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SEPARABLE ALGEBRAIC CLOSURE IN A TOPOS

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0. Introduction

In this paper we construct the separable algebraic closure of any field in a topos $\mathscr{E}$. (A separable closure is a field extension in which every separable polynomial splits and which is generated by the roots of these polynomials. P is a separable polynomial if P and its derivative P' are relatively prime.) The separable closure lives in a new topos, over $\mathscr{E}$, and has the universal property which makes it a spectrum in the sense of Cole, see [1], [9] and [4, Theorem 6.58].

Recall that if $\mathscr{E}$ is the topos of Sets there are two ways to construct this spectrum. One is to use the étale topos which is the home of the generic separable closure of an arbitrary commutative ring in Sets (see [3, 12]). As shown by Hakim [3, pp. 77–84], this construction can be applied within any Grothendieck topos by setting up the étale topology (object by object on the defining site of the topos).

Alternatively, we can describe the generic separable closure of a field as a field extension in the topos of continuous G-Sets where G is the profinite Galois group. In this paper, we generalize the profinite Galois group approach. We feel that the value of this paper lies not so much in the alternative, more internal, construction of the separable closure but in the topos theoretic structure we develop along the way.

We define a profinite group and more generally profinite groupoid, and profinite category in any topos. If Γ is a profinite category in $\mathscr{E}$ then we construct the topos $\mathscr{E}^{\Gamma}$ of continuous Γ-actions. We generalize the profinite Galois group of the algebraic closure of a field. In general it is a connected profinite groupoid. As shown in Section 3, below, it is Morita equivalent to a profinite group precisely when the separable closure of K in $\mathscr{E}$ can be constructed within the topos $\mathscr{E}$ itself. For example, any field in Sets has an algebraic closure in Sets so the profinite Galois groupoid is effectively equivalent to a profinite group. When there does exist a separable closure of K in $\mathscr{E}$ itself then the Galois groupoid is equivalent to the profinite group constructed in [6].

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Profinite categories in Sets are filtered limits of finite categories and so are categories with Compact, Hausdorff, totally disconnected topologies. The continuous actions form a topos of 'functor-sheaves' over the category. (Details are in Example 5.1 below. Profinite categories in spatial topoi are discussed in 5.2.) It is our contention that several instances of spectra can be described as profinite categories. For example, the localization spectrum (= the space of prime ideals with the Zariski topology) can be thought of as a Boolean Space (the prime ideals with the *patch* topology, see [2, p. 234]) together with a category structure given by the ordering $P \leq Q$ iff $P \supseteq Q$. The 'functor-sheaves' (i.e. the continuous actions) then coincide with the sheaves over the Zariski topology. Curiously, the integral domain spectrum (see [5]) is the *dual* profinite category (with the same Boolean space of objects but with the ordering reversed.)

Conventions. (1) Throughout this paper 'field' shall mean 'geometric field' (as in [4, p. 215]).

(2) The topos $\mathscr{E}$ will always be assumed to have *countable limits.* It follows that $\mathscr{E}$ also has countable colimits.

(3) When there is an equivalence relation on $A \times B$ then $[a, b]$ denotes the equivalence class containing (a, b).

(4) Other terminology is generally based on [4], and, in the case of field extensions, [6].

(5) The proof of Theorem 1.1 is postponed until Section 4. I hope this will make the paper more readable. The basic constructions and some lemmas are included in the earlier sections so that the main results and the technical flavor can be obtained from reading Sections 1, 2 and 3. Section 5 gives some examples.

We shall make frequent use of internal colimits, particularly over filtered categories. It is convenient at this point to make a definition and list three basic properties of colimits.

Definition. Let $p: F \to C$ be a diagram over the filtered category C in a topos $\mathscr{E}$. Recall that F_0, C_0 are the objects of F and C and $p_0: F_0 \to C_0$ the object part of p. Let Q be the colimit and $\phi: F_0 \to Q$ the canonical map.

We say that F is a *monodiagram,* or that Q is a *filtered union,* if $(\phi, p_0): F_0 \to Q \times C_0$ is mono. [In Sets this means that for each map $i \to j$ in C, the corresponding map $F_i \to F_j$ is mono where $F_i = p^{-1}(i)$, so $Q = \bigcup F_i$.]

Lemma 0.1 (properties of internal colimits).

(i) *Internal colimits are preserved by inverse image functors.*

(ii) *Internal filtered colimits are left exact (i.e. if C is filtered then the colimit functor from $\mathscr{E}^C$ to $\mathscr{E}$ is left exact).*

(iii) *Let $p: F \to C$ be a monodiagram with colimit Q, and let $t: X \to Q$ be given. Then there exists a monodiagram $\bar{X}$ over C and map $\bar{t}: \bar{X} \to F$ such that $X =$* Colimit $\bar{X}$ *and $t =$* Colimit $\bar{t}$. *[In* Sets, *$Q = \bigcup F_i$ and $\bar{X}_i = t^{-1}(F_i)$ etc.]*

Proof. (i) follows from the definition of an internal colimit as a coequalizer, see [4, 2.24 or p. 51].

(ii) is known, see [4, 2.58 or p. 70].

(iii) Since $(\phi, p_0): F_0 \to Q \times C_0$ is mono, we can let $\bar{X} \to X \times C_0$ be the pullback of F_0 along $t \times C_0$. It is straightforward to verify that $\bar{X}$ has the required properties.

1. Profinite groups, groupoids, categories and actions

The purpose of this section is to generalize the notion of 'profinite group action' to 'profinite category action'.

We start with

Notation. By a *finite category* we mean a category with a finite set of maps. A *groupoid* is a category in which each map is invertible. A *group* is a groupoid with exactly one object. We let:

Fin Cat = The category of all finite categories and functors,

Fin Grpoid = The full subcategory of Fin Cat comprised of the groupoids,

Fin Grp = The category of finite groups.

Definition. By a *profinite category, Γ*, in a topos $\mathscr{E}$, we mean a left exact functor Γ from Fin Cat to $\mathscr{E}$. We say that Γ is a *profinite groupoid* if it factors through the forgetful functor from Fin Cat to Fin Grpoid which 'forgets about' all non-invertible maps.

Recall that in [6] a profinite group in $\mathscr{E}$ is defined to be a left exact functor Γ_0 from Fin Grp to $\mathscr{E}$. Every such Γ_0 gives rise to a profinite groupoid Γ where $\Gamma(C)$ is the disjoint union $\bigcup \Gamma_0(\mathrm{Aut}(c))$ where c ranges over the objects of C and $\mathrm{Aut}(c)$ is the group of Automorphisms. A profinite category Γ in $\mathscr{E}$ is said to be a *profinite group* if it arises in this way, i.e. if $\Gamma(C)$ is $\bigcup \Gamma(\mathrm{Aut}(c))$.

Examples. (1) In Sets, a profinite category Γ can be represented by a category W with a compatible Boolean topology and enough continuous functors into finite discrete categories. Then $\Gamma(C)$ is the set of continuous functors from W to C. (See Example 5.1 for further details.)

(2) In any topos $\mathscr{E}$, each internal category D has a *profinite completion, $\hat{D}$* where $\hat{D}(C)$ is the object of functors from D to C (see Lemma 1.2 for further details).

Definition. Let $\emptyset$ be the empty category in Fin Cat. Then a profinite category Γ is *regular* (resp. *trivial*) if $\Gamma(\emptyset) = 0$, (resp. $\Gamma(\emptyset) = 1$). Any profinite category can be represented as a regular profinite category glued to a trivial one.

Definition. An *action* (in Sets) is a pair (A, C) where C is a category and A is a functor from C to Sets. (We use terminology which emphasizes that actions generalize the notion of group action). (A, C) is a *finite action* if C is a finite category and $A(c)$ is a finite set for all c in C. We sometimes write A_c for $A(c)$ and call it the *fiber of A over c*. The action (A, C) is a *group action* (resp. *groupoid action*) if C is a group (resp. groupoid). We also say that (A, C) is an *action over C*.

If (A, C) and (B, C) are actions over C then a *C-equivariant map, $m: A \to B$,* is a natural transformation.

Definition. If $f: D \to C$ is a functor and if (A, C) is an action over C then $f^0(A, C) = (Af, D)$ is the *induced action* over D.

An *action map* from (A, C) to (B, D) is a pair (m, f) where $f: D \to C$ is a functor and $m: Af \to B$ is D-equivariant. We also say that (m, f) is an *action map over f*. We let:

$$\text{Fin Act} = \text{The category of all finite actions and action maps.}$$

The notation Fin Act/Grp (resp. Fin Act/Grpoid) denotes the category of finite actions over groups (resp. groupoids).

Definition. (A, C) is an *internal action* in the topos E if C is an internal category and A is an internal diagram. The *object of all action maps* between two internal actions is easily defined.

Definition. By a *profinite action* in a topos $\mathscr{E}$ we mean a left exact functor:

$$X: \text{Fin Act}^{\text{op}} \to \mathscr{E}.$$

There is a canonical left exact embedding:

$$I: \text{Fin Cat} \to \text{Fin Act}^{\text{op}}$$

given by $I(C) = (\emptyset, C)$ where $\emptyset$ is the *empty action* (or functor sending each object to the empty set). Therefore if X is a profinite action then $\Gamma = XI$ is a profinite category. We say that X is a *profinite action over Γ*, or even that (X, Γ) is a profinite action, when $XI = \Gamma$.

X is said to be a *profinite groupoid action* (resp. *profinite group action*) when $\Gamma = XI$ is a profinite groupoid (resp. profinite group).

Finally, if X and Y are profinite actions over Γ then $m: X \to Y$ is a *Γ-equivariant map* if m is a natural transformation such that mI is the identity.

Notation. Let X be a profinite action over Γ and let (A, C) be in Fin Act. Then

$$\pi: X(A, C) \to \Gamma(C)$$

shall generally denote the *canonical projection*, $X(i)$, arising from $i: (\phi, C) \to (A, C)$.

Definition. Let Γ be a profinite category in a topos $\mathscr{E}$. Then $\mathscr{E}^{\Gamma}$ is the category of profinite actions over Γ and Γ-equivariant maps.

Theorem 1.1. *Let Γ be a profinite category in the topos $\mathscr{E}$. (Recall that $\mathscr{E}$ is assumed to have countable limits.) $\mathscr{E}^{\Gamma}$ is then a topos over $\mathscr{E}$. The geometric functor $\mathscr{E}^{\Gamma} \to \mathscr{E}$ sends X to $X(1,1)$.*

Proof. The proof is banished to Section 4 of this paper.

Comment. It is known that if Γ is a profinite group in Sets then the category of continuous Γ-sets is a topos. Theorem 1.1 generalizes this in two ways: Γ can be a profinite groupoid, or profinite category and the underlying topos need not be Sets. If Γ is a profinite category in Sets then Sets$^{\Gamma}$ is the category of functor-sheaves discussed in the introduction (see also 5.1 below). If D is an internal category in a topos $\mathscr{E}$ then there are topoi $\mathscr{E}^{D}$ and $\mathscr{E}^{\hat{D}}$ (see example 2, above). Generally, these topoi differ, Lemma 1.2, below, shows one case where they coincide.

Definition. Two profinite categories Γ and Γ' in $\mathscr{E}$ are *Morita equivalent* if $\mathscr{E}^{\Gamma}$ is naturally equivalent to $\mathscr{E}^{\Gamma'}$. In Sets every connected profinite groupoid is Morita equivalent to a profinite group, but this is not true for all topoi. See 5.4 below. (A profinite category is *connected* if it preserves coproducts.)

Example 5.6 shows that Morita equivalence is weaker than natural equivalence.

Lemma 1.2. *Let D be an internal category in $\mathscr{E}$ and let $\hat{D}$ be its profinite completion (as in example 2, above). Suppose that D is a finite category in $\mathscr{E}$ in the sense of having 'at most n morphisms' where n is an 'ordinary' positive integer (in* Sets*). (That is, the object of morphisms of D is to satisfy the geometric condition, for all $x_0, x_1, \dots, x_n$, of $\bigvee\{x_i = x_j \mid i<j\}$.) Then $\mathscr{E}^{D}$ is equivalent to $\mathscr{E}^{\hat{D}}$.*

Proof. Let F be a diagram over D. We have to construct a profinite action $\hat{F}$ over $\hat{D}$. Since (F,D) is an internal action in $\mathscr{E}$ we may define $\hat{F}(A,C)$ as the internal object of action maps from (A,D) to (F,D). Conversely, let X be a profinite action over $\hat{D}$. Define $\bar{X}$ as the object of all $[\zeta,d,a]$ where $d \in \mathrm{Obj}(D)$, $\zeta \in X(A,C)$, for some (A,C) and $a \in A(c)$ where $c = \pi(\zeta)(d)$. Further regard $[\zeta,d,a] = [\zeta',d,a']$ if $\zeta \in X(A,C)$, $\zeta' \in X(A',C')$ and there exists $(m,f):(A,C) \to (A',C')$ with $m(a) = a'$ and $X(m,f)(\zeta') = \zeta$. Then $\bar{X}$ is a diagram over D with $[\zeta,d,a]$ lying over d.

There is a canonical map $(\hat{F})^{-} \to F$ given by $[\zeta,d,a] \to f(a)$ where $\zeta = (m,f)$. There is an analogous map $X \to (\bar{X})\hat{}$. Given $\zeta \in X(A,C)$ we must find a member of $\mathrm{Hom}[(A,C),(\bar{X},D)]$. Let $f: D \to C$ be $\pi(\zeta)$, then for each $d \in D$ let $c = f(d)$ and map $a \in A(c)$ to $[\zeta,d,a]$. That these maps are isomorphisms can be verified in Sets, which suffices in view of [8].

2. The profinite Galois groupoid and the separable closure

Let K be a field in a topos $\mathscr{E}$. We wish to construct the separable closure of K. The main idea is to construct the generic separable polynomial over K, for K a field in $\mathscr{E}$ and split it by the procedure, given in [6], which we now review:

Splitting Construction. Let P be a separable polynomial of degree n over a field K in a topos $\mathscr{E}$. Let $K\langle \bar{r}\rangle$ be the 'splitting ring' of P, obtained by formally adjoining to K a complete set of roots $\bar{r}=\{r_1,\dots,r_n\}$ for P. Then S_n, the group of permutations of $\{1,\dots,n\}$, acts on $K\langle \bar{r}\rangle$ so $K\langle \bar{r}\rangle$ lives in $\mathscr{E}^{S_n}$ (and is the generic regular ring extension of K for which there is a distinguished root set for P).

$K\langle \bar{r}\rangle$ is always regular when P is separable (see [6]) so we make it into a field in the topos of sheaves over B where B is the Boolean algebra of idempotents of $K\langle \bar{r}\rangle$. Note that the topos of B-sheaves, denoted $\mathrm{Shv}(B, \mathscr{E}^{S_n})$ is constructed in $\mathscr{E}^{S_n}$ where B lives. We need to interpret this construction in terms of a profinite groupoid.

Lemma 2.1. *Let B be a decidable Boolean algebra in $\mathscr{E}$. Let H be a finite group which acts on B as a group of Boolean homomorphisms (so B is still a Boolean algebra in $\mathscr{E}^H$.) There then exists a profinite groupoid Γ on $\mathscr{E}$ for which $\mathscr{E}^\Gamma$ is equivalent to* $\mathrm{Shv}(B, \mathscr{E}^H)$.

Proof. By an *H-partition of length k* we mean a k-tuple $\bar{b}=(b_1,\dots,b_k)$ where each $b_i\in B$ is nonzero, $\bigvee b_i=1$, $b_i\wedge b_j=0$ for $i\neq j$ and for each $h\in H$ and each i there is a j with $h(b_i)=b_j$.

Let $W(k)$ be the object of all H-partitions of length k (so $W(k)\subseteq B^k$), and let W be the object of all H-partitions, so $W=\bigcup W(k)$.

Each H-partition $\bar{b}$ can be regarded as a groupoid with $\mathrm{Hom}(b_i,b_j)$ being the set of all (h,b_i,b_j) with $h\in H$ and $h(b_i)=b_j$. Therefore G, the generic H-partition, is also a groupoid (in $\mathscr{E}_W$, which denotes the topos of objects in $\mathscr{E}$ over W, since G is an H-partition there). By using Lemma 1.2, we can identify G with its profinite completion $\hat{G}$ [since $\mathscr{E}_W$ is the disjoint union of $\mathscr{E}_{W(k)}$].

Therefore, for each finite category C, we have that $\hat{G}(C)$ is the object of functors from G to C hence the object in $\mathscr{E}_W$ with $\mathrm{Fun}(\bar{b},C)$ 'lying over $\bar{b}\in W$'. Further note that W is filtered by refinement and if $\bar{b}$ if refined by $\bar{b}'$ there is an associated functor $\bar{b}'\to\bar{b}$ and hence a map $\mathrm{Fun}(\bar{b},C)$ to $\mathrm{Fun}(\bar{b}',C)$. In this way $\hat{G}(C)$ becomes a diagram over the filtered W. We define $\Gamma(C)$ as the filtered colimit of $\hat{G}(C)$. Then Γ is left exact [p. 70 of [4], or Lemma 0.1(ii)].

Let X be a profinite action over Γ. Then for every finite action (A,C) there is a projection $\pi: X(A,C)\to\Gamma(C)$. By Lemma 0.1 (iii), $X(A,C)$ can be regarded as a diagram over W. That is $\Gamma(C)=\bigcup \mathrm{Hom}(\bar{b},C)$ and we can write $X(A,C)=\bigcup X(\bar{b},A,C)$ where $X(\bar{b},A,C)$ is $\pi^{-1}(\mathrm{Hom}(\bar{b},C))$. So $X(A,C)$ is an object over W with 'fiber' $X(\bar{b},A,C)$ over $\bar{b}\in W$ and X becomes a profinite action in E_W over $\hat{G}$. By applying Lemma 1.2, to each $W(k)$, we can regard X as a diagram over the category G, with $X(\bar{b})$ a diagram over $\bar{b}$ for each $\bar{b} \in W$. (We let $X(b_i)$ be the fiber over $b_i \in \bar{b}$.) Finally, note that the family of diagrams, $X(\bar{b})$ over $\bar{b}$ is filtered and has the following special property: Suppose that $\bar{b}'$ refines $\bar{b}$ with $i : \bar{b}' \to \bar{b}$ the refinement functor. Then any action map (m, f) from (A, C) to $(X(\bar{b}'), \bar{b}')$ must factor through $(X(\bar{b}), \bar{b})$ iff $f : \bar{b}' \to C$ factors through i. It is readily shown that filtered families of diagrams with this special property correspond to profinite actions over Γ. On the other hand let $P(\bar{b}, X) = \prod \{X(b_i) \mid b_i \in \bar{b}\}$. That the canonical map $P(\bar{b}, X)$ to $P(\bar{b}', X)$ be an isomorphism is readily seen to be equivalent to the special property. [Note that $P(\bar{b}, x)$ is the set of action maps from $(H \times \bar{b}, \bar{b})$ to $(X(\bar{b}), \bar{b})$ where group multiplication makes $H \times \bar{b}$ into an action over $\bar{b}$.] Let $P(X)$ be the colimit of $P(\bar{b}, X)$ for $\bar{b} \in W$. Then $P(X)$ is easily seen to be the global sections of a sheaf over B. For if $d \in B$ let W_d be the object of H-partitions which refine $\{d, -d\}$ and let:

$$P(\bar{b}, X) \mid d = \prod \{X(b_i) \mid b_i \in \bar{b}, b_i \leq d \text{ for } \bar{b} \in W_d\}$$

and $P(X) \mid d = \operatorname{Colim} P(\bar{b}, X) \mid d$. This construction of $P(X) \mid d$ (with 'd' as index) takes place over B and produces the required sheaf. Conversely, given the sheaf, the steps can be reversed to produce the action.

Corollary. *If P is a separable polynomial over K in a topos $\mathscr{E}$ then there is a profinite groupoid Γ_P, constructed as above, such that the generic splitting field of P lives in $\mathscr{E}^{\Gamma_P}$.*

Proof. It suffices to show that B is decidable where B is the Boolean Algebra of idempotents of $K\langle r \rangle$. This follows since $b \in B$ is nonzero iff.

$$\bigvee \{\sigma(b) \mid \sigma \in S_n\} = 1.$$

(Here the sup is taken in the Boolean algebra B).

Construction of the profinite Galois groupoid – The generic separable polynomial

Now that we have discussed how to split a single polynomial we can determine how to split all separable polynomials simultaneously. Recall that K is a field in a topos $\mathscr{E}$. We let *n-POLY* be the object of *separable, monic* polynomials of degree n in one indeterminate ξ, with coefficients in K. Note that n-POLY can be viewed as a subobject of K^n with $(a_1, \ldots a_n)$ corresponding to $\xi^n + a_1\xi^{n-1} + \cdots + a_n$. [The condition that $(a_1, \ldots, a_n)$ corresponds to a separable polynomial, P, is geometric, since the Euclidean algorithm applied to P and P' must produce a constant g.c.d. (greatest common denominator) in n steps.] We define *POLY* as $\bigcup n$-POLY (for positive integers n). Then POLY is a directed, partially ordered set with $P \leq Q$ iff P divides Q. We can construct the sup, $P \vee Q$ (or least common multiple of P and Q) since $P \vee Q$ is PQ/D where D is the g.c.d. constructible by the Euclidean algorithm. Note that $P \vee Q$ is separable if P and Q are separable.

The generic separable polynomial lives in $\mathscr{E}_{\text{POLY}}$. [We identify K with the 'constant field' $K \times \mathrm{POLY}$ in $\mathscr{E}_{\mathrm{POLY}}$. The ring of polynomials with coefficients in K is then $K[\xi] \times \mathrm{POLY}$. A typical element of $\mathscr{E}_{\mathrm{POLY}}$ can be envisioned as an indexed collection $\{X_p\}$ with P varying in POLY. The distinguished polynomial is the element of $K[\xi] \times \mathrm{POLY}$ which 'looks like P' over P. To be precise it is the diagonal map $\mathrm{POLY} \to K[\xi] \times \mathrm{POLY}$ which is an element (as POLY is the terminal object of $\mathscr{E}_{\mathrm{POLY}}$). This polynomial can be shown to be separable, since the Euclidean Algorithm can be described geometrically.]

Splitting the generic polynomial. The splitting field construction discussed at the beginning of this section can now be applied in the topos $\mathscr{E}_{\mathrm{POLY}}$ to the generic separable polynomial. By the corollary to Lemma 2.1, there is a profinite group Γ and a splitting field $K^{\#}$ in $\mathscr{E}_{\mathrm{POLY}})^{\Gamma}$. (To meet the hypotheses, the corollary should be applied independently to each component, $\mathscr{E}_{n\text{-POLY}}$.)

To get the separable closure of K and the complete profinite Galois action we want to 'glue together' all the splitting fields of each separable polynomial and 'glue together' all of the actions of Γ. But POLY is a directed set and once we show that Γ and $K^{\#}$ are diagrams over POLY we can form colimits. To do this, regard Γ as an 'indexed collection' $\{\Gamma_P\}$ of profinite groupoids for $P \in \mathrm{POLY}$. We must find a canonical natural transformation $\Gamma_R \to \Gamma_Q$ whenever R divides Q. Let R be a polynomial of degree n with root set $\bar{r} = \{r_1, \ldots, r_n\}$ and Q of degree m with root set $\bar{q} = \{q_1, \ldots, q_m\}$. Let B_R (resp. B_Q) be the Boolean algebra of idempotents of the splitting ring $K\langle \bar{r} \rangle$ (resp $K\langle \bar{q} \rangle$). Then S_n acts on $\bar{r}$ and B_R while S_m acts on $\bar{q}$ and B_Q. Let t range over the set of order-preserving one-to-one maps (in Sets) of $\{1, \ldots, n\}$ into $\{1, \ldots, m\}$. Since $K\langle \bar{q} \rangle$ becomes a field over B_Q and since R divides Q and R and Q are both separable, the roots of R may be found among the roots of Q. For each map t, let $b(t)$ be the idempotent in B_Q which is the truth value of "$q_{t(1)}, \ldots, q_{t(n)}$ are the roots of R". Then, clearly, $\{b(t)\}$ is an S_m-partition of B_Q. This enables us to define the desired map $\Gamma_R \to \Gamma_Q$. Let C be a finite category and let $\zeta \in \Gamma_R(C)$ be represented by a functor $\bar{b} \to C$ where $\bar{b}$ is an S_n-partition of B_R. Each t induces a natural isomorphism from $K\langle \bar{r} \rangle$ to $K\langle \bar{q} \rangle b_t$ (sending r_i to $q_{t(i)}$) and so $\bar{b}$ gives rise to a partition of each b_t, hence, letting t vary, to an S_m-partition $\bar{\bar{b}}$ of B_Q. The functor $\bar{b} \to C$ can readily be lifted to a functor $\bar{\bar{b}} \to C$. This gives us the map from $\Gamma_R(C)$ to $\Gamma_Q(C)$. These maps compose compatibly because we always choose order-preserving injections of the roots of R to those of Q. We define $\bar{\Gamma}$ to be the internal colimit of this filtered diagram. $\bar{\Gamma}$ is then a profinite groupoid.

The separable closure. We must locate the field $\bar{K}$ in $E^{\bar{\Gamma}}$ which is to be the separable closure of K. For each separable polynomial P, let K_P be the field in $\mathrm{Shv}(B_P, \mathscr{E}^{S_n}) = \mathscr{E}^{\Gamma_P}$ which splits P (so $\{K_P\}$ lives in $\mathscr{E}_{\mathrm{POLY}}$). Suppose that R divides Q and that (A, C) is a finite action. There is a natural map $K_R(A, C) \to K_Q(A, C)$. But $\zeta \in K_R(A, C)$ consists of an S_n-partition $\bar{b}$, a functor $f : \bar{b} \to C$ and, for each b_i, a map $A(c_i) \to K\langle \bar{r} \rangle b_i$ (where $f(b_i) = c_i$). Using the above partition $\{b(t)\}$ of B_Q and isomorphisms $K\langle \bar{r} \rangle \to K\langle \bar{q} \rangle b(t)$ there is an obvious lift of ζ from $K_R(A, C)$ to $K_Q(A, C)$. As before, this defines a diagram over POLY and we let $\bar{K}$ be its filtered colimit. Then:

Theorem 2.2. *$(\bar{K}, \mathscr{E}^{\Gamma})$ is the separable closure of K where $\bar{K}, \Gamma$ are as constructed above. This is an example of a spectrum in the sense of Cole (see* [4, Theorem 6.58]).

Proof. $\bar{K}$ is readily shown to be a separably closed, separably algebraic extension of $i^*(K)$ where $i^*: \mathscr{E} \to \mathscr{E}^{\Gamma}$ is the obvious functor. For example to show that $\bar{K}$ is separably closed (given that it is separably algebraic over K) it suffices to show that every separable polynomial P in K splits in $\bar{K}$. But if U is the subobject of 1 where P splits, then interpreting U as a subobject of Γ in $\mathscr{E}_{\text{POLY}}$ we see that $U = \Gamma$ for all $R \in \text{POLY}$ for which P divides R. In the colimit, $U = 1$.

We must obtain the following adjointness property: Let $p^*: \mathscr{E} \to \mathscr{F}$ be an inverse image functor and let $p^*(K) \subseteq L$ be a separably closed extension of K. We need to find a lift $\bar{p}^*: \mathscr{E}^{\Gamma} \to \mathscr{F}$ with the right properties, and show that it is unique to within natural equivalence. The idea is to use the fact that each K_R in $\mathscr{E}^{\Gamma_R}$ is a spectrum and to glue together all of the resulting lifts. This immediately gives us uniqueness since the lifts must be unique over n-POLY (by the splitting of the generic polynomial of degree n) and inverse image functors preserve internal colimits [Lemma 0.1(i)]. For the existence we must take some care to show that the lifts can be compatibly glued together. For convenience we first consider the case where there exists a separable closure L of K in $\mathscr{E}$. We must find $g^*: \mathscr{E}^{\Gamma} \to \mathscr{E}$ and $g_*: \mathscr{E} \to \mathscr{E}^{\Gamma}$ with the right properties. It suffices to give a specific geometric description of $g_*(\mathscr{E})$ for each $R \in \text{POLY}$. For example:

$$(g_*)(E)_R = \{(h, b) \mid h: M_b \to E, b \in B_R\}$$

where M_b is the object of homomorphisms m from $K\langle r \rangle$ to L for which $m(b) = 1$.

For the general case, given $p^*: \mathscr{E} \to \mathscr{F}$ we can lift p^* to an inverse image functor from $\mathscr{E}^{\Gamma}$ to $\mathscr{F}^{\Gamma}$ and apply the above method.

3. The case when a separable closure exists in $\mathscr{E}$

Let K be a field in a topos $\mathscr{E}$. The profinite groupoid, Γ, of the previous section is then called the profinite Galois groupoid of the separable closure of K. In Sets, we expect to get a profinite Galois *group* and the question arises, "In what sense can Γ be regarded as a group?" First of all, Γ is clearly a connected groupoid (since S_n acts transitively on each S_n-partition $\bar{b}$ of B). In Sets, every profinite connected groupoid is Morita equivalent to a profinite group.

From another point of view, suppose that there were a separable closure L of K already existing in $\mathscr{E}$. Then as shown in [6], there is a 'Galois theater of action' associated to L and a profinite group Γ associated to this theater. The relation between Γ and Γ is that they are Morita equivalent. In this section we prove that Γ is Morita equivalent to a profinite group iff such a separable closure L of K does exist in $\mathscr{E}$.

Proposition 3.1. *Let P be a separable polynomial over K and let $\Gamma(=\Gamma_P)$ be the profinite groupoid of the splitting field of P so that* $\mathrm{Shv}(B, \mathscr{E}^{S_n})$ *is isomorphic to $\mathscr{E}^{\Gamma}$ as in Lemma* 2.1.

Suppose that there exists in $\mathscr{E}$ a field extension L of K in which P splits. Then there is a profinite group Γ_0 which is Morita equivalent to Γ. Conversely L exists if Γ_0 exists.

Proof. Let $K\langle \bar{r}\rangle$ be the splitting *ring* of P, as before, so that S_n acts on the left on $K\langle \bar{r}\rangle$ where n is the degree of P. We may as well assume that L is generated as a field extension of K by the roots of P. Let M be the object in $\mathscr{E}$ of K-ring homomorphisms from $K\langle \bar{r}\rangle$ to L. Then S_n acts on M on the *right* by composition. It is easy to verify (as it can be done in Sets) that for every m_1 and m_2 in M there is a unique $\sigma \in S_n$ with $m_1\sigma = m_2$. Also note that $\exists\, m \in M$ (or that M has 'global support' or that $M \to 1$ is epi).

Recall that B is the Boolean algebra of idempotents of $K\langle \bar{r}\rangle$ and that S_n acts on B on the left. Let $\bar{b}$ be an S_n-partition of B. Then, for each $m \in M$ there is a unique $b_i \in \bar{b}$ for which $m(b_i) = 1$. We write $b_i = \mathrm{Supp}(m)$ and so, for each $\bar{b}$ we have defined:

$$\mathrm{Supp}: M \to \bar{b}.$$

Note that $\mathrm{Supp}(m\lambda^{-1}) = \lambda\, \mathrm{Supp}(m)$ for $\lambda \in S_n$.

We shall regard S_n as a constant object in E and let S_n act on itself by conjugation [so $(\lambda, \sigma) \to \lambda\sigma\lambda^{-1}$; the notation $\lambda\sigma$ shall always denote composition, so this conjugation action will be denoted by $\lambda\sigma\lambda^{-1}$]. This produces an object $\bar{S}_n$ in E^{S_n}. Observe that $\bar{S}_n$ is a group in E^{S_n}. Then:

$$M \otimes \bar{S}_n = \{m \otimes \sigma \mid m \in M, \sigma \in S_n \text{ with } m\lambda \otimes \sigma = m \otimes \lambda\sigma\lambda^{-1}\}.$$

For each S_n-partition $\bar{b}$ we define

$$H(\bar{b}) = \{m \otimes \sigma \in M \otimes \bar{S}_n \mid \sigma \text{ fixes } \mathrm{Supp}\, m\}.$$

An easy check shows that $M \otimes \bar{S}_n$ is a group under $(m \otimes \sigma)(m \otimes \tau) = m \otimes (\sigma\tau)$ and that $H(\bar{b})$ is a subgroup (the verification uses the fact that S_n acts in a simple transitive manner on M). We shall continue this proof after proving the following lemma:

Lemma 3.2. *Let $\bar{b}$ and $H(\bar{b})$ be as above. Regard $\bar{b}$ as a groupoid (as in the proof of Lemma* 2.1). *Then the groupoid $\bar{b}$ is Morita equivalent to the group $H(\bar{b})$.*

Proof. [In this proof we shall treat $\bar{b}$ as a *constant* S_n-partition, meaning that $\bar{b} = \{b_1, \ldots, b_k\}$ where each b_i is an actual element or map $b_i: 1 \to B$. As usual, this does not affect any geometric consequences since the generic S_n-partition of length k, which lives in $\mathscr{E}_{W(k)}$, is a constant S_n-partition. Furthermore, we shall apply the lemma directly to $\mathscr{E}_{W(k)}$.]

Let X be a diagram over $\mathfrak{b}$ and let $X(b_i)$ be the part of X lying over b_i. So $X = \bigcup X(b_i)$, and, since X is fibered over $\mathfrak{b}$, it follows that S_n acts on X on the left. [If $\sigma(b_i) = b_j$ then σ maps $X(b_i)$ to $X(b_j)$.] Let

$$M \otimes X = \{m \otimes x \mid m \in M, x \in X \text{ with } m\lambda \otimes x = m \otimes \lambda x\}$$

and define

$$X^* = \{m \otimes m \mid x \in X(b_i) \text{ for } b_i = \mathrm{Supp}(m)\}.$$

Then $H(\mathfrak{b})$ acts on X^* by $(m \otimes \sigma)(m \otimes x) = m \otimes \sigma x$.

Conversely, suppose that $H(\mathfrak{b})$ acts on Y in $\mathscr{E}$. Then $H(\mathfrak{b})$ acts on $Y \times M$ so that $(m \otimes \sigma)(y, m)$ is $((m \otimes \sigma)y, m\sigma)$. [It is an exercise manipulating conjugations to show that all of the above operations really are well defined.] Let $Y^\#$ be the object of $H(\mathfrak{b})$-orbits of $Y \times M$. [That is $Y \times M$ modulo the equivalence relation induced by $H(\mathfrak{b})$.] So $Y^\# = \{[y, m] \mid y \in Y, m \in M \text{ and } [y, m] = [(m \otimes \sigma)y, m\sigma]\}$. Define $p: Y^\# \to \mathfrak{b}$ by $p([y, m]) = \mathrm{Supp}\, m$. If $\lambda \in S_n$ then $\lambda[y, m] = [y, m\lambda^{-1}]$. A straightforward verification shows that $Y^\#$ is a well-defined diagram over $\mathfrak{b}$.

To prove that $X = (X^*)^\#$ let $x \in X(b_i)$ be given and choose $m \in M$ with $\mathrm{Supp}(m) = b_i$ and define $\phi(x) = [m \otimes x, m]$. This is well defined since such m exist and if m_1, m_2 in M have $\mathrm{Supp}(m_1) = \mathrm{Supp}(m_2) = b_i$ then $[m_1 \otimes x, m_1] = [m_2 \otimes x, m_2]$. (We can find λ with $m_1 = m_2\lambda$, then $m_1 \otimes \lambda \in H(\mathfrak{b})$ etc.) So ϕ is uniquely determined and can be readily shown to be an isomorphism from X to $(X^*)^\#$.

Similarly $\psi: Y \to (Y^\#)^*$ is an isomorphism where $\psi(y) = m \otimes [y, m]$.

Proof of Proposition 3.1 (continued). Let W be the object of all S_n-partitions of B with $\mathfrak{b} \leq \mathfrak{b}'$ iff $\mathfrak{b}$ is refined by $\mathfrak{b}'$. For each finite group G let $\Gamma(G)$ be the filtered colimit of $\mathrm{Hom}(H(\mathfrak{b}), G)$. If X is in $\mathscr{E}^\Gamma$ then X is a filtered colimit of objects of $\mathscr{E}^{H(\mathfrak{b})}$ (by applying (iii) of Lemma 0.1) which is equivalent to $\mathscr{E}^{\mathfrak{b}}$. But a filtered colimit of objects of $\mathscr{E}^{\mathfrak{b}}$ is an S_n-sheaf over B (as in the proof of Lemma 2.1). The steps can be reversed to set up the required equivalence between $\mathscr{E}^\Gamma$ and $\mathrm{Shv}(B, \mathscr{E}^{S_n})$. The converse, that if $\mathrm{Shv}(B, \mathscr{E}^{S_n})$ is equivalent to a category of profinite actions over a profinite group is not needed now and will follow from Lemma 3.4.

Theorem 3.3. *Let K be a field in a topos $\mathscr{E}$. Let $\bar{K}$ in $\mathscr{E}^\Gamma$ be the separable closure K as in Theorem 2.2. Then the profinite groupoid $\bar{\Gamma}$ is Morita equivalent to a profinite group iff there exists in $\mathscr{E}$ a separable closure L of K. (In this case Γ is Morita equivalent to H, the profinite group associated with Galois theater of L/K, as in* [6], *and L lives in $\mathscr{E}^H$ and can be regarded as the generic separable closure of K.)*

Proof. Suppose that such an L exists. For each separable polynomial P let $L_P \subsetneq L$ be the K-subfield generated by the roots of P. Let Γ_P be the profinite groupoid for which $\mathscr{E}^{\Gamma_P}$ is equivalent to $\mathrm{Shv}(B_P, \mathscr{E}^{S_n})$. Then each Γ_P is Morita equivalent to a group by Lemma 3.2 and the colimit over POLY of $\{\Gamma_P\}$ is still equivalent to the colimit of the profinite groups as constructed in Lemma 3.2.

Conversely, suppose that Γ is Morita equivalent to a profinite group H. Then there is a separable closure $\bar{L}$ of K in $\mathscr{E}^H$, and, by Lemma 3.4 below, an inverse image functor U from $\mathscr{E}^H$ to $\mathscr{E}$ for which $L=U(\bar{L})$ has the required properties. Finally the Galois theater of L/K is represented by a left exact functor L': Fun(T, Sets) to $\mathscr{E}$ (as in [6]). This produces a profinite action by the restriction functor r from Fun[Fin Act, Sets] to Fun[T, Sets]. The composition is equivalent to the functor from Fun[Fin Act, Sets] induced by the profinite action $\bar{L}$ in $\mathscr{E}^H$. (Given a theater map from (A,G) to L there is a partition fine enough to preserve all distinctions which are polynomial determined as shown in [6].)

Lemma 3.4. *Let Γ be a regular profinite group on $\mathscr{E}$. Then there is an 'underlying object functor' $U: \mathscr{E}^\Gamma \to \mathscr{E}$ which is an inverse image functor. Moreover, if $i^*: \mathscr{E} \to \mathscr{E}^\Gamma$ is the 'constant' functor then Ui^* is equivalent to the identity on $\mathscr{E}$. [If Γ is an ordinary finite group then an object of $\mathscr{E}^\Gamma$ can be regarded as a pair (X,α) with $X\in\mathscr{E}$ and $\alpha: \Gamma\times X$ an action. In this case, $U(X,\alpha)=X$.]*

Proof. Let $X\in\mathscr{E}^\Gamma$ be given. Let $U(X)$ be the canonical colimit of all $X(A,G)_a$ where: $a\in A$, $(A,G)\in$ Fin Act/Grp and $X(A,G)_a=X(A,G)$. We can regard $U(X)$ as the object of all $[a,x]$ where $a\in A$, $x\in X(A,G)$ and with $[a,x]=[b,y]$ if there is an action map (m,f) from (A,G) to (B,H) such that $m(a)=b$ and $X(m,f)(y)=x$. So $U(X)$ is a quotient of $\bigcup(A\times X(A,G))$.

To define a right adjoint V for U, let $E\in\mathscr{E}$ and $(A,G)\in$ Fin Act/Grp be given. Define

$$\hat{E}(A,G)=\{(\zeta,f)\mid \zeta\in\Gamma(G), f:A\to Y\}.$$

If $(m,d):(B,H)\to(A,G)$ is an action map then define $\hat{E}(m,d)(\zeta,f)=(\Gamma/d)(\zeta,fm)$. To show that $\hat{E}$ is in $\mathscr{E}^\Gamma$ it suffices to show that $\hat{E}$ is left exact and that $\hat{E}(\phi,G)$ is $\Gamma(G)$. Both of these verifications are straightforward. We define $V(E)=\hat{E}$. Clearly $V:\mathscr{E}\to\mathscr{E}^\Gamma$. [If $j:E\to F$ is a map in $\mathscr{E}$ then $V(j)(\zeta,f)=(z,jf)$]. A 'member' of $UV(E)$ is an equivalence class $[a,\zeta,f]$ with $a\in A$, $\zeta\in\Gamma(G)$ and $f:A\to E$. Define $\varepsilon: UV(E)\to E$ so that $\varepsilon[a,\zeta,f]=f(a)$. Each map $t:U(X)\to E$ lifts to a map $\bar{t}:X\to V(E)$. We must define $\bar{t}(A,G):X(A,G)\to\hat{E}(A,G)$. If $x\in X(A,G)$ then $\pi(x)\in\Gamma(G)$ and x determines a map $f:A\to E$ by $f(a)=t[a,x]$. Let $\bar{t}(x)=(\pi(x),f)$. A direct check now shows that ε is a back adjunction. Since Ui^* is easily shown to be equivalent to the identity functor, the lemma will be proven when we show that U is left exact. To do this we shall represent U as a colimit over an appropriate filtered diagram. Recall that limits in $\mathscr{E}^\Gamma$ are taken 'over Γ' – they do *not* coincide with limits in the category of all functors from (Fin Act/Grp)$^{\mathrm{op}}$ to $\mathscr{E}$. This gives us a clue to finding the right filtered diagram, Δ. We let the objects of Δ be the disjoint union of all $\Gamma(G)$ for (a representative set of) all finite groups G. A map from $\zeta\in\Gamma(G)$ to $\zeta'\in\Gamma(H)$ is a 3-triple (f,ζ,ζ') where $f:H\to G$ and $\Gamma(f)(\zeta')=\zeta$. (So Δ is a comma category).

For each finite group G let $(\tilde{G},G)$ be the action where $\tilde{G}=G$ and where G acts on $\tilde{G}$ by multiplication.

Let $X \in \mathscr{E}^{\Gamma}$ be given. Regard X as a diagram over Δ with $X(\tilde{G}, G)$ lying over $\Gamma(G)$. If $x \in X(\tilde{G}, G)$ with $\pi(x) = \zeta \in \Gamma(G)$ and if $f: H \to G$ and $\zeta' \in \Gamma(H)$ are such that $(f, \zeta, \zeta'): \zeta \to \zeta'$. We must construct $(f, \zeta, \zeta')(x) \in X(\tilde{H}, H)$. But $(\tilde{G}, H)$ is an action, via $f: H \to G$, and is the pullback (in Fin Act^{op}) of $(\tilde{G}, G)$, $(\emptyset, H)$ and $(\emptyset, G)$. Since X preserves this pullback there exists $x' \in X(\tilde{G}, H)$ lying over ζ' and x. But there is also an action map $\tilde{f}$ from $(\tilde{H}, H)$ to $(\tilde{G}, H)$ and we define $(f, \zeta, \zeta')(x)$ as $X(\tilde{f})(x')$.

We claim that $U(X)$ is the colimit of this diagram over U. The main step is that if $[a, x] \in U(X)$ with $a \in A$ and $x \in X(A, G)$ then there exists a unique G-equivariant map t from $(\tilde{G}, G)$ to (A, G) which sends the identity, e (in $\tilde{G}$) to $a \in A$. Then $[a, x]$ is equivalent to $[e, y]$ where $y = X(t)(x)$ and $[a, x]$ can be represented by $y \in X(\tilde{G}, G)$. Finally, limits in $\mathscr{E}^{\Gamma}$ correspond to limits over Δ so U is left exact and the lemma is true.

4. Proof of Theorem 1.1

Recall that Γ is a left exact functor from Fin Cat to $\mathscr{E}$. To show that $\mathscr{E}^{\Gamma}$ is a topos we shall 'glue together' all the finite actions that are involved. We need the following theorem of Wraith's from [10].

Wraith glueing construction. Recall that $\mathscr{E}$ has countable limits. Let Δ be a countable category (i.e. Δ has a countable number of morphisms.) For each object $d \in \Delta$ let $\mathscr{E}(d)$ be a topos over $\mathscr{E}$. For each map $h: c \to d$ in Δ let $\mathscr{E}(h): \mathscr{E}(c) \to \mathscr{E}(d)$ be a functor. Assume that each $\mathscr{E}(d)$ has, and each $\mathscr{E}(h)$ preserves, countable limits. Then the topoi $\{\mathscr{E}(d)\}$ can be glued together to form a topos Gl(Δ) where a glued object X (in Gl(Δ)) consists of an object $X(d)$ in $\mathscr{E}(d)$ for all d and for each $h: c \to d$ a map ζ_h from $X(d)$ to $\mathscr{E}(h)[X(c)]$ such that if $k = jh$ then $\xi_k = \mathscr{E}(j)(\xi_h)\xi_j$. If each functor $\mathscr{E}(h)$ has a left adjoint, h^*, then by adjointness ξ_h corresponds to ξ_h from $h^*(X(d))$ to $X(c)$. The conditions on the ξ's corresponds to $\zeta_k = \zeta_h h^*(\zeta_j)$ whenever $k = jh$. If the $\mathscr{E}(h)$ functors are geometric, then Gl(Δ) is a topos over $\mathscr{E}$. [The proof of this is given in [10], particularly the last paragraph. The essential steps are also given in [4, p. 109–110]. As noted in [10], the result extends to uncountable cardinals.]

Proof of Theorem 1.1. Given Γ: Fin Cat $\to \mathscr{E}$ let Δ = Fin Cat. For each $C \in \Delta$ let $\mathscr{E}(C)$ be $(\mathscr{E}_{\Gamma(C)})^C$, that is the topos of all functors from C to $\mathscr{E}_{\Gamma(C)}$, the category of objects over $\Gamma(C)$. For each $h: C \to D$ in Δ let $h_*: \mathscr{E}(C) \to \mathscr{E}(D)$ be the corresponding geometric functor. It is easier to work with the inverse image part, h^*, which is essentially pulling back along $\Gamma(h)$. So we can apply the Wraith glueing construction to get a topos Gl(Δ). So $X \in$ Gl(Δ) means $X(C) \in \mathscr{E}(C)$ for all C and there are maps ζ_h from $h^*(X(D))$ to $X(C)$ for each $h: C \to D$ etc. We need to construct a topology on Gl(Δ) for which we need:

Definition. Let $h: C \to D$ be a functor between finite categories. Let A be a C-action (i.e. diagram over C) and B a D-action in a topos $\mathscr{F}$. Let $h^0(B)$ be the induced C-action. Then a C-equivariant map $\zeta: h^0(B) \to A$ is h-universal if for every D-action E in $\mathscr{F}$ and every C-equivariant map $f: h^0(E) \to A$ there is a unique D-equivariant map $\bar{f}: E \to B$ for which $\zeta h^0(\bar{f}) = f$.

The glued object X has the *action lifting property* (or *ALP*) iff ζ_h is h-universal for every $h: C \to D$. [Note that $\zeta_h: h^*(X(D)) \to X(C)$ and h^* can be regarded as $h^0(h')$ where h' is pulling back along $\Gamma(h)$.] To complete the proof of Theorem 1.1 we shall find a topology on $\mathrm{Gl}(\Delta)$ whose sheaves are precisely the glued object with ALP and show that these in turn are equivalent to profinite actions over Γ.

Lemma 4.1. *Let C and D be finite categories and let $H: C \to D$ be a functor. Let A be a C-action and B a D-action in a topos $\mathscr{F}$. Let $\zeta: h^0(B) \to A$ be C-equivariant. Let $U \subseteq A$ be a C-subaction. Then there exists a largest D-subaction V of B for which $\zeta(h^0(V)) \subseteq U$.*

Proof. $y \in V(d)$ iff for all $\lambda: d \to h(c)$ in D we have $\zeta_c(\lambda(y)) \in U(c)$. Since D is finite, this is easily constructed.

The topology J. Let X be a glued object and let $U \subseteq X$ be a subobject. For every $h: C \to D$ let V_h be the image in $X(D)$ of the largest subobject of $h'(X(D))$ mapping onto $U(C)$. Define $\tilde{U}$ so that $\tilde{U}(D) = \bigcup V_h$ where h ranges over all functors into D. Then $\tilde{U}$ is a subobject of X and $U \subseteq \tilde{U}$ and the passage from U to $\tilde{U}$ is preserved by pullbacks. Let $U^{(n)}$ be defined by iterating this process n times and let $\bar{U}$ be the $\bigcup U^{(n)}$. Because the categories C and D are finite, it follows that $\tilde{\bar{U}} = \bar{U}$ and $U \to \bar{U}$ is a closure operator corresponding to a topology J on $\mathrm{Gl}(\Delta)$ (see [4, p. 77]). Moreover U is J-closed iff $U = \bar{U}$.

Lemma 4.2. *Let A be an object in $\mathscr{E}(C)$. [So A is a C-action over $\Gamma(C)$.] Then A freely generates a glued object U together with a C-equivariant map $A \to U(C)$ such that C-equivariant maps from A to $X(C)$ correspond exactly to maps in* $\mathrm{Gl}(\Delta)$ *from U to X.*

Proof. Let $U(K)$ be the disjoint union of $f^*(A)$ where f ranges over the functors from K to C and f^* is essentially pulling back along $\Gamma(f)$.

Lemma 4.3. *Let X be in* $\mathrm{Gl}(\Delta)$. *Then X has ALP iff X is a J-sheaf.*

Proof. If X has ALP then a straightforward argument shows that (for $U \subseteq Y$) maps $U \to X$ have unique lifts to maps $\bar{U} \to X$.

Conversely, let X be a J-sheaf. Let $h: C \to D$ be a functor. Let W be a D-action in $\mathscr{E}_{\Gamma(C)}$ with projection $\pi: W \to \Gamma(C)$. Let $h^0(W)$ be the induced C-action. Let $t: h^0(W) \to X(C)$ be C-equivariant. Let U be the glued object freely generated by $h^0(W)$ as a C-action in $\mathscr{E}_{\Gamma(C)}$. Let V be the glued object freely generated by W, the D-

action in $\mathscr{E}_{\Gamma(D)}$ with projection $\Gamma(h)\pi$ from W to $\Gamma(D)$. Then t corresponds a map $U \to X$. Also there is a canonical map $U \to V$. It is readily shown that $U \subsetneq V$ and $V = \tilde{U}$. So there is a unique map $V \to X$ extending the map corresponding to T. This gives rise to the required map from W to $X(D)$.

Canonical limit construction. Let C be a finite category. For each object c in C let $C(c)$ be the action over C for which $C(c)(d)$ is $\mathrm{Hom}(c,d)$. If $k: c_1 \to c_2$ is a morphism then $k^*: C(c_2) \to (c_1)$ given by composing with k on the right is C-equivariant.

Let A be any finite action over C. We define a *canonical diagram* in Fin Act with *vertices* $C(c)_a = (C(c), C)$ for each $a \in A(c)$ and arrows $k^*: C(c_2)_b \to C(c_1)_a$ whenever $k: c_1 \to c_2$ with $k(a) = b$. We add a *base vertex* $(\emptyset, C)$ together with the unique C-equivariant arrrow. $(\emptyset, C) \to C(c)_a$ for each $a \in A(c)$. Then (A, C) is the *colimit* in Fin Act of the canonical diagram. So if X a profinite action then $X(A, C)$ is the *limit* of X of the canonical diagram. It follows that X is determined by the values of $X(C(c), C)$ and $X(\emptyset, C)$.

Proof of Theorem 1.1 (continued). In view of Lemma 4.3 it suffices to equate profinite actions over Γ with glued objects having ALP. Let X be a profinite action over Γ. We construct a diagram $X(C)$ over C in $\mathscr{E}_{\Gamma(c)}$ so that for each $c \in C$ the fiber $X(C)(c)$ of this diagram is $X(C(c), C)$. The projection onto $\Gamma(C)$ is obtained by applying X to the unique C-equivariant map from $(\emptyset, C)$ to $(C(c), C)$. If $\lambda: c_1 \to c_2$ in C then $X(\lambda^*)$ goes from $X(C)(c_1)$ to $X(C)(c_2)$. Clearly, $X(C) \in \mathscr{E}(C)$ [as used in the definition of $\mathrm{Gl}(\Delta)$].

Now let $h: C \to D$ be a functor. For each $d \in D$ we have a D-action $D(d)$ which gives rise to the C-action $h^0(D(d))$. $h^*(X(D))$ is the C-action in $\mathscr{E}_{\Gamma(c)}$ which has fiber $X[h^0(D(d))]$ over c (where $d = h(c)$) since $h^0(D(d))$ together with $(\emptyset, C)$, $(\emptyset, D)$ form a pullback in Fin $\mathrm{Act}^{\mathrm{op}}$ and preserves pullbacks. There is a map from $C(c)$ to $h^0(D(d))$, given by h, and applying X to this action map for each $c \in C$ we obtain a map we shall call:

$$\zeta_h: H^*(X(D)) \to X(C).$$

This defines a glued object (the details are straightforward.) Before finishing the proof, we note a critical property of the diagram $X(C)$:

Lemma 4.4. *Let X be a profinite action over Γ and let $X(C)$ be the C-action in $\mathscr{E}_{\Gamma(c)}$ defined above. Let (A, C) be any finite action over C. Then $X(A, C)$ is equivalent to the object of C-equivariant maps in $\mathscr{E}_{\Gamma(c)}$ from (A, C) to $X(C)$.*

Proof. $X(A, C)$ is the limit of the canonical diagram and so is equivalent to a subobject of the product $\prod \{X(C(c)) \mid a \in A(c)\}$. The subobject corresponds precisely to the subobject determined by the C-equivariant maps from (A, C) to $X(C)$.

Proof of Theorem 1.1 (concluded). The glued object $\{X(C)\}$ associated to each profinite action X, over Γ, is now shown to have ALP. Let (B,D) be a finite action over D and let t from $h^0(B,D)$ to $X(C)$ be C-equivariant. Then t corresponds to an 'element' of $X(h^0(B,D))$ in the sense of Lemma 4.4, with $(A,C)=h^0(B,D)$. But $X(h^0(B,D))=h^0(X(B,D))$, as X preserves pullbacks, and this is h^0 of the object of D-equivariant maps from (B,D) to $X(D)$ (by Lemma 4.4 again). This sets up the required equivalence, showing that $\{X(C)\}$ has ALP.

Conversely, let Y be a J-sheaf, that is, a glued object with ALP. We regard Y as a profinite action by defining $Y(A,C)$ as the object of C-equivariant maps from (A,C) to $Y(C)$ as suggested by Lemma 4.4. [Note that $Y(A,C)$ is an object of E with a projection to $\Gamma(C)$.] Using the ALP, we can show that Y is functorial. A straightforward verification (using Lemma 4.4) shows that the passages from profinite action to J-sheaf to profinite action and from J-sheaf to profinite action to J-sheaf are naturally isomorphic to the identities.

5. Examples and observations

5.1. *Profinite categories and actions in sets.* Let Γ: Fin Cat$\to$Sets be a profinite category. In the usual way, form the comma category diagram with the following set of vertices:

$$\{(C,x)\mid C\in \text{Fin Cat};\ x\in\Gamma(C)\}.$$

(along with the obvious maps). Let $\hat{\Gamma}$ denote the filtered limit of this diagram. It is readily shown that $\hat{\Gamma}$ is a category with a compact, totally disconnected, Hausdorff topology, and is a category object in compact spaces. As a near converse, any category object in compact spaces, *with enough continuous functors into finite, discrete categories to separate morphisms* is uniquely of the form $\hat{\Gamma}$ where $\Gamma(C)$ is the set of continuous functors from $\hat{\Gamma}$ to C.

Let X: Fin Act$^{\mathrm{op}}\to$Sets be an action over Γ. As above, we represent X by its canonical limit in Sets. Then X is a filtered limit of finite actions over finite categories. But as shown in [7], any filtered limit of sheaves with finite discrete bases is a sheaf over the compact limit of the finite discrete bases. So X can be shown to be both a diagram and a sheaf over the profinite category $\hat{\Gamma}$. Moreover, X must satisfy the following 'continuous action' condition: Let G be an ultrafilter of morphisms of $\hat{\Gamma}$ and let F_0 (resp. F_1) be the ultrafilter of domains (resp. codomains). Let $f=\operatorname{Lim} F$, $d=\operatorname{Lim} F_0$, $r=\operatorname{Lim} F_1$, so $f:d\to r$ in C. Let $x\in X(d)$ be given. By choosing a local section through x at d we can get an ultrafilter $F(x)$ of values $f_\alpha(x_\alpha)$ for $f_\alpha\in F$ and x_α in the local section through x. It is required that $f(x)=\operatorname{Lim} F(x)$. It can be shown that a profinite action over $\hat{\Gamma}$ is precisely the same thing as a functor-sheaf over $\hat{\Gamma}$ satisfying the continuous action condition.

5.2. *Spatial profinite categories.* Let $\mathscr{E}=\operatorname{Shv}(X)$ where X is a topological space. We say that E is a 'compact space over X' if E is a topological space with projection

$p: E \to X$ such that E is 'as compact as X is' (see [6]). This means, that given an ultrafilter U on E and a limit point x of $p(U)$ there must then be a unique $e \in E$ which is a limit of U and for which $p(e) = x$. In particular each fiber $p^{-1}(x)$ must be compact and Hausdorff. Then a profinite category over X is a category object E in the category of all compact spaces over X for which each fiber $p^{-1}(x)$ is a profinite category (i.e. has enough continuous functors to finite discrete categories.)

5.3. *Profinite categories and Boolean algebras.* Every profinite category $\Gamma: \text{Fin Cat} \to \mathscr{E}$ has an underlying Boolean algebraic structure. We can extend Γ uniquely to a functor from Fun(Fin Cat$^{\text{op}}$, Sets) to $\mathscr{E}$ (i.e. see [2] or [8]) and map Fin Set to this functor category as follows: If $F \in \text{Fin Set}$ then define $F(C) = \text{Hom}(C_0, F)$ where C_0 is the set of objects of C. This gives us a left exact composite functor:

$$\text{Fin Set} \to \text{Fun}[\text{Fin Cat}^{\text{op}}, \text{Sets}] \to \mathscr{E}$$

which determines a Boolean algebra in $\mathscr{E}$, the Boolean algebra of objects of Γ. Similarly, using C_1 instead of C_0, we can define the Boolean algebra of morphisms of Γ.

5.4. *Maximally algebraic fields.* Let C be the category consisting of two objects 0 and 1 with non-identity maps $m: 0 \to 1$ and $\gamma: 1 \to 1$ such that $\gamma m = m$ and $\gamma^2 = 1$. Let $\mathscr{E} = \text{Fun}(C, \text{Sets})$ and represent an object of $\mathscr{E}$ as a pair (X_0, X_1) with maps $m: X_0 \to X_1$ and $\gamma: X_1 \to X_1$. Consider the field (R, C) where R = the reals, C = the complex field, $m: R \to C$ the usual embedding and γ is complex conjugation. Then there is no algebraic extension of (R, C) within the topos $\mathscr{E}$. We might say that (R, C) is maximally algebraic. The profinite Galois groupoid is therefore not Morita equivalent to any profinite group.

5.5. *Theaters and profinite actions.* A theater was defined in [6] to axiomatize the notion of a profinite group acting continuously on a 'theater of action'. As shown in [6], every theater gives rise to a profinite group Γ. As we might expect a theater also gives rise to a profinite action over Γ. [Using the generic theater, from [6], this means finding a left exact functor from Fin Act$^{\text{op}}$ to Fun(T_0, Sets) where T_0 is the category of finite theaters. But each finite theater t can be viewed as a finite action and the desired functor associated to (A, G) in Fin Act sends t to the set of action maps from (A, G) to t.] The converse is not true as theaters are too rigid. For example every theater must be decidable (as shown in [6]).

5.6. *Morita equivalence is weaker than natural equivalence.* This is not surprising, but for the record here is a simple example. Let G be the trivial group and H the groupoid with two objects and a single isomorphism in each hom set. Then G and H are profinite groups is Sets (e.g. by 5.1), and are easily seen to be Morita equivalent. But as functors from Fin Cat to Sets, G and H are not equivalent, even their restrictions to finite groups differ. For example $G(Z_2) = \mathrm{Hom}(G, Z_2)$ has only one element, but $H(Z_2) = \mathrm{Fun}(H, Z_2)$ has 2 (essentially different) elements.

5.7. *Full algebraic closure.* Let L be a separably closed field in a topos $\mathscr{E}$. It is relatively easy to make L into an algebraically closed field, and stay within $\mathscr{E}$. We sketch the construction. If L is a field over Z_p, we simply add pth roots. In the general case, there is subobject U of 1 of $\mathscr{E}$ where L satisfies the condition that $p=0$ for some prime p. Let $\mathscr{E}_0$ be the open topos and $\mathscr{E}_1$ the closed topos as in the Wraith glueing (e.g. [4, p. 112]). Further decompose $\mathscr{E}_0$ into the $\mathscr{E}_0(p)$, where $p=0$ and add pth roots to the part of the field over $\mathscr{E}_0(p)$. Leave the part of L in $\mathscr{E}_1$ alone. Then the pieces can be re-glued to get an algebraically closed extension.

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