $\bar{u}: T \rightarrow S$. We will show that $\bar{u}_C \circ \eta_C = 1_C$ for every object C of \mathcal{C} which implies, by [7, Theorem 1.2], that F is separable. We will first prove that for every set of $\{Q_i\}_{i \in I}$ of objects of \mathcal{Q} , the equality $\eta_{\oplus Q_i} = \oplus \eta_{Q_i}$ holds. Let $u_j: Q_j \rightarrow \oplus Q_i$ be the j th canonical map, for $j \in I$. We have

$$\eta_{\oplus Q_i} \circ u_j = T(u_j) \circ \eta_{Q_j} = (\oplus \eta_{Q_i}) \circ u_j.$$

This implies that $\eta_{\oplus Q_i} = \oplus \eta_{Q_i}$. Now, since $S = 1_{\mathcal{C}}$, we have that $\bar{u}_{\oplus P_i} = \oplus u_{P_i}$. Let C be an object of \mathcal{C} . There is an epimorphism $f: \oplus Q_i \rightarrow C$, where $\{Q_i\}$ is a set of objects of \mathcal{Q} . We have

$$\bar{u}_{\oplus Q_i} \circ \eta_{\oplus Q_i} = (\oplus u_{Q_i}) \circ (\oplus \eta_{Q_i}) = \oplus (u_{Q_i} \circ \eta_{Q_i}) = 1_{\oplus Q_i}$$

and, since $\bar{u} \circ \eta$ is natural,

$$1_C \circ f = f = f \circ (\bar{u}_{\oplus Q_i} \circ \eta_{\oplus Q_i}) = (\bar{u}_C \circ \eta_C) \circ f.$$

This implies that $\bar{u}_C \circ \eta_C = 1_C$. The converse is a direct consequence of [7, Theorem 1.2]. \square

Let $\varphi: R \rightarrow S$ be a ring homomorphism. The functor restriction of scalars $\varphi_*: {}_S\text{Mod} \rightarrow {}_R\text{Mod}$ is right adjoint to the induction functor $S \otimes_R - = \text{Ind}(-): {}_R\text{Mod} \rightarrow {}_S\text{Mod}$. Taking in the foregoing theorems $\mathcal{P} = \{S\}$ and $\mathcal{Q} = \{R\}$, we obtain the following.

Corollary 1.4 (Năstăsescu et al. [4 Proposition 1.3]). *Let $\varphi : R \rightarrow S$ be a ring homomorphism. Then*

1. φ_* is separable if and only if the canonical epimorphism $S \otimes_R S \rightarrow S$ splits as (S, S) -bimodule map.
2. $\text{Ind}(-)$ is separable if and only if φ is a splitting monomorphism of (R, R) -bimodules.

We will denote by $\varepsilon : \text{hom}_R(S, R) \rightarrow R$ the homomorphism of (R, R) -bimodules defined by $\varepsilon(f) = f(1)$ for every $f \in \text{hom}_R(S, R)$.

Corollary 1.5. *Let $\varphi : R \rightarrow S$ be a ring homomorphism. The functor $\text{hom}_R(S, -) : {}_R\text{Mod} \rightarrow {}_S\text{Mod}$ is separable if and only if the evaluation map $\varepsilon : \text{hom}_R(S, R) \rightarrow R$ is a splitting epimorphism of (R, R) -bimodules.*

Proof. Apply Theorem 1.2 taking $\mathcal{P} = \{R\}$. \square

2. Separable functors for graded ring homomorphisms

Let G be a group with neutral element e . For a G -graded ring R , we will denote by $R - gr$ the category of all G -graded unital left R -modules. A reference for the general theory of graded modules is [5].

There is a number of interesting functors relating the categories $R - gr$, ${}_R\text{Mod}$ and ${}_{R_e}\text{Mod}$, whose separability is investigated in [4]. In this section, we will investigate the separability of the functors associated to a homomorphism of graded rings.

Definition 2.1. Let $\varphi : R \rightarrow S$ be a ring homomorphism, where R and S are G -graded rings. We will say that φ is an homomorphism of graded rings if $\varphi(R_x) \subseteq S_x$ for every $x \in G$.

For every graded left S -module M , write $\varphi_*^{gr}(M)$ for M considered as R -module graded in the obvious way. This gives an exact functor

$$S - gr \xrightarrow{\varphi_*^{gr}} R - gr.$$

For a graded left R -module M , put $\text{IND}(M) = S \otimes_R M$. This left S -module can be graded by putting

$$(\text{IND}(M))_x = \left\{ \sum_{yz=x} s_y \otimes m_z : s_y \in S_y, m_z \in M_z \right\}$$

and consider also the graded left S -module $\text{HOM}(S, M)$ whose x th homogeneous component is

$$(\text{HOM}(S, M))_x = \{ f \in \text{hom}_R(S, M) : f(S_y) \subseteq M_{yx} \forall y \in G \}.$$

This provides two functors

$$R - gr \begin{matrix} \xrightarrow{\text{IND}(-)} \\ \xrightarrow{\text{HOM}_R(S,-)} \end{matrix} S - gr.$$

A straightforward argument shows that $\text{IND}(-)$ is a left adjoint functor to φ_*^{gr} and $\text{HOM}(S, -)$ is a right adjoint functor to φ_*^{gr} . Our aim is to characterize the separability of these functors.

Let A be a G -graded ring. For $x \in G$, let us denote by $T_x^A : A - gr \rightarrow A - gr$ the isomorphism of categories defined as follows. This functor is the identity on morphisms. For $M \in A - gr$, takes $T_x^A(M) \in A - gr$ as the underlying A -module M with the new grading given by $(T_x^A(M))_y = M_{yx}$ for every $y \in G$. We will use also the notation $M(x) = T_x^A(M)$.

Definition 2.2. Let A, B be G -graded rings. A functor $F : A - gr \rightarrow B - gr$ is said to be a *graded functor* if $F \circ T_x^A = T_x^B \circ F$ for every $x \in G$.

It is very easy to show that the functors $\text{IND}(-)$, $\text{HOM}(S, -)$ and φ_*^{gr} are graded functors.

Proposition 2.3. Let $\varphi : R \rightarrow S$ be a homomorphism of G -graded rings. The functor $\text{IND}(-)$ is separable if and only if there is a family $\{u_x : S \rightarrow R : x \in G\}$ of morphisms in $R - gr$ such that

- (a) $u_x \circ \varphi = 1_R$ for every $x \in G$.
- (b) $u_{xy}(sr) = u_x(s)r$ for every $x, y \in G$, every $s \in S$ and every $r \in R_y$.

Proof. Let $\mathcal{Q} = \{R(x) : x \in G\}$. Observe that, for every $x \in G$, we have a natural isomorphism $\varphi_*^{gr}(\text{IND}(R(x))) \cong \varphi_*^{gr}(S(x))$. With this identification, the unit of the adjunction takes in $R(x)$ the value $\varphi(x) : R(x) \rightarrow S(x)$. If $\text{IND}(-)$ is separable, then, by Theorem 1.3, there are natural homomorphisms $u_{R(x)} : S(x) \rightarrow R(x)$ ($x \in G$) in $R - gr$ such that $u_{R(x)} \circ \varphi(x) = 1_{R(x)}$ for every $x \in G$. Let $u_x : S \rightarrow R$ be equal to $u_{R(x)}$ provided that the shift functors are understood. This set $\{u_x : S(x) \rightarrow R(x)\}$ of morphisms satisfies the conditions stated. Conversely, assume such a family of maps given. For $R(x) \in \mathcal{Q}$, define $u_{R(x)} : \varphi_*^{gr} S(x) \rightarrow R(x)$ given by u_x when the x th shift functor is considered. This gives a natural transformation in the conditions of Theorem 1.3, whence $\text{IND}(-)$ is separable. \square

The following two propositions can be deduced from Theorem 1.2 likewise Proposition 2.3 has been deduced from Theorem 1.3.

Proposition 2.4. Let $\varphi : R \rightarrow S$ be a homomorphism of G -graded rings. Consider $\varepsilon : S \otimes S \rightarrow S$ the canonical multiplication map. The functor φ_*^{gr} is separable if and only

if there is a family $\{u_x : S \rightarrow S \otimes_R S : x \in G\}$ of morphisms in $S - gr$ such that

- (a) $\varepsilon \circ u_x = 1_S$ for every $x \in G$.
- (b) $u_{xy}(st) = u_x(s)t$ for every $s \in S, t \in S_y$ and $x, y \in G$.

Proposition 2.5. Let $\varphi : R \rightarrow S$ be a homomorphism of G -graded rings. Consider $\varepsilon : \text{HOM}(S, R) \rightarrow R$ be defined by $\varepsilon(f) = f(1)$. The functor $\text{HOM}(S, -)$ is separable if and only if there is a family $\{u_x : R \rightarrow \text{HOM}(S, R) : x \in G\}$ of morphisms in $R - gr$ such that

- (a) $\varepsilon \circ u_x = 1_R$ for every $x \in G$.
- (b) $u_{xy}(rt) = u_x(r)t$ for every $r \in R, t \in R_y$ and $x, y \in G$.

3. Strongly separable functors

Let $\varphi : R \rightarrow S$ be an homomorphism of G -graded rings. We will analyze the relationship between the separability of the graded functors

$$\text{IND}(-), \text{HOM}(S, -) : R - gr \rightarrow S - gr,$$

$$\varphi_*^{gr} : S - gr \rightarrow R - gr$$

and the ungraded functors

$$\text{Ind}(-), \text{hom}_R(S, -) : {}_R\text{Mod} \rightarrow {}_S\text{Mod},$$

$$\varphi_* : {}_S\text{Mod} \rightarrow {}_R\text{Mod}.$$

Roughly speaking, we will prove that ungraded separability implies graded separability and we will identify the additional condition needed to have the converse. This condition is stated in the following definition.

Definition 3.1. Let A, B be G -graded rings. A graded functor $F : A - gr \rightarrow B - gr$ is said to be *strongly separable* if it is separable and the separability mapping v satisfies that the following diagram commutes

$$\begin{array}{ccc} \text{hom}_{B-gr}(F(M), F(N)) & \xrightarrow{v_{M,N}} & \text{hom}_{A-gr}(M, N) \\ \downarrow T_x & & \downarrow T_x \\ \text{hom}_{B-gr}(F(M)(x), F(N)(x)) & \xrightarrow{v_{M(x),N(x)}} & \text{hom}_{A-gr}(M(x), N(x)) \end{array}$$

for every $M, N \in R - gr$ and every $x \in G$.

Definition 3.2. Let $F, G : A - gr \rightarrow B - gr$ be graded functors. A natural transformation $\alpha : F \rightarrow G$ is said to be graded if $T_x(\alpha_M) = \alpha_{M(x)}$ for every $M \in A - gr$ and for every $x \in G$.

Proposition 3.3. *Let*

$$F : R - gr \xrightleftharpoons{\quad} S - gr : G$$

be graded functors such that F is left adjoint to G . If the unit η and the counit ρ are graded natural transformations, then

1. *F is strongly separable if and only if there is a graded natural transformation $u : GF \rightarrow 1_{R-gr}$ such that $u_M \circ \eta_M = 1_M$ for every $M \in R - gr$.*
2. *G is strongly separable if and only if there is a graded natural transformation $u : 1_{S-gr} \rightarrow FG$ such that $\rho_N \circ u_N = 1_N$ for every $N \in S - gr$.*

Proof. We will only prove (1) since the proof of (2) is similar. Assume that F is strongly separable. If v is a separability mapping then put $u_M = v_{GF, M}(\rho_{FM})$ for every graded left R -module M . From [7, Theorem 1.2] and [2, Theorem 4.1], this gives a natural transformation $u : GF \rightarrow 1_{R-gr}$ such that $u_M \circ \eta_M = 1_M$ for every graded left R -module M . Now it is routine to check that u is graded.

Conversely, assume a graded natural transformation $u : GF \rightarrow 1_{S-gr}$ is given such that $u_M \circ \eta_M = 1_M$ for every $M \in R - gr$. Following [2, Theorem 4.1] we define a separability mapping by $v_{M, M'}(f) = u_{M'} \circ G(f) \circ \eta_M$ for $M, M' \in S - gr$ and $f \in \text{hom}_{R-gr}(FM, FM')$. A straightforward computation shows that F is strongly separable. \square

Theorem 3.4. *Let*

$$F : R - gr \xrightleftharpoons{\quad} S - gr : G$$

be right exact graded functors such that F is left adjoint to G . Consider the subcategories $\mathcal{Q} = \{R(x) : x \in G\}$ and $\mathcal{P} = \{S(x) : x \in G\}$ of $R - gr$ and $S - gr$, respectively. If the unit η and the counit ρ are graded natural transformations, then

1. *The functor F is strongly separable if and only if there is a natural transformation $u : GF \nu_{\mathcal{Q}} \rightarrow 1_{\mathcal{Q}}$ such that $u_R \circ \eta_R = 1_R$ and $T_x(u_R) = u_{R(x)}$ for every $x \in G$.*
2. *The functor G is strongly separable if and only if there is a natural transformation $u : 1_{\mathcal{P}} \rightarrow FG \nu_{\mathcal{P}}$ such that $\rho_S \circ u_S = 1_S$ and $T_x(u_S) = u_{S(x)}$ for every $x \in G$.*

Proof. We will only prove part (1) since part (2) is analogue. The necessity is a consequence of Proposition 3.3. Therefore, assume that there is a natural transformation $u : GF \nu_{\mathcal{Q}} \rightarrow 1_{\mathcal{Q}}$ such that $u_R \circ \eta_R = 1_R$ and $T_x(u_R) = u_{R(x)}$ for every $x \in G$. It is an immediate consequence that $u_{R(x)} \circ \eta_{R(x)} = 1_{R(x)}$ for every $x \in G$. By Theorem 1.3, the functor F is separable. Moreover, by the proof of that theorem, there is a natural transformation $\bar{u} : GF \rightarrow 1_{R-gr}$ extending u and such that $\bar{u}_C \circ \eta_C = 1_C$ for every $C \in R - gr$. By

Proposition 3.3, if we prove that $T_x(\bar{u}_C) = \bar{u}_{C(x)}$ for every $x \in G$, then F is strongly separable. Given $C \in R - gr$, take an epimorphism $f : \bigoplus R(x_i) \rightarrow C$ in $R - gr$, where $x_i \in G$. Using that \bar{u} is natural we obtain

$$f \circ \bar{u}_{\bigoplus R(x_i)} = \bar{u}_C \circ GF(f)$$

whence

$$T_x(f) \circ T_x(\bar{u}_{\bigoplus R(x_i)}) = T_x(\bar{u}_C) \circ T_x(GF(f))$$

for every $x \in G$. Since G and F are graded functors, we have

$$T_x(f) \circ T_x(\bar{u}_{\bigoplus R(x_i)}) = T_x(\bar{u}_C) \circ GF(T_x(f)).$$

But \bar{u} is natural and, thus,

$$T_x(f) \circ T_x(\bar{u}_{\bigoplus R(x_i)}) = \bar{u}_{C(x)} \circ GF(T_x(f)).$$

Therefore,

$$T_x(\bar{u}) \circ GF(T_x(f)) = \bar{u}_{C(x)} \circ GF(T_x(f))$$

for every $x \in G$. Finally, $GF(T_x(f))$ is an epimorphism since F, G and T_x are right exact functors. Hence, $T_x(\bar{u}_C) = \bar{u}_{C(x)}$ and the proof is finished. \square

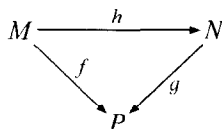
Next, we will apply the foregoing general results to characterize when the canonical functors associated to an homomorphism of graded rings $\varphi : R \rightarrow S$ are strongly separable. We start with the restriction of scalars graded functor.

Theorem 3.5. *Let $\varphi : R \rightarrow S$ be a morphism of G -graded rings. The graded functor $\varphi_*^{gr} : S - gr \rightarrow R - gr$ is strongly separable if and only if the canonical multiplication map $\mu : S \otimes_R S \rightarrow S$ is a splitting epimorphism of graded (S, S) -bimodules.*

Proof. This is a consequence of Theorem 3.4.2 together with [4, Proposition 1.3]. \square

The proof of [5, Lemma I.2.1] can be easily adapted to prove the following one.

Lemma 3.6. *Let M, N, P be graded (R, R) -bimodules. Consider the commutative diagram*



where f is a graded morphism of (R, R) -bimodules. If g (resp. h) is a morphism of graded (R, R) -bimodules then there is a graded morphism of bimodules h' (resp. g') such that $f = g \circ h'$ (resp. $f = g' \circ h$).

The following result relates the separability of the graded restriction of scalars functor and the ungraded one.

Theorem 3.7. *Let $\varphi : R \rightarrow S$ be a morphism of graded rings. The functor $\varphi_* : {}_S\text{Mod} \rightarrow {}_R\text{Mod}$ is separable if and only if $\varphi_*^{gr} : S - gr \rightarrow R - gr$ is strongly separable.*

Proof. If φ_* is separable, then, by [4, Proposition 1.3], there is a (S, S) -bimodule map $u : S \rightarrow S \otimes_R S$ such that $\mu \circ u = 1_S$. By Lemma 3.6, there is a morphism of graded (S, S) -bimodules $\bar{u} : S \rightarrow S \otimes S$ such that $\mu \circ \bar{u} = 1_S$. By Theorem 3.5, φ_*^{gr} is a separable functor. Conversely, if φ_*^{gr} is strongly separable then it follows from Theorem 3.5 and [4, Proposition 1.3] that φ_* is separable. \square

The next example shows that a graded functor can be separable but not strongly separable.

Example 3.8. Let R be a strongly graded ring, i.e., $R_x R_y = R_{xy}$ for every $x, y \in G$. If R_e is endowed with the trivial G -grading, then the inclusion $\varphi : R_e \rightarrow R$ is a homomorphism of graded rings. We claim that the functor $\varphi_*^{gr} : R - gr \rightarrow R_e - gr$ is separable. To prove this claim, consider the functors $(-)_e^R : R - gr \rightarrow {}_{R_e}\text{Mod}$ and $(-)_e^{R_e} : R_e - gr \rightarrow {}_{R_e}\text{Mod}$ which, for a G -graded module, take its homogeneous e -component. By [1, Theorem 3.8], $(-)_e^R$ is an equivalence of categories. Moreover, $(-)_e^{R_e} \circ \varphi_*^{gr} = (-)_e^R$ which implies, by [4, Lemma 1.1.(3)], that φ_*^{gr} is separable. Now, if φ_*^{gr} is strongly separable, then $\varphi_* : {}_R\text{Mod} \rightarrow {}_{R_e}\text{Mod}$ is separable (Theorem 3.7). But this is not true if the group G is infinite (see [4, Proposition 2.1]).

Recall that for the homomorphism of graded rings $\varphi : R \rightarrow S$, there are two ungraded functors, $\text{hom}_R(S, -), \text{Ind}(-) : \text{mod } R \rightarrow \text{mod } S$ and two graded functors $\text{HOM}(S, -), \text{IND}(-) : R - gr \rightarrow S - gr$. Our last theorem states that if one of these functors is (strongly) separable, then so is for the other three. Moreover, this characterizes when φ is a splitting monomorphism of (R, R) -bimodules.

Theorem 3.9. *The following conditions are equivalent for a morphism of graded rings $\varphi : R \rightarrow S$.*

- (i) φ is a splitting monomorphism of (R, R) -bimodules.
- (ii) φ is a splitting monomorphism of graded (R, R) -bimodules.
- (iii) The functor $\text{IND}(-) : R - gr \rightarrow S - gr$ is strongly separable.
- (iv) The functor $\text{Ind} : {}_R\text{Mod} \rightarrow {}_S\text{Mod}$ is separable.
- (v) The functor $\text{hom}_R(S, -) : {}_R\text{Mod} \rightarrow {}_S\text{Mod}$ is separable.
- (vi) The functor $\text{HOM}(S, -) : R - gr \rightarrow S - gr$ is strongly separable.
- (vii) The evaluation map $\omega : \text{HOM}(S, R) \rightarrow R$ is a splitting epimorphism of graded (R, R) -bimodules.
- (viii) The evaluation map $\varepsilon : \text{hom}_R(S, R) \rightarrow R$ is a splitting epimorphism of (R, R) -bimodules.

Proof. (i) \Rightarrow (ii) Let $\beta: S \rightarrow R$ be an R -bimodule map such that $\beta \circ \varphi = 1_R$. By Lemma 3.6, there is a graded homomorphism of R -bimodules $\bar{\beta}: S \rightarrow R$ such that $\bar{\beta} \circ \varphi = 1_R$.

(ii) \Rightarrow (i) Evident.

(iii) \Rightarrow (ii) If $\text{IND}(-)$ is separable, then, by Proposition 3.3, there is a graded natural transformation $u: \varphi_*^{gr} \text{IND}(-) \rightarrow 1_{R-gr}$ such that $u_M \circ \eta_M = 1_M$ for every $M \in R-gr$. Now, it is easy to prove that $u_R: S \rightarrow R$ is a morphism of graded (R, R) -bimodules. Since $\eta_R = \varphi$, we have that u_R is a splitting epimorphism for φ .

(ii) \Rightarrow (iii) Assume that there exists an homomorphism of graded (R, R) -bimodules $u_R: S \rightarrow R$ such that $u \circ \varphi = 1_R$. Consider $\mathcal{Q} = \{R(x): x \in G\}$. For $x \in G$, define $u_{R(x)} = T_x(u_R)$. A routine argument shows that this gives a natural transformation $u: \varphi_*^{gr} \text{IND}(-) \rightarrow 1_{\mathcal{Q}}$. Now, Theorem 3.4 applies here.

(i) \Leftrightarrow (iv) This is Corollary 1.4.2.

(i) \Rightarrow (viii) Assume that $\beta: S \rightarrow R$ is an R -bimodule homomorphism such that $\beta \circ \varphi = 1_R$. Define $\phi: R \rightarrow \text{hom}_R(S, R)$ by $\phi(r) = r\beta$, where we use the canonical structure of R -bimodule on $\text{hom}_R(S, R)$. It is a routine matter to show that ϕ is an R -bimodule map. Moreover, for $r \in R$,

$$\begin{aligned} (\varepsilon \circ \phi)(r) &= \phi(r)(1) = (r\beta)(1) = \beta(1r) \\ &= \beta(r1) = \beta(\varphi(r)) = r. \end{aligned}$$

(viii) \Leftrightarrow (v) This is Corollary 1.5.

(viii) \Rightarrow (i) There is an R -bimodule homomorphism

$$\phi: R \rightarrow \text{hom}_R(S, R)$$

such that $\varepsilon \circ \phi = 1_R$. Write $\beta = \phi(1)$. Now it is routine to check that β is a homomorphism of R -bimodules such that $\beta \circ \varphi = 1_R$.

(vi) \Rightarrow (vii) Assume that $\text{HOM}(S, -)$ is a strongly separable functor. Since φ_*^{gr} is a left adjoint functor to $\text{HOM}(S, -)$, it follows from Proposition 3.3 that there exists a graded natural transformation $u: 1_{R-gr} \rightarrow \varphi_*^{gr} \circ \text{HOM}(S, -)$ that splits the counit of the adjunction. In particular, $\omega \circ u_R = 1_R$ and $T_x(u_R) = u_{R(x)}$ for every $x \in G$. It is easy to check that u_R is a graded (R, R) -bimodule map.

(vii) \Rightarrow (vi) Conversely, assume that there is a graded R -bimodule homomorphism $u_R: R \rightarrow \text{HOM}(S, R)$ such that $\omega \circ u_R = 1_R$. Let $M \in R-gr$. We obtain a homomorphism of graded left R -modules u_M as the composite map

$$M \xrightarrow{i} R \otimes_R M \xrightarrow{u_R \otimes 1} \text{HOM}_R(S, R) \otimes_R M \xrightarrow{\gamma} \text{HOM}_R(S, M).$$

A routine computation shows that this defines a graded natural transformation

$$u: 1_{R-gr} \rightarrow \varphi_*^{gr} \text{HOM}(S, -).$$

Moreover, if $\rho: \varphi_*^{gr} \text{HOM}(S, -) \rightarrow 1_{R-gr}$ is the counit of the adjunction, then $\rho \circ u = 1_{R-gr}$. By Proposition 3.3, $\text{HOM}(S, -)$ is a strongly separable functor.

(vii) \Rightarrow (v) Let $u_R : R \rightarrow \text{HOM}(S, R)$ be a graded R -bimodule map such that $\omega \circ u_R = 1_R$. Consider the ungrading functor $U : R\text{-gr} \rightarrow R\text{Mod}$ and let $\varepsilon : \text{hom}_R(S, R) \rightarrow R$ be the canonical evaluation map. It is clear that $\varepsilon \circ i = \omega$, where i denotes the inclusion map $UHOM(S, R) \subseteq \text{hom}_R(S, R)$. Therefore, the map $v_R = i \circ U(u_R) : R \rightarrow \text{hom}_R(S, R)$ is an R -bimodule map such that $\varepsilon \circ v_R = 1_R$. By Corollary 1.5, $\text{hom}_R(S, -)$ is separable.

(v) \Rightarrow (vii) Let $v_R : 1_R \rightarrow \text{hom}_R(S, R)$ be an R -bimodule morphism such that $\varepsilon \circ v_R = 1_R$. Let $f = v_R(1) \in \text{hom}_R(S, R)$. Since v_R is a homomorphism of R -bimodules, we have that $rf = fr$ for every $r \in R$. Define $f_e : S \rightarrow R$ by $f_e(s) = f(s)_x$ for $s \in S_x$. By [5, Lemma I.2.1], $f_e \in \text{HOM}(S, R)$. Now, let $u_R : R \rightarrow \text{HOM}(S, R)$ given by $u_R(r) = rf_e$ for every $r \in R$. It is easy to check that u_R is a graded R -bimodule map such that $\omega \circ u_R = 1_R$. \square

Corollary 3.10. *If $\varphi : R \rightarrow S$ is any ring homomorphism, then the following conditions are equivalent.*

- (i) φ is a splitting monomorphism of (R, R) -bimodules.
- (ii) The functor $\text{Ind}(-) : {}_R\text{Mod} \rightarrow {}_S\text{Mod}$ is separable.
- (iii) The functor $\text{hom}_R(S, -) : {}_R\text{Mod} \rightarrow {}_S\text{Mod}$ is separable.
- (iv) The evaluation map $\varepsilon : \text{hom}_R(S, R) \rightarrow R$ is a splitting epimorphism of (R, R) -bimodules.

Proof. φ can be considered as homomorphism of G -graded rings for the one-element group G . Therefore, the foregoing theorem applies. \square

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