O’ZBEKISTON RESPUBLIKASI OLIY VA O’RTA MAXSUS TA’LIM VAZIRLIGI

AJINIYOZ NOMIDAGI NUKUS DAVLAT PEDAGOGIKA INSTITUTI

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MATEMATIK ANALIZ FANIDAN MUSTAQIL ISHLARNI BAJARISHGA DOIR USLUBIY QO’LLANMA

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Kirish.

Ushbu qo'llanma 1 kurs talabalar uchun mo'ljallangan bo'lib, 1 kurs o'quv dasturi asosida tuzilgan va o'quv adabiyoti davlat ta'lim standartining bakalavr mutaxassisligi «Matematika va informatika» yonalishiga mos keladi.

Qo'llanma birinchi o'quv yilida «Matematik analiz» fanidan mustaqil ishlari uchun mo'ljallangan bo'lib, to'plam haqida tushuncha, sonli ketma –ketliklar, funksiya, funksiya limiti, funksiya uzluksizligi va tekis uzluksizligi, funksiya hosilasi va differensiali ham da differensial hisobning ba'zi bir tadbiqlarini , aniqmas integral, aniq integral, aniq integralning ba'zi bir tadbiqlari mavzularini o`z ichiga oladi.

Bu qo'llanmada har bir mustaqil ishi uch qismga ajratilgan, ya'ni 1 –qismda mustaqil ishini bajarish uchun lozim bo'lgan asosiy tushuncha va teoremalar keltirilgan, 2 –qismda talaba mustaqil ishini oson o'zlashtirishi uchun misol va masalalar to'liq yechib ko'rsatilgan, 3 –qismda esa mustaqil yechish uchun misollar tavsiya qilingan.

1 – mustaqil ish

Mavzu: Ratsional sonlar to'plami. Haqiqiy sonlar to'plami. Haqiqiy sonlarni to'g`ri chiziqda tasvirlash. Haqiqiy sonning moduli.

1. To'plam haqida tushuncha.

To'plam tushunchasi matematikani boshlang'ich tushunchalaridan bo'lib, unga ta'rif berilmaydi. To'plam tushunchasi nimalardan iborat ekanligini tushunish uchun quyidagi misollarga murojaat qilamiz.
1) Shu auditoriyadagi studentlar to'plami.
2) Hamma butun sonlar to'plami.
3) Tekislikdagi biror nuqtadan o'tuvchi to'g'ri chiziqlar to'plami
4) Markazi berilgan nuqtada bo'lgan aylanalar to'plami.
5) N natural sonlar to'plami va hokazo.

Matematikada to'plam haqida so'z yuritilganda, bir qancha narsalar bittaga birlashtirilib qaraladi va *A, B, C, D*, ... harflar bilan belgilanadi. Yuqoridagi misollardan ko'rinadiki, har bir to'plam nomining o'zi qaysi elementlar bu to'plamga kiritilganini ko'rsatib turibdi. To'plam elementlari kichik *a,b,c,d,...* harflar bilan belgilanadi. Agar *A* to'plam *a,b,c* elementlardan tashkil topgan bo'lsa, $A=\{a,b,c\}$ kabi yoziladi. Agar *A* to'plamni ixtiyoriy elementini *X* harfi bilan belgilasak, uni $A=\{x\}$ kabi yozamiz. Masalan, barcha natural sonlar to'plamini N desak, $N=(1,2,3,4,...)$ kabi belgilanadi, buni yana $A=\{n\}$ kabi ham yozish mumkin.

Agar biror *a* narsa *A* to'plamning elementi bo'lsa, $a \in A$ ko'rinishida yoziladi. $a \notin A$ belgilash esa a element *A* to'plamga tegishli emasligini bildiradi. Masalan, natural sonlar to'plamini *N* bilan belgilasak, u holda $5 \in N,\ 7 \in N,\ 0 \notin N,\ 5{,}2 \notin N$ ko'rinishlarda yozish mumkin. Birorta elementga ega bo'lmagan to'plam bo'sh to'plam deyiladi.

Masalan, parallel to'g'ri chiziqlarning kesishish nuqtalari to'plami, $x^2+1=0$ tenglamaning haqiqiy ildizlari to'plami, kvadrati ikkiga teng bo'lgan ratsional sonlar to'plami va hokazo. Bo'sh to'plam odatda $\varnothing$ simvol bilan belgilanadi. *A* va *B* to'plamlar bir xil elementlardan iborat bo'lsa, teng to'plamlar deyiladi va $A=B$ kabi yoziladi. Bundan tashqari matematikada yana quyidagi belgilashlar ham ishlatiladi.
$\forall$ - har qanday degan belgi, $\exists$ - mavjudki degan belgidir.
$\wedge$ - va belgisi, $\vee$ - yoki belgisidir.
$\Leftrightarrow$ - bo'lganda faqat shundagina, $\Rightarrow$ kelib chiqadi. Bu belgilashlarga ko'ra *A* va *B* to'plamlar tengligini quyidagicha yozish mumkin:

$$(A=B) \Leftrightarrow ((\forall x \in A \Rightarrow x \in B) \wedge (\forall x \in B \Rightarrow x \in A)).$$

A va *B* to'plamlar bir xil elementlarni o'z ichiga olganda va faqat shundagina tengdir.

Masalan, 1 dan 10 gacha bo'lgan natural sonlar to'plamlari bu sonlar qaysi tartibda joylashganligidan qat'iy nazar o'zaro tengdir. Agar *A* to'plamning har bir elementi B to'plamning ham elementi bo'lsa, u holda A to'plam B to'plamning qism to'plami deyiladi va $A\subset B$ kabi yoziladi. Bu ta'rifga ko'ra har qanday to'plam o'z-o'zining qism to'plami hisoblanadi.

Masalan, $N\subset Z$, $Q\subset R$, A - sinfdagi o'quvchilar to'plami, B - bir to'garakka qatnashuvchi o'quvchilar to'plami bo'lsa, $B\subset A$ kabi yoziladi.

Ko'pincha matematikada tadqiqot maqsadlariga qarab berilgan A to'plamdan barcha elementlari biror umumiy xossaga ega bo'lgan qism to'plam ajratiladi, unda A to'plamning hamma elementlari shu xossaga ega bo'lavermaydi. Uni quyidagicha yoziladi:

$\{x\in A \ ...\}$ bu degan so'z A to'plamga tegishli va *". . . "* xossaga ega bo'lgan barcha x lar to'plami. Masalan, 3 dan kichik natural sonlar to'plami B ni quyidagicha yozish mumkin: $B=\{x\in N: x<3\}=\{1,2\}$

Endi, $M=\{x:...\}$ belgilash $M=\{x\in R:...\}$ kabi belgilashga teng kuchlidir, ya'ni M to'plam *". . ."* xossaga ega bo'lgan haqiqiy sonlar to'plami deganidir. Yuqoridagi belgilashlarga ko'ra ratsional sonlar to'plami Q ni quyidagicha ta'riflash mumkin.

$$Q=\{x:x=\frac{m}{n},\ m\in Z,\quad n\in N\}$$

2. To'plam ustida amallar.

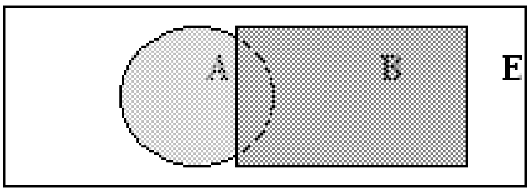
Ta'rif: Barcha elementlari A va B to'plamlarning kamida biriga tegishli bo'lgan elementlardan tuzilgan to'plam A va B to'plamlarning birlashmasi yoki ularning yig'indisi deyiladi va $A\cup B$ kabi belgilanadi.

Bu ta'rifni matematik tilda quyidagicha yozish mumkin:

$$(x\in(A\cup B))\Leftrightarrow((x\in A)\vee(x\in B))$$

Misollar:

a) $A=\{1,2,3,4,5\}$, $B=\{1,3,5,7,9\}$ bo'lsa, u holda $A\cup B=\{1,2,3,4,5,7,9\}$ dan iborat bo'ladi.



$A\cup B=C$

b) A- barcha manfiy bo'lmagan butun sonlar to'plami bo'lsin. B- barcha butun manfiy sonlar to'plami bo'lsin, u holda $A\cup B=Z$ barcha butun sonlar to'plami bo'ladi.

Ta'rif: Barcha elementlari A va B to'plamlarning har biriga tegishli bo'lgan elementlardan tuzilgan to'plamga A va B to'plamlarning kesishmasi deyiladi hamda $A\cap B$ kabi belgilanadi.

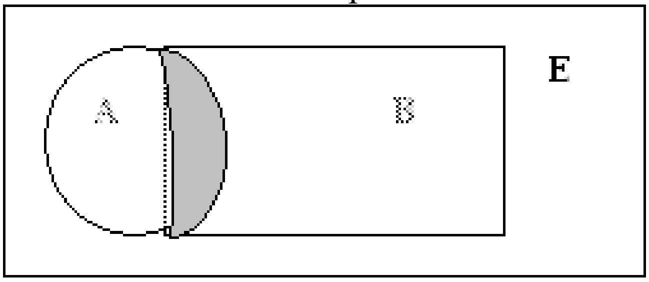
Bu ta'rifni matematik tilda quyidagicha yozish mumkin.

$$(x\in(A\cap B))\Leftrightarrow((x\in A)\wedge(x\in B)) \quad \text{yoki } A\cap B=\{x:(x\in A)\wedge(x\in B)\}$$

Misol. *1)* $N\cap Z=N$ bo'ladi.

2) $A=\{1,3,5,7,9\}$, $B=\{4,6,7,8,9\}$ bo'lsa, $A\cap B=\{7,9\}$ bo'ladi.

3) A- hamma romblar to'plami, B - hamma to'g'ri to'rtburchaklar to'plami bo'lsin, u holda $A\cap B$ hamma kvadratlar to'plamidan iborat bo'ladi.



$A\cap B$

To'plamlarning birlashmasi, kesishmasi sonlarning yig'indisi va ko'paytmalarining ko'p xossalariga o'xshash bo'ladi. Masalan, o'rin almashtirish, gruppalash va taqsimot qonunlari sonlar va to'plamlar uchun ham bir xil bo'lishligini quyidagicha ko'rsatish mumkin:

1) $a+b=b+a$ bo'lsa, $A\cup B=B\cup A$

2) $a\cdot b=b\cdot a$ bo'lsa, $A\cap B=B\cap A$

3) $(a+b)+c=a+(b+c)$ bo'lsa, $(A\cup B)\cup C=A\cup(B\cup C)$

4) $(a+b)\cdot c=a\cdot s+b\cdot c$ bo'lsa, $(A\cup B)\cap C=(A\cap C)\cup(B\cap C)$

Bunday o'xshashlik har doim ham o'rinli emas. Masalan, to'plamlarning quyidagi xossalari uchun to'g'ri emas.

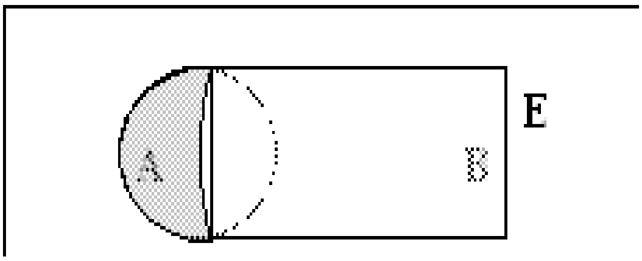
1) $(A\cup B)\cap(B\cup C)=(C\cap B)\cup A$

2) $A\cup A=A$

3) $A\cap A=A$

Ta'rif: A to'plamning B to'plamda bo'lmagan hamma elementlariga A va B to'plamlarning ayirmasi deyiladi va $A\backslash B$ kabi belgilanadi.

Misollar.



1) Agar $A=\{1,2,3,4\}$, $B=\{1,2\}$ bo'lsa, u holda $A\backslash B=\{3,4\}$ bo'ladi.

2) Agar $A=\{1,2,5\}$, $B=\{3,4\}$ bo'lsa, u holda $A\backslash B=\{1,2,5\}$ bo'ladi.

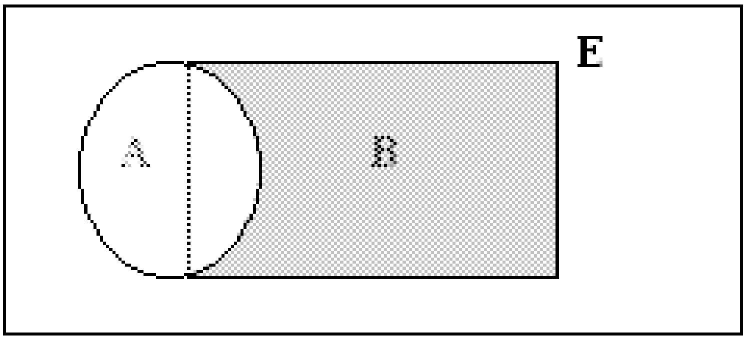
3) Agar $A=\{1,2\}$, $B=\{1,2,3\}$ bo'lsa, u holda $A\backslash B=\varnothing$ bo'ladi.

Bu to'plamning ayirmasi ta'rifini matematik tilda quyidagicha yozish mumkin:

$$x\in(A\backslash B)\Leftrightarrow((x\in A)\wedge(x\notin B)) \text{ yoki}$$

$$(A\backslash B)=\{x: x\in A,\ x\notin B\}$$

Agar $B\subset A$ bo'lsa, u holda A va B to'plamlarning ayirmasi B to'plamning A to'plamgacha to'ldiruvchisi deyiladi va S_AB kabi belgilanadi.



Misol:

1) Irratsional sonlar to'plami $\frac{p}{q}$ ratsional sonlar to'plamining haqiqiy sonlar to'plamigacha to'ldirmasidir.

2) A - barcha to'g'ri to'rtburchaklar to'plami, B - kvadratlar to'plami, C - turli tomonli to'g'ri to'rtburchaklar to'plami bo'lsin, u holda $A\backslash B = C$ va $A\backslash C=B$ bo'ladi.

3) $Q\backslash R = \varnothing$

To'g'ri chiziqdagi istalgan bir nuqtani 0 nuqta deb olib uni O harfi bilan belgilaymiz. 0 dan o'ng tomonga musbat yo'nalish chap tomonga esa manfiy yo'nalish deb ma'lum bir kesmani o'lchov birligi sifatida qabul qilamiz. O'lchov birligini 0 dan o'ngga va chapga o'lchab joylashtirganda to'g'ri chiziqda $\pm 1,\ \pm 2,\ \pm 3,...$ sonlarga mos nuqtalarni hosil qilamiz, bu nuqtalar butun nuqtalar, ularga mos keluvchi sonlarni esa butun sonlar deb ataladi va u Z harfi bilan belgilanadi. $Z = \{...-3, -2, -1, 0, 1, 2, 3, ...\}$

3. Ratsional sonlar to'plami

Ta'rif: Cheksiz davriy o'nli kasr ko'rinishida yozish mumkin bo'lgan sonlar ratsional sonlar deyiladi. Barcha musbat va manfiy butun va kasr sonlar nol soni bilan birgalikda ratsional sonlar to'plamini hosil qiladi. Ratsional sonlar to'plamini yana quyidagicha ta'riflash mumkin. Barcha $\frac{p}{q}$ ko'rinishidagi sonlarga ratsional sonlar to'plami deyiladi. Bu yerda $p,\ q\neq 0$ butun sonlar. Ratsional sonlar Q harfi bilan belgilanadi. Ratsional sonlar to'plami quyidagi muhim xossaga ega:

I. Q ratsional sonlar to'plami tartiblangan to'plamdir. Ixtiyoriy ikkita a va b ratsional sonlar olinsa, ular uchun $a=b$, $a>b$ yoki $a<b$ munosabatdan faqat bittasigina o'rinlidir.

II. Q ratsional sonlar to'plami zich joylashgan to'plamdir. Ixtiyoriy a va b ratsional son olinsa, bu ratsional sonlar orasida yotuvchi bitta yoki cheksiz ko'p ratsional son yotadi. Masalan, $c = \frac{a+b}{2}$ ratsional son uchun $a<c<b$ bo'ladi. Ixtiyoriy ikkita a va b ratsional son orasida kamida bitta ratsional son mavjudligidan bu ratsional sonlarning orasida cheksiz ko'p ratsional sonlarni mavjudligi kelib chiqadi.

4. Haqiqiy sonlar to'plami

Ta'rif: Quyidagi haqiqiy sonlar aksiomatikasi deb yuritiladigan shartlar majmuasiga bo'ysunadigan $\mathbb{R}$ to'plam haqiqiy sonlar to'plami deyiladi, uning elementlariga esa haqiqiy sonlar deb aytiladi:

I. Qo'shish aksiomalari.

$\mathbb{R}$ da har bir $x, y \in \mathbb{R}$ juftlikka shu sonlarning yig'indisi deyiladigan $x + y \in \mathbb{R}$ element mos quyilib, quyidagilar bajariladi:

1^0. $\forall x, y \in \mathbb{R}:\ x + y = y + x$ - yig'indining kommutativligi;

2^0. $\forall x, y, z \in \mathbb{R}:\ (x + y) + z = (x + z) + y$ - yig'indining assotsiativligi;

3^0 . $\exists 0 \in \mathbb{R}:\ \forall x \in \mathbb{R}:\ x + 0 = x$ - nolning mavjudligi;

4^0. $\forall x \in \mathbb{R}\ \ \exists(-x) \in \mathbb{R}: x + (-x) = 0$ - qarama-qarshi sonning mavjudligi.

II. Ko'paytirish aksiomalari.

$\mathbb{R}$ da har bir $x, y \in \mathbb{R}$ juftlikka shu sonlarning ko'paytmasi deyiladigan $x \cdot y \in \mathbb{R}$ element mos quyilib, quyidagilar bajariladi:

5^0. $\forall x, y \in \mathbb{R}\ :\ x \cdot y = y \cdot x$ - ko'paytmaning kommutativligi;

6^0. $\forall x, y, z \in \mathbb{R}\ :\ (x \cdot y) \cdot z = (x \cdot y) \cdot z$ - ko'paytmaning assotsiativligi;

7^0. $\exists 1 \in \mathbb{R} \setminus \{0\}\ \forall x \in \mathbb{R}:\ x \cdot 1 = x$ - birning mavjudligi;

8^0. $\forall x \in \mathbb{R} \setminus \{0\}\ \ \exists x^{-1} \in \mathbb{R}:\ x \cdot x^{-1} = 1$ - teskari sonning mavjudligi;

9^0. $\forall x, y, z \in \mathbb{R}:\ (x + y) \cdot z = x \cdot z + y \cdot z$ - distributivlik.

*A*gar $x \in \mathbb{R}$, $y \in \mathbb{R}$ bo'lsa, u holda $x+(-y)$ soniga x va y sonlarning ayirmasi deyiladi va u uchun $x-y$ belgilash qabul qilingan. *A*gar $y \neq 0$ bo'lsa, u holda $x \cdot y^{-1}$ soni $\frac{x}{y}$ orqali belgilanib, unga x va y sonlarning nisbati deyiladi.

III. Tartib aksiomalari .

$\mathbb{R}$ da $\leq$ tartib munosabati aniqlanib, quyidagi aksiomalar bajariladi:

10^0. $\forall x \in \mathbb{R}\ :\ x \leq x$

11^0. $(x \leq y) \wedge (y \leq x) \Rightarrow x = y$

12^0. $(x \leq y) \wedge (y \leq z) \Rightarrow x \leq z$

13^0. $\forall x, y \in \mathbb{R}:\ (x \leq y) \vee (y \leq x)$

14^0. $\forall x, y, z \in \mathbb{R}\ : x \leq y \Rightarrow x + z \leq y + z$

15^0. $(0 \leq x) \wedge (0 \leq y) \Rightarrow 0 \leq xy$

IV. To'lalik (uzluksizlik, zichlik) aksiomasi.

16^0. *A*gar $\mathbb{R}$ ning bo'shmas X va Y qism to'plamlari $\forall x \in X, \forall\ y \in Y \Rightarrow x \leq y$ xossaga ega bo'lsa, u holda $x \leq c \leq y$ shartni qanoatlantiradigan $c \in R$ mavjud.

*A*gar biror to'plam mazkur aksiomatikaga buysinsa, bunday to'plam haqiqiy sonlar aksomatikasining modeli deyiladi. Bunday modellardan biri – bu cheksiz davriymas o'nli kasrlar to'plamidir. Ikkinchisi esa Dedekind tomonidan XIX asr o'rtasida taklif qilingan maxsus konstruksiyalar – Dedekind kesimlaridan iborat.

5. Haqiqiy sonlarning xossalari

Yuqorida keltirilgan aksiomalardan quyidagi xossalarni hosil qilamiz (tekshiring).

1. Nol soni yagonadir.
2. Ixtiyoriy x soni uchun yagona qarama-qarshi $-x$ mavjud.
3. $a+x = b$ tenglama yagona $x = b + (-a)$ yechimga ega.
4. Bir soni yagonadir.
5. Ixtiyoriy nolmas x soni uchun yagona teskari x^{-1} mavjud.
6. $a\cdot x = b$, $a \neq 0$ tenglama yagona $x = b\cdot a^{-1}$ yechimga ega.
7. $\forall x\in \mathbb{R} \Rightarrow x\cdot 0 = 0.$
8. $x\cdot y = 0 \Rightarrow (x = 0)\vee (y = 0).$
9. $\forall x\in \mathbb{R} \Rightarrow -x = (-1)\cdot x.$
10. $\forall x\in \mathbb{R}$ $(-1)\cdot(-x) = x.$
11. $\forall x\in \mathbb{R}$ $(-x)\cdot(-x) = x\cdot x.$

Haqiqiy sonning absolyut qiymati va uning xossalari.

Ta'rif: $x\in \mathbb{R}$ *sonning absolyut qiymati* yoki *moduli* deb, $x \geq 0$ bo'lganda x ga va $x< 0$ bo'lganda $-x$ ga teng bo'lgan $|x|$ soniga aytiladi.

Sonning moduli quyidagi xossalarga ega:

1. $\forall x\in \mathbb{R}$ $-|x| \leq x \leq |x|.$
2. $\forall x,y\in \mathbb{R}: |xy| = |x|\ |y|;$
3. $|x| \leq a \Leftrightarrow -a \leq x \leq a;$
4. $\forall x, y\in \mathbb{R}$ $|x+y| \leq |x|+|y|.$
5. $\forall x, y\in \mathbb{R}$ $||x| - |y|| \leq |x - y|.$

Nazorat savollari

1. To'plam deganda nimani tushunasiz?
2. To'plamlar ustida amallarni tushuntiring?
3. Natural sonlar to'plamini tushuntiring?
4. Butun sonlar to'plamida qanday amallar o'rinli?
5. Ratsional sonlar to'plamida bajarilgan kesim turlarini ayting?
6. Qanday kesimga irratsional son deyiladi?

Mustaqil yechish uchun misollar

1 - topshiriq. İxtiyoriy A,B,C,D to'plamlar uchun quyidagi munosabatlar isbotlansin.

1. $(A\cup B)\cup C = A\cup(B\cup C).$

2. $(A\cap B)\cap C = A\cap(B\cap C).$

3. $(A\cup B)\cap C = (A\cap C)\cup(B\cap C).$

4. $(A\setminus B)\cap C=(A\cap C)\setminus(B\cap C)$.

5. $(A\setminus B)\setminus C=(A\setminus C)\setminus(B\setminus C)$.

6. $(A\setminus B)\cup(B\setminus C)\cup(C\setminus A)\cup(A\cap B\cap C)=A\cup B\cup C$.

7. $(A\cap C)\cup(B\cap D)\subset(A\cup B)\cap(C\cup D)$.

8. $(B\setminus C)\setminus(B\setminus A)\subset A\setminus C$.

9. $A\setminus(B\cup C)=(A\setminus B)\setminus C$.

10. $(A\cup B)\setminus C\subset A\cup(B\setminus C)$.

11. $(A\cup C)\setminus B\subset(A\setminus B)\cup C$.

12. $A\cap B\subset A\subset A\cup B$.

13. $A\cap(A\cup B)=A$.

14. Agar $A\setminus B=C$ bo'lsa, $A=B\cup C$ ning o'rinli bo'lishi kelib chiqadimi?

15. Agar $A=B\cup C$ bo'lsa, $A\setminus B=C$ ning o'rinli bo'lishi kelib chiqadimi?

16. $(A\cup B)\setminus(A\cup C)\subset A\cup(B\setminus C)$.

17. $(A\cap B)\setminus C=(A\setminus C)\cap(B\setminus C)$.

18. $A\setminus(B\cup C)\subset(A\setminus B)\cup(A\setminus C)$.

19. Agar A va B chekli to'plamlar bo'lib, ularning elementlari soni mos ravishda $n(A)$, $n(B)$ bo'lsa,

$$n(A\cup B)=n(A)+n(B)-n(A\cap B)$$

bo'lishi isbotlansin.

20. Agar A chekli to'plam bo'lib, uning elementlari soni m ga teng bo'lsa, bu to'plamning barcha qismiy to'plamlari to'plamining elementlari soni 2^m ta bo'lishi isbotlansin.

21. Kvadrati 3 ga teng bo'lgan ratsional sonning mavjud emasligi isbotlansin.

22. Agar r - ratsional son, α - irratsional son bo'lsa, $\alpha+r$ ning irratsional son bo'lishi isbotlansin.

23. Agar α va β irratsional son bo'lsa, $\alpha+\beta$, $\alpha-\beta$ sonlar irratsional bo'ladimi?

24. Agar α va β irratsional son bo'lib, $\alpha+\beta$ esa ratsional son bo'lsa, $\alpha-\beta$ va $\alpha+2\beta$ sonlarning irratsional bo'lishi isbotlansin.

25. Ushbu

$$\alpha=\sqrt{1+\sqrt{2+\ldots+\sqrt{n}}} \qquad (n\in N,\ \ n\geq 2)$$

sonning irratsional bo'lishi isbotlansin.

25. Ushbu $\log 2,\ \ \log_2 3$ sonlarning irratsional son bo'lishi isbotlansin.

2 – mustaqil ish
Mavzu: Chegaralangan va chegaralanmagan sonli to'plamlar. Sonli to'plamlarning chegaralari.

1. Haqiqiy sonning absolyut qiymati va uning xossalari.

Ta'rif: $x\in\mathbb{R}$ *sonning absolyut qiymati* yoki *moduli* deb, $x\geq 0$ bo'lganda x ga va $x<0$ bo'lganda $-x$ ga teng bo'lgan $|x|$ soniga aytiladi.
Sonning moduli quyidagi xossalarga ega:

1. $\forall x\in\mathbb{R}$ $-|x|\leq x\leq |x|$.
2. $\forall x,y\in\mathbb{R}: |xy|=|x|\,|y|;$
3. $|x|\leq a \Leftrightarrow -a\leq x\leq a;$
4. $\forall x,y\in\mathbb{R}$ $|x+y|\leq |x|+|y|$.
5. $\forall x,y\in\mathbb{R}$ $||x|-|y||\leq |x-y|$.

2. Quyidan (yuqoridan) chegaralangan to'plam. *A*niq quyi (yuqori) chegara.

Ta'rif: $a\in\mathbb{R}$ va $b\in\mathbb{R}$, $a<b$ – ixtiyoriy sonlar bo'lsin. Quyidagi to'plamlar uchlari *a va b* larda bo'lgan oraliqlar deyiladi:

$(a,b)=\{x\in\mathbb{R}\ |a<x<b\}$ – ochiq oraliq yoki interval,

$[a,b]=\{x\in\mathbb{R}\ |a\leq x\leq b\}$ – yopiq oraliq yoki segment,

$[a,b)=\{x\in\mathbb{R}\ |a\leq x<b\}$, $(a,b]=\{x\in\mathbb{R}\ |a<x\leq b\}$ – yarim intervallar,

$(a,\infty)=\{x\in\mathbb{R}\ |a<x\}$, $(-\infty,a)=\{x\in\mathbb{R}\ |x<a\}$ – sonli o'qlar, $(-\infty,\infty)=\{x\in\mathbb{R}\ \}$ – sonli to'g'ri chiziq.

Ta'rif: $X\subset\mathbb{R}$ to'plam *yuqoridan (*quyi*dan) chegaralangan* deyiladi, agar $\forall\ x\in X$ $x\leq c$ $(c\leq x)$ shartni qanoatlantiruvchi $c\in\mathbb{R}$ mavjud bo'lsa. Bu holda c soni X to'plamning *yuqori (*quyi*) chegarasi* deyiladi.

Ta'rif: Bir vaqtda ham yuqoridan, ham quyidan chegaralangan to'plamlar *chegaralangan* deyiladi.

Ta'rif: $a\in X$ soni $X\subset\mathbb{R}$ to'plamning eng katta, maksimal (mos ravishda eng kichik, minimal) elementi deyiladi, agar barcha $x\in X$ uchun $x\le a$ ($a\le x$). Belgilash: $a = \max X$ ($a = \min X$).

11° aksiomadan maksimal (minimal) elementlarning soni birdan katta emasligi kelib chiqadi. *A*yrim, xatto chegaralangan to'plamlar maksimal elementga ega bo'lmasligi mumkin.

Misol : $X = [0,1)$ to'plam maksimal elementga ega emas.

Ta'rif: $X\subset\mathbb{R}$ to'plamning yuqori chegaralardan eng kichigi (quyi chegaralardan eng kattasi) X to'plamning aniq yuqori chegarasi (aniq quyi chegarasi) deyiladi va *supX (infX)* orqali belgilanadi.

Boshqacha aytganda, $(a=\sup X)\Leftrightarrow(\forall x\in X: x\le a)\cap(\forall\varepsilon>0\,\exists x_1\in X, x_1>a-\varepsilon)$, $(a=\inf X)\Leftrightarrow(\forall x\in X: x\ge a)\cap(\forall\varepsilon>0\,;\exists x_1\in X, x_1\le a+\varepsilon)$.

*3. A*niq yuqori chegaraning mavjudligi

Quyidagi lemma uzluksizlik aksiomasidan kelib chiqadi:

Lemma. (*aniq yuqori chegara haqida*). Yuqoridan chegaralangan bo'sh bo'magan sonli to'plam yagona aniq yuqori chegaraga ega.

Isbot: $X\subset\mathbb{R}$ **-** yuqoridan chegaralangan to'plam bo'lsin**.** Uning yuqori chegaralarda iborat bo'lgan $Y = \{y\in\mathbb{R}\ |\forall x\in X\ \ x\le y\}$ – to'plamni qaraymiz. 16° aksiomaga ko'ra shunday $c\in\mathbb{R}$ mavjudki, $\forall x\in X\ \forall y\in Y$ uchun $x\le c\le y$ qo'sh tengsizlik o'rinli. Demak c soni Xning yuqori chegarasi bo'lib, $c\in Y$ bo'ladi. Ya'ni $c = \min Y$, demak, ta'rifga ko'ra c soni X ning aniq yuqori chegarasi bo'ladi: $c = \sup X$. Uning yagonaligi 11° aksiomadan kelib chiqadi.

*A*niq quyi chegara haqida lemma xudi shunday isbotlanadi.

Izoh. 16° aksioma mazkur lemma bilan tengkuchli.

Tayanch tushunchalar: Haqiqiy sonlar absolyut qiymati, quyidan (yuqoridan) chegaralangan to'plam, aniq quyi (yuqori) chegara, oraliqlar.

Nazorat savollari

1. Haqiqiy sonning absolyut qiymatiga ta'rif bering va uning xossalarini bayon qiling.
2. Qanday to'plamga yuqoridan (quyidan) chegaralangan deyiladi? Misollar keltiring.
3. Qanday to'plamga chegaralangan deyiladi? Misollar keltiring.
4. Qanday to'plamga chegaralanmagan deyiladi? Misollar keltiring.
5. Sonli to'plamning aniq yuqori va aniq quyi chegarasi qanday ta'riflanadi?
6. Yuqoridan chegaralangan to'plamning aniq yuqori chegarasi mavjudligi haqidagi teorema qanday isbotlanadi?
7. Quyidan chegaralangan to'plamning aniq quyi chegarasi mavjudligi haqidagi teorema qanday isbotlanadi?

Mustaqil yechish uchun misollar.

1 - topshiriq. Quyidagi to'plamlar chegaralanganlikka tekshirilsin:

1. $E=\{x=1-n^2-6n: \ n\in N\}.$

2. $E=\left\{x=\dfrac{n}{1+n^2}: \ n\in N\right\}.$

3. $E=\left\{x=\dfrac{(-1)^n\cdot n+10}{\sqrt{n^2+1}}: \ n\in N\right\}.$

4. $E=\left\{x=\dfrac{n}{a^n}: \ n\in N, \ a>1\right\}.$

5. $E=\left\{x=\left[1+(-1)^n\right]\cdot n+\dfrac{1-(-1)^n}{n}: \ n\in N\right\}.$

6. Ushbu

$$E\left\{x=\frac{1}{n}: \ n\in N\right\}$$

to'plamning aniq yuqori h'amda aniq quyi chegaralari topilsin.

7. Ushbu

$$E=\left\{x=1+\frac{(-1)^n}{n}: \ n\in N\right\}$$

to'plamning aniq yuqori va aniq quyi chegaralari topilsin.

8. $E\subset R$ to'plam uchun $SupE$ va $\inf E$ lar mavjud bo'lib,

$$SupE=\inf E$$

bo'lsa, E to'plam to'g'risida nima deyish mumkin.

9. Agar $E\subset R$, $F\subset E$ to'plamlar uchun:

1) $\forall x\in E, \forall y\in F: x\le y,$

2) $\forall\varepsilon>0, \exists x_0\in E, \exists y_0\in F: y_0-x_0<\varepsilon$

bo'lsa, u holda

$$SupE=\inf F$$

bo'lishi isbotlansin.

10. Agar $E\subset R$ to'plam chegaralangan bo'lib, $E_1\subset E$ bo'lsa, u h'olda

$$SupE_1\le SupE, \quad \inf E_1\ge\inf E$$

bo'lishi isbotlansin.

11. Agar $E \subset R$ to'plam chegaralangan bo'lib, $a \in R$ bo'lsa, u h'olda

$$Sup\{a+E\} = a + SupE$$

bo'lishi isbotlansin.

12. Agar $E \subset R$ to'plam chegaralangan bo'lib, $a > 0$ bo'lsa, u h'olda

$$Sup\{a \cdot E\} = a \cdot SupE$$

bo'lishi isbotlansin.

13. Aytaylik, chegaralangan $E = \{x\} \subset R$ to'plam h'ar bir x elementining qarama-qarshisi $-x$ lardan tuzilgan to'plam F bo'lsin: $F = \{-x : x \in E\}$. U h'olda

$$SupF = -\inf E, \quad \inf F = -SupE$$

bo'lishi isbotlansin.

14. Aytaylik, $E = \{x\} \subset R$, $F = \{y\} \subset R$ chegaralangan to'plamlar bo'lib, $E + F = \{x + y : x \in E, y \in F\}$ bo'lsin. U h'olda

$$Sup(E+F) = SupE + SupF,$$

$$\inf(E+F) = \inf E + \inf F$$

bo'lishi isbotlansin.

15. Aytaylik, $E = \{x\} \subset R, F = \{y\} \subset R$ chegaralangan to'plamlar bo'lib, $E - F = \{x - y : x \in E, y \in F\}$ bo'lsin. U h'olda

$$Sup(E-F) = SupE - \inf F$$

bo'lishi isbotlansin

16. Ushbu $E_+ = \{x : x > 0\}$, $F_+ = \{y : y > 0\}$ to'plamlar yordamida tuzilgan $E_+ \cdot F_+ = \{x \cdot y : x \in E_+, y \in F_+\}$ to'plam uchun

$$\inf(E_+ \cdot F_+) = \inf E_+ \cdot \inf F_+,$$

$$Sup(E_+ \cdot F_+) = SupE_+ \cdot SupF_+$$

bo'lishi isbotlansin.

17. Aytaylik, $E \subset R$, $F \subset R$ to'plamlar yuqoridan chegaralangan bo'lsin. Unda

$$Sup(E \cup F) = \max(SupE,\ \ SupF)$$

bo'lishi isbotlansin.

($\max(a,\ \ b) - a$ va b larning kattasi)

18. $x \in R$ sonning absolyut qiymati $|x|$ quyidagicha tariflanadi:

$$|x| = \begin{cases} x, & агар\ \ x \geq 0 \\ -x, & агар\ \ x < 0 \end{cases}$$

İxtiyoriy x h'aqiqiy son uchun

$$|x| \geq 0, \qquad |x| = |-x|, \quad -|x| \leq x \leq |x|$$

bo'lishi isbotlansin.

19. İxtiyoriy x va y h'aqiqiy sonlari uchun

1) $|x + y| \leq |x| + |y|$,

2) $||x| - |y|| \leq |x - y|$,

3) $|x \cdot y| = |x| \cdot |y|$

bo'lishi isbotlansin.

20. İxtiyoriy x va y h'aqiqiy sonlari ushbu

$$|x \cdot y| \leq \frac{1}{2}\left(x^2 + y^2\right)$$

tengsizlikni qanoatlantirishi isbotlansin.

21. Ushbu

$$E = \{x:\ \ |x| < a,\ \ a > 0\},$$

$$F = \{x:\ \ -a < x < a,\ \ a > 0\}$$

to'plamlarning o'zaro tengligi isbotlansin.

22. Ushbu

$$E = \{x:\ \ |x| > a\},$$

$$F = \{x:\ \ x > a,\ \ x < -a\}$$

to'plamlarning tengligi isbotlansin.

2 - topshiriq. Quyidagi tengsizliklarning echimlar to'plami topilsin:

1. $|3x-1| \le |2x-1| + |x|$
2. $|x+3| > 2$
3. $\left|\dfrac{x}{1+x}\right| > \dfrac{x}{x+1}$
4. $|x^2-2x-3| > x^2-2x-3$
5. $|x^2-2x| > x^2-|2x|$
6. $|x-4| + |x+4| \le 10$
7. $|x-2| < 3$
8. $|x| < x+1$
9. $|x^2-5| > 2$
10. $|x+3| - |x+1| < 2$
11. $|x+2| + |x-2| \ge 12$

3 - topshiriq. Quyidagi tenglamalarning echimlar to'plami topilsin:

1. $|2x+3| = x^2$.
2. $|\sin x| = \sin x + 2$.
3. $\left|\dfrac{x-1}{x+1}\right| = \dfrac{x-1}{x+1}$.
4. $|x^2-5x+6| = -(x^2-5x+6)$.
5. $|x| = x^2-6$.
6. $|(x^2+2x+5)+(x-5)| = |x^2+2x+5| + |x-5|$.
7. $|(x^4-4)-(x^2+2)| = |x^4-4| - |x^2+2|$.

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3 – **mustaqil ish**
Mavzu: Sonli ketma – ketlik va uning limiti. Asosiy tushuncha va teoremalar.

N va R to'plamlar berilgan bo'lib, f–har bir natural n $(n\in N)$songa biror haqiqiy $x_n(x_n\in R)$ sonni mos qo'yuvchi akslantirish bo'lsin:

$f:N\to R$ yoki $f:n\to x_n$. Bu holda $x_n=f(n)$ kabi belgilanadi.

1 – ta'rif. $f(n)$ o'zgaruvchining qiymatlaridan tuzilgan $x_1,x_2,...,x_n,...$ to'plam sonlar ketma – ketligi deyiladi va $\{x_n\}$ kabi belgilanadi.

x_n $(n=1,2,...)$ miqdorlar $\{x_n\}$ ketma - ketlikning hadlari deyiladi.

$\{x_n\}$ va $\{y_n\}$ ketma - ketliklar berilgan bo'lsa,

$\{x_n+y_n\}=\{x_1+y_1,\,x_2+y_2,\,...\}$,

$\{x_n-y_n\}=\{x_1-y_1,\,x_2-y_2,\,...\}$,

$\{x_n\cdot y_n\}=\{x_1\cdot y_1,\,x_2\cdot y_2,\,...\}$,

$$\left\{\frac{x_n}{y_n}\right\}=\left\{\frac{x_1}{y_1},\,\frac{x_2}{y_2},\,...\right\},\qquad (y_n\neq 0,\ n=1,2,...)$$

ketma - ketliklarga mos ravishda $\{x_n\}$ va $\{y_n\}$ ketma - ketliklarning **yig`indisi, ayirmasi, ko'paytmasi va nisbati** deyiladi,

2 - ta'rif. Agar $\exists M$ $(\exists m)$ son mavjud bo'lsaki, $\forall n\in N$ uchun $x_n\le M$ $(x_n\ge m)$ tengsizlik o'rinli bo'lsa, $\{x_n\}$ ketma - ketlik yuqoridan (quyidan) chegaralangan deyiladi. Aks holda esa, ya'ni $\forall M(\forall m)$ son olinganda ham $\exists n\in N$ son mavjud bo'lsaki, $x_n>M$ $(x_n<m)$ bo'lsa, $\{x_n\}$ ketma - ketlik **yuqoridan (quyidan) chegaralanmagan** deyiladi.

3 - ta'rif. Agar $\exists M>0$ son mavjud bo'lsaki, $\forall n\in N$ uchun $|x_n|\le M$ bo'lsa, $\{x_n\}$ ketma - ketlik **chegaralangan** deyiladi. Aks holda esa, ya'ni $\forall M>0$ son olinganda ham $\exists n_0\in N$ son topilsaki $|x_{n_0}|>M$ bo'lsa, $\{x_n\}$ **chegaralanmagan ketma - ketlik** deyiladi.

4 - ta'rif. Berilgan $\{x_n\}$ ketma - ketlik uchun shunday a son topilib, $\forall\varepsilon>0$ son olinganda ham $\exists n_0=n_0(\varepsilon,a)\in N$ con mavjud bo'lsaki, $n>n_0$ tengsizlikni qanoatlantiruvchi barcha natural sonlar uchun $|x_n-a|<\varepsilon$ tengsizlik o'rinli bo'lsa, a son $\{x_n\}$ ketma - ketlikning **limiti** deyiladi va $\lim\limits_{n\to\infty}x_n=a$ ko'rinishda belgilanadi.

Agar 4 - ta'rifdagi shartni qanoatlantiruvchi a son mavjud bo'lmasa, $\{x_n\}$ ketma - ketlik **limitga ega emas** deyiladi.

5 - ta'rif (4 - ta'rifning inkori). Agar $\forall n_0 \in N$ son olinganda ham $\exists \varepsilon > 0$, $\exists n > n_0$ son topilsaki, $|x_n - a| \geq \varepsilon$ bo'lsa, a son $\{x_n\}$ ketma - ketlikning **limiti emas** deyiladi va $\lim\limits_{n\to\infty} x_n \neq a$ ko'rinishda belgilanadi.

6 - ta'rif. Agar $\{x_n\}$ ketma - ketlik chekli limitga ega bo'lsa, bu ketma - ketlik **yaqinlashuvchi** deyiladi. Aks holda bu ketma - ketlik **uzoqlashuvchi** deyiladi.

Cheksiz kichik va cheksiz katta ketma - ketliklar

1 - ta'rif. Agar $\{x_n\}$ ketma - ketlikning limiti nolga teng, ya'ni $\lim\limits_{n\to\infty} x_n = 0$ bo'lsa, $\{x_n\}$ ketma - ketlik **cheksiz kichik** ketma - ketlik deyiladi.

2 - ta'rif. Agar $\forall M > 0$ son olinganda ham $\exists n_0 \in N$ son mavjud bo'lsaki, $\forall n > n_0$ natural sonlar uchun $|x_n| > M$ tengsizlik o'rinli bo'lsa, $\{x_n\}$ ketma - ketlik **cheksiz katta** ketma - ketlik deyiladi.

Agar $\{x_n\}$ cheksiz katta ketma - ketlik bo'lsa, $\lim\limits_{n\to\infty} x_n = \infty$ ko'rinishda yoziladi. Agar $\{x_n\}$ cheksiz katta ketma - ketlik bo'lib, biror nomerdan boshlab barcha hadlari **musbat (manfiy)** bo'lsa, $\lim\limits_{n\to\infty} x_n = +\infty$ ($\lim\limits_{n\to\infty} x_n = -\infty$) ko'rinishda yoziladi.

Har qanday cheksiz katta ketma - ketlik chegaralanmagan bo'ladi, lekin bu tasdiqning teskarisi har doim ham o'rinli bo'lavermaydi.

1 - teorema. Chekli sondagi cheksiz kichik ketma - ketliklar yigindisi cheksiz kichik ketma - ketlik bo'ladi.

2 - teorema. Chegaralangan ketma - ketlik bilan cheksiz kichik ketma - ketlik ko'paytmasi cheksiz kichik ketma - ketlik bo'ladi.

3 - teorema. Agar $\forall n \in N$ uchun $x_n \neq 0$ bo'lib, $\{x_n\}$ - cheksiz katta (cheksiz kichik) ketma - ketlik bo'lsa, u holda $\left\{\frac{1}{x_n}\right\}$ cheksiz kichik (cheksiz katta) ketma - ketlik bo'ladi.

4 - teorema. $\lim\limits_{n\to\infty} x_n = a$ bo'lishi uchun $\{\alpha_n\} = \{x_n - a\}$ ketma - ketlikning cheksiz kichik ketma - ketlik bo'lishi zarur va yetarlidir.

Yaqinlashuvchi ketma - ketliklarning xossalari.

5 - teorema. Agar $\{x_n\}$ ketma - ketlik yaqinlashuvchi bo'lsa, uning limiti yagona bo'ladi.

6 - teorema. Agar $\{x_n\}$ ketma - ketlik yaqinlashuvchi bo'lsa, u chegaralangan bo'ladi.

7 - teorema. Agar $\{x_n\}$ va $\{y_n\}$ ketma - ketliklar yaqinlashuvchi bo'lsa, u holda $\{x_n + y_n\}$, $\{x_n \cdot y_n\}$ ketma - ketliklar ham yaqinlashuvchi bo'ladi va

$$\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n,$$

$$\lim_{n\to\infty}(x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$$

formulalar o'rinli bo'ladi.

8 - teorema. Agar $\{x_n\}$ va $\{y_n\}$ ketma – ketliklar yaqinlashuvchi bo'lib, $\forall n \in N$ uchun $y_n \neq 0$ va $\lim\limits_{n\to\infty} y_n \neq 0$ bo'lsa, $\left\{\dfrac{x_n}{y_n}\right\}$ ketma - ketlik ham yaqinlashuvchi bo'ladi va

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$$

formula o'rinli bo'ladi.

9 - teorema. Agar $\lim\limits_{n\to\infty} x_n = a$ bo'lib, biror nomerdan boshlab $x_n \geq c$ $(x_n \leq c)$ bo'lsa, u holda $a \geq c$ $(a \leq c)$ bo'ladi.

10 - teorema. Agar $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = a$ bo'lib, biror nomerdan boshlab $x_n \leq z_n \leq y_n$ tengsizlik o'rinli bo'lsa, u holda $\lim\limits_{n\to\infty} z_n = a$ bo'ladi.

Agar $\lim\limits_{n\to\infty} x_n = 0$, $\lim\limits_{n\to\infty} y_n = 0$ bo'lsa, $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n}$ - ga $\dfrac{0}{0}$ ko'rinishdagi aniqmaslik deyiladi. $\dfrac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$ va boshqa ko'rinishdagi aniqmasliklar ham shu kabi ta'riflanadi.

* **1- misol.** $\lim\limits_{n\to\infty} x_n = a$ ekanligi tarif yordamida ko'rsatilsin $(n_0(\varepsilon) - ?)$.

$$x_n = \frac{2n^2}{n^2 - 2}, \quad a = 2$$

◀ $(\lim\limits_{n\to\infty} x_n = a) \Leftrightarrow (\forall\varepsilon > 0\, \exists n_0 = n_0(\varepsilon) \in N : \forall n > n_0\ |x_n - a| < \varepsilon)$

$$|x_n - a| = \left|\frac{2n^2}{n^2-2} - 2\right| = \left|\frac{2n^2 - 2n^2 + 4}{n^2 - 2}\right| = \left|\frac{4}{n^2-2}\right| = \left|\frac{4}{(n-2)(n+2)}\right| < \frac{4}{n+2} < \frac{4}{n} < \varepsilon$$

$$\Rightarrow n > \frac{4}{\varepsilon} \Rightarrow n_0 = \left[\frac{4}{\varepsilon}\right] + 1$$

Demak, $\forall \varepsilon > 0$ son olinganda ham $n_0 = \left[\frac{4}{\varepsilon}\right] + 1$ deb olsak, $\forall n > n_0$ uchun $|x_n - a| < \varepsilon$ bo'ladi. $\Rightarrow \lim\limits_{n\to\infty} x_n = a$ ►

2– misol. a soni $\{x_n\}$ ketma – ketlikning limiti emasligi ta'rif yordamida ko'rsatilsin.

$$x_n = \sqrt{n^3 + 1} - n, \quad a = 1$$

◄ $(\lim\limits_{n\to\infty} x_n \neq a) \Leftrightarrow (\forall n_0 \in N \ \exists \varepsilon > 0, \ \exists n > n_0 : |x_n - a| \geq \varepsilon)$

$$|x_n - a| = \left|\left(\sqrt{n^3 + 1} - n\right) - 1\right| = \left|\sqrt{n^3 + 1} - (n+1)\right| = \left|\frac{n^3 + 1 - (n+1)^2}{\sqrt{n^3 + 1} + n + 1}\right| =$$

$$= \left|\frac{n^3 - n^2 - 2n}{\sqrt{n^3 + 1} + n + 1}\right| = \left|\frac{n - 1 - \frac{2}{n}}{\sqrt{\frac{1}{n} + \frac{1}{n^4}} + \frac{1}{n} + \frac{1}{n^2}}\right| > \frac{1}{4} = \varepsilon, n > 4$$

Demak, $\varepsilon = \frac{1}{4}$ deb olsak, $\forall n_0 \in N \ \exists n_1 > n_0, n_1 > 4$ uchun $|x_n - a| \geq \varepsilon$ tengsizlik bajarilar ekan. Bu esa $\lim\limits_{n\to\infty} x_n \neq a$ ekanligini anglatadi. ►

3– misol. Tengsizliklarda limitga o'tish haqidagi 10 – teoremadan foydalanib $\{x_n\}$ ketma – ketlikning yaqinlashuvligi ko'rsatilsin.

$$x_n = \frac{2n + 3\cos\frac{n\pi}{4}}{n^3}$$

◄ $\dfrac{2n - 3}{n^3} \leq \dfrac{2n + 3\cos\frac{n\pi}{4}}{n^3} \leq \dfrac{2n + 3}{n^3}$,

Agar $y_n = \dfrac{2n - 3}{n^3}$ va $z_n = \dfrac{2n + 3}{n^3}$ deb belgilasak, unda $\forall n \in N$ uchun $y_n \leq x_n \leq z_n$ qo'sh tengsizlik bajariladi. $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} z_n = 0$ va «ikki mirshab haqidagi teorema» ga ko'ra $\lim\limits_{n\to\infty} x_n = 0$ bo'ladi. ►

4 – misol. Sonli ketma – ketlikning limiti hisoblansin. $\left(\lim\limits_{n\to\infty} - ?\right)$

$$x_n = n + \sqrt[3]{4 - n^3}$$

$$\lim_{n\to\infty}\left(n+\sqrt[3]{4-n^3}\right)=\lim_{n\to\infty}\left(n-\sqrt[3]{n^3-4}\right)=\lim_{n\to\infty}\frac{n^3-n^3+4}{n^2+n\cdot\sqrt[3]{n^3-4}+\sqrt[3]{\left(n^3-4\right)^2}}=$$

$$=\lim_{n\to\infty}\frac{4}{n^2+n\cdot\sqrt[3]{n^3-4}+\sqrt[3]{\left(n^3-4\right)^2}}=0$$

Mustaqil yechish uchun misollar.

1 - topshiriq. $\lim\limits_{n\to\infty} x_n = a$ **ekanligi ta'rif yordamida ko'rsatilsin** $\left(n_0(\varepsilon)-?\right)$.

1.1. $x_n=\dfrac{3n+5}{2n}, a=\dfrac{3}{2}.$

1.2. $x_n=\dfrac{6n}{2n+1}, a=3.$

1.3. $x_n=\dfrac{2n^2-1}{3n^2+2}, a=\dfrac{2}{3}.$

1.4. $x_n=\dfrac{9-3n^3}{1+2n^3}, a=-\dfrac{3}{2}.$

1.5. $x_n=\dfrac{-2n^2}{2+4n^2}, a=-\dfrac{1}{2}.$

1.6. $x_n=-\dfrac{5n}{n+1}, a=-5.$

1.7. $x_n=\dfrac{5n+1}{1-2n}, a=-\dfrac{5}{2}.$

1.8. $x_n=\dfrac{2n+1}{3n-5}, a=\dfrac{2}{3}.$

1.9. $x_n=\dfrac{1-2n^2}{n^2+3}, a=-2.$

1.10. $x_n=\dfrac{3n^2}{2-n^2}, a=-3.$

1.11. $x_n=\dfrac{4+2n}{1-3n}, a=-\dfrac{2}{3}.$

1.12. $x_n=\dfrac{5n}{1+2n}, a=\dfrac{5}{2}$

1.13. $x_n=\dfrac{13-n^2}{1+2n^2}, a=-\dfrac{1}{2}.$

1.14. $x_n=-\dfrac{2n-1}{2-3n}, a=\dfrac{2}{3}$

1.15. $x_n=\dfrac{3n-1}{5n+1}, a=\dfrac{3}{5}.$

1.16. $x_n=\dfrac{5n+1}{10n-3}, a=\dfrac{1}{2}.$

1.17. $x_n=\dfrac{1+3n}{6-n}, a=-3.$

1.18. $x_n=-\dfrac{4n+3}{n+5}, a=-4$

1.19. $x_n=\dfrac{3n^2+2}{4n^2-1}, a=\dfrac{3}{4}.$

1.20. $x_n=\dfrac{2-n^2}{3+2n^2}, a=-\dfrac{1}{2}.$

2 – topshiriq. a **soni** $\{x_n\}$ **ketma-ketlikning limiti emasligi ta'rif yordamida ko'rsatilsin.**

2.1. $x_n=(-1)^n \quad a=0.$

2.2. $x_n=\cos\dfrac{\pi n}{3},\ a=1.$

2.3. $x_n=\dfrac{1}{2}\sin\dfrac{\pi n}{6},\ a=\dfrac{1}{2}.$

2.4. $x_n=2\cos\dfrac{\pi n}{10},\ a=-1.$

2.5. $x_n=2^{(-1)^n n}+2,\ a=0.$

2.6. $x_n=n\cdot\left[1+(-1)^n\right],\ a=0.$

2.7. $x_n=(-1)^n,\ a=-1.$

2.8. $x_n=(-1)^{n+1},\ a=1.$

2.9. $x_n = \frac{-\cos \pi n}{2+\cos \pi n}$, $a=3$.

2.10. $x_n = \left(\frac{1}{3}\right)^n$, $a=1$.

2.11. $x_n = \frac{n^2-1}{n^3}$, $a=1$.

2.12. $x_n = \frac{2n+3}{n^2}$, $a=2$.

2.13. $x_n = \frac{1}{2n+1}$, $a=\frac{1}{2}$.

2.14. $x_n = \sin\frac{\pi n}{2}$, $a=1$.

2.15. $x_n = \frac{(-1)^2}{n}$, $a=-1$.

2.16. $x_n = \sin\frac{\pi n}{2}$, $a=1$.

2.17. $x_n = \frac{n+1}{3-2n^2}$, $a=-\frac{1}{2}$

2.18. $x_n = (-1)^n n^2$, $a=-1$

2.19. $x_n = n^{(-1)^{n+1}}$, $a=0$

2.20. $x_n = \frac{n^2}{2n+1}$, $a=1$

3 – topshiriq.

Yaqinlashuvchi ketma – ketlikning chegaralanganligi haqidagi teoremadan foydalanib $\{x_n\}$ ketma – ketlikning uzoqlashuvchi ekanligi ko'rsatilsin.

3.1 $x_n = n$

3.2 $x_n = n\sin\frac{\pi n}{4}$

3.3 $x_n = \sqrt{n}\sin\frac{\pi n}{2}$

3.4 $x_n = (-1)^n \ln n$

3.5 $x_n = (-1)^{n+1}\ln\frac{2}{n}+2$

3.6 $x_n = \frac{1}{n}+n^{(-1)^n}$

Cheksiz kichik ketma – ketlikning chegaralangan ketma – ketlikka ko'paytmasi haqidagi teoremadan foydalanib $\{x_n\}$ ketma – ketlikning yaqinlashuvchi ekanligi ko'rsatilsin.

3.7 $x_n = \frac{\operatorname{sgn}(tgn)}{n}$

3.8 $x_n = \frac{\sin n}{n}$

3.9 $x_n = \frac{2+(-1)^n}{n^2}$

3.10 $x_n = \frac{\cos \pi n}{n\sqrt{n}}$

3.11 $x_n = \frac{\sin\frac{\pi n}{4}}{3n^3}$

3.12 $x_n = \frac{\cos n}{(n+1)}$

Tengsizliklarda limitga o'tish haqidagi 10 - teoremadan foydalanib $\{x_n\}$ ketma – ketlikning yaqinlashuvchiligi ko'rsatilsin.

3.13 $x_n = \frac{3^{(-1)^n}+2}{n}$

3.14 $x_n = \frac{1+(-1)^n n}{n^2}$

3.15 $x_n = \frac{2+(-1)^{\frac{n(n+1)}{2}}}{n!}$

3.16 $x_n = \frac{n+1}{n-1}\sin\frac{\pi n}{2}$

3.17 $x_n = \left(\frac{2}{3n}\right)^n$

3.18 $x_n = \left(\frac{n+\cos n}{2n-1}\right)^n$

3.19 $x_n = \frac{1}{n!}$

3.20 $x_n = \frac{\sqrt{n}+\sin n}{n\sqrt{n}}$

4 – topshiriq. Sonli ketma – ketlikning limiti hisoblansin. $\left(\lim\limits_{n\to\infty} - ?\right)$

4.1 $x_n = 2\left[\sqrt{n(n^2-2)} - \sqrt{n^3-1}\right]$

4.2 $x_n = \frac{(n-1)^{10} + (n-1)^{15}}{3n^{15}}$

4.3 $x_n = \sqrt{(n^2+2)(n^2-4)} - \sqrt{n^4-9}$

4.4 $x_n = \frac{\sqrt{n^5-8} - n\sqrt{n^3}}{\sqrt{n}}$

4.5 $x_n = \sqrt{n^2-3n+2} - n$

4.6 $x_n = n + \sqrt[3]{4-n^2}$

4.7 $x_n = \sqrt{n(n+2)} - \sqrt{n^2-2n+3}$

4.8 $x_n = \sqrt{(n+2)(n+1)} - \sqrt{(n-1)(n+3)}$

4.9 $x_n = n^2\left[\sqrt{n(n^4-1)} - \sqrt{n^5-8}\right]$

4.10 $x_n = \frac{n^3-3n^2+4}{n^3-5n^2}$

4.11 $x_n = \sqrt{n^2+3n-2} - \sqrt{n^2-3}$

4.12 $x_n = \sqrt{n}\left(\sqrt{n+2} - \sqrt{n-3}\right)$

4.13 $x_n = \sqrt{n(n+5)} - n$

4.14 $\sqrt{n^3+8}(\sqrt{n^3+2} - \sqrt{n^3-1})$

4.15 $x_n = \frac{\sqrt{(n^2+1)(n^2+2)} - \sqrt{n^4+2}}{2\sqrt{n^3}}$

4.16 $x_n = n - \sqrt{n(n-1)}$

4.17 $x_n = \left(n - \sqrt[3]{n^3-5}\right)$

4.18 $x_n = \sqrt[5]{n}\left[\sqrt[5]{n^2} - \sqrt[5]{n(n+2)}\right]$

4.19 $\sqrt{n+2}(\sqrt{n+3} - \sqrt{n-4})$

4.20 $x_n = \frac{\sqrt{n^2+1} - \sqrt{n^2+n}}{n - \sqrt{n^2-n}}$

4 - mustaqil ish

Mavzu: Monoton ketma - ketliklar va ularning limiti. Fundamental ketma – ketliklar. Qismiy ketma - ketliklar.

Monoton ketma - ketliklar

1 - ta'rif. Agar $\{x_n\}$ ketma - ketlikning hadlari $\forall n \in N$ uchun $x_n \le x_{n+1}$ $(x_n \ge x_{n+1})$ tengsizlikni qanoatlantirsa $\{x_n\}$ o'**suvchi (kamayuvchi)** ketma - ketlik deyiladi.

2 - ta'rif. O`suvchi va kamayuvchi ketma - ketliklar **monoton** ketma - ketliklar deb ataladi.

1 – teorema. Agar $\{x_n\}$ ketma—ketlik o'suvchi bo'lib, yuqoridan chegaralangan bo'lsa, u chekli limitga ega; agar $\{x_n\}$ ketma –ketlik yuqoridan chegaralanmagan bo'lsa, u holda $\{x_n\}$ ketma –ketlikning limiti $+\infty$ bo'ladi.

2 - teorema. Agar $\{x_n\}$ ketma—ketlik kamayuvchi bo'lib, quyidan chegaralangan bo'lsa, u chekli limitga ega; agar $\{x_n\}$ ketma –ketlik quyidan chegaralanmagan bo'lsa, u holda $\{x_n\}$ ketma –ketlikning limiti $-\infty$ bo'ladi.

Fundamental ketma - ketliklar

3-ta'rif. Agar $\forall \varepsilon > 0$ son olinganda ham $\exists n_0 = n_0(\varepsilon) \in N$ son mavjud bo'lsaki, $\forall n > n_0$ va $p \in N$ sonlar uchun $|x_{n+p} - x_n| < \varepsilon$ tengsizlik bajarilsa, $\{x_n\}$ **fundamental** ketma - ketlik deyiladi.

4 - ta'rif. (3 - ta'rifning inkori). $\forall n_0 \in N$ son olinganda ham shunday $n > n_0$, $p \in N$, $\varepsilon > 0$ sonlar mavjud bo'lib, $|x_{n+p} - x_n| \ge \varepsilon$ tengsizlik o'rinli bo'lsa, $\{x_n\}$ ketma – ketlik **fundamental emas** deyiladi.

3 –teorema (Koshi). Ketma - ketlikning yaqinlashuvchi bo'lishi uchun uning fundamental bo'lishi zarur va yetarlidir.

Qismiy ketma - ketliklar. Ketma - ketlikning yuqori va quyi limitlari

$\{x_n\}$ ketma - ketlik berilgan bo'lib, $n_1, n_2, ..., n_k, ...$ o'suvchi natural sonlar ketma - ketligi bo'lsin. $\{x_n\}$ ketma - ketlikning $n_1, n_2, ..., n_k, ...$ nomerli hadlaridan $x_{n_1}, x_{n_2}, ..., x_{n_k}, ...$ ketma - ketlikni tuzamiz. Hosil bo'lgan $\{x_{n_k}\}$ sonli ketma - ketlik $\{x_n\}$ ketma - ketlikning q**ismiy ketma - ketligi** deb ataladi.

5 –ta'rif. $\{x_n\}$ ketma -ketlik hadlaridan tuzilgan to'plamning aniq yuqori chegarasi (aniq quyi chegarasi) $\{x_n\}$ ketma –ketlikning aniq yuqori (aniq quyi) chegarasi deyiladi va $\sup\{x_n\}(\inf\{x_n\})$ kabi belgilanadi.

4 - teorema. Agar $\lim\limits_{n\to\infty} x_n = a$ bo'lsa, u holda uning har qanday qismiy ketma – ketligining limiti ham a teng bo'ladi.

5 - teorema. (Bolsano –Veyershtrass). Agar $\{x_n\}$ ketma - ketlik chegaralangan bo'lsa, u holda bu ketma - ketlikdan yaqinlashuvchi qismiy ketma - ketlik ajratish mumkin.

6 - ta'rif. $\{x_n\}$ ketma - ketlikning qismiy ketma - ketligi limiti $\{x_n\}$ ketma - ketlikning qismiy limiti deb ataladi.

7 - ta'rif. $\{x_n\}$ ketma - ketlik qismiy limitlarining eng kattasi (eng kichigi) berilgan ketma – ketlikning **yuqori (quyi)** limiti deyiladi va $\overline{\lim\limits_{n\to\infty}} x_n \left(\underline{\lim\limits_{n\to\infty}} x_n \right)$ ko'rinishda belgilanadi.

6 - teorema. $\lim\limits_{n\to\infty} x_n = a$ bo'lishi uchun $\overline{\lim\limits_{n\to\infty}} x_n = \underline{\lim\limits_{n\to\infty}} x_n = a$ bo'lishi zarur va yetarli.

*** 1– misol.** $x_n = \dfrac{2^n}{n!}$ **ketma – ketlik monoton ketma – ketlikning limiti haqidagi teoremadan foydalanib, yaqinlashishga tekshirilsin.**

◄ Monoton ketma – ketlikning limiti haqidagi teoremadan foydalanib, berilgan $\{x_n\}$ ketma – ketlikning yaqinlashuvchi ekanligini ko'rsatamiz.

$$\frac{x_{n+1}}{x_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} \le 1$$

Demak, bu ketma –ketlik monoton kamayuvchi.

Endi bu ketma – ketlikning quyidan chegaralanganligini ko'rsatamiz:

$$x_n = \frac{2^n}{n!} > 0, \qquad \forall n \in N$$

Shunday qilib, $\{x_n\}\downarrow$ va $\forall n \in N$ uchun $x_n > 0 \Rightarrow \lim\limits_{n\to\infty} x_n - \exists \Rightarrow \{x_n\}$- yaqinlashuvchi ►

2– misol. $\lim\limits_{n\to\infty} x_n - ?$

$$x_n = \left(\frac{2n^2 + 2n - 7}{2n^2 + n + 9} \right)^{3n+1}$$

◄ $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \left(\dfrac{2n^2 + 2n - 7}{2n^2 + n + 9} \right)^{3n+1} = \lim\limits_{n\to\infty} \left(\dfrac{2n^2 + n + 9 + n - 16}{2n^2 + n + 9} \right)^{3n+1} =$

$= \lim\limits_{n\to\infty} \left(1 + \dfrac{n-16}{2n^2 + n + 9} \right)^{3n+1} = ((\lim\limits_{n\to\infty} (1 + \dfrac{1}{n})^n = e$ tenglikdan foydalanamiz.)

$$\lim_{n\to\infty}\left(1+\frac{1}{\frac{2n^2+n+9}{n-16}}\right)^{\frac{2n^2+n+9}{n-16}\cdot\frac{n-16}{2n^2+n+9}\cdot(3n+1)}=e^{\lim\limits_{n\to\infty}\frac{n-16}{2n^2+n+9}\cdot(3n+1)}=e^{\frac{3}{2}}$$ ►

3– misol. $\{x_n\}$ **ketma – ketlikning yuqori va quyi limitlari topilsin ($\overline{\lim\limits_{n\to\infty}}\, x_n-?$ $\underline{\lim\limits_{n\to\infty}}\, x_n-?$)**

$$x_n=\frac{n+1}{n}\sin\frac{n\pi}{4}$$

◄ $\sin\frac{n\pi}{4}$ ning qiymatlarini tekshiramiz.

$n=1$ $\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$, $n=2$ $\sin\frac{2\pi}{4}=1$, $n=3$ $\sin\frac{3\pi}{4}=\frac{\sqrt{2}}{2}$

$n=4$ $\sin\pi=0$, $n=5$ $\sin\frac{5\pi}{4}=-\frac{\sqrt{2}}{2}$ $n=6$ $\sin\frac{3\pi}{2}=-1$

$n=7$ $\sin\frac{7\pi}{4}=-\frac{\sqrt{2}}{2}$ $n=8$ $\sin 2\pi=0$ $n=9$ $\sin\frac{9\pi}{4}=\frac{\sqrt{2}}{2}$

Berilgan ketma – ketlikning qiymatlari to'plamida $\sin\frac{n\pi}{4}$ ning $\frac{\sqrt{2}}{2},1,\frac{\sqrt{2}}{2},0,-\frac{\sqrt{2}}{2},-1,-\frac{\sqrt{2}}{2},0,...$ qiymatlari cheksiz ko'p uchraydi, Demak, $\{x_{8k}\},\{x_{8k-1}\},\{x_{8k-2}\},\{x_{8k-3}\},\{x_{8k-4}\},\{x_{8k-5}\},\{x_{8k-6}\},\{x_{8k-7}\}$ ning har biri berilgan ketma – ketlikning yaqinlashuvchi qismiy ketma-ketligi bo'ladi. Bu ketma-ketliklarning limitlarini hisoblaymiz

$$x_{8k}=\frac{8k+2}{8k}\cdot 0=0,\quad x_{8k-1}=\frac{8k}{8k-1}\cdot\left(-\frac{\sqrt{2}}{2}\right)=-\frac{4\sqrt{2}k}{8k-1},\quad x_{8k-2}=-\frac{8k-1}{8k-2}$$

$$x_{8k-3}=-\frac{\sqrt{2}}{2}\cdot\frac{8k-2}{8k-3},\quad x_{8k-4}=\frac{8k-3}{8k-4}\cdot 0=0,\quad x_{8k-5}=\frac{\sqrt{2}}{2}\cdot\frac{8k-4}{8k-5}$$

$$x_{8k-6}=\frac{8k-5}{8k-6}\qquad x_{8k-7}=\frac{\sqrt{2}}{2}\frac{8k-6}{8k-7}$$

$$\lim_{k\to\infty}x_{8k}=0,\ \lim_{k\to\infty}x_{8k-1}=-\frac{\sqrt{2}}{2},\ \lim_{k\to\infty}x_{8k-2}=-1,\ \lim_{k\to\infty}x_{8k-3}=-\frac{\sqrt{2}}{2},\ \lim_{k\to\infty}x_{8k-4}=0,\ \lim_{k\to\infty}x_{8k-5}=\frac{\sqrt{2}}{2},$$

$$\lim_{k\to\infty}x_{8k-6}=1,\ \lim_{k\to\infty}x_{8k-7}=\frac{\sqrt{2}}{2},$$

Demak, $\overline{\lim\limits_{n\to\infty}}\, x_n=1,\qquad \underline{\lim\limits_{n\to\infty}}\, x_n=-1$ ►

4- misol. Ketma –ketlikning aniq yuqori va aniq quyi chegaralarini, yuqori va quyi limitlarini toping.

$$x_n = \frac{2n+1}{n} \cdot \cos\frac{2n\pi}{3}$$

$\cos\frac{2n\pi}{3}$ ning qiymatlarini tekshiramiz.

$n=1 \quad \cos\frac{2\pi}{3} = -\frac{1}{2}, \qquad n=2 \quad \cos\frac{4\pi}{3} = -\frac{1}{2}, \qquad n=3 \quad \cos 2\pi = 1$

$n=4 \quad \cos\frac{8\pi}{3} = -\frac{1}{2}$

Berilgan ketma – ketlikning qiymatlari to'plamida $\cos\frac{2n\pi}{3}$ ning $-\frac{1}{2}, -\frac{1}{2}, 1, -\frac{1}{2}, -\frac{1}{2}, 1, \ldots$ qiymatlari cheksiz ko'p uchraydi, Demak,

$\{x_{3k}\}, \{x_{3k-1}\}, \{x_{3k-2}\}$ ketma-ketliklarning har biri berilgan ketma – ketlikning yaqinlashuvchi qismiy ketma-ketligi bo'ladi. Bu ketma- ketliklarning limitlarini hisoblaymiz

$$x_{3k} = 2 + \frac{1}{3k}, \quad x_{3k-1} = -1 - \frac{1}{2(3k-1)}, \quad x_{3k-2} = -1 - \frac{1}{2(3k-2)},$$

$$\lim_{k\to\infty} x_{3k} = 2, \quad \lim_{k\to\infty} x_{3k-1} = -1, \quad \lim_{k\to\infty} x_{3k-2} = -1,$$

$$\inf\{x_{3k}\} = 2, \quad \inf\{x_{3k-1}\} = -\frac{5}{4}, \inf\{x_{3k-2}\} = -\frac{3}{2}$$

$$\sup\{x_{3k}\} = \frac{7}{3}, \quad \sup\{x_{3k-1}\} = -1, \quad \sup\{x_{3k-2}\} = -1 .$$

Demak $\inf\{x_n\} = -\frac{3}{2}, \sup\{x_n\} = \frac{7}{3}, \quad \overline{\lim_{n\to\infty}}\, x_n = 2, \quad \underline{\lim_{n\to\infty}}\, x_n = -1$ ▶

Misol. $x_n = \frac{\cos 1}{3} + \frac{\cos 2}{3^2} + \ldots + \frac{\cos n}{3^n}, \quad n \in N$ ketma-ketlik yaqinlashuvchi ekanini isbotlang.

Bu ketma-ketlikni Koshi kriteriyasi bo'yicha yaqinlashishga tekshiramiz. Quydagi ayirma modilini baholaymiz:

$$|x_{n+p} - x_n| - \left|\frac{\cos(n+1)}{3^{n+1}} + \ldots + \frac{\cos(n+p)}{3^{n+p}}\right| \le \frac{1}{3^{n+1}} + \ldots + \frac{1}{3^{n+p}} = \frac{1}{3^{n+1}} \cdot \frac{1-\frac{1}{3^p}}{1-\frac{1}{3}} < \frac{1}{2\cdot 3^n} < \frac{1}{3^n}$$

ε -ixtiyoriy son bo'lsin $\lim\limits_{n\to\infty}\frac{1}{3^n}=0$ ligidan, yuqoridagi ε son uchun shunday N mavjudki, istalgan $n\geq N$ da $\frac{1}{3^n}<\varepsilon$ o'rinli . demak , agar $n\geq N$ va p - ixtiyoriy natural son bo'lsa, $|x_{n+p}-x_n|<\frac{1}{3^n}<\varepsilon$ bajariladi.

Shunday qilib , Koshi kriteriyasi sharti bajarildi va shuning uchun ketma-ketlik yaqinlashuvchi.

Mustaqil yechish uchun misollar.

1 –topshiriq. Koshi kriteriyasidan foydalanib $\{x_n\}$ ketma - ketlik yaqinlashishga tekshirilsin.

1.1 $x_n=\frac{\operatorname{sgn}(\sin n)}{n}$ 1.2 $x_n=\frac{1}{(n+1)^2}$

1.3 $x_n=\frac{n^2}{n-n^2}$ 1.4 $x_n=\frac{|\cos 1|}{2}+\frac{|\cos 2|}{2^2}+...+\frac{|\cos n|}{2^n}$

1.5 $x_n=\frac{1+\sin\left(\frac{\pi n}{4}\right)}{n^3}$ 1.6 $x_n=\frac{n+1}{n^2\cdot(4+\cos n)}$

1.7 $x_n=\cos 1+\frac{\cos 2}{2^2}+...+\frac{\cos n}{n^2}$ 1.8 $x_n=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{n}}$

1.9 $x_n=\frac{\sin\alpha}{2}+\frac{\sin 2\alpha}{2^2}+...+\frac{\sin n\alpha}{2^n}$ 1.10 $x_n=\frac{arctgn}{n}$

1.11 $x_n=\frac{1}{2^2}+\frac{2}{3^2}+...+\frac{n}{(n+1)^2}$ 1.12 $x_n=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}$

Monoton ketma – ketlikning limiti haqidagi teoremadan foydalanib $\{x_n\}$ ketma - ketlik yaqinlashishga tekshirilsin.

1.13 $x_n=\frac{n!}{n^n}$ 1.14 $x_n=\left(1+\frac{1}{n}\right)^n$

1.15 $x_n=\left(1-\frac{1}{2}\right)\cdot\left(1-\frac{1}{2^2}\right)\cdot...\cdot\left(1-\frac{1}{2^n}\right)$ 1.16 $x_n=\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{2^2}\right)\cdot...\cdot\left(1+\frac{1}{2^n}\right)$

1.17 $x_n=\frac{a^n}{n!},\ a>0$ 1.18 $x_n=p_0+\frac{p_1}{10}+\frac{p_2}{10^2}+...+\frac{p_n}{10^n},\ |p_k|<10, k=0,1,2,...$

1.19 $x_n=\frac{3^n}{n!}$ 1.20 $x_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{n(n+1)}$

2 – topshiriq. $\lim\limits_{n\to\infty}x_n-?$

2.1 $x_n=\left(\frac{2n^2-2n+1}{2n^2+1}\right)^n$ 2.2 $x_n=\left(\frac{3n^2+2n}{3n^2+1}\right)^{n^2}$

2.3 $x_n = \left(\frac{3n-1}{3n+1}\right)^{n+1}$

2.4 $x_n = \left(\frac{2n^2+1}{2n^2}\right)^{n}$

2.5 $x_n = \left(\frac{2n+3}{2n+1}\right)^{n+1}$

2.6 $x_n = \left(\frac{n+1}{n-1}\right)^{n}$

2.7 $x_n = \left(\frac{n^2-3n+6}{n^2+5n+1}\right)^{n^2}$

2.8 $x_n = \left(\frac{n-10}{n+1}\right)^{3n+1}$

2.9 $x_n = \left(\frac{6n-7}{6n+4}\right)^{3n+2}$

2.10 $x_n = \left(\frac{3n^2+4n-1}{3n^2+2n+7}\right)^{2n+5}$

2.11 $x_n = \left(\frac{n^2+n+1}{n^2+n-1}\right)^{-n^2}$

2.12 $x_n = \left(\frac{n^2+n+1}{n^2+n-1}\right)^{n}$

2.13 $x_n = \left(\frac{n-1}{n+1}\right)^{n^2}$

2.14 $x_n = \left(\frac{4n^2+3n-1}{4n^2+2n+3}\right)^{n}$

2.15 $x_n = \left(\frac{3n+1}{3n-1}\right)^{2n+3}$

2.16 $x_n = \left(\frac{2n^2+7n-1}{2n^2+3n-1}\right)^{-n^2}$

2.17 $x_n = \left(\frac{n+3}{n+5}\right)^{n+4}$

2.18 $x_n = \left(\frac{n^3+1}{n^3-1}\right)^{2n-n^3}$

2.19 $x_n = \left(\frac{5n-1}{5n+1}\right)^{-n}$

2.20 $x_n = \left(\frac{3n^2-5n}{3n^2-5n+1}\right)^{n^2+1}$

3 – topshiriq. $\{x_n\}$ ketma – ketlikning yuqori va quyi limitlari topilsin ($\overline{\lim\limits_{n\to\infty}}\, x_n - ?,\ \underline{\lim\limits_{n\to\infty}}\, x_n - ?$).

3.1 $x_n = 2 + \frac{n}{n+1}\cos\frac{\pi n}{2}$

3.2 $x_n = \frac{1+(-1)^n n}{n}$

3.3 $x_n = (-1)^{\sin\frac{\pi n}{2}} n$

3.4 $x_n = \frac{1}{n^2} + 2\sin\frac{\pi n}{3}$

3.5 $x_n = \frac{\left|1-(-1)^n\right|\cdot 2^n + 1}{2^n + 3}$

3.6 $x_n = \sqrt[n]{4^{(-1)^n} + 2}$

3.7 $x_n = 2^{(-1)^n} n$

3.8 $x_n = (-1)^{n-1}\left(1 + \frac{2}{n+1}\right)$

3.9 $x_n = (\cos\frac{\pi n}{2})^{n+1} + \frac{1}{n}$

3.10 $x_n = \frac{n}{n+1}\sin^2\frac{\pi n}{4}$

3.11 $x_n = \frac{n-1}{n+1}\cos n\pi$

3.12 $x_n = \frac{n^2+1}{n^2-1}\sin^n \frac{\pi n}{2}$

3.13 $x_n = \frac{3n-2}{n}\sin\frac{\pi n}{2}$

3.14 $x_n = \left(1{,}5\cdot\cos\frac{2\pi n}{3}\right)^n$

3.15 $x_n = \frac{n^2\sin\frac{\pi n}{2}+1}{n+1}$

3.16 $x_n = (-1)^{\frac{n(n+1)}{2}}\frac{n^2+1}{1-n^2}$

3.17 $x_n = \frac{n^2-n+1}{2n^2-1}\cos\frac{2\pi n}{3}$

3.18 $x_n = \frac{2n^2}{2-n+n^2}\sin\left(\frac{2\pi n}{3}\right)$

3.19 $x_n = \left(\frac{n+1}{n}\right)^n(-1)^{n+1}+\cos\frac{\pi n}{2}$

3.20 $x_n = \frac{n}{n+1}\cos\frac{\pi n}{4}$

4- topshiriq. **Ketma –ketlikning aniq yuqori va aniq quyi chegaralarini, yuqori va quyi limitlarini toping.**

4.1 $x_n = 1-\frac{(-1)^n}{n^2}$

4.2 $x_n = 2-\frac{(-1)^{n-1}}{n+1}$

4.3 $x_n = (-1)^{\frac{(n+1)n}{2}}\frac{n^2+1}{n^3}$

4.4 $x_n = 1+2\cdot(-1)^n-4\cdot(-1)^{\frac{n(n+1)}{2}}$

4.5 $x_n = \frac{n^2}{2^n}$

4.6 $x_n = \frac{\sqrt{n}}{n+64}$

4.7 $x_n = \frac{n}{n+1}\cdot\cos\frac{n\pi}{2}$

4.8 $x_n = \frac{n+1}{n^2}\cdot\cos\frac{n\pi}{3}$

4.9 $x_n = \sqrt[n]{2+3^{n\cdot(-1)^{n+1}}}$

4.10 $x_n = \sqrt[n]{1-3^{n\cdot(-1)^{n+1}}}$

4.11 $x_n = \left(1+\frac{1}{n}\right)^n\cdot(-1)^n$

4.12 $x_n = \left(1+\frac{1}{n}\right)^n\cdot(-1)^n+\sin\frac{n\pi}{4}$

4.13 $x_n = \left(1+\frac{1}{n}\right)^{-n}\cdot(-1)^n+\sin\frac{2n\pi}{3}$

4.14 $x_n = \left(\frac{n+1}{n}\right)^n\cdot\cos n\pi$

4.15 $x_n = 1+\frac{n+1}{n+2}\cdot\sin\frac{n\pi}{2}$

4.16 $x_n = 1-\frac{3n+4}{n+3}\cdot\cos n\pi$

4.17 $x_n = 1+n\cdot\sin\frac{n\pi}{2}$

4.18 $x_n = \frac{4n+2}{3n-1}\cdot\cos\frac{n\pi}{4}$

4.19 $x_n = \left(4+\frac{5}{n^2}\right)\cdot\sin\left(\frac{\pi}{2}+n\pi\right)$

4.20 $x_n = \left(1-\frac{2}{n^3}\right)\cdot\sin\frac{2n\pi}{3}$

5 –mustaqil ish.
Mavzu: Funksiya tushunchasi. Funksiya limiti

Bizga biror $X \subset R$ to'plam berilgan bo'lib, x o'zgaruvchi miqdor X to'plamdan olingan bo'lsin. Agar har bir $x \in X$ songa biror qonun yoki qoidaga ko'ra bitta y son mos qo'yilsa, u holda X to'plamda **funksiya** aniqlangan deyiladi va $y = f(x)$ kabi belgilanadi, x o'zgaruvchiga **erkli o'zgaruvchi** (yoki funksiyaning argumenti), X to'plam $f(x)$ funksiyaning **aniqlanish sohasi**, x soniga mos keluvchi y soniga esa funksiyaning x nuqtadagi **xususiy qiymati** deb ataladi. $f(x)$ funksiyaning barcha xususiy qiymatlar to'plami Y ga $f(x)$ funksiyaning **qiymatlar to'plami** (yoki **o'zgarish sohasi**) deyiladi. Shunday qilib,

$$Y = \{y \in R : y = f(x), x \in X\}$$

Agar a $(a \in X$ yoki $a \notin X)$ nuqtaning ixtiyoriy atrofida X to'plamniig a dan farqli kamida bitta nuqtasi bo'lsa, u holda a nuqta X to'plamning **limit nuqtasi** deyiladi. Bundan keyin $X - f(x)$ funksiyaning aniqlanish sohasi, a nuqta X to'plamning limit nuqtasi deb tushuniladi.

1-ta'rif (Koshi). Agar $\forall \varepsilon > 0$ uchun $\exists \delta = \delta(\varepsilon, a) > 0$ topilsaki, $0 < |x - a| < \delta$ tengsizlikni qanoatlantiruvchi $\forall x \in X$ uchun $|f(x) - b| < \varepsilon$ tengsizlik bajarilsa, u holda b soni $f(x)$ funksiyaning a **nuqtadagi limiti** deyiladi va $\lim\limits_{x \to a} f(x) = b$ kabi belgilanadi.

2 - ta'rif (Geyne). Agar X to'plamning nuqtalaridan tuzilgan, a ga intiluvchi $\forall \{x_n\}$ $(x_n \neq a, n = 1,2,,...,)$ ketma - ketlik olinganda ham $\{f(x_n)\}$ ketma - ketlik hamma vaqt yagona b soniga intilsa, shu b soni $f(x)$ funksiyaning a **nuqtadagi limiti** deb ataladi.

Keltirilgan ta'riflardan ko'rinib turibdiki, funksiyaning a nuqtadagi limiti mavjud bo'lishi uchun funksiya a nuqtada aniqlangan bo'lishi, ya'ni $a \in X$ bo'lishi, mutlaqo shart emas (a nuqtaning X to'plam uchun limit nuqta bo'lishi yetarli, ya'ni, umuman olganda, $a \notin X$)

Endi 1 - va 2 - ta'riflarga teskari ta'riflarni keltiramiz.

1 - ta'rifning inkori. Agar $\exists \varepsilon > 0$ topilsaki, $\forall \delta > 0$ uchun $0 < |x - a| < \delta$ tengsizlikni qanoatlantiruvchi $\exists x \in X$ mavjud bo'lib, $|f(x) - b| \geq \varepsilon$ tengsizlik bajarilsa, b soni $f(x)$ funksiyaning a **nuqtadagi limiti emas** deyiladi $(\lim\limits_{x \to a} f(x) \neq b)$

2 - ta'rifning inkori. Agar a nuqtaga intiluvchi $\exists \{x_n\}$ $(x_n \in X,\, x_n \neq a,\, n = 1,2,...)$ ketma - ketlik topilsaki, unga mos $\{f(x)\}$ ketma - ketlik b ga intilmasa, u holda b son $f(x)$ funksiyaning a **nuqtadagi limiti emas** deyiladi.

1 - teorema. Funksiya limitining 1 - va 2 – ta'riflari ekvivalentdir.

Biz 1 - teoremadan quyidagi xulosani chiqaramiz:
funksiyaning limitini hisoblayotganda qaysi ta'rif bo'yicha hisoblash oson va qulay bo'lsa, shu ta'rifdan foydalanish kerak.

Ba'zi bir hollarda $f(x)$ funksiyaning a nuqtadagi limiti mavjud bo'lmaydi. Ana shunday hollarda funksiyaning nuqtadagi **bir tomonli (o'ng va chap)** limitlari to'g`risida gap yuritiladi.

3 – ta'rif (Koshi). $\forall\varepsilon>0$ uchun $\exists\delta=\delta(a,\varepsilon)>0$ topilsaki, $a<x<a+\delta$ $(a-\delta<x<a)$ tengsizlikni qanoatlantiruvchi $\forall x\in X$ uchun $|f(x)-b|<\varepsilon$ tengsizlik bajarilsa, b son $f(x)$ funksiyaning a nuqtadagi **o'ng (chap) limiti** deb ataladi va

$$\lim_{x\to a+0} f(x)=f(a+0)=b \ \ (\lim_{x\to a-0} f(x)=f(a-0)=b)$$

kabi belgilanadi.

4 - ta'rif (Geyne), a nuqtaga intiluvchi $\forall\{x_n\},\ x_n\in X, x_n>a\ \ (x_n<a)$ ketma - ketlik olinganda ham unga mos $\{f(x_n)\}$ ketma – ketlik hamma vaqt yagona b soniga intilsa, shu b soni $f(x)$ funksiyaning a nuqtadagi **o'ng (chap) limiti** deyiladi.

2 - teorema. $\lim\limits_{x\to a} f(x)=b$ bo'lishi uchun $f(a+0)=f(a-0)=b$ tenglikning bajarilishi zarur va yetarli

Endi funksiyaning $x\to+\infty$ dagi limiti ta'rifini beramiz. $f(x)$ funksiya $(c,+\infty)$ cheksiz oraliqda aniqlangan bo'lsin.

5 - ta'rif. (Koshi). $\forall\varepsilon>0$ uchun $\exists A>0$ topilsaki, $\forall x>A$ uchun $|f(x)-b|<\varepsilon$ tengsizlik bajarilsa, b son $f(x)$ funksiyaning $x\to+\infty$ **dagi limiti** deyiladi va $\lim\limits_{x\to+\infty} f(x)=b$ kabi belgilanadi.

6 - ta'rif. (Geyne). $+\infty$ ga intiluvchi $\forall\{x_n\}$ $(x_n>c)$ ketma - ketlik uchun unga mos $\{f(x_n)\}$ ketma - ketlik b soniga intilsa, b soni $f(x)$ funksiyaning $x\to+\infty$ **dagi limiti** deb ataladi.

3 - va 4 - ta'riflar ham da 5 - va 6 - ta'riflar bir - biriga ekvivalent, $\lim\limits_{x\to-\infty} f(x)=b$ ning ta'rifi ham yuqoridagiga o'xshash aniqlanadi. Agar $\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to+\infty} f(x)=b$ bo'lsa, u holda $\lim\limits_{x\to\infty} f(x)=b$ deb yoziladi.

7 - ta'rif. Agap $\lim\limits_{x\to a} f(x)=\infty$ $(\lim\limits_{x\to a} f(x)=0)$ bo'lsa, $f(x)$ funksiya a nuqtada **cheksiz katta (cheksiz kichik) funksiya** deyiladi.

Cheksiz katta va cheksiz kichik funksiyalar ham cheksiz katta va cheksiz kichik ketma - ketliklar uchun keltirilgan xossalarga ega.

Limitga ega bo'lgan funksiyalarning xossalari

8 - ta’rif. Ushbu $\dot{U}_\delta(a) = \{x \in R : 0 < |x - a| < \delta\}$ to’plam a nuqtaning **o’yilgan δ atrofi** deb ataladi.

3– teorema. $f(x)$ va $g(x)$ funksiyalar a nuqtaning biror o’yilgan atrofida aniqlangan bo’lib, $\lim\limits_{x\to a} f(x) = b$ va $\lim\limits_{x\to a} g(x) = c$ bo’lsin. U holda

1) $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = b \pm c$

2) $\lim\limits_{x\to a}[f(x) \cdot g(x)] = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = b \cdot c$

3) agar $c \neq 0$ bo’lsa, $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{b}{c}$ bo’ladi.

4 - teorema. Agar $f(x)$, $g(x)$ va $h(x)$ funksiyalar a nuqtaning biror o’yilgan atrofida aniqlangan bo’lib, shu atrofda $f(x) \le g(x) \le h(x)$ tengsizlikni qanoatlantirsa va $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = b$ tenglik bajarilsa, u holda $\lim\limits_{x\to a} g(x) = b$ bo’ladi.

Funksiya limitini hisoblashda quyidagi ajoyib limitlar katta ahamiyatga ega.

Birinchi ajoyib limit:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Ikkinchi ajoyib limit:

$$\lim_{x\to\infty} (1 + \frac{1}{x})^x = e$$

1) Birinchi ajoyib limitni isbotlaymiz:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad (1)$$

Avvalo $0 < x < \frac{\pi}{2}$ intervaldan olingan barcha x lar uchun $\sin x < x < tgx$ tengsizliklar o’rinli. Bu maktab matematikasidan ma’lum. $\sin x > 0$ bo’lgani uchun bu tengsizliklarni $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ ko’rinishda yozish mumkin. Undan

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x \qquad (2)$$

tengsizliklar kelib chiuqadi.

Biz (2) tengsizlikni ixtiyoriy $x \in \left(0, \frac{\pi}{2}\right)$ uchun isbot qildik. $\frac{\sin x}{x} (x \neq 0)$ va $\cos x$ funksiyalarning juftligidan bu tengsizlikning barcha $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}$ uchun to’g’riligini topamiz. Shu bilan birga $0 < |x| < \frac{\pi}{2}$ da

$1-\cos x = 2\sin^2\frac{x}{2} \le 2\cdot\frac{|x|}{2} = |x|$ tengsizlikning o'rinli bo'lishini e'tiborga olsak, yuqoridagi (2) tengsizlik quyidagi $0 < \left|1-\frac{\sin x}{x}\right| < |x|$ ko'rinishga kelishini topamiz.

Agar $\forall\varepsilon > 0$ son berilganda ham δ. deb ε va $\frac{\pi}{2}$ sonlarning kichigi olinsa, argument x ning $0<|x|<\delta$ tengsizliklarni qanoatlantiruvchi barcha qiymatlarida

$$\left|1-\frac{\sin x}{x}\right| = \left|\frac{\sin x}{x}-1\right| < \varepsilon$$

tengsizlik o'rinli bo'ladi. Bu esa funksiya limitining Koshi ta'rifiga ko'ra (1) limitning to'g'riligini anglatadi.

2) Ikkinchi ajoyib limitni isbotlaymiz:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \qquad (3)$$

(bunda *e=2.71…*).

Buning uchun $+\infty$ ga intiluvchi ixtiyoriy $\{x_k\}$ ketma-ketlikni olaylik. Bu holda barcha *k=1,2,3,…* lar uchun $x_k > 1$ deb qarash mumkin. Har bir x_k ning butun qismini n_k orqali belgilab, ushbu $\{x_k\} = n_k (k=1,2,3,...)$ $+\infty$ ga intiluvchi $n_1, n_2, ..., n_k, ...$ natural sonlar ketma-ketligini hosil qilamiz.

Ma'lumki,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \quad .$$

Bu munosabatdan

$$\lim_{x_k\to\infty}\left(1+\frac{1}{n_k}\right)^{n_k} = e$$

ekani kelib chiqadi.

Endi ushbu

$\{x_k\} = n_k \Rightarrow n_k \le x_k < n_k+1 \Rightarrow \frac{1}{n_k+1} < \frac{1}{x_k} \le \frac{1}{n_k} \Rightarrow 1+\frac{1}{n_k+1} < 1+\frac{1}{x_k} \le 1+\frac{1}{n_k}$ munosabatlar o'rinli bo'lishini e'tiborga olib, topamiz:

$$\left(1+\frac{1}{n_k+1}\right)^{n_k} < \left(1+\frac{1}{x_k}\right)^{x_k} < \left(1+\frac{1}{n_k}\right)^{n_k+1} \qquad (4)$$

Biroq

$$\lim_{x_k\to+\infty}\left(1+\frac{1}{n_k+1}\right)^{n_k} = \lim_{x_k\to+\infty}\left[\left(1+\frac{1}{n_k+1}\right)^{n_k+1}\left(1+\frac{1}{n_k+1}\right)^{-1}\right] = e,$$

$$\lim_{x_k\to+\infty}\left(1+\frac{1}{n_k}\right)^{n_k+1} = \lim_{x_k\to+\infty}\left[\left(1+\frac{1}{n_k}\right)^{n_k}\left(1+\frac{1}{n_k}\right)\right] = e$$

limitlar o'rinli bo'lgani uchun (4) tengsizliklarda (bunda $x_k \to +\infty$) limitga o'tsak izlangan (3) limit hosil bo'ladi.

Endi $-\infty$ ga intiluvchi ixtiyoriy $\{x_k\}$ ketma-ketlikni olaylik. Bunda $x_k, (-1, k = 1,2,...)$ deb qarash mumkin. Agar $y_k = -x_k$ deb belgilasak, unda $y_k \to +\infty$ va $y_k > 1 (k = 1,2,...)$ bo'ladi. Ravshanki,

$$\left(1+\frac{1}{x_k}\right)^{x_k} = \left(1-\frac{1}{y_k}\right)^{-y_k} = \left(\frac{y_k}{y_k-1}\right)^{y_k} = \left(1+\frac{1}{y_k-1}\right)^{y_k}.$$

Undan

$$\lim_{x_k\to\infty}\left(1+\frac{1}{x_k}\right)^{x_k} = \lim_{y_k\to+\infty}\left[\left(1+\frac{1}{y_k-1}\right)^{y_k-1}\left(1+\frac{1}{y_k-1}\right)\right] = e.$$

Shunday qilib $-\infty$ ga intiluvchi har qanday $\{x_k\}$ ketma-ketlik olganda ham $f(x) = \left(1+\frac{1}{x}\right)^x$ funksiya qiymatlaridan tuzilgan $\{f(x_k)\} = \left\{\left(1+\frac{1}{x_k}\right)^{x_k}\right\}$ ketma-ketlik hamma vaqt e limitga ega ekani isbotlandi. Funksiya limitining Geyne ta'rifiga ko'ra $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ limit ham o'rinli bo'ladi.

3) Quyidagi

$$\lim_{x\to 0}(1+\mu x)^{\frac{1}{x}}, \quad 0 \neq \mu \in R$$

limitni hisoblang. Biz buni $\lim\limits_{x\to 0}(1+\mu x)^{\frac{1}{x}} = \lim\limits_{x\to 0}\left[(1+\mu x)^{\frac{1}{x}}\right]^{\mu}$ ko'rinishda yozib olamiz.

Ravshanki, $x \to 0$ da $y = \mu x \to 0$ bo'ladi. Bundan quyidagini topamiz:

$$\lim_{x\to 0}\left[(1+\mu x)^{\frac{1}{\mu x}}\right]^{\mu} = \left[\lim_{x\to 0}(1+y)^{\frac{1}{y}}\right]^{\mu} = e^{\mu}.$$

Shu misoldan foydalanib $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ limitni ham hisoblash mumkin. Unda $0 \neq x \in R$. Ravshanki, $\frac{x}{n} \in R$ da $n \to \infty$ da $\frac{x}{n} = y \to 0$.

Shuning uchun

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{y\to 0}\left[(1+y)^{\frac{1}{y}}\right]^x = \left[\lim_{y\to 0}(1+y)^{\frac{1}{y}}\right]^x = e^x.$$

4) Quyidagi limitlarni isbotlaymiz.

a) $\lim\limits_{x\to 0}\frac{\log_a(1+x)}{x} = \log_a e \quad (a > 0, a \neq 1);$

b) $\lim\limits_{x\to 0}\frac{a^x - 1}{x} = \ln a \quad (a > 0, a = 1);$

c) $\lim\limits_{x\to 0}\frac{(1+x)^{\alpha} - 1}{x} = \alpha$

Bu munosabatlarni isbotlashda logarifimik,ko'rsatkichli va darajali funksiyalarning uzluksizligidan foydalanamiz. Darhaqiqat,

a) holda

$$\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \lim_{x\to 0}\log_a(1+x)^{\frac{1}{x}} = \log_a\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right] = \log_a e$$

b) holda esa $a^x - 1 = t$ deb, $x \to 0$ da $t \to 0$ bo'lishini e'tiborga olib topamiz:

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{x\to 0}\frac{t}{\log_a(1+t)}=\frac{1}{\log_a\left[\lim\limits_{t\to 0}(1+t)^{1/t}\right]}=\frac{1}{\log_a e}=\ln a;$$

c) holda $(1+x)^\alpha - 1 = t$ deb, so'ngra $\alpha = \dfrac{\ln(1+t)}{\ln(1+x)}$ va $x \to 0$ da $t \to 0$ bo'lishini hisobga olsak,

$$\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\lim_{x\to 0}\frac{t}{x}=\lim_{x\to 0}\left[\frac{t}{\ln(1+t)}\cdot\frac{\ln(1+t)}{\ln(1+x)}\cdot\frac{\ln(1+x)}{x}\right]=\alpha$$

kelib chiqadi.

Funksiya limiti uchun Koshi teoremasi

$f(x)$ funksiya X to'plamda berilgan bo'lib, a nuqta X to'plamning limit nuqtasi bo'lsin.

9 -ta'rif. Agar $\forall\varepsilon > 0$ uchun $\exists\delta > 0$ topilsaki, argument x ning $0 < |x' - a| < \delta$, $0 < |x'' - a| < \delta$ tengsizlikni qanoatlantiruvchi $\forall x', x''(x' \in X, x'' \in X)$ qiymatlarida $|f(x'') - f(x')| < \varepsilon$ tengsizlik o'rinli bo'lsa, *f(x)* funksiya uchun *a* **nuqtada Koshi sharti bajariladi** deyiladi.

9-ta'rifning inkori. Agar $\forall\delta > 0$ son olganimizda ham $\exists\varepsilon > 0$ va $0 < |x' - a| < \delta$, $0 < |x'' - a| < \delta$ tengsizliklarni qanoatlantiruvchi $x', x'' \in X$ lar mavjud bo'lib, $|f(x'') - f(x')| \geq \varepsilon$ tengsizlik bajarilsa, *f(x)* funksiya uchun *a* **nuqtada Koshi sharti bajarilmaydi** deyiladi.

5 -teorema. (Koshi). *f(x)* funksiya *a* nuqtada chekli limitga ega bo'lishi uchun bu funksiyaning *a* nuqtada Koshi shartini bajarishi zarur va yetarlidir.

*** 1– misol. $y = f(x)$ funksiyaning aniqlanish sohasi topilsin ($D(f) - ?$).**

$$y=\frac{\sin(2x-1)}{\sqrt[3]{x^2-3x+1}}$$

$$D(f):\left\{x^2-3x+1\neq 0\Rightarrow\left\{\left(x-\frac{3-\sqrt5}{2}\right)\left(x-\frac{3+\sqrt5}{2}\right)\neq 0\Rightarrow\right.\right.$$

$$\Rightarrow\left\{x\in\left(-\infty,\frac{3-\sqrt5}{2}\right)\cup\left(\frac{3-\sqrt5}{2},\frac{3+\sqrt5}{2}\right)\cup\left(\frac{3+\sqrt5}{2},+\infty\right)\Rightarrow\right.$$

$$\Rightarrow\left(D(f)=\left(-\infty,\frac{3-\sqrt5}{2}\right)\cup\left(\frac{3-\sqrt5}{2},\frac{3+\sqrt5}{2}\right)\cup\left(\frac{3+\sqrt5}{2},+\infty\right)\right.$$

2– misol. Quyidagi $\lim\limits_{x\to 4}\dfrac{x^2-16}{x^2-4x}=2$ tenglik ta'rif yordamida isbotlansin ($\delta(\varepsilon)$ topilsin).

◀ $(\lim_{x\to a} f(x)=b) \Leftrightarrow (\forall\varepsilon>0\ \exists\delta(\varepsilon)>0 : 0<|x-a|<\delta \Rightarrow |f(x)-b|<\varepsilon)$

$f(x)$ funksiyani $x=4$ nuqtaning biror atrofida, masalan (3; 5) intervalda qaraymiz: $\forall\varepsilon>0$ son olamiz va $|f(x)-2|$ ayirmani $x\neq 4$ da quyidagi ko'rinishga keltiramiz:

$$|f(x)-2|=\left|\frac{x^2-16}{x^2-4x}-2\right|=\left|\frac{x+4}{x}-2\right|=\left|\frac{4-x}{x}\right|=\frac{|x-4|}{|x|}<\frac{|x-4|}{3}$$

chunki $x\in(3;5)$ edi.

Oxiri tengsizlikdan ko'rinib turibdiki, agar $\delta=3\varepsilon$ deb olsak, $0<|x-4|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x\in(3;5)$ uchun

$$|f(x)-2|=\frac{|x-4|}{3}<\frac{\delta}{3}=\frac{3\varepsilon}{3}=\varepsilon$$

bo'ladi. Bu yerdan ta'rifga ko'ra $\lim_{x\to 4}\frac{x^2-16}{x^2-4x}=2$ ekanligini hosil qilamiz.▶

3– misol. $\lim_{x\to 0}\frac{\sqrt[3]{1-x}-\sqrt[3]{1+x}}{\sqrt{1-x}-\sqrt{1+x}}$ **hisoblansin.**

◀ $$\lim_{x\to 0}\frac{\sqrt[3]{1-x}-\sqrt[3]{1+x}}{\sqrt{1-x}-\sqrt{1+x}}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{-2x\left(\sqrt{1-x}+\sqrt{1+x}\right)}{-2x\left(\sqrt[3]{(1-x)^2}+\sqrt[3]{1-x^2}+\sqrt[3]{(1+x)^2}\right)}=$$

$$\lim_{x\to 0}\frac{\left(\sqrt{1-x}+\sqrt{1+x}\right)}{\left(\sqrt[3]{(1-x)^2}+\sqrt[3]{1-x^2}+\sqrt[3]{(1+x)^2}\right)}=\frac{2}{3}$$ ▶

4– misol. $\lim_{x\to 0}\frac{1-\cos x}{e^{x^2}-1}$ **hisoblansin.**

◀ $$\lim_{x\to 0}\frac{1-\cos x}{e^{x^2}-1}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{e^{x^2}-1}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{e^{x^2}-1}\cdot\frac{x^2}{\frac{x^2}{4}4}=\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{2\frac{x^2}{4}}\cdot\frac{x^2}{e^{x^2}-1}=\frac{1}{2}$$

$\left(\begin{array}{l}\lim_{t\to 0}\frac{e^t-1}{t}=1\\ \lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=1\end{array}\right)$ dan foydalandik.▶

5 – misol. $\lim_{x\to\pi}(\cos 2x)^{\frac{1}{ctgx}}$ **hisoblansin.**

◀ $$\lim_{x\to\pi}(\cos 2x)^{\frac{1}{ctgx}}=\left(1^\infty\right)=\lim_{x\to\pi}\left(1+(\cos 2x-1)\right)^{\frac{1}{ctgx}}=\lim_{x\to\pi}\left(1+(\cos 2x-1)\right)^{\frac{1}{\cos 2x-1}\cdot\frac{\cos 2x-1}{ctgx}}=$$

$$=(\lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e \text{ дан фойдаланамиз})=e^{\lim_{x\to\pi}\frac{\cos 2x-1}{ctgx}}=e^{\lim_{x\to\pi}\frac{-2\sin^3 x}{\cos x}}=e^0=1$$ ▶

6 – misol.

$$\lim_{x\to 0}\frac{4^x-2^{3x}}{\arcsin 2x-7x}$$

limiti hisoblansin.

$$◀ \lim_{x\to 0}\frac{4^x-2^{3x}}{\arcsin 2x-7x}=\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\frac{4^x-1}{x}-3\cdot\frac{2^{3x}-1}{3x}}{2\cdot\frac{\arcsin 2x}{2x}-7}= \quad ▶$$

$$=\left(\lim_{t\to 0}\frac{a^t-1}{t}=\ln a \text{ ва } \lim_{t\to 0}\frac{\arcsin t}{t}=1\right)=\frac{\ln 4-3\ln 2}{2-7}=\frac{\ln 2}{5}$$

Mustaqil yechish uchun misollar.

1 – topshiriq. $y=f(x)$ **funksiyaning aniqlanish soxasi topilsin** $(D(f)-?)$**.**

1.1 $y=\ln(x^2-5x+16)$

1.2 $y=\log_{0,5}\left(\frac{1}{2}-2^{x-1}\right)$

1.3 $y=\arcsin(x^2-x+2)$

1.4 $y=\frac{\sqrt{x^2-9}}{\log_2(x^2+2x-1)}$

1.5 $y=\frac{\sqrt{x+5}}{\lg(9-5x)}$

1.6 $y=\sqrt{\log_3\frac{2x-3}{x-1}}$

1.7 $y=\lg\frac{3x-x^2}{x-1}$

1.8 $y=\sqrt[3]{\frac{x+2}{\lg\sin x}}$

1.9 $y=\lg(16-x^2)+ctgx$

1.10 $y=(8-2x-x^2)^{-\frac{3}{2}}$

1.11 $y=\sqrt{x^2-|x|}$

1.12 $y=\sqrt[3]{\frac{x}{1-|x|}}$

1.13 $y=\sqrt{3-5x-2x^2}$

1.14 $y=\sqrt{\frac{x}{6-x}}$

1.15 $y=\frac{tgx}{\cos 2x}$

1.16 $y=\sqrt{\frac{\sin x+\cos x}{\sin x-\cos x}}$

1.17 $y=\arccos(0,5x-1)$

1.18 $y=\arccos x-\arcsin(3-x)$

1.19 $y=arctg\frac{x}{x^3-8}$

1.20 $y=\arccos\frac{x^2-1}{x}$

2 – topshiriq. Quyidagi tengliklar ta'rif yordamida isbotlansin ($\delta(\varepsilon)$- topilsin).

2.1 $\lim_{x\to 1}\frac{x^4+4x^2-5}{x^3+2x^2-x-2}=\frac{12}{5}$

2.2 $\lim_{x\to 1}\frac{5x^2-4x-1}{x-1}=6$

2.3 $\lim_{x\to -2}\frac{3x^2+5x-2}{x+2}=-7$

2.4 $\lim_{x\to 3}\frac{4x^2-14x+6}{x-3}=10$

2.5 $\lim\limits_{x\to-\frac{1}{2}}\frac{6x^2+x-1}{x+\frac{1}{2}}=-5$

2.6 $\lim\limits_{x\to\frac{1}{2}}\frac{6x^2-x-1}{x-\frac{1}{2}}=5$

2.7 $\lim\limits_{x\to-\frac{1}{3}}\frac{9x^2-1}{x+\frac{1}{3}}=-6$

2.8 $\lim\limits_{x\to2}\frac{3x^2-5x-2}{x-2}=7$

2.9 $\lim\limits_{x\to-\frac{1}{3}}\frac{3x^2-2x-1}{x+\frac{1}{3}}=-4$

2.10 $\lim\limits_{x\to-1}\frac{7x^2+8x+1}{x+1}=-6$

2.11 $\lim\limits_{x\to3}\frac{x^2-4x+3}{x-3}=2$

2.12 $\lim\limits_{x\to\frac{1}{2}}\frac{2x^2+3x-2}{x-\frac{1}{2}}=5$

2.13 $\lim\limits_{x\to\frac{1}{3}}\frac{6x^2-5x+1}{x-\frac{1}{3}}=-1$

2.14 $\lim\limits_{x\to-\frac{7}{5}}\frac{10x^2+9x-7}{x+\frac{7}{5}}=-19$

2.15 $\lim\limits_{x\to-\frac{7}{2}}\frac{2x^2+13x+21}{2x+7}=-\frac{1}{2}$

2.16 $\lim\limits_{x\to\frac{5}{2}}\frac{2x^2-9x+10}{2x-5}=\frac{1}{2}$

2.17 $\lim\limits_{x\to\frac{1}{2}}\frac{6x^2+x-1}{x-\frac{1}{3}}=5$

2.18 $\lim\limits_{x\to-\frac{1}{2}}\frac{6x^2-75x-39}{x+\frac{1}{2}}=-81$

2.19 $\lim\limits_{x\to11}\frac{2x^2-21x-11}{x-11}=23$

2.20 $\lim\limits_{x\to5}\frac{5x^2-24x-5}{x-5}=26$

3 – topshiriq. Limitlar hisoblansin.

3.1 $\lim\limits_{x\to4}\frac{\sqrt[3]{x^2-16}}{\sqrt{x+12}-\sqrt{3x+4}}$

3.2 $\lim\limits_{x\to\frac{1}{2}}\frac{\sqrt[3]{2-3x}-\sqrt[3]{6-x}}{\sqrt[3]{8+x^2}}$

3.3 $\lim\limits_{x\to8}\frac{\sqrt{9+2x}-5}{\sqrt[3]{x^2}-4}$

3.4 $\lim\limits_{x\to4}\frac{\sqrt[3]{16x}-4}{\sqrt{4+x}-\sqrt{2x}}$

3.5 $\lim\limits_{x\to-2}\frac{\sqrt[3]{x-6}+2}{x+2}$

3.6 $\lim\limits_{x\to3}\frac{\sqrt[3]{9x}-3}{\sqrt{3+x}-\sqrt{2x}}$

3.7 $\lim\limits_{x\to1}\dfrac{\sqrt{x}-1}{x^2-1}$

3.8 $\lim\limits_{x\to2}\dfrac{\sqrt[3]{4x}-2}{\sqrt{2+x}-\sqrt{2x}}$

3.9 $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$

3.10 $\lim\limits_{x\to1}\dfrac{\sqrt[3]{x}-1}{\sqrt{1+x}-\sqrt{2x}}$

3.11 $\lim\limits_{x\to0}\dfrac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{x+2\cdot\sqrt[3]{x^4}}$

3.12 $\lim\limits_{x\to0}\dfrac{\sqrt[3]{8+3x+x^2}-2}{x+x^2}$

3.13 $\lim\limits_{x\to0}\dfrac{\sqrt[3]{1-2x+x^2}-(1+x)}{x}$

3.14 $\lim\limits_{x\to8}\dfrac{\sqrt[3]{9+2x}-5}{\sqrt[3]{x}-2}$

3.15 $\lim\limits_{x\to16}\dfrac{\sqrt[4]{x}-2}{\sqrt{x}-4}$

3.16 $\lim\limits_{x\to-2}\dfrac{\sqrt[3]{x-6}+2}{x^3+8}$

3.17 $\lim\limits_{x\to3}\dfrac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}$

3.18 $\lim\limits_{x\to1}\dfrac{\sqrt{x-1}}{\sqrt[3]{x^2-1}}$

3.19 $\lim\limits_{x\to1}\dfrac{6x^3-5x^2-1}{5x^3+2x^2-4x-3}$

3.20 $\lim\limits_{x\to1}\dfrac{8x^4-6x^2-x-4}{x^3-3x^2+2}$

4 – topshiriq. Limitlar hisoblansin.

4.1 $\lim\limits_{x\to0}\dfrac{\ln(1+\sin x)}{\sin 4x}$

4.2 $\lim\limits_{x\to0}\dfrac{1-\cos x}{\cos 2x-\cos x}$

4.3 $\lim\limits_{x\to0}\dfrac{e^{2x}-e^x}{x+tgx^3}$

4.4 $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{1-\sin 2x}{(\pi-4x)^2}$

4.5 $\lim\limits_{x\to\pi}\dfrac{3+3\cos x}{\sin^2 x}$

4.6 $\lim\limits_{x\to0}\dfrac{\sin 2x}{\sin 5x}$

4.7 $\lim\limits_{x\to0}\dfrac{1+x\sin x-\cos 2x}{\sin^2 x}$

4.8 $\lim\limits_{x\to1}\dfrac{\sqrt{x^2-1}}{\sin \pi x}$

4.9 $\lim\limits_{x\to\pi}\dfrac{\cos 2x-\cos 6x}{\sin^2 2x}$

4.10 $\lim\limits_{x\to2\pi}\dfrac{\sin 5x-\sin 2x}{x-2\pi}$

4.11 $\lim\limits_{x\to2}\dfrac{\ln(5-2x)}{\ln(x-1)}$

4.12 $\lim\limits_{x\to1}\dfrac{\sqrt{x^2-3x+3}-1}{\sin \pi x}$

4.13 $\lim\limits_{x\to\pi}\dfrac{x^2-\pi^2}{\sin x}$

4.14 $\lim\limits_{x\to1}\dfrac{3^{5x-3}-3^{2x^2}}{tg\pi x}$

4.15 $\lim\limits_{x\to4}\dfrac{2^x-16}{\sin \pi x}$

4.16 $\lim\limits_{x\to1}\dfrac{3^{5x-3}-3^{2x^2}}{tg\pi x}$

4.17 $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\ln ctgx}{\sin 4x}$

4.18 $\lim\limits_{x\to2}\dfrac{\ln(9-2x^2)}{\sin 2\pi x}$

4.19 $\lim\limits_{x\to 0}\dfrac{5^{2x}-2^{5x}}{\cos x-\cos x^3}$ 4.20 $\lim\limits_{x\to\frac{4\pi}{3}}\dfrac{1+2\cos x}{4\pi-3x}$

5 - topshiriq. Limitlar hisoblansin.

5.1 $\lim\limits_{x\to 1}\left(\dfrac{4x-1}{2x+1}\right)^{\frac{1}{\sqrt[3]{x}-1}}$ 5.2 $\lim\limits_{x\to a}\left(\dfrac{\cos x}{\cos a}\right)^{\frac{1}{x-a}}$

5.3 $\lim\limits_{x\to 1}\left(\dfrac{2x}{3x-1}\right)^{\frac{1}{\sqrt{x}-1}}$ 5.4 $\lim\limits_{x\to 2}\left(\dfrac{\sin x}{\sin 2x}\right)^{\frac{1}{x-2}}$

5.5 $\lim\limits_{x\to 8}\left(\dfrac{3x-9}{x+7}\right)^{\frac{1}{\sqrt[3]{x}-2}}$ 5.6 $\lim\limits_{x\to\frac{\pi}{4}}(tgx)^{\frac{2}{\sin(\frac{\pi}{4}-x)}}$

5.7 $\lim\limits_{x\to 16}\left(\dfrac{x-5}{\frac{x}{2}+3}\right)^{\frac{1}{\sqrt[4]{x}-2}}$ 5.8 $\lim\limits_{x\to a}\left(2-\dfrac{x}{a}\right)^{tg\frac{\pi x}{2a}}$

5.9 $\lim\limits_{x\to 2\pi}(\cos x)^{\frac{1}{\sin 2x}}$ 5.10 $\lim\limits_{x\to 2\pi}(\cos x)^{\frac{1}{\sin^2 2x}}$

5.11 $\lim\limits_{x\to 3}\left(\dfrac{6-x}{3}\right)^{tg\frac{\pi x}{6}}$ 5.12 $\lim\limits_{x\to 4\pi}(\cos x)^{\frac{ctgx}{\sin 4x}}$

5.13 $\lim\limits_{x\to 1}(3-2x)^{tg\frac{\pi x}{2}}$ 5.14 $\lim\limits_{x\to 4\pi}(\cos x)^{\frac{5}{tg5x\cdot\sin 2x}}$

5.15 $\lim\limits_{x\to 3}\left(\dfrac{6-x}{3}\right)^{tg\frac{\pi x}{6}}$ 5.16 $\lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{6tgx\cdot tg3x}$

5.17 $\lim\limits_{x\to 1}(2e^{x-1}-1)^{\frac{x}{x-1}}$ 5.18 $\lim\limits_{x\to\frac{\pi}{2}}\left(ctg\dfrac{x}{2}\right)^{\frac{1}{\cos x}}$

5.19 $\lim\limits_{x\to 1}(\sin\pi x+1)^{\frac{3x-1}{x-1}}$ 5.20 $\lim\limits_{x\to\pi}(1+\sin 3x)^{\cos ecx}$

6- topshiriq. Limitlar hisoblansin.

6.1 $\lim\limits_{x\to 0}\dfrac{5^{2x}-2^{3x}}{x+arctg3x}$ 6.2 $\lim\limits_{x\to 0}\dfrac{e^{3x}-e^{-2x}}{2\arcsin x-\sin x}$

6.3 $\lim_{x\to 0}\frac{6^{2x}-7^{-2x}}{\sin 3x-2x}$

6.4 $\lim_{x\to 0}\frac{e^{5x}-e^{3x}}{\sin 2x-\sin x}$

6.5 $\lim_{x\to 0}\frac{3^{2x}-5^{3x}}{arctgx+x^{3}}$

6.6 $\lim_{x\to 0}\frac{e^{2x}-e^{3x}}{arctgx-x^{2}}$

6.7 $\lim_{x\to 0}\frac{3^{5x}-2^{x}}{x-\sin 9x}$

6.8 $\lim_{x\to 0}\frac{e^{4x}-e^{-2x}}{2arctgx-\sin x}$

6.9 $\lim_{x\to 0}\frac{12^{x}-5^{-3x}}{2\arcsin x-x}$

6.10 $\lim_{x\to 0}\frac{e^{7x}-e^{-2x}}{\sin x-2x}$

6.11 $\lim_{x\to 0}\frac{3^{5x}-2^{7x}}{\arcsin 2x-x}$

6.12 $\lim_{x\to 0}\frac{e^{5x}-e^{x}}{\arcsin x+x^{3}}$

6.13 $\lim_{x\to 0}\frac{4^{x}-2^{7x}}{tg3x-x}$

6.14 $\lim_{x\to 0}\frac{e^{x}-e^{-x}}{tg2x-\sin x}$

6.15 $\lim_{x\to 0}\frac{10^{2x}-7^{-x}}{2tgx-arctgx}$

6.16 $\lim_{x\to 0}\frac{e^{2x}-e^{x}}{\sin 3x-\sin 5x}$

6.17 $\lim_{x\to 0}\frac{7^{3x}-3^{2x}}{tgx+x^{3}}$

6.18 $\lim_{x\to 0}\frac{e^{4x}-e^{2x}}{2tgx-\sin x}$

6.19 $\lim_{x\to 0}\frac{3^{2x}-7^{x}}{\arcsin 3x-5x}$

6.20 $\lim_{x\to 0}\frac{e^{2x}-e^{-5x}}{2\sin x-tgx}$

6–mustaqil ish.
Mavzu: Funksiyaning uzluksizligi va uzilishi. Funksiyaning tekis uzluksizligi

f(x) funksiya *a* nuqtaning biror to'liq atrofida aniqlangan bo'lsin.

1-ta'rif. Agar

$$\lim_{x\to a} f(x) = f(a) \qquad (3)$$

bo'lsa, *f(x)* funksiya *a* **nuqtada uzluksiz** deyiladi.
Funksiya uzluksizligi ta'rifini Koshi va Geyne ta'riflari yordamida ham berish mumkin.

2-ta'rif (Koshi). Agar $\forall\varepsilon > 0$ son olinganda ham shunday $\delta > 0$ son topilsaki, funksiya argumenti x ning $|x-a| < \delta$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida $|f(x) - f(a)| < \varepsilon$ tengsizlik bajarilsa, $f(x)$ funksiya *a* nuqtada uzluksiz deyiladi.

3-ta'rif(Geyne). Agar $X = \{x\}$ to'plamning elementlaridan tuzilgan *a* ga intiluvchi har qanday $\{x_n\}$ ketma-ketlik olinganida ham funksiya qiymatlaridan tuzilgan mos $\{f(x_n)\}$ ketma ketlik hamma vaqt $f(a)$ ga intilsa, $f(x)$ funksiya *a* nuqtada uzluksiz deyiladi.

Endi *f(x)* funksiya *a* nuqtaning biror o'ng (chap) yarim atrofida, ya'ni $[a, a+\delta)$ (mos ravishda, $(a-\delta, a]$ yarim oraliq aniqlangan bo'lsin.

4-ta'rif. Agar $\lim_{x\to a+0} f(x) = f(a)$ $(\lim_{x\to a-0} f(x) = f(a))$

bo'lsa, *f(x)* funksiya *a* nuqtada o'ngdan (chapdan) uzluksiz deyiladi.

1-teorema. *f(x)* funksiyaning *a* nuqtada uzluksiz bo'lishi uchun uning shu nuqtada o'ngdan va chapdan uzluksiz bo'lishi zarur va yyetarlidir.

Faraz qilaylik, *f(x)* funksiya *a* nuqtada uzluksiz bo'lsin. U holda $\lim_{x\to a} f(x) = f(a)$ bo'ladi. bundan $\lim_{x-a\to 0}[f(x) - f(a)] = 0$ Agar $\Delta x = x - a$ argument orttirmasi va $\Delta y := \Delta f(a) = f(x) - f(a)$ - funksiyaning *a* nuqtadagi orttirmasi belgilashlarini kiritsak, $x = a + \Delta x$ va $\Delta y = \Delta f(a) = f(a+\Delta x) - f(a)$ bo'ladi. Natijada, biz

$$\lim_{x-a\to 0}[f(x) - f(a)] = \lim_{\Delta x\to 0}[f(a+\Delta x) - f(a)] = \lim_{\Delta x\to 0}\Delta y = 0$$

ekanligini hosil qilamiz. Shunday qilib,

$$\lim_{\Delta x\to 0}\Delta y = 0 \qquad (4)$$

tenglik bajarilsa, *f(x)* funksiya *a* nuqtada uzluksiz bo'ladi.

5-ta'rif. *f(x)* funksiya (*c, d*) oraliqning hap bir nuqtasida uzluksiz bo'lsa, funksiya (*c, d*) intervalda uzluksiz deyiladi.
f(x) funksiya (*c, d*) da uzluksiz bo'lib, *c* nuqtada o'ngdan, *d* nuqtada chapdan uzluksiz bo'lsa, unda u [*c, d*] kesmada uzluksiz deyiladi.
X to'plamda uzluksiz funksiyalar sinfi *C(X)* kabi belgilanadi.

6–ta'rif. Agar

$\lim\limits_{x\to a} f(x) = b \neq f(a)$ (1 - hol)

$\lim\limits_{x\to a} f(x) - \not\exists$ (2 - hol)

$\lim\limits_{x\to a} f(x) = \infty$ (3 - hol)

bo'lsa, unda *f(x)* funksiya *a* **nuqtada uzilishga ega** deyiladi.

Funksiyaning *a* nuqtada uzilishga ega bo'ladigan hollarini alohida - alohida ko'rib chiqaylik,

a) $\lim\limits_{x\to a} f(x) = b \neq f(a)$ bo'lsin.

Bu holda $\lim\limits_{x\to a+0} f(x) = f(a+0)$ va $\lim\limits_{x\to a-0} f(x) = f(a-0)$ lar mavjud bo'lib, $f(a+0) = f(a-0) \neq f(a)$ bo'ladi. Bunday nuqta **bartaraf qilish mumkin bo'lgan uzilish nuqtasi** deb ataladi.

Misollar.

1. $f(x) = \begin{cases} x^3 + x, & \text{агар } x \neq 0 \text{ булса,} \\ 1, & \text{агар } x = 0 \text{ булса} \end{cases}$

funksiya uchun $x = 0$ nuqta bartaraf qilish mumkin bo'lgan uzilish nuqtasi bo'ladi, chunki

$$\lim_{x\to a+0} f(x) = \lim_{x\to a-0} f(x) = 0 \text{ va } f(0) = 1$$

Agar $f(0) = 0$ deb qabul qilsak, funksiya uzluksiz bo'lib qoladi.

2. $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & \text{агар } x \neq 0 \text{ булса,} \\ 1, & \text{агар } x = 0 \text{ булса} \end{cases}$

funksiya uchun ham $x = 0$ nuqta bartaraf qilish mumkin bo'lgan uzilish nuqtasi bo'ladi, chunki

$\lim\limits_{x\to 0} f(x) = 0$ va $f(0) = 1$

b) $\lim\limits_{x\to a} f(x)$ mavjud emas.

Bunda quyidagi uchta hol bo'lishi mumkin.

1-hol. $\lim\limits_{x\to a-0} f(x) = f(a-0)$ va $\lim\limits_{x\to a+0} f(x) = f(a+0)$ lar mavjud va

$$f(a-0) \neq f(a+0)$$

Funksiyaning bunday nuqtadagi uzilishi birinchi tur uzilish va $|f(a+0) - f(a-0)|$ ayirmaga funksiyaning *a* nuqtadagi sakrashi deyiladi.

Masalan, $f(x) = \begin{cases} \dfrac{1}{1+2^{\frac{1}{x}}}, & \text{агар } x \neq 0 \text{ булса,} \\ 0, & \text{агар } x = 0 \text{ булса} \end{cases}$ funksiya uchun $x = 0$ nuqta 1 - tur uzilish nuqtasi bo'ladi va funksiyaning bu nuqtadagi sakrashi 1 ga teng:

$$|f(a+0) - f(a-0)| = |f(+0) - f(-0)| = |0 - 1| = 1$$

2-hol. $x \to a$ da *f(x)* funksiyaning o'ng va chap limitlaridan hech bo'lmaganda biri mavjud emas.. Funksiyaning *a* nuqtadagi bunday uzilishi ikkinchi tur uzilish deyiladi.

Misollar**.**

1. $f(x) = \begin{cases} \sin\frac{1}{x}, & x > 0 \\ \text{x}, & x \le 0 \end{cases}$

funksiya $x = 0$ nuqtada ikkinchi tur uzilishga ega, chunki, lekin.

$\lim\limits_{x \to +0} f(x) = \lim\limits_{x \to 0} \sin\frac{1}{x}$

2. $D(x) = \begin{cases} 0, & \text{агар } x - ratsional\ bo'lsa, \\ 1, & \text{агар } x - irratsional\ bo'lsa \end{cases}$

funksiya $\forall a \in R$ nuqtada ikkinchi tur uzilishga ega, chunki $x \to a$ da $D(x)$ funksiyaning o'ng limiti ham , chap limita ham mavjud emas.

3-hol $x \to a$ da $f(x)$ funksiyaning o'ng va chap limitlaridan biri cheksiz yoki o'ng va chap limitlar turli ishorali cheksiz. Funksiyaning *a* nuqtadagi bunday uzilishi ham **ikkinchi tur uzilish** deyiladi.

c) $\lim\limits_{x \to a} f(x) = \infty$ bo'lsa, $f(x)$ funksiya $x = a$ nuqtada **ikkinchi tur uzilishga** ega deyiladi.

Uzluksiz funksiyalarning xossalari

2-teorema. Agar $f(x)$ va $g(x)$ funksiyalar $X \subset R$ to'plamda aniqlangan bo'lib, ularning har biri $a \in X$ nuqtada uzluksiz bo'lsa, u holda

1) $f(x) \pm g(x)$,

2) $f(x) \cdot g(x)$,

3) $\frac{f(x)}{g(x)}$ ($\forall x \in X$ uchun $g(x) \ne 0$)

funksiyalar ham shu nuqtada uzluksiz bo'ladi.

Izoh: 2 - teoremaning aksi har doim ham o'rinli bo'lavermaydi. Masalan, $f(x) = x$ va $g(x) = \begin{cases} \cos\frac{1}{x}, & x \ne 0, \\ 0, & x = 0 \end{cases}$ funksiyalar ko'paytmasi $f(x) \cdot g(x) = x \cdot \cos\frac{1}{x}$ funksiya *R* da uzluksiz, lekin $g(x)$ funksiya $x = 0$ nuqtada uzilishga ega.

Aytaylik, $y=f(x)$ funksiya X to'plamda, $z=\varphi(y)$ funksiya esa $Y=\{y=f(x):x\in X\}$ to'plamda aniqlangan bo'lib, ular yordamida X to'plamda aniqlangan: $z=\varphi[f(x)]$ murakkab funksiya tuzilgan bo'lsin.

3-teorema. Agar $y=f(x)$ funksiya $a\in X$ nuqtada, $z=\varphi(y)$ funksiya esa, unga mos $y_a=f(a)$ nuqtada uzluksiz bo'lsa, $z=\varphi[f(x)]$ murakkab funksiya a nuqtada uzluksiz bo'ladi.

Bu teorema limit hisoblashda juda muhim rol o'ynaydi.

4-teorema. Agar $\lim\limits_{x\to a} f(x)=b$ $(b>0)$ va $\lim\limits_{x\to a} g(x)=c$ bo'lsa, $\lim\limits_{x\to a}[f(x)]^{g(x)}=b^c$ bo'ladi.

$[f(x)]^{g(x)}$ ko'rinishdagi funksiyaga **darajali – ko'rsatkichli funksiya** deb ataladi.

Funksiyaning tekis uzluksizligi

Biror $y=f(x)$ funksiya X to'plamda berilgan bo'lsin.

7-ta'rif. Agar $\forall\varepsilon>0$ son uchun $\exists\delta=\delta(\varepsilon)>0$ son topilsaki, X to'plamning $|x''-x'|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x'$ va x' $(x', x''\in X)$ nuqtalarida $|f(x'')-f(x')|<\varepsilon$ tengsizlik bajarilsa, $f(x)$ funksiya X to'plamda **tekis uzluksiz** deb ataladi.

7-ta'rifning inkori. $\exists\varepsilon>0$ son topilsaki, $\forall\delta>0$ son olinganda ham $|x''-x'|<\delta$ tengsizlikni qanoatlantiruvchi shunday $\forall x', x''\in X$ nuqtalar mavjud bo'lib $|f(x'')-f(x')|\ge\varepsilon$ tengsizlik bajarilsa, $f(x)$ funksiya X to'plamda **tekis uzluksiz** emas deyiladi.

5-teorema (Kantor). Agar $f(x)$ funksiya $[a,b]$ kesmada aniqlangan va uzluksiz bo'lsa, u shu kesmada tekis uzluksiz bo'ladi.

***1– misol. $y=f(x)$ funksiyaning $x=x_0$ nuqtadagi o'ng va chap limitlari topilsin. ($f(x_0+0)-?, f(x_0-0)-?$)**

$$f(x)=\frac{x^2-4}{x+2}, x_0=-2$$

$$◄\ \lim_{x\to-2}\frac{x^2-4}{x+2}=\lim_{x\to-2}\frac{(x-2)(x+2)}{x+2}=\lim_{x\to-2}(x-2)=-4$$

$$f((-2)-0)=f((-2)+0)=\lim_{x\to-2}f(x)=\lim_{x\to-2}\frac{x^2-4}{x+2}=-4\ ►$$

2– misol. $y=f(x)$ funksiya $x=x_0$ nuqtada uzluksiz ekanligi ta'rif yordamida isbotlansin ($\delta(\varepsilon)$ topilsin).

$$f(x)=2x^2-4x+1,\ x_0=1$$

◄ $f(x)$ funksiyani $x_0 = 1$ nuqtaning biror atrofida, masalan, (0;2) intervalda qaraymiz. $\forall\varepsilon > 0$ son olamiz va $|f(x) - f(x_0)| = |f(x) - f(1)|$ ayirmani baholaymiz:

$$|f(x) - f(1)| = |2x^2 - 4x + 1 - (-1)| = |2x^2 - 4x + 2| = 2|x-1|^2 <$$

$$< 2\delta^2 = \varepsilon$$

Bu tenglikdan ko'rinib turibdiki, agar $\delta = \sqrt{\dfrac{\varepsilon}{2}}$ deb olsak, $|x-1| < \delta$ tengsizlikni qanoatlantiruvchi $\forall x \in (0;2)$ uchun $|f(x) - f(1)| = 2|x-1|^2 < 2\delta^2 = \varepsilon$ bo'ladi $\Rightarrow f(x) = 2x^2 - 4x + 1$ funksiya $x_0 = 1$ nuqtada uzluksiz►

3– misol. $f(x) = \operatorname{sgn}\left(\cos\dfrac{1}{x}\right)$ **funksiya uzluksizlikka tekshirilsin.**

◄ $f(x) = \operatorname{sgn}\left(\cos\dfrac{1}{x}\right)$ funksiya $x = 0$ va $\cos\dfrac{1}{x} = 0$ tenglikni qanoatlantiruvchi nuqtalarda uzilishga ega. Bu nuqtalardagi uzilish turini aniqlaymiz.

Nolga intiluvchi ikkita $x_n' = \dfrac{1}{\frac{\pi}{2} + 2\pi n}$ >0 va $x_n'' = \dfrac{1}{2\pi n}$ >0 ketma –ketliklarni olamiz.

Bu ketma –ketliklarga mos funksiya hadlaridan tuzilgan ketma –ketliklar $f(x_n') = \operatorname{sgn}\left(\cos\left(\dfrac{\pi}{2} + \pi n\right)\right) = 0,\ f(x_n'') = \operatorname{sgn}(\cos 2\pi n) = 1$. Funksiya limitining Geyne ta'rifiga ko'ra $x = 0$ nuqtada funksiyaning o'ng limiti mavjud emas. Funksiya bu nuqtada II –tur uzilishga ega.

Endi biz $\cos\dfrac{1}{x} = 0$ tenglamaning yechimi bo`lgan $x = \dfrac{1}{\frac{\pi}{2} + \pi n}$, $n \in Z$ nuqtalarida funksiyaning o'ng va chap limitlarini tekshiramiz.

$$x_k = \frac{1}{\frac{\pi}{2} + 2\pi k},\quad \lim_{x \to \frac{1}{\frac{\pi}{2}+2\pi k}+0} \operatorname{sgn}\left(\cos\frac{1}{x}\right) = 1,\quad \lim_{x \to \frac{1}{\frac{\pi}{2}+2\pi k}-0} \operatorname{sgn}\left(\cos\frac{1}{x}\right) = -1,$$

$$x_k = \frac{1}{\frac{3\pi}{2} + 2\pi k},\quad \lim_{x \to \frac{1}{\frac{3\pi}{2}+2\pi k}+0} \operatorname{sgn}\left(\cos\frac{1}{x}\right) = -1,\quad \lim_{x \to \frac{1}{\frac{3\pi}{2}+2\pi k}-0} \operatorname{sgn}\left(\cos\frac{1}{x}\right) = 1,$$

Demak, funksiya bu nuqtalarda I –tur uzilishga ega.►

4– misol. $y = f(x)$ **funksiya** X **to`plamda tekis uzluksiz ekanligi ta'rif yordamida ko'rsatilsin (**$\delta = \delta(\varepsilon)$ **topilsin).**

$$f(x) = \sin x + \cos x,\quad x \in R$$

◄($f(x)$ funksiya X to'plamda tekis uzluksiz) $\Leftrightarrow$

$(\forall\varepsilon>0\ \exists\delta=\delta(\varepsilon)>0,\ \forall x',\ x''\in X: |x''-x'|<\delta \Rightarrow |f(x'')-f(x')<\varepsilon|$.

$\forall\varepsilon>0$ son olib $|f(x'')-f(x')|$ ni baholaymiz:

$$|f(x'')-f(x')|=|\sin x''+\cos x''-(\sin x'+\cos x')|=$$

$$=|(\sin x''-\sin x')+(\cos x''-\cos x')|=\left(\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}\text{ ва}\right.$$

$$\left.\cos\alpha-\cos\beta=-2\sin\frac{\alpha-\beta}{2}\cdot\sin\frac{\alpha+\beta}{2}\text{ формулалардан фойдаланамиз}\right)=$$

$$=\left|2\sin\frac{x''-x'}{2}\cdot\cos\frac{x''+x'}{2}-2\sin\frac{x''-x'}{2}\cdot\sin\frac{x''+x'}{2}\right|=$$

$$=2\cdot\left|\sin\frac{x''-x'}{2}\right|\cdot\left|\cos\frac{x''+x'}{2}-\sin\frac{x''+x'}{2}\right|\le 2\cdot\frac{|x''-x'|}{2}\cdot\left(\left|\cos\frac{x''+x'}{2}\right|+\left|\sin\frac{x''+x'}{2}\right|\right)\le$$

$$\le|x''-x'|\cdot(1+1)=2\cdot|x''-x'|$$

Bu tenglikdan ko'rinib turibdiki $\delta=\frac{\varepsilon}{2}$ deb olsak, $|x''-x'|<\delta$ tengsizlikni qanoatlantiruvchi $\forall x',x''\in R$ uchun $|f(x'')-f(x')|<\varepsilon$ bo'ladi. $\Rightarrow f(x)$ funksiya R da tekis uzluksiz.▶

Misol. $y=\sqrt{x}$ funksiya har bir $x_0>0$ nuqtada uzluksiiz va $x_0=0$ nuqtada o'ngdan uzluksiz ekanligini isbotlang.

Uzluksizlik ta'rifidan foydalanib, $\sqrt{x}$ ning $x_0>0$ nuqtada uzluksizligini isbotlaymmiz.

Quydagi ayirma modulini shakl almashtirib baholaymiz:

$$0\le\left|\sqrt{x}-\sqrt{x_0}\right|=\left|\frac{x-x_0}{\sqrt{x}+\sqrt{x_0}}\right|\le\frac{|x-x_0|}{\sqrt{x_0}}$$

$\lim\limits_{x\to x_0}|x-x_0|=0,\quad \frac{1}{\sqrt{x_0}}$ - o'zgarmas son ekanligidan $\lim\limits_{x\to x_0}\frac{|x-x_0|}{\sqrt{x_0}}=0$

Bu erda $\lim\limits_{x\to x_0}\left|\sqrt{x}-\sqrt{x_0}\right|=0$ kelib chiqadi va $\lim\limits_{x\to x_0}\sqrt{x}=\sqrt{x_0}$, ya'ni $\forall x_0>0$ da $y=\sqrt{x}$.

Funksiya uzluksiz.

Endi $\sqrt{x}$ ni $x_0=0$ nuqtada o'ngdan uzluksizligini ko'rsatamiz $\varepsilon-$ixtiyoriy musbat son bo'lsin.

$\left|\sqrt{x}-0\right|<\varepsilon$ tengsizlik $0\le x<\varepsilon^2$ tengsizlikka teng kuchli . $\delta=\varepsilon^2$ deb olamiz , u holda $0\le x<\delta$ tengsizlikdan $\sqrt{x}<\varepsilon$ tengsizlik kelib chiqadi . Demak $\lim\limits_{x\to+0}\sqrt{x}=0=y(0)$ va shuning uchun $\sqrt{x}$ funksiya $x_0=0$ nuqtada o'ngdan uzluksiz.

2-misol $y=\frac{1}{x}$ funksiya

1) istalgan $[a;+\infty)$ oraliqda tekis uzluksiz , bu erda $a>0$;
2) istalgan $(0;a]$ oraliqda tekis uzluksiz emasligni isbotlang.
Δ 1) $x', x'' \in [0;+\infty)$, $a>0$ bo'lsin , u holda

$$\left|\frac{1}{x'}-\frac{1}{x''}\right| = \frac{|x'-x''|}{x' \cdot x''} \le \frac{1}{a^2}\cdot|x'-x''|$$ tengsizlik $x' \ge a > 0$, $x'' \ge a > 0$ lar uchun o'rinli.

ε -ixtiyoriy musbat son bo'lsin . $\delta = a^2\varepsilon$ deb olamiz u holda $[a;+\infty)$ dan olingan $|x'-x''|<\delta$ tengsizlikni qanoatlantiruvchi istalgan x', x'' lar uchun

$$\left|\frac{1}{x'}-\frac{1}{x''}\right| < \frac{1}{a^2}\cdot\delta = \varepsilon$$ kelib chiqadi.

Bu $\frac{1}{x}$ funksiyaning $(0;a]$, $a>0$ oraliqda tekis uzluksiz ekanini bildiradi.

2) $x', x'' \in (0;a]$, $a>0$ bo'lsin $\left|\frac{1}{x'}-\frac{1}{x''}\right| = \left|\frac{x'-x''}{x' \cdot x''}\right|$

tenglikdan ko'rinadiki $\left|\frac{1}{x'}-\frac{1}{x''}\right|$ qiymat etarlicha kichik x', x'' larda o'sadi, lekin tayinlangan $|x'-x''|$ da x' yoki x'' ga qaraganda nolga sekin yaqinlashadi.

$x'' = \frac{x'}{2}$ deb olamiz , u holda $|x'-x''| = \frac{x'}{2}$ $\left|\frac{1}{x'}-\frac{1}{x''}\right| = \frac{1}{x'}$ bo'ladi.

$x' < a$ va $|x'-x''| = \frac{x'}{2} < \delta$ tengsizliklar o'rinli bo'lishi uchun $x' = \frac{\delta a}{\delta + a}$ deb olish etarli . Demak , $\varepsilon = a$ deb olib, ixtiyoriy δ musbat son uchun $x' = \frac{\delta a}{\delta + a}$, $x'' = \frac{\delta a}{2(\delta + a)}$ deb olamiz .U holda

$|x'-x''| = \frac{\delta a}{2(\delta + a)} < \delta$, $\left|\frac{1}{x'}-\frac{1}{x''}\right| > a = \varepsilon$ bo'ladi Demak , $y = \frac{1}{x}$ funksiya $(0;a]$ tekis uzluksiz emas.

Mustaqil yechish uchun misollar.

1 – topshiriq. $y = f(x)$ funksiyaning $x = x_0$ nuqtadagi o'ng va chap limitlari topilsin $(f(x_0+0)-?, f(x_0-0)-?)$.

1.1 $f(x) = arctg\frac{2}{x}$, $x_0 = 0$

1.2 $f(x) = \frac{1}{1+2^{\frac{1}{x-1}}}$, $x_0 = 1$

1.3 $f(x) = \frac{x-1}{x}$, $x_0 = 0$

1.4 $f(x) = 5^{\frac{2}{2-x}}$, $x_0 = 2$

1.5 $f(x) = 2^{\frac{1}{x-5}}$, $x_0 = 5$

1.6 $f(x) = \frac{x}{1+x^2}$, $x_0 = 0$

1.7 $f(x) = \mathrm{sgn}(\sin x)$, $x_0 = \pi$

1.8 $f(x) = ctg 2x$, $x_0 = \frac{\pi}{2}$

1.9 $f(x)=\dfrac{1}{x+3^{\frac{x}{2-x}}}$, $x_0=2$

1.10 $f(x)=\dfrac{1}{x-[x]}$, $x_0=1$

1.11 $f(x)=x+\lfloor x^2\rfloor$, $x_0=4$

1.12 $f(x)=\dfrac{x}{\cos x}$, $x_0=\dfrac{\pi}{2}$

1.13 $f(x)=\operatorname{sgn}\left(\cos\dfrac{1}{x}\right)$, $x_0=0$

1.14 $f(x)=e^{-\frac{1}{2x}}$, $x_0=0$

1.15 $f(x)=\dfrac{\sin x}{|x|}$, $x_0=0$

1.16 $f(x)=\dfrac{x^2-9}{(x-3)}$, $x_0=3$

1.17 $f(x)=\dfrac{x^3-1}{|x-1|}$, $x_0=1$

1.18 $f(x)=\dfrac{2}{1+2^{\frac{1}{2-x}}}$, $x_0=2$

1.19 $f(x)=\dfrac{\sqrt{1-x^2}}{x}$, $x_0=0$

1.20 $f(x)=\sin^2\dfrac{1}{x}$, $x_0=0$

2 – topshiriq. $y=f(x)$ funksiyaning $x=x_0$ nuqtada uzluksiz ekanligi ta'rif yordamida isbotlansin ($\delta(\varepsilon)$- topilsin).

2.1 $f(x)=x^2-1$, $x_0=4$

2.2 $f(x)=3x^2-2$, $x_0=1$

2.3 $f(x)=x^2-3$, $x_0=4$

2.4 $f(x)=x^2-4x$, $x_0=3$

2.5 $f(x)=x^2-5x$, $x_0=2$

2.6 $f(x)=x^2-6x$, $x_0=2$

2.7 $f(x)=-4x^2+2x-3$, $x_0=1$

2.8 $f(x)=-2x^2-8$, $x_0=2$

2.9 $f(x)=-5x^2-9$, $x_0=3$

2.10 $f(x)=-4x^2+9$, $x_0=4$

2.11 $f(x)=-3x^2+8$, $x_0=5$

2.12 $f(x)=-2x^2+7$, $x_0=6$

2.13 $f(x)=2x^2+6$, $x_0=7$

2.14 $f(x)=3x^2+5$, $x_0=8$

2.15 $f(x)=4x^2+4$, $x_0=9$

2.16 $f(x)=5x^2+3$, $x_0=8$

2.17 $f(x)=5x^2+1$, $x_0=7$

2.18 $f(x)=4x^2-1$, $x_0=6$

2.19 $f(x)=x^2-2$, $x_0=3$

2.20 $f(x)=2x^2-3x+1$, $x_0=4$

3 – topshiriq. Quyidagi funksiyalar a ning qanday qiymatlarida uzluksiz ekanligi aniqlansin (3.1-3.10 misollar).

3.1 $\begin{cases} x\cos\dfrac{2}{x}, & x\neq 0,\\ a, & x=0\end{cases}$

3.2 $\begin{cases} ax^3+2, & x>0,\\ -x, & x\leq 0\end{cases}$

3.3 $\begin{cases} \sin x, & x\leq 0,\\ a(x-2), & x>0\end{cases}$

3.4 $\begin{cases} x^2+a, & x>0,\\ 1-x^2, & x\leq 0\end{cases}$

3.5 $\begin{cases} 2^x, & x\geq 0,\\ a(x-1), & x<0\end{cases}$

3.6 $\begin{cases} \dfrac{\cos x}{2x+\pi}, & x\neq -\dfrac{\pi}{2},\\ a, & x=-\dfrac{\pi}{2}\end{cases}$

3.7 $\begin{cases}(\arcsin x)ctgx, & x \neq 0, \\ a, & x = 0\end{cases}$ 3.8 $\begin{cases}\dfrac{c^x - 1}{x}, & x \neq 0, \\ a, & x = 0,\ c > 0,\end{cases}$

3.9 $\begin{cases}\dfrac{x}{(1+2x)}, & x \neq -\dfrac{1}{2} \\ a, & x = -\dfrac{1}{2}\end{cases}$ 3.10 $\begin{cases}e^{-\frac{1}{x^2}}, & x \neq 0, \\ a, & x = 0\end{cases}$

Quyidagi funksiyalar uzluksizlikka tekshirilsin (3.11-3.20 misollar).

3.11 $f(x) = x^2 - \lfloor x^2 \rfloor$ 3.12 $f(x) = \lim\limits_{n\to+\infty} \dfrac{\ln(1+e^{xn})}{\ln(1+e^n)}$

3.13 $f(x) = \lim\limits_{n\to\infty} \dfrac{1}{1+x^{2n+1}}$ 3.14 $f(x) = \lim\limits_{n\to\infty} \dfrac{x}{1+(2\sin x)^{2n}}$

3.15 $f(x) = \lim\limits_{n\to\infty} \cos^{2n} x$ 3.16 $f(x) = [x]\cos \pi x$

3.17 $f(x) = \lim\limits_{n\to\infty} \dfrac{x+e^{nx}}{1+xe^{nx}}$ 3.18 $f(x) = \lim\limits_{n\to\infty} \sqrt[2n]{\cos^{2n} x + \sin^{2n} x}$

3.19 $f(x) = \lim\limits_{n\to\infty} \sqrt[n]{1+2x^n}$ 3.20 $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n}-1}{x^{2n}+1}$

4 – topshiriq. Quyidagi funksiyalar berilgan oraliqda tekis uzluksizlikka tekshirilsin. (4.1-4.8 misollar)

4.1 $f(x) = \dfrac{2x}{9-x^2},\ -1 \le x \le 1$ 4.2 $f(x) = \ln x,\ 0 < x < 1$

4.3 $f(x) = \dfrac{\sin x}{x},\ 0 < x < \pi$ 4.4 $f(x) = e^x \cos\dfrac{1}{x},\ 0 < x < 1$

4.5 $f(x) = arctgx,\ -\infty < x < +\infty$ 4.6 $f(x) = x\sin x,\ 0 \le x < +\infty$

4.7 $f(x) = \dfrac{x-2}{x^2},\ 1 \le x \le 2$ 4.8 $f(x) = x\cos x,\quad 0 \le x < +\infty$

$y = f(x)$ **funksiya** X **to'plamda tekis uzluksiz emasligi isbotlansin (4.9-4.14 misollar).**

4.9 $f(x) = \sin\dfrac{1}{x},\ X = (0,1)$ 4.10 $f(x) = \dfrac{1}{x},\ X = (0,1)$

4.11 $f(x) = x^2,\ X = R$ 4.12 $f(x) = \sin\dfrac{1}{x-1},\ X = (0,1)$

4.13 $f(x) = x^3 + 1,\ X = R$ 4.14 $f(x) = \dfrac{1}{x^2-4},\ X = (2,3)$

$y = f(x)$ **funksiya** X **to'plamda tekis uzluksiz ekanligi ta'rif yordamida ko'rsatilsin (**$\delta = \delta(\varepsilon)$ **topilsin) (4.15-4.20 misollar).**

4.15 $f(x) = x^2 + 1,\ X = (-\infty, -\infty)$ 4.16 $f(x) = \sqrt[3]{x+1},\ X = [0;2]$

4.17 $f(x)=\dfrac{1}{x+1}$, $X=[0{,}1;2]$

4.18 $f(x)=x^2+5x$, $X=[1;3]$

4.19 $f(x)=5x^2-2x-1$, $X=[1;5]$

4.20 $f(x)=x^3-1$, $X=[2;3]$

7- mustaqil ish.
Mavzu: Funksiya hosilasi va differensiali.
Hosila va differensial ta'riflari. Hosilaning geometrik va mexanik ma'nolari

$y = f(x)$ funksiya (a,b) oraliqda aniqlangan bo'lib, $x \in (a,b)$ bo'lsin. Bu x nuqtaga shunday Δx orttirma beraylikki, $x + \Delta x \in (a,b)$ bo'lsin.

1-ta'rif. Agar $\Delta x \to 0$ da $\frac{\Delta y}{\Delta x}$ nisbatning limiti

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad (1)$$

mavjud va chekli bo'lsa, bu limit $y = f(x)$- funksiyaning x nuqtadagi hosilasi deyiladi va $f^{'}(x)$ kabi belgilanadi.

2-ta'rif. Agar $\Delta x \to +0$ $(\Delta x \to -0)$da $\frac{\Delta y}{\Delta x}$ nisbatning limiti

$$\lim_{\Delta x \to +0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to +0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \left(\lim_{\Delta x \to -0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to -0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \right)$$

mavjud va chekli bo'lsa, bu limit $y = f(x)$- funksiyaning x nuqtadagi ung hosilasi (chap hosila si) deyiladi va $f'(x+0)$ ($f'(x-0)$)kabi belgilanadi.

1 va 2 - ta'riflardan quyidagilar kelib chiqadi:

1) Agar $y = f(x)$ funksiya x nuqtada $f'(x)$ hosilaga ega bo'lsa, u holda $f'(x+0)$ va $f'(x-0)$ lar mavjud va $f'(x+0) = f'(x-0) = f'(x)$ bo'ladi.

2) Agar $f'(x+0)$ va $f'(x-0)$ lar mavjud bo'lib, $f'(x+0) = f'(x-0)$ bo'lsa, unda $f'(x)$ ham mavjud va $f'(x) = f'(x+0) = f'(x-0)$ bo'ladi.

1-teorema. Agar x_0 nuqtada $f'(x_0)$ mavjud bo'lsa, u holda $y = f(x)$ funksiya grafigining $(x_0, f(x_0))$ nuqtasiga urinma o'tkazish mumkin va bu urinmaning burchak koeffitsienti $f'(x_0)$ ga teng bo'ladi.

$y = f(x_0) + f'(x_0) \cdot (x - x_0)$ - (2) - urinma tenglamasi.

$y = f(x_0) - \frac{1}{f'(x_0)} \cdot (x - x_0)$ - (3) - normal tenglamasi.

Agar $S = f(t)$ moddiy nuqtaning sonlar o'qidagi t vaqtga mos keluvchi o'rnini bildirsa, unda $\Delta f = f(t+\Delta t) - f(t)$-nuqtaning Δt vaqt oralig\`idagi ko'chishi, $\frac{f(t+\Delta t) - f(t)}{\Delta t}$ - o'rtacha tezlik, $f'(t)$ esa t momentdagi **oniy tezlik** bo'ladi.

3 - ta'rif. Agar Δy ni ushbu

$$\Delta y = f(x+\Delta x) - f(x) = A(x) \cdot \Delta x + \alpha(x, \Delta x) \cdot \Delta x \quad (4)$$

bu yrda $\Delta x \to 0$ da $\alpha(x,\Delta x) \to 0$ ko'rinishda ifodalash mumkin bo'lsa, unda $y = f(x)$ funksiya x nuqtada **differensiallanuvchi** deyiladi.

$A(x)\cdot\Delta x$ ifoda funksiya orttirmasining chiziqli bosh qismi yoki funksiya **differensiali** deb ataladi va dy kabi belgilanadi.

$\alpha(x,\Delta x)\cdot\Delta x$ ifoda funksiya orttirmasining **qoldiq hadi** deb ataladi. Agar 0-simvolikadan foydalansak, $\Delta x \to 0$ da $\Delta y = A\cdot\Delta x + o(\Delta x)$ tenglikni hosil qilamiz.

2-teorema. Agar $y = f(x)$ funksiya $x \in (a;b)$ nuqtada chekli hosilaga ega bo'lsa, funksiya shu nuqtada uzluksiz bo'ladi.

3-teorema. $y = f(x)$ funksiya $x \in (a;b)$ nuqtada differensiallanuvchi bo'lishi uchun shu nuqtada chekli $f'(x)$ mavjud bo'lishi zarur va yetarli.

Agar 3-teorema shartlari bajarilsa $df(x) = f'(x)\cdot\Delta x = f'(x)\cdot dx$ bo'ladi.

Murakkab funksiyaning hosilasi.

Aytaylik, $y = f(u)$ funksiya $(a;b)$da $u = \varphi(x)$ funksiya esa $(c;d)$ da berilgan bo'lib, ular yordamida $y = f[\varphi(x)]$ murakkab funksiya tuzilgan bo'lsin.

4-teorema. Agar $u = \varphi(x)$ funksiya $x_0 \in (a;b)$ nuqtada $\varphi'(x_0)$hosilaga ega bo'lib, $y = f(u)$ funksiya x_0 nuqtaga mos keluvchi $u_0\,(u_0 = f(x_0))$ nuqtada $f'(u_0)$ hosilaga ega bo'lsa, u holda murakkab funksiya $y = f[\varphi(x)]$ ham x_0 nuqtada hosilaga ega va

$$y'(x_0) = f'(u_0)\cdot\varphi'(x_0) \qquad (5)$$

tenglik o'rinli bo'ladi.

Differensialning taqribiy hisoblashga tatbiqi.

Ma'lumki, $y = f(x)$ funksiya x_0 nuqtada differensiallanuvchi bo'lsa, unda

$$\Delta f(x_0) = df(x_0) + o(\Delta x)$$

tenglik o'rinli bo'ladi. Agar $df(x_0) \neq 0$ bo'lsa, bu tenglikdan yyetarlicha kichik Δx lar uchun $\Delta f(x_0) \approx df(x_0)$ yoki

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\cdot\Delta x \qquad (10)$$

taqribiy hisoblash formulasini hosil qilamiz.

Differensiallashning umumiy qoidalari.

1. $y = c = const, \quad y' = 0$ 2. $y = c\cdot u\ (c = const), \quad y' = c\cdot u'$

3. $y = u \pm v, \quad y' = u' \pm v'$ 4. $y = u\cdot v, \quad y' = u'\cdot v + u\cdot v'$

5. $y = \dfrac{u}{v}\ (v(x) \neq 0), \quad y' = \dfrac{u'\cdot v - u\cdot v'}{v^2}$ 6. $y = f(u)\ (u = u(x)), \quad y' = f'\cdot u'_x$

7. $y = f(x), \quad x = f^{-1}(y) \quad y'_x = \dfrac{1}{x'_y}$ 8. $y = (u^v) \quad y' = u^v\cdot v'\cdot\ln u + u^{v-1}\cdot v\cdot u'$

Asosiy elementar funksiyalarning hosilalari.

1. $\left(x^n\right)=nx^{n-1}$ 2. $\left(x^x\right)=x^x\cdot(1+\ln x)$

3. $(\sin x)'=\cos x$ 4. $(\cos x)'=-\sin x$

5. $(tgx)'=\dfrac{1}{\cos^2 x}$ 6. $(ctgx)'=-\dfrac{1}{\sin^2 x}$

7. $(\ln x)'=\dfrac{1}{x}$ 8. $(\log_a x)'=\dfrac{1}{x\ln a}\quad (a>0,\ a\neq 1)$

9. $\left(e^x\right)=e^x$ 10. $\left(a^x\right)=a^x\ln a\ (a>0)$

11. $(\arcsin x)'=\dfrac{1}{\sqrt{1-x^2}}$ 12. $(\arccos x)'=-\dfrac{1}{\sqrt{1-x^2}}$

13. $(arctgx)'=\dfrac{1}{1+x^2}$ 14. $(arcctgx)'=-\dfrac{1}{1+x^2}$

15. $(shx)'=chx$ 16. $(chx)'=shx$

17. $(thx)'=\dfrac{1}{ch^2x}$ 18. $(cthx)'=-\dfrac{1}{sh^2x}$

19. $(arcshx)'=\dfrac{1}{\sqrt{x^2+1}}$ 20. $(arcchx)'=\dfrac{1}{\sqrt{x^2-1}}$

21. $(arcthx)'=\dfrac{1}{1-x^2}$ 22. $(arccthx)'=-\dfrac{1}{1-x^2}$

1 –misol. Hosila ta'rifidan foydalanib $f'(0)$ topilsin.

$$f(x)=\begin{cases} x^2cos\dfrac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$$

◄ $\lim\limits_{\Delta x\to 0}\dfrac{f(0+\Delta x)-f(0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{(\Delta x)^2\cos\dfrac{1}{\Delta x}-0}{\Delta x}=\lim\limits_{\Delta x\to 0}\Delta x\cos\dfrac{1}{\Delta x}=0.$

Demak $f'(0)=0$ ►

2– misol. Funksiya grafigining abtsissasi x_0 bulgan nuqtasiga o'tkazilgan normal tenglamasi topilsin. ▷

$$y=\frac{\sqrt[3]{x^2}}{x^2+1},\quad x_0=-1$$

◀Ma'lumki, normal tenglamasi $y=f(x_0)-\frac{1}{f'(x_0)}\cdot(x-x_0)$

ko'rinishga ega. $f(x_0)=f(-1)=\frac{1}{(-1)^2+1}=\frac{1}{2}$

$$f'(x)=\left(\frac{\sqrt[3]{x^2}}{x^2+1}\right)'=\frac{\frac{2}{3}x^{-\frac{1}{3}}(x^2+1)-2x\sqrt[3]{x^2}}{(x^2+1)^2}=\frac{2(x^2+1)-6x^2}{3\sqrt[3]{x}(x^2+1)^2}=\frac{1-4x^2}{3\sqrt[3]{x}(x^2+1)^2},$$

$=f'(-1)=\frac{1-4}{-12}=\frac{1}{4}\Rightarrow y=\frac{1}{2}-4(x+1)\Rightarrow$ normal tenglamasi:

$8x+2y+7=0$ ▶

3 – misol. Hosila hisoblansin.

$$y=\frac{\arcsin\sqrt[3]{x^2}}{\sqrt{x^3+1}}$$

$$◀\ y'=\left(\frac{\arcsin\sqrt[3]{x^2}}{\sqrt{x^3+1}}\right)'=\frac{\frac{1}{\sqrt{1-\sqrt[3]{x^4}}}\cdot\frac{2}{3}\cdot\frac{\sqrt{x^3+1}}{\sqrt[3]{x}}-\frac{3x^2}{2\sqrt{x^3+1}}\arcsin\sqrt[3]{x^2}}{x^3+1}=$$

$$=\frac{4x^3+4-9x^{\frac{7}{3}}\sqrt{1-\sqrt[3]{x^4}}\arcsin\sqrt[3]{x^2}}{6\sqrt[3]{x}\sqrt{\left(1-\sqrt[3]{x^4}\right)\left(x^3+1\right)^3}}\ ▶$$

4– misol. Differensial yordamida ifodaning taqribiy qiymati hisoblansin.

$$y=\sqrt[3]{x^5-5},\quad x=2{,}002$$

◀Taqribiy qiymat $f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\cdot\Delta x$ (1)

formula yordamida hisoblanadi.

Bizda

$f(x)=\sqrt[3]{x^5-5},\ \ x=2{,}002$

$x_0=2,\ \ \Delta x=0{,}002\Rightarrow f'(x)=(\sqrt[3]{x^5-5})'=[(x^5-5)^{\frac{1}{3}}]'=$

$\frac{1}{3}(x^5-5)^{-\frac{2}{3}}(x^5-5)'=\frac{5x^4}{3\sqrt[3]{(x^5-5)^2}}\Rightarrow f(2)=\sqrt[3]{32-5}=\sqrt[3]{27}=3,$

$$f'(2)=\frac{80}{3\cdot 9}=\frac{80}{27}$$

Topilgan ifodalarni (1) tenglikka olib borib qo'yamiz:

$$\sqrt[3]{(2{,}002)^5-5}\approx 3+\frac{80}{27}\cdot 0{,}002=3\frac{4}{675}$$ ►

Mustaqil yechish uchun misollar.

1 - topshiriq. Hosila ta'rifidan foydalanib $f'(0)$ topilsin.

1.1 $f(x)=\begin{cases}\left(2x^2+x^3 cos\frac{1}{x}\right), & x\neq 0,\\ 0, & x=0\end{cases}$

1.2 $f(x)=\begin{cases}\arcsin\left(x\sin\frac{1}{x}\right), & x\neq 0,\\ 0, & x=0\end{cases}$

1.3 $f(x)=\begin{cases}\arccos\left(x^2\cos\frac{1}{x}\right)+\frac{2}{3}x, & x\neq 0,\\ 0, & x=0\end{cases}$

1.4 $f(x)=\begin{cases}0, & x=0,\\ \sqrt{\left(1+x^2\sin\frac{1}{x}\right)^2-1}, & x\neq 0\end{cases}$

1.5 $f(x)=\begin{cases}\arcsin\left(x\cos\frac{5x}{x}\right), & x\neq 0,\\ 0, & x=0\end{cases}$

1.6 $f(x)=\begin{cases}\frac{x}{1+e^{\frac{1}{x}}}, & x\neq 0,\\ 0, & x=0\end{cases}$

1.7 $f(x)=\begin{cases}\frac{|\sin x|}{|x|}, & x\neq 0,\\ 0, & x=0\end{cases}$

1.8 $f(x)=\begin{cases}0, & x=0,\\ x^2\cdot\cos\frac{4}{3x}+\frac{x^2}{2}, & x\neq 0\end{cases}$

1.9 $f(x)=\begin{cases}arctg\left(x^3-x^{\frac{3}{2}}\sin\frac{1}{3x}\right), & x\neq 0,\\ 0, & x=0\end{cases}$

1.10

$f(x)=\begin{cases}x^2\cdot\cos^2\frac{11}{x}, & x\neq 0,\\ 0, & x=0\end{cases}$

1.11 $f(x)=\begin{cases}\sin x\cdot\cos\frac{5}{x}, & x\neq 0,\\ 0, & x=0\end{cases}$

1.12 $f(x)=\begin{cases}2x^2+x^2\cos\frac{1}{x}, & x\neq 0,\\ 0, & x=0\end{cases}$

1.13 $f(x)=\begin{cases} x+\arcsin\left(x^2\sin\frac{6}{x}\right), & x\neq 0,\\ 0, & x=0\end{cases}$ 1.14 $f(x)=\begin{cases} \frac{1-\cos x}{x}, & x\neq 0,\\ 0, & x=0\end{cases}$

1.15 $f(x)=\begin{cases} tg\left(2^{x^{2\cos\frac{1}{8x}}}-1+x\right), & x\neq 0,\\ 0, & x=0\end{cases}$ 1.16 $f(x)=\begin{cases} 6x+x\sin\frac{1}{x}, & x\neq 0,\\ 0, & x=0\end{cases}$

1.17 $f(x)=\begin{cases} arctgx\cdot\sin\frac{7}{x}, & x\neq 0,\\ 0, & x=0\end{cases}$ 1.18 $f(x)=\begin{cases} x\sin\frac{1}{x^2}, & x\neq 0,\\ 0, & x=0\end{cases}$

1.19 $f(x)=\begin{cases} x^2+x\sin\frac{1}{x^2}, & x\neq 0,\\ 0, & x=0\end{cases}$ 1.20 $f(x)=\begin{cases} e^{x^2\sin\frac{1}{x}}-1, & x\neq 0,\\ 0, & x=0\end{cases}$

2 – topshiriq. Funksiya grafigining abtsissasi x_0 nuqtasiga o'tkazilgan urinma (2.1 – 2.12 misollarda) yoki normal (2.13 – 2.21 misollarda) tenglamasi topilsin.

2.1 $y=\frac{x^2}{4}+2x-4,\ x_0=2$ 2.2 $y=x^2+4x-2,\ x_0=2$

2.3 $y=2x^2-x^3,\ x_0=1$ 2.4 $y=x^2+8x^4-32,\ x_0=4$

2.5 $y=x^2+\sqrt{x},\ x_0=1$ 2.6 $y=\sqrt[3]{x^2}+2,\ x_0=-4$

2.7 $y=\frac{1+x\sqrt{x}}{2},\ x_0=9$ 2.8 $y=\sqrt[3]{x}-5,\ x_0=8$

2.9 $y=2x^2-3x+1,\ x_0=1$ 2.10 $y=\frac{x^2-3x+6}{x^2},\ x_0=3$

2.11 $y=\sqrt{x}-3\sqrt[3]{x}-1,\ x_0=64$ 2.12 $y=\frac{x^3+3}{x^3-2},\ x_0=2$

2.13 $y=2x^2+3,\ x_0=-1$ 2.14 $y=\frac{x^{29}+6}{x^4+1},\ x_0=1$

2.15 $y=2x^3+\frac{1}{x^2}+1,\ x_0=1$ 2.16 $y=5x\sin x,\quad x_0=\frac{\pi}{4}$

2.17 $y=\frac{x^5+1}{5}\cos x,\ x_0=\pi$ 2.18 $y=\frac{x^4+3x^2-1}{2-x^2},\ x_0=-1$

2.19 $y=(x^2-2x)e^x$, $x_0=0$

2.20 $y=\frac{1}{3\sqrt{x}+1}$, $x_0=4$

3– topshiriq. Hosila xisoblansin.

31 $y=\frac{x^2+3\sqrt{x}}{\sqrt{x^3}}$

3.2 $y=2\sqrt{\frac{1-\sqrt{x}}{x^2}}$

33 $y=\frac{(x^2+2x)\sqrt{x^2-1}}{2x}$

3.4 $y=\frac{(x^3+2)\sqrt{x^2-4x}}{x^2+1}$

35 $y=\frac{\sin x}{x^2+x}$

3.6 $y=\frac{x\sin\frac{x}{2}}{\sqrt{x^2+5}}$

37 $y=\frac{\sin\sqrt{x+1}}{x^2}$

3.8 $y=\frac{(2x^2+3)\cos\sqrt{x}}{x^3+1}$

39 $y=\frac{x^2\cos x}{\sqrt{x^2+2x}}$

3.10 $y=e^x\sqrt[5]{x^3+\frac{1}{x}}$

311 $y=\sqrt{\frac{x+1}{(x-1)^2}e^x+2}$

3.12 $y=\frac{\arcsin(x-2)}{x^2}$

313 $y=\frac{\sqrt[3]{x^2+1}}{x+1}\ln\left(x^2+1\right)$

3.14 $y=\frac{x^5\arccos\sqrt{x}}{\sqrt{4-x^3}}$

315 $y=\frac{arctg(x+2)}{\sqrt{x^2+4x+5}}$

3.16 $y=\frac{\cos\sqrt{(1+x)^3}}{x^2}$

317 $y=\frac{\sqrt{x-1}tg(x+2)}{4x^2-4}$

3.18 $y=arctg\frac{\sqrt{4+x^2}}{x^3}$

319 $y=\frac{e^{x^2}-1}{\sqrt{1+2x^2}}$

3.20 $y=\frac{2^x\ln x}{x\sqrt[3]{(2+x)^2}}$

4 – topshiriq. Differensial yordamida ifodaning taqribiy qiymati hisoblansin.

4.1 $y=\sqrt[5]{x}-1$, $x=1,26$

4.2 $y=\sqrt[3]{x}+2$, $x=64,14$

4.3 $y=\frac{\sqrt{5x^2-1}}{2}$, $x=1,28$

4.4 $y=arctgx$, $x=0,08$

4.5 $y=\sqrt[3]{x^3+2x}$, $x=0,15$

4.6 $y=\sqrt{x^3+x^2+4}$, $x=1,98$

4.7 $y=\sqrt[3]{x^2+4},\ x=2,05$ 4.8 $y=\sqrt{x+\cos x},\ x=0,02$

4.9 $y=x^9+1,\ x=0,892$ 4.10 $y=x^8,\ x=2,02$

4.11 $y=\sqrt{4x-1},\ x=2,56$ 4.12 $y=\sqrt[5]{x^2},\ x=1,03$

4.13 $y=\dfrac{1}{\sqrt{2x^2+x+1}},\ x=1,016$ 4.14 $y=x^{21},\ x=0,998$

4.15 $y=\dfrac{1}{\sqrt{x}},\ x=4,16$ 4.16 $y=\sqrt{1+\sin x},\ x=0,01$

4.17 $y=2x^8+\sqrt{2x},\ x=2,01$ 4.18 $y=\sqrt[3]{2x-1},\ x=1,02$

4.19 $y=\sqrt{3x-\cos\pi x},\ x=1,05$ 4.20 $y=\sqrt{3x^2+\sin\dfrac{\pi x}{2}},\ x=1,06$

8 - mustaqil ish.

Mavzu: Turli ko'rinishda berilgan funksiyalarning hosilalari. Yuqori tartibli hosilalar.

a) Teskari funksiyaning hosilasi.

1 –teorema. $f(x)$ funksiya $(a;b)$ da aniqlangan, uzluksiz va qat'iy o'suvchi (qat'iy kamayuvchi) bo'lsin. Agar $f(x)$ funksiya $x_0\in(a;b)$ nuqtada $f'(x_0)\neq 0$ hosilaga ega bo'lsa, bu funksiyaga teskari $x=f^{-1}(y)$ funksiya x_0 nuqtaga mos bo'lgan y_0 nuqtada hosilaga ega va

$$\left[f^{-1}(y)\right]'_{y=y_0}=\frac{1}{f'(x_0)} \qquad (1)$$

tenglik o'rinli.

b) Parametrik ko'rinishda berilgan funksiyaning hosilasi.

Faraz qilaylik, $y=y(x)$ funksiya parametrik ko'rinishda.

$$\begin{cases} x=\varphi(t) \\ y=\Psi(t) \end{cases} \quad \alpha<t<\beta \qquad (2)$$

sistema yordamida aniqlangan bo'lsin. Agar $\varphi(t)$ va $\Psi(t)$ funksiyalar differensiallanuvchi bo'lib, $\varphi'(t)\neq 0$ bo'lsa, unda (2)-sistema differensiallanuvchi $y=\Psi\left[\varphi^{-1}(x)\right]$ funksiyani aniqlaydi va

$$y'_x=\frac{y'_1}{x'_1}=\frac{\Psi'(t)}{\varphi'(t)} \qquad (3)$$

tenglik o'rinli bo'ladi.

c) Oshkormas funksiyaning hosilasi.

Agar biror oraliqda differensiallanuvchi bo'lgan $y=y(x)$ funksiya $F(x,y)=0$ tenglik yordamida aniqlansa, unda oshkormas ko'rinishda berilgan funksiyaning $y'=y'(x)$ hosilasini ushbu

$$\frac{d}{dx}F(x,y)=0 \qquad (4)$$

tenglikdan topish mumkin.

Masalan, ushbu $y^5 \sin x + y^2 + x = 0$ tenglik yordamida oshkormas ko'rinishda berilgan $y = y(x)$ funksiyaning y' hosilasini topaylik.

(4) tenglikka ko'ra

$$(y^5 \sin x + y^2 + x)'_x = 0 \Rightarrow 5y^4 \cdot y' \sin x + y^5 \cdot \cos x + 2yy' + 1 = 0 \Rightarrow$$

$$\Rightarrow y' = \frac{-1 - y^5 \cos x}{5y^4 \sin x + 2y}$$

d) darajali ko'rsatkichli funksiya hosilasi.

$y = [u(x)]^{v(x)}$ $(u(x) > 0)$ bo'lib, $u(x)$ va $v(x)$ funksiyalar $u'(x)$ va $v'(x)$ hosilalarga ega bo'lsin. U holda darajali ko'rsatkichli funksiya hosilasi

$$\left([u(x)]^{v(x)}\right)' = \left([u(x)]^{v(x)}\right) \cdot \left[v'(x) \cdot \ln u(x) + \frac{v(x)}{u(x)} u'(x) \right] \qquad (5)$$

ga teng bo'ladi.

Yuqori tartibli hosila va differensiallar.

a) $y = y(x)$ funksiyaning yuqori tartibli hosila va differensiallari ushbu

$$f^{(n)}(x) = \left\{ f^{(n-1)}(x) \right\}' \qquad (n = 2,3,...),$$

$$d^n y = d(d^{n-1} y) \qquad (n = 2,3,...),$$

tengliklar yordamida aniqlanadi.

Asosiy formulalar

1) $(a^x)^n = a^x \cdot \ln^n a \quad (a > 0);\ (e^x)^{(n)} = e^x$

2) $(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2} \right)$

3) $(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2} \right)$

4) $(x^\alpha)^{(n)} = \alpha(\alpha - 1)...(\alpha - n + 1)x^{\alpha - n},\ \alpha \in R$

5) $(\ln x)^{(n)} = \frac{(-1)^{n-1} \cdot (n-1)}{x^n}$

Leybnis formulasi.

Agar $u = u(x)$ va $v = v(x)$ funksiyalar n - tartibli hosilalarga ega bo'lsa, unda $y = u(x) \cdot v(x)$ funksiya ham n - tartibli hosilaga ega bo'ladi va

$$y^{(n)}(u\cdot v)^{(n)}=\sum_{k=0}^{n}C_n^k u^{(k)}v^{(n-k)} \qquad (6)$$

tenglik o'rinli bo'ladi. Bu yerda $u^{(0)}=u, \quad v^{(0)}=v$ va $C_n^k=\dfrac{n!}{k!(n-k)!}$

(6) - formulaga n - tartibli hosilani hisoblash uchun **Leybnis formulasi** deyiladi.

$u(x)\cdot v(x)$ funksiyaning n - tartibli differensiali $d^n(u\cdot v)$ uchun ham Leybnis formulasi o'rinli.

1–misol. Hosila hisoblansin.

$$y=x^{x\cos x}\cdot 2^x$$

◀

$$y'=\left(x^{x\cos x}\cdot 2^x\right)=\left(x^{x\cos x}\right)\cdot 2^x+x^{x\cos x}\left(2^x\right)=\left(e^{\ln x^{x\cos x}}\right)\cdot 2^x+x^{x\cos x}\cdot\left(2^x\ln 2\right)=$$

$$=e^{\ln x^{x\cos x}}\left((\cos x-x\sin x)\ln x+\cos x\right)\cdot 2^x+x^{x\cos x}\cdot\left(2^x\ln 2\right)=$$

$$=x^{x\cos x}\cdot 2^x\left(\cos x\cdot\ln x-x\cdot\sin x\cdot\ln x+\cos x+\ln 2\right)$$

▶

2– misol. Funksiya grafigining abtsissasi $x_0=x(t_0)$ bo'lgan nuqtasiga o'tkazilgan urinma va normal tenglamalari topilsin.

$$\begin{cases} x=t-t^2, \\ y=1-t^3, \ \ t_0=1 \end{cases}$$

◀Biz $y=f(x_0)+f'(x_0)\cdot(x-x_0)$ - (1) (urinma tenglamasi),

$y=f(x_0)-\dfrac{1}{f'(x_0)}\cdot(x-x_0)$ - (2) (normal tenglamasi),

va $y'_x=\dfrac{y'_t}{x'_t}$ - (3) (parametrik ko'rinishda berilgan funksiyaning hosilasi)

formulalardan foydalanamiz:

$$x_0=1-1^2=0;\ f(x_0)=1-1^3=0;\ \ y'_x=\frac{(1-t^3)'}{(t-t^2)'}=\frac{-3t^2}{1-2t}\Rightarrow f'(x_0)=\frac{-3}{-1}=3$$

Topilgan qiymatlarni (1) va (2) tenglamalarga olib borib qo'yib urinma va normal tenglamalarni topamiz:

$$\begin{cases} y=3(x-0) \\ y=-\dfrac{1}{3}(x-0) \end{cases}\Rightarrow\begin{cases} y=3x\ -urinma \\ y=-\dfrac{1}{3}x-normal \end{cases} \qquad ▶$$

3– misol. Parametrik ko'rinishda berilgan funksiyaning ikkinchi tartibli hosilasi hisoblansin.

$$\begin{cases} x = t + \cos t \\ y = 2 + \sin t \end{cases}$$

◄Bu masalani (3) – formuladan ikki marta foydalanish yordamida yechamiz.

$$y'_x = \frac{y'_t}{x'_t} = \frac{(2+\sin t)'}{(t+\cos t)'} = \frac{\cos t}{1-\sin t} = \frac{1+\sin t}{\cos t}$$

$$y'_{x^2} = \frac{(y'_t)'_t}{x'_t} = \frac{\left(\frac{1+\sin t}{\cos t}\right)'}{1-\sin t} = \frac{\cos^2 t + \sin t(1+\sin t)}{\cos^2 t(1-\sin t)} = \frac{1}{(1-\sin t)^2}$$ ►

4– misol. n **- tartibli hosila hisoblansin.**

$$y = \frac{1}{x^2-3x+2}$$

◄ $$y = \frac{1}{x^2-3x+2} = \frac{1}{(x-2)(x-1)} = \frac{1}{x-2} - \frac{1}{x-1}$$

$$y^{(n)} = \left(\frac{1}{x-2} - \frac{1}{x-1}\right)^{(n)} = \left(\frac{1}{x-2}\right)^{(n)} - \left(\frac{1}{x-1}\right)^{(n)} = \left[(x-2)^{-1}\right]^{(n)} - \left[(x-1)^{-1}\right]^{(n)} =$$

$$= (-1)(-1-1)\ldots(-1-n+1)(x-2)^{-1-n} - (-1)(-1-1)\ldots(-1-n+1)(x-1)^{-1-n} =$$

$$= (-1)^n n!\left[(x-2)^{-1-n} - (x-1)^{-1-n}\right] = (-1)^n n!\left(\frac{1}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right)$$

Demak, $\forall n \in N$ uchun

$$y^{(n)} = (-1)^n n!\left(\frac{1}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right)$$ bo'ladi.►

Mustaqil yechish uchun misollar.

1 – topshiriq. Hosila hisoblansin.

1.1 $y = (\sin x)^{\frac{1}{2}arctgx}$ 1.2 $y = (\cos\sqrt{x})^{\ln\cos\sqrt{x}}$

1.3 $y = (\sin x)^{5\ln(\sin x)}$ 1.4 $y = (\arccos x)^{\cos x}$

1.5 $y = (arctgx)^{e^x-1}$ 1.6 $y = x^{\ln\cos x}$

1.7 $y = (tg3x)^{2arctgx}$ 1.8 $y = (x^2+1)^{arctgx}$

1.9 $y = (arctgx)^{4e^x}$ 1.10 $y = (\arccos 5x)^{e^x+1}$

1.11 $y=(\sin x)^{\cos(x^2+1)}$

1.12 $y=(x^5+4)^{\arcsin x}$

1.13 $y=x^{\sin(\cos x)}$

1.14 $y=(x^3+\cos x)^{tgx}$

1.15 $y=(\arcsin x)^{\ln\frac{5x}{2}}$

1.16 $y=\sin(x^2+1)^{\cos x}$

1.17 $y=e^{x^{19}}\cdot x^2$

1.18 $y=x^x\cdot 3^{x^2}$

1.19 $y=(\cos\sqrt{x})^{\ln(\cos x)}$

1.20 $y=x^{e^{\sin x}}$

2 – topshiriq. Funksiya grafigining abtsissasi $x_0=x(t_0)$ bo'lgan nuqtasiga o'tkazilgan urinma va normal tenglamalari topilsin.

2.1 $\begin{cases} x=4t+t^2 \\ y=2t-t^3,\ t_0=1 \end{cases}$

2.2 $\begin{cases} x=2(t\cos t-2\sin t) \\ y=t(t\sin t+2\cos t),\ t_0=\dfrac{\pi}{4} \end{cases}$

2.3 $\begin{cases} x=\sqrt{t}-\cos t \\ y=\sin t,\ t_0=\dfrac{\pi}{6} \end{cases}$

2.4 $\begin{cases} x=\dfrac{t+1}{1+t^2} \\ y=\dfrac{t^2+1}{1+t^2},\ t_0=2 \end{cases}$

2.5 $\begin{cases} x=3\sin^2 t \\ y=3\cos^2 t,\ t_0=\dfrac{\pi}{4} \end{cases}$

2.6 $\begin{cases} x=2(ctgt) \\ y=tgt+ctgt,\ t_0=\dfrac{\pi}{3} \end{cases}$

2.7 $\begin{cases} x=3(t-\sin t) \\ y=3(1-\cos t),\ t_0=\dfrac{\pi}{3} \end{cases}$

2.8 $\begin{cases} x=t^2\cos t \\ y=t^2\sin t,\ t_0=\dfrac{\pi}{2} \end{cases}$

2.9 $\begin{cases} x=\sin^2 t \\ y=\cos^2 t,\ t_0=\dfrac{\pi}{6} \end{cases}$

2.10 $\begin{cases} x=\dfrac{t+1}{t} \\ y=\dfrac{t-1}{t},\ t_0=-1 \end{cases}$

2.11 $\begin{cases} x=\arcsin\dfrac{t}{\sqrt{1+t^2}} \\ y=\arccos\dfrac{1}{\sqrt{1+t^2}},\ t_0=1 \end{cases}$

2.12 $\begin{cases} x=\ln(1+t^2) \\ y=t-arctgt,\ t_0=1 \end{cases}$

2.13 $\begin{cases} x=\dfrac{1+\ln t}{t^2} \\ y=\dfrac{3+2\ln t}{t},\ t_0=1 \end{cases}$

2.14 $\begin{cases} x=t(1-\sin t) \\ y=t\cos t,\ t_0=0 \end{cases}$

2.15 $\begin{cases} x = \dfrac{t-1}{t^2} \\ y = \dfrac{1}{2t^3} + \dfrac{2}{t}, \ t_0 = 2 \end{cases}$

2.16 $\begin{cases} x = \dfrac{1+t^3}{t^2} \\ y = \dfrac{t+1}{t^2}, \ t_0 = 1 \end{cases}$

2.17 $\begin{cases} x = a\sin^3 t \\ y = a\cos^3 t, \ t_0 = \dfrac{\pi}{4} \end{cases}$

2.18 $\begin{cases} x = \cos^3 t \\ y = 4\sin^2 t, \ t_0 = \dfrac{\pi}{4} \end{cases}$

2.19 $\begin{cases} x = 2(t\sin t + \cos t) \\ y = 2(\sin t - t\cos t), \ t_0 = \dfrac{\pi}{6} \end{cases}$

2.20 $\begin{cases} x = -t^4 + t^2 \\ y = t^2 - t^3, \ t_0 = -1 \end{cases}$

3 – topshiriq. Parametrik ko'rinishda berilgan funksiyaning ikkinchi tartibli hosilasi hisoblansin.

3.1 $\begin{cases} x = \sin 2t \\ y = 2\cos t \end{cases}$

3.2 $\begin{cases} x = \sqrt{t^3 - 1} \\ y = t^2 - 1 \end{cases}$

3.3 $\begin{cases} x = \sqrt{t^2 - 1} \\ y = \dfrac{1}{\sqrt{t}} \end{cases}$

3.4 $\begin{cases} x = \sqrt{t} - 1 \\ y = \dfrac{1}{\sqrt{t}} \end{cases}$

3.5 $\begin{cases} x = e^t \cos t \\ y = e^t \sin t \end{cases}$

3.6 $\begin{cases} x = \cos^2 t \\ y = \sin^2 t \end{cases}$

3.7 $\begin{cases} x = 1 + \sin t \\ y = 1 - \cos t \end{cases}$

3.8 $\begin{cases} x = \sqrt{t} + t \\ y = \sin(t-2) \end{cases}$

3.9 $\begin{cases} x = \ln\dfrac{1}{t} \\ y = \dfrac{1}{1+t^2} \end{cases}$

3.10 $\begin{cases} x = \ln(\sin t) \\ y = \ln(\cos t) \end{cases}$

3.11 $\begin{cases} x = \sqrt{t} \\ y = \dfrac{1}{\sqrt{1-t}} \end{cases}$

3.12 $\begin{cases} x = t - \sin t \\ y = 2 - \cos t \end{cases}$

3.13 $\begin{cases} x = \sin t \\ y = \sec t \end{cases}$

3.14 $\begin{cases} x = \cos t \\ y = \ln(\sin t) \end{cases}$

3.15 $\begin{cases} x = tgt \\ y = \dfrac{1}{\sin 2t} \end{cases}$

3.16 $\begin{cases} x = \cos t + t\sin t \\ y = \sin t - t\cos t \end{cases}$

3.17 $\begin{cases} x = \ln\sqrt{t-1} \\ y = \dfrac{t}{\sqrt{t-1}} \end{cases}$

3.18 $\begin{cases} x = t + 1 \\ y = \arcsin t \end{cases}$

3.19 $\begin{cases} x = \sqrt{t} + t^2 \\ y = \sqrt[3]{t} \end{cases}$

3.20 $\begin{cases} x = 4(t^2 - \cos t) \\ y = 4(t + \sin t) \end{cases}$

4 – topshiriq. n **- tartibli hosila hisoblansin.**

4.1 $y = \dfrac{x+2}{3(x+1)}$

4.2 $y = \sqrt[5]{x+1}$

4.3 $y = \dfrac{2x+3}{8x+6}$

4.4 $y = \ln(x+2)$

4.5 $y = 3^{3x} + 1$

4.6 $y = \sin ax + \cos bx$

4.7 $y = \sin^4 x + \cos^4 x$

4.8 $y = 2^{3x+1}$

4.9 $y = \sin 2x + \cos 2x$

4.10 $y = \sqrt[3]{e^{2x+1}}$

4.11 $y = \dfrac{4x+15}{5x+1}$

4.12 $y = \lg(3x+1)$

4.13 $y = 7^{5x}$

4.14 $y = \dfrac{x}{9(4x+9)}$

4.15 $y = \dfrac{4}{x}$

4.16 $y = \dfrac{5x+1}{13(2x+3)}$

4.17 $y = 3^{2x+5}$

4.18 $y = \dfrac{1+x^2}{1-x^2}$

4.19 $y = (x-1)\cdot 2^{x-1}$

4.20 $y = \dfrac{x}{\sqrt{1-5x}}$

9 –mustaqil ish.
Mavzu: Differensial hisobning tadbiqlari.
Differensial xisobning asosiy teoremalari.

Aytaylik , $y = y(x)$ funksiya $[a,b]$ da aniqlangan bo'lsin.

1 - teorema. (Ferma teoremasi). Agar $f(x)$ funksiya $(a;b)$ da aniqlangan va bu oraliqning ichki $c \in (a,b)$ nuqtasida o'zining eng katta (eng kichik) qiymatiga erishsin. Agar bu nuqtada funksiya chekli $f'(c)$ hosilaga ega bo'lsa, u holda $f'(c) = 0$ bo'ladi.

2 - teorema. (Roll teoremasi). Agar
1) $f(x) \in C[a,b]$
2) $\forall x \in (a,b)$ uchun chekli $f'(x)$ mavjud
3) $f(a) = f(b)$ bo'lsa $\exists x_0 \in (a,b)$ nuqta topiladiki, $f'(x_0) = 0$
bo'ladi.

3 - teorema. (Lagranj teoremasi). Agar
1) $f(x) \in C[a,b]$
2) $\forall x \in (a,b)$ uchun chekli $f'(x)$ mavjud
bo'lsa $\exists x_0 \in (a,b)$ nuqta topiladiki,

$$f(a) - f(b) = f'(x_0) \cdot (b - a)$$

bo'ladi.

1 - natija. Arap $\forall x \in (a,b)$ uchun $f'(x) = 0$ bo'lsa, unda (a,b) da $f(x) = const$ bo'ladi.

2 - natija. Agar $f(x)$ funksiya (a,b) intervalda chegaralangan $f'(x)=0$ hosilaga ega bo'lsa, u holda $f(x)$ (a,b) da tekis uzluksiz bo'ladi.

Lagranj teoremasini ba'zi bir tengsizliklarni isbotlashda qo'llash mumkin. Masalan, $(1 + x^{\alpha}) \geq 1 + \alpha x$ **Bernulli tengsizligi** $\forall x > -1$ va $a > 1$ da o'rinli ekanligi isbotlansin.

◄ **1 – hol.** $x > 0$ bo'lsin. Unda $f(u) = (1 + u)^{\alpha}$, $u \in [0, x]$ funksiya uchun Lagranj teoremasiga ko'ra $\exists x_0 \in (0, x)$ nuqta topiladiki

$f(x) - f(0) = (1 + x)^{\alpha} - 1 = \alpha \cdot (1 + x_0)^{\alpha - 1} \cdot x > \alpha x$ bo'ladi $\Rightarrow (1 + x)^{\alpha} > 1 + \alpha x$

2 – hol. $-1 < x < 0$ bo'lsin. Unda $f(u) = (1 + u)^{\alpha}$, $u \in [x,0]$ funksiya uchun Lagranj teoremasini qo'llaymiz. $\Rightarrow \exists x_0 \in (x,0)$

$f(0) - f(x) = 1 - (1 + x)^{\alpha} = \alpha \cdot (1 + x_0)^{\alpha - 1} \cdot (0 - x) = (1 + x_0 < 1) < -\alpha x \Rightarrow$

$\Rightarrow (1 + \alpha x)^{\alpha} > 1 + \alpha x$

3 – hol. $x = 0$ bo'lsin. Unda $(1 + x)^{\alpha} = 1 + \alpha x = 1$ bo'ladi. Endi 3 ta holni umumlashtirsak, isbot qilishimiz kerak bo'lgan Bernulli tengsizligini hosil qilamiz. ►

4 – teorema (Koshi teoremasi). Agar
1) $f(x), g(x) \in C[a,b]$,

2) $\forall x \in (a,b)$ uchun chekli $f'(x)$ va $g'(x)$ mavjud hamda $g'(x) \neq 0$ bo'lsa, unda $\forall x_0 \in (a,b)$ nuqta topiladiki,

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

tenglik o'rinli bo'ladi.

Funksiya limitini hisoblashda $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 1^{∞}, 0^0 va shu kabi aniqmasliklarga duch kelinadi. Bu aniqmasliklarni ochishda Lopital qoidalari katta yordam beradi.

5-teorema. $f(x)$ va $g(x)$ funksiyalar uchun quyidagi shartlar o'rinli bo'lsin.
1) $f(x)$ va $g(x)$ funksiyalar a nuqtaning biror atrofida aniqlangan va chekli hosilaga ega,

2) $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0,$

3) a nuqtaning shu atrofida $[f'(x)]^2 + [g'(x)]^2 \neq 0,$

4) $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ **-** chekli yoki cheksiz.

U holda $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ tenglik o'rinli bo'ladi.

Izoh: Agar bu teoremaning shartlari Δx nuqtaning chap (yoki o'ng) yarim atrofida bajarilsa, unda teorema $\frac{f(x)}{g(x)}$ ning a nuqtadgi chap (yoki o'ng) limitiga nisbatan o'rinli bo'ladi. Yuqoridagi $\frac{0}{0}$ ko'rinishidagi aniqmasliklar uchun keltirilgan Lopital teoremasi $\frac{\infty}{\infty}$ ko'rinishidagi aniqmasliklar uchun ham o'rinli bo'ladi. Boshqa ko'rinishdagi aniqmasliklar esa $\frac{0}{0}$ va $\frac{\infty}{\infty}$ ko'rinishidagi aniqmasliklarga keltiriladi.

Funksiyalarni tekshirish.

a) Funksiyaning monotonligi.

Faraz qilaylik, $y = f(x)$ funksiya (a,b) oraliqda berilgan bo'lsin.

1-ta'rif. $x_2 > x_1$ tengsizlikni qanoatlantiruvchi $\forall x_1, x_2 \in (a,b)$ uchun $f(x_2) \geq f(x_1)$ $(f(x_2) \leq f(x_1))$ bo'lsa, $f(x)$ funksiya (a,b) oraliqda **o'suvchi** $\uparrow$ **(kamayuvchi** $\downarrow$**)** deyiladi.

Agar funksiya o'suvchi yoki kamayuvchi bo'lsa, bunday funksiya **monoton** funksiya deyiladi.

1 - teorema. $f(x)$ funksiya (a,b) intervalda chekli $f'(x)$ hosilaga ega bo'lsin. Bu funksiya shu intervalda o'suvchi (kamayuvchi) bo'lishi uchun (a,b) da $f'(x) \geq 0$ $(f'(x) \leq 0)$ bo'lishi zarur va yyetarli.

b) Funksiyaning ekstremumlari.

$y = f(x)$ funksiya (a,b) intervalda berilgan bo'lib, $x_0 \in (a,b)$ bo'lsin.

2 - ta'rif. Agar x_0 nuqtaning shunday $\bigcup_\delta (x_0)$ atrofi mavjud bo'lsaki, $\forall x \in \bigcup_\delta (x_0)$ uchun $f(x) \leq f(x_0)$ $(f(x) \geq f(x_0))$ tengsizlik o'rinli bo'lsa, $f(x)$ funksiya x_0 nuqtada **maksimumga (minimumga)** erishadi deyiladi. $f(x_0)$ qiymat $f(x)$ ning maksimum (minimum) qiymati deyiladi va

$$f(x_0) = \max_{x \in \bigcup_\delta x_0} \{f(x)\} \left(f(x_0) = \min_{x \in \bigcup_\delta x_0} \{f(x)\} \right)$$

kabi belgilanadi.

Funksiyani maksimum va minimumi umumiy nom bilan uning **ekstremumi** deyiladi.

2 - teorema. (Ekstremumning zaruriy sharti). Agar $f(x)$ funksiya x_0 nuqtada $(x_0 \in (a,b))$ chekli $f'(x_0)$ hosilaga ega bo'lib, bu nuqtada $f(x)$ funksiya ekstremumga erishsa, u holda $f'(x_0) = 0$ bo'ladi.

Endi funksiya ekstremumga erishishining yyetarli shartlarini keltiramiz.

Faraz qilaylik, $y = f(x)$ funksiya x_0 nuqtada uzluksiz bo'lib, $\bigcup_\delta (x_0) \setminus \{x_0\}$ da chekli $f'(x)$ hosilaga ega bo'lsin.

3 - teorema. Agar $f'(x)$ hosila x_0 nuqtadan o'tishda o'z ishorasini musbatdan (manfiydan) manfiyga (musbatga) o'zgartirsa, unda $f(x)$ funksiya x_0 nuqtada maksimumga (minimumga) erishadi. Agar $f'(x)$ ishorasini o'zgartirmasa, u holda $f(x)$ funksiya x_0 nuqtada ekstremumga erishmaydi.

4 - teorema. $f(x)$ funksiya x_0 nuqtada $f', f'', \ldots f^{(n)}$ hosilalarga ega bo'lib,

$$f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0, \quad f^{(n)}(x_0) \neq 0,$$

bo'lsin. Unda

1) agar n juft son bo'lib,

$$f^{(n)}(x_0)<0 \quad (f^{(n)}(x_0)>0)$$

bo'lsa, $f(x)$ funksiya x_0 nuqtada **maksimumga (minimumga)** erishadi.

2) Agar n toq son bo'lsa, $f(x)$ funksiya x_0 nuqtada ekstremumga erishmaydi.
Funksiyaning hosilasi nolga aylanadigan yoki hosilasi mavjud bo'lmagan nuqtalariga uning **kritik** nuqtalari deyiladi.

Izox: Funksiya hosilasi mavjud bo'lmagan nuqtalarda ham funksiya ekstremumga erishishi mumkin. Masalan, $f(x)=|x|$ funksiya uchun $f'(0)$ - mavjud emas, lekin funksiya $x=0$ nuqtada minimumga erishadi.
$[a,b]$ kesmada uzluksiz bo'lgan $f(x)$ funksiya o'zining shu kesmadagi eng katta (eng kichik) qiymatiga kritik nuqtada yoki kesmaning chegaraviy nuqtasida erishadi.

c) Funksiyaning qavariqligi, egilish nuqtalari.

3 - ta'rif. Agar (a,b) oraliqda berilgan $y=f(x)$ funksiya grafigi $\forall[x_1,x_2]\subset(a,b)$ kesmaning chetki nuqtalarini tutashtiruvchi vatardan **yuqorida (pastda)** yotsa, unda $y=f(x)$ funksiya $[a,b]$ oraliqda q**avariq (botiq)** deb ataladi.

5 - teorema. $y=f(x)$ funksiya (a,b) intervalda aniqlangan va bu intervalda chekli $f'(x)$ hosilaga ega bo'lsin. $f(x)$ funksiyaning (a,b) da q**avariq**$\cap$ **(botiq**$\cup$**)** bo'lishi uchun $f'(x)$ ning (a,b) da **kamayuvchi (o'suvchi)** bo'lishi zarur va yetarli.

6 - teorema. $y=f(x)$ funksiya (a,b) intervalda aniqlangan va bu intervalda ikkinchi tartibli $f''(x)$ hosilaga ega bo'lsin. $f(x)$ ning (a,b) intervalda q**avariq**$\cap$ **(botiq**$\cup$**)** bo'lishi uchun shu intervalda $f''(x)\le 0$ $\left(f''(x)\ge 0\right)$ tengsizlikning bajarilishi zarur va yetarli.

4 - ta'rif. Agar $x=a$ nuqtadan o'tishda $y=f(x)$ funksiyaning grafigi qavariqligi yoki botiqligini o'zgartirsa, u holda $x=a$ nuqta funksiya grafigining egilish nuqtasi deyiladi.

1– misol. Quyidagi

$$\frac{2}{\pi}x<\sin x<x,\ 0<x<\frac{\pi}{2}$$

tengsizlik isbotlansin.

◄ Oldin $\frac{2}{\pi}x<\sin x$ tengsizlikni isbotlaymiz. Buning uchun $\varphi(x)=\frac{2}{\pi}x-\sin x$ funksiya tuzib olamiz. $\varphi'(x)=\frac{2}{\pi}-\cos x=0$ $\quad \frac{2}{\pi}=\cos x$ $\ x=\arccos\frac{2}{\pi}$

$0 < x < \arccos\frac{2}{\pi}$ da $\varphi'(x) < 0$ bundan $\varphi(x)$ funksiyaning bu oraliqda kamayuvchiligi ko'rinadi. $\varphi(0) = 0$ ligidan $0 < x < \arccos\frac{2}{\pi}$ da $\varphi(x) < 0$ ya'ni $\frac{2}{\pi}x < \sin x$ kelib chikadi.

$\arccos\frac{2}{\pi} < x < \frac{\pi}{2}$ da $\varphi'(x) > 0$ bundan $\varphi(x)$ funksiyaning bu oraliqda o'suvchiligi ko'rinadi. $\varphi\left(\frac{\pi}{2}\right) = 0$ ligidan $\arccos\frac{2}{\pi} < x < \frac{\pi}{2}$ da $\varphi(x) < 0$ ya'ni $\frac{2}{\pi}x < \sin x$ kelib chiqadi. Demak, $\frac{2}{\pi}x < \sin x,\ 0 < x < \frac{\pi}{2}$ o'rinli. Endi $\sin x < x,\ 0 < x < \frac{\pi}{2}$ ni ham xuddi shunday usul bilan isbotlash mumkin. ►

2 – misol. Limit hisoblansin. $\lim\limits_{x\to 0}\frac{1-\cos x^2}{x^2 \sin x^2}$

◄ $\lim\limits_{x\to 0}\frac{1-\cos x^2}{x^2 \sin x^2} = \left(\frac{0}{0}\right)$ (Lopital teoremasidan foydalanamiz) =

►

$$\lim_{x\to 0}\frac{1-\cos x^2}{x^2\sin x^2} = \lim_{x\to 0}\frac{\left(1-\cos x^2\right)'}{\left(x^2\sin x^2\right)'} = \lim_{x\to 0}\frac{2x\sin x^2}{2x\sin x^2 + 2x^3\cdot\cos x^2} = \lim_{x\to 0}\frac{\sin x^2}{\sin x^2 + x^2\cdot\cos x^2} =$$

$$= \left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{\left(\sin x^2\right)'}{\left(\sin x^2 + x^2\cdot\cos x^2\right)'} = \lim_{x\to 0}\frac{2x\cos x^2}{2x\cdot\cos x^2 + 2x\cos x^2 - 2x^3\sin x^2} =$$

$$= \lim_{x\to 0}\frac{\cos x^2}{2\cos x^2 - x^2\cdot\sin x^2} = \frac{1}{2}$$

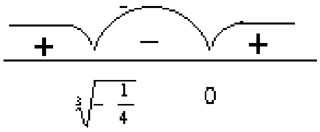
3– misol. Quyidagi funksiyalarni qavariqlik va botiqlik oraliqlari topilsin.

$$f(x) = 4x^2 + \frac{1}{x}$$

◄Bu funksiyaning ikkinchi tartibli hosilasini hisoblaymiz.

$f'(x) = 8x - \frac{1}{x^2}$, $f''(x) = 8 + \frac{2}{x^3}$. Ikkinchi tartibli hosilani nolga tenglaymiz.

$8 + \frac{2}{x^3} = 0$ bundan $x^3 = -\frac{1}{4}$ $x = \sqrt[3]{-\frac{1}{4}}$. Ikkinchi tartibli hosila ishoralarini tekshiramiz.



Demak, funksiya $\left(\sqrt[3]{-\frac{1}{4}};0\right)$ da qavariq, $\left(-\infty;\sqrt[3]{-\frac{1}{4}}\right)$ va $(0;+\infty)$ da botiq.►

$f(x)=\cos\frac{\pi}{2}$ funksiyaning o'sishi va kamayishi oraliqlarini toping.

Δ Berilgan funrsiya juft funrsiya bo'lgani uchun $x>0$ da monotonlik orlig'ini topish etarli. $x>0$ da $f'(x)=\frac{\pi}{x^2}\sin\frac{\pi}{x}>0$ tengsizlikni echib, $0<\frac{\pi}{x}<\pi$ yoki $2\pi k<\frac{\pi}{x}<\pi+2\pi k$, $k\in N$ ni olamiz, bu erda $x>1$ yoki $\frac{1}{2k+1}<x<\frac{1}{2k}$, $k\in N$

SHunday qilib , $(0;+\infty)$ va $\left(\frac{1}{2k+1},\frac{1}{2k}\right)$, $k\in N$ oraliqlarda funksiya qat'iy o'sadi. $\left(\frac{1}{2k+1},\frac{1}{2k}\right)$, $k\in N$ oraliqlarda $f'(x)<0$ tengsizlik o'rinli , va shuning uchun bu oraliqlarda funksiya qat'iy kamayuvchi . Agar $x<0$ bo'lsa , u holda funksiyaning juftligidan fodalanib $\left(-\frac{1}{2k},-\frac{1}{2k-1}\right)$, $k\in N$ oraliqlarda funksiya qat'iy o'sadi, $(-\infty;-1)$ va $\left(-\frac{1}{2k},-\frac{1}{2k-1}\right)$, $k\in N$ oraliqlarda qat'iy kamayuvchi ekanini olamiz.
Berilgan funksiya $x=0$ nuqtaning hech qanday atrofida manaton emas . Bu nuqtaning istalgan atrofi berilgan funksiyaning sanoqli sondagi o'sish va kamayish oraliqlarini o'z ichiga oladi.

Mustaqil yechish uchun misollar.

1 – topshiriq. Quyidagi tengsizliklar isbotlansin.

1.1 $|\sin x-\sin y|\le|x-y|$

1.2 $tgx>x+\frac{x^3}{3},\ 0<x<\frac{\pi}{2}$

1.3 $e^x>1+\ln(1+x)$

1.4 $e^x\ge ex,\ x\in R$

1.5 $b^n-a^n>n(b-a)a^{n-1},\ 0<a<b,\ n\in N$

1.6 $(a+b)^n\le a^p+b^p,\ 0\le p\le1$

1.7 $\cos x\ge1-\frac{x^2}{2},\ x\in R$

1.8 $\sqrt[n]{x}-\sqrt[n]{y}\le\sqrt[n]{x-y},\ x\ge y\ge0.$

1.9 $\sin x>x-\frac{x^3}{6},\ x>0$

1.10 $arctgx>x-\frac{x^3}{3},\ 0<x\le1$

1.11 $\ln(1+x)>\frac{x}{1+x},\ x>0$

1.12 $arctgx<x-\frac{x^3}{6},\ 0<x\le1$

1.13 $e^x > 1 + x,\ x \in R$

1.14 $\ln(1+x) < x,\ x > 0$

1.15 $\frac{x-y}{x} < \ln\frac{x}{y},\ x > y > 0$

1.16 $\frac{\ln x}{x-1} \le \frac{1}{\sqrt{x}},\ x > 0,\ x \ne 1$

1.17 $x^p - y^p \le px^{p-1}(x-y),\ 0 < y < x,\ \ p > 1$

1.18 $\ln(1+x) \ge x - \frac{x^2}{2},\ \ x \ge 0$

1.19 $|arctga - arctgb| \le |a-b|$

1.20 $\ln\frac{x}{y} < \frac{x-y}{y},\ 0 < y < x$

2 – topshiriq. Limit hisoblansin.

2.1 $\lim\limits_{x\to 0}\frac{\ln(1+x)-x}{x^2}$

2.2 $\lim\limits_{x\to 0}\frac{e^x - 1 - x}{x^2}$

2.3 $\lim\limits_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$

2.4 $\lim\limits_{x\to\frac{1}{2}}(2-2x)^{tg\pi x}$

2.5 $\lim\limits_{x\to +0}\frac{e^{-\frac{1}{x}}}{x^{10}}$

2.6 $\lim\limits_{x\to\frac{\pi}{2}}[tgx - (1-\sin x)^{-1}]$

2.7 $\lim\limits_{x\to 0}\frac{(1+x)^{\frac{1}{x}} - e}{x}$

2.8 $\lim\limits_{x\to 0}\frac{(1+x)^x - 1}{x^2}$

2.9 $\lim\limits_{x\to +\infty}\frac{x^2 + \sin x}{e^x + \cos x}$

2.10 $\lim\limits_{x\to +0} x^{\sin x}$

2.11 $\lim\limits_{x\to +\infty}\frac{x^3 + \ln x}{x^3 + \cos x}$

2.12 $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin x}}$

2.13 $\lim\limits_{x\to +\infty}\frac{x^2 + e^x}{\sin x + e^{2x}}$

2.14 $\lim\limits_{x\to 0}(x^{-2} - \sin^{-2} x)$

2.15 $\lim\limits_{x\to +\infty}\frac{x^4 + \cos x}{e^x + \sin x}$

2.16 $\lim\limits_{x\to +0} x^{-2tgx}$

2.17 $\lim\limits_{x\to 0}(x^{-2} - ctg^2 x)$

2.18 $\lim\limits_{x\to\frac{\pi}{4}}(tgx)^{tg2x}$

2.19 $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-\sqrt{1+x^2-x\cos x}}{\ln^3(1-x)}$ 2.20 $\lim\limits_{x\to 0}\dfrac{sh(tgx)-x}{x^3}$

3- topshiriq.

Quyidagi funksiyalarning o'sish va kamayish oraliqlarini toping.

3.1 $f(x)=\dfrac{2x}{1+x^2}$ 3.2 $f(x)=x+\sin x$

3.3 $f(x)=xe^{-3x}$ 3.4 $f(x)=x^2e^{-x^2}$

3.5 $f(x)=x+|\sin 2x|$ 3.6 $f(x)=e^{\pi x}\cos\pi x$

3.7 $f(x)=x^2-\ln x^2$

Quyidagi funksiyalarni ekstremumga tekshiring.

3.8 $f(x)=2x^2-x^4$ 3.9 $f(x)=\dfrac{\ln^2 x}{x}$

3.10 $f(x)=\dfrac{1}{x^2-x}$ 3.11 $f(x)=e^x\sin x$

3.12 $f(x)=x+\sqrt{3-x}$ 3.13 $f(x)=arctgx-\dfrac{1}{2}\ln\left(1+x^2\right)$

3.14 $f(x)=\ln\cos x-\cos x$

Quyidagi funksiyalarni qavariqlik va botiqlik oraliqlari topilsin.

3.15 $f(x)=x\ln x$ 3.16 $f(x)=e^{-x^2}$

3.17 $f(x)=\dfrac{\sqrt{x}}{1+x}$ 3.18 $f(x)=2x^2+\ln x$

3.19 $f(x)=\dfrac{x^4}{1+x^3}$ 3.20 $f(x)=e^{-2x}\sin^2 x$

10-mustaqil ish.

Mavzu: Funksiyalarni to'liq tekshirish va grafiklarini chizish.

Funksiya grafigining asimptotalari.

1 - ta'rif. Agar $\lim\limits_{x\to a} f(x)=\infty$ bo'lsa, $x=a$ to'g`ri chiziq $y=f(x)$ funksiya grafigining **vertikal asimptotasi** deyiladi.

2 - ta'rif. Agar $\lim\limits_{x\to\infty} f(x)=b$ bo'lsa, $y=b$ to'g`ri chiziq $y=f(x)$ funksiya grafigining **gorizontal asimptotasi** deyiladi.

3 - ta'rif. Agar $\lim\limits_{x\to\infty}[f(x)-(ax+b)]=0$ bo'lsa, $y=ax+b$ to'g`ri $y=f(x)$ funksiya grafigining **og`ma asimptotasi** deyiladi.

4 - teorema. $y=f(x)$ funksiya grafigi $x\to+\infty$ da $y=\kappa x+b$ og`ma asimptotaga ega bo'lishi uchun

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\kappa,\qquad \lim_{x\to+\infty}[f(x)-\kappa x]=b$$

bo'lishi zarur va yetarlidir.

Bu teorema $x \to -\infty$ da ham o'rinlidir.

Funksiyalarni to'liq tekshirish va grafiklarini chizish.

Funksiyani to'la tekshirish va grafigini yasash quyidagilarni aniqlash yordamida amalga oshiriladi.

1) Funksiyani aniqlanish sohasini topish.
2) Aniqlanish sohasining chegaraviy nuqtalaridagi harakterini aniqlash.
3) Funksiyaning juft yoki toqligini va agar imkon bo'lsa, boshqa markaz va simmetriya o'qlarini aniqlash.
4) Davriylikka tekshirish.
5) Uzilish nuqtalarini topish va ularning turini aniqlash (2 - punktni to'ldiradi).
6) Koordinata o'qlari bilan kesishish nuqtalarini topish.
7) Funksiyaning ishorasi o'zgarmaydigan oraliqlarni aniqlash.
8) Monotonlik va ekstremumga tekshirish.
9) Egilish nuqtalari, qavariqlik va botiqlik oraliqlarini topish.
10) Asimptotalarni aniqlash.
11) Tekshirish natijalarini yo'llari $x, y, f(x), f'(x), f''(x)$ larga mos bo'lgan jadval ko'rinishida ifodalash (oxirgi yo'lda faqat ishora aniqlanadi).
12) Jadvaldagi nuqtalarni tekislikda ifodalash.
13) Asimtotalarni yasash.
14) Yuqoridagi tekshirish natijalarini hisobga olgan holda tekislikdagi nuqtalarni chiziq yordamida tutashtirish.

Izox: Agar funksiya parametrik ko'rinishda yoki qutb koordinatalar sistemasida berilgan bo'lsa ham u yuqoridagi sxema yordamida tekshiriladi.

1 – misol. Birinchi tartibli hosiladan foydalanib

$$y = -x^3 + 2x^2 - x$$

funksiyani grafigini chizing.

Berilgan funksiyaning hosilasini hisoblaymiz:

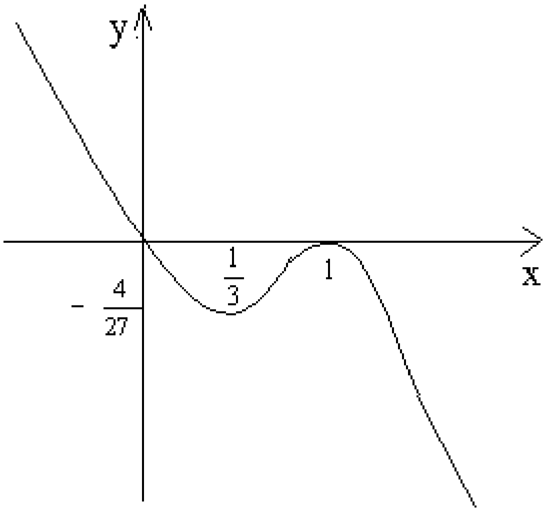
$$y' = -3x^2 + 4x - 1$$

$$-3x^2 + 4x - 1 = 0 \Rightarrow -3(x - \frac{1}{3})(x-1) = 0 \Rightarrow x = \frac{1}{3},\ x = 1$$

Intervallar usulidan foydalanib bu hosilaning ishorasi saqlanadigan oraliqlarni topamiz va quyidagi jadvalni tuzamiz.

x	$(-\infty;\frac{1}{3})$	$\frac{1}{3}$	$\left(\frac{1}{3};1\right)$	1	$(1;+\infty)$
y'	-	0	+	0	-
y	↘	$\min -\frac{4}{27}$	↗	max 0	↘

Jadvaldagi ma’lumotlardan foydalanib berilgan funksiyaning grafigini chizamiz (1 - chizma). ►



1-chizma

2 – misol. $y=\dfrac{4x^3-3x}{4x^2-1}$ **funksiyaning asimptotalarini toping va grafigini yasang.**

$$◄\ y=\frac{4x^3-3x}{4x^2-1}=\frac{x\cdot(4x^2-3)}{(2x+1)(2x-1)}=\frac{x\left(x+\frac{\sqrt{3}}{2}\right)\left(x-\frac{\sqrt{3}}{2}\right)}{\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}\ ►$$

a) Vertikal asimptota: $x=-\dfrac{1}{2}$ va $x=\dfrac{1}{2}$ to’g`ri chiziqlar vertikal asimptota bo’ladi, chunki $\lim\limits_{x\to-\frac{1}{2}} f(x)=\infty$ va $\lim\limits_{x\to\frac{1}{2}} f(x)=\infty$.

Funksiyaning shu nuqtadagi o’ng va chap limitlarini ham hisoblaymiz:

$$\lim_{x\to-\frac{1}{2}+0} f(x)=-\infty,\qquad \lim_{x\to-\frac{1}{2}-0} f(x)=+\infty$$

$$\lim_{x\to\frac{1}{2}+0} f(x)=-\infty\qquad \lim_{x\to\frac{1}{2}-0} f(x)=+\infty$$

b) Gorizontal asimptota: $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\dfrac{4x^3-3x}{4x^2-1}=\infty\Rightarrow$ gorizontal asimptota yo’q.

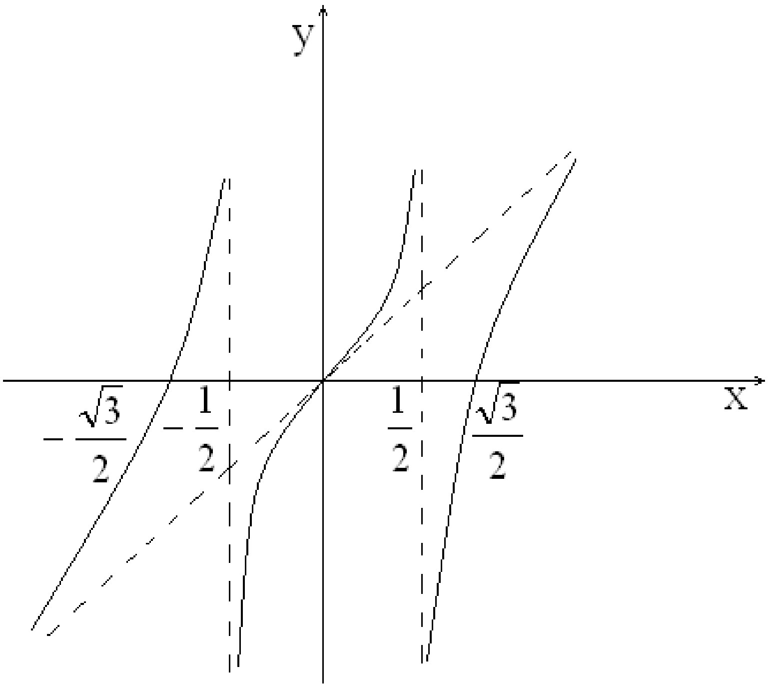
c) Og`ma asimptota: $a=\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to\infty}\dfrac{4x^3-3x}{x(4x^2-1)}=1,$

$$b = \lim_{x\to\infty} [f(x) - ax] = \lim_{x\to\infty}\left[\frac{4x^3 - 3x}{4x^2 - 1} - x\right] = \lim_{x\to\infty}\frac{4x^3 - 3x - 4x^3 + x}{4x^2 - 1} =$$

$$\lim_{x\to\infty}\frac{-2x}{4x^2 - 1} = 0 \Rightarrow y = x$$

- og`ma asimptota.

Bu asimptotalardan foydalanib funksiya grafigini chizamiz (2 - chizma).



2 - □□□□□

3– misol. $y = x + e^{-x}$ **funksiyani to'liq tekshiring va grafigini chizing.**

Funksiyani yuqorida ko'rsatilgan sxema asosida to'liq tekshiramiz.

Funksiyaning aniqlanish sohasi haqiqiy sonlar to'plami.

Funksiya juft ham, toq ham, davriy ham emas.

Funksiya uzluksiz. OY o'qi bilan kesishish nuqtasi: $y = f(0) = 1$

OX o'qi bilan kesishmaydi. Endi funksiyani monotonlik va ekstremumga tekshiramiz:

$$y' = \left(x + e^{-x}\right)' = 1 - e^{-x} = 0 \Rightarrow x = 0$$

Intervallar usulidan foydalanib bu ifodaning ishorasi saqlanadigan oraliqlarni topamiz va quyidagi jadvalni tuzamiz:

x	$(-\infty;0)$	0	$(0;+\infty)$
y'	-	0	+
y	↘	min 1	↗

Qavariqlikka tekshirish uchun y'' ni hisoblaymiz:

$$y'' = (y')' = \left(1 - e^{-x}\right)' = e^{-x} \neq 0 \Rightarrow \frac{1}{e^{x}} > 0$$ Funksiya hamma joyda botiq.

Funksiya asimptotalarini topamiz:

a) Vertikal asimptota: yo'q

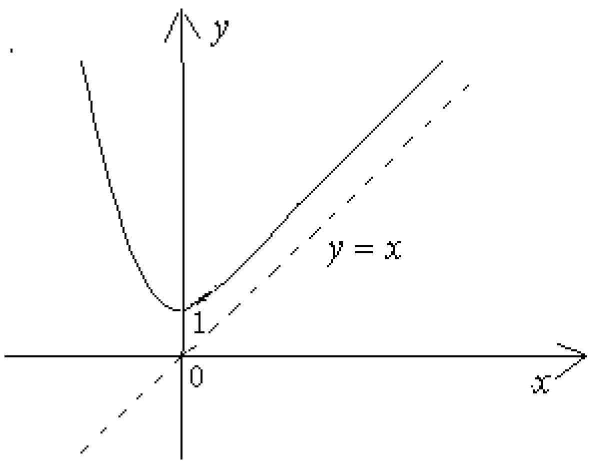
b) Gorizontal asimptota: $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} x + e^{-x} = \infty \Rightarrow$ Gorizontal asimptota yo’q.

v) Og`ma asimptota: $k = \lim\limits_{x\to\infty} \frac{f(x)}{x} = \lim\limits_{x\to\infty} \frac{x + e^{-x}}{x} = 1$

$b = \lim\limits_{x\to\infty} [f(x) - kx] = \lim\limits_{x\to\infty} \left(x + e^{-x} - x\right) = 0 \Rightarrow y = x$

og`ma asimptota.

Endi topilgan ma’lumotlardan foydalanib funksiya grafigini chizamiz (3 - chizma).



Mustaqil yechish uchun misollar.

1- topshiriq. Birinchi tartibli hosiladan foydalanib funksiyaning grafigini chizing.

1.1 $y = x^2(x-1)$ 1.2 $y = \frac{x - 2x^2 + 1}{4}$

1.3 $y = x^3 + 3x^2 - 1$ 1.4 $y = x^2(x-1)^2$

1.5 $y = 3x^3 - 2x^2 - 2$ 1.6 $y = 5x^2 - 2x - x^3$

1.7 $y = x^2(x-3)^2$ 1.8 $y = x^3 + 3x^2 - 5$

1.9 $y = 6x - 8x^3 + x^2$ 1.10 $y = (x-2)^2(x-1)^2$

1.11 $y = x^3 + 3x^2 - 1$ 1.12 $y = 2 - 12x^2 - 8x^3$

1.13 $y = (x+1)^2(2x-1)^2$ 1.14 $y = 2x^3 + 5x^2 + 2x$

1.15 $y = 6x^3 - 4x - 2$ 1.16 $y = \frac{x^3 - x^2 - 3}{4} - x$

1.17 $y = x^4 + 2x^2 + 1$ 1.18 $y = \frac{x(x^2 - 4)}{2}$

1.19 $y=-x^4+3x^3+2$ 1.20 $y=5x^4-4x^3-4$

2 – topshiriq. Funksiya asimptotalarini toping va grafigini yasang.

2.1 $y=\frac{1}{1+x}-\frac{10}{3x^2}+\frac{1}{1-x}$ 2.2 $y=\frac{x^2-4}{\sqrt{x^2-1}}$

2.3 $y=\frac{x^2+x-11}{2x-4}$ 2.4 $y=\frac{x^3}{1-x^2}$

2.5 $y=\frac{x^2+1}{\sqrt{x^2-4}}$ 2.6 $y=\frac{x^4}{(x+1)^3}$

2.7 $y=\left(\frac{x+1}{x-1}\right)^3$ 2.8 $y=\frac{x^2+9}{\sqrt{4x^2-6}}$

2.9 $y=\frac{3x}{\sqrt[3]{2x+1}}$ 2.10 $y=\frac{x^2-4x+4}{x-1}$

2.11 $y=\frac{2+x^2}{\sqrt{9x^2-1}}$ 2.12 $y=\frac{x^2+1}{\sqrt{4x^2-3}}$

2.13 $y=\frac{1-x^2+x}{4x-5}$ 2.14 $y=\frac{4x^2+9}{4x+8}$

2.15 $y=\frac{x^3-4x}{3x^2-4}$ 2.16 $y=\frac{x^2-3}{\sqrt{3x^2-2}}$

2.17 $y=\frac{x}{\sqrt[3]{x^2-1}}$ 2.18 $y=\frac{x-2}{\sqrt{x^2+1}}$

2.19 $y=\frac{x^2\sqrt{x^2-1}}{2x^2-1}$ 2.20 $y=\frac{|1+x|^{\frac{3}{2}}}{\sqrt{x}}$

3 – topshiriq. Funksiyani to'liq tekshiring va grafigini yasang.

3.1 $y=\sin x+\cos^2 x$ 3.2 $y=(7+2\cos x)\sin x$

3.3 $y=\sin x+\frac{1}{3}\sin 3x$ 3.4 $y=\cos x-\frac{1}{2}\cos 2x$

3.5 $y=\sin^4 x+\cos^4 x$ 3.6 $y=\frac{\cos 2x}{\cos x}$

3.7 $y=x^2-2\ln x$ 3.8 $y=x\ln^2 x$

3.9 $y=x^2\ln x$ 3.10 $y=2x-tgx$

3.11 $y=(1+x^2)e^{-x^2}$ 3.12 $y=e^{2x-x^2}$

3.13 $y = x^{\frac{2}{3}} e^{-x}$ 3.14 $y = \frac{x}{2} arctgx$

3.15 $y = \frac{1}{arctgx}$ 3.16 $y = \arccos\frac{1-x^2}{1+x^2}$

3.17 $y = \arccos\frac{2x}{1+x^2}$ 3.18 $y = \arccos\frac{1-x}{1-2x}$

3.19 $y = \ln\left(x + \sqrt{x^2+1}\right)$ 3.20 $y = \frac{\arcsin x}{\sqrt{1-x^2}}$

11-mustaqil ish.
Mavzu: Aniqmas integral tushunchasi. Integrallash usullari.

$f(x)$ funksiya biror (a,b) intervalda aniqlangan bo`lsin.

1 - ta'rif. Agar (a,b) da $f(x)$ funksiya shu intervalda differentsiallanuvchi $F(x)$ funksiyaning hosilasiga teng , ya'ni $\forall x \in (a,b)$ uchun $F'(x) = f(x)$ bo`lsa, u holda $F(x)$ funksiya (a,b) intervalda $f(x)$ funksiyaning **boshlang'ich funksiyasi** deyiladi.

Ma'lumki, $F(x)$ funksiya boshlang'ich funksiya bo`lsa, $F(x)+c$ xam boshlang'ich funksiya bo`ladi.

2 - Ta'rif. (a,b) intervalda berilgan $f(x)$ funksiya boshlang'ich funksiyalarining umumiy ifodasi $F(x)+c$ shu - $f(x)$ funksiyaning **aniqmas integrali** deb ataladi va

$$\int f(x)dx$$

kabi belgilanadi.

Demak,

$$\int f(x)dx = F(x) + c \qquad (1)$$

Aniqmas integralning sodda xossalari.

1. $d\left[\int f(x)dx\right] = f(x)dx$
2. $\int dF(x) = F(x) + c \quad (c = const)$
3. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
4. $\int k \cdot f(x)dx = k \cdot \int f(x)dx \qquad (k \neq 0)$

Elementar funksiyalarning aniqmas integrali.

1. $\int 0 \cdot dx = C,\ C = const;$ 2. $\int 1 \cdot dx = \int dx = x + C;$

3. $\int x^{\mu} dx = \frac{x^{\mu+1}}{\mu+1} + C \ (\mu \neq -1);$ 4. $\int \frac{1}{x} dx = \int \frac{dx}{x} = \ln|x| + C \ (x \neq 0)$

5. $\int \frac{1}{1+x^2}dx = \int \frac{dx}{1+x^2} = arctgx + C;$ 6. $\int \frac{1}{\sqrt{1-x^2}}dx = \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C;$

7. $\int a^x dx = \frac{a^x}{\ln a} + C;$ 8. $\int \sin x dx = -\cos x + C;$

9. $\int \cos x dx = \sin x + C;$ 10. $\int \frac{1}{\sin^2 x}dx = \int \frac{dx}{\sin^2 x} = -ctgx + C$

11. $\int \frac{dx}{\cos^2 x} = \int \frac{dx}{\cos^2 x} = tgx + C$ 12. $\int shx\, dx = chx + C$

13. $\int chx\, dx = shx + C$ 14. $\int \frac{dx}{sh^2 x} = -cthx + C$

15. $\int \frac{dx}{ch^2 x} = thx + C$

O`zgaruvchilarni almashtirib integrallash usuli.

$f(x)$ funksiya biror (a,b) intervalda aniqlangan va $f(x) = \varphi(g(x))g'(x)$ ko`rinishda yozilishi mumkin bo`lsin. Agap $\varphi(x)$ funksiya (t_1, t_2) intervalda $\Phi(t)$ boshlang'ich funksiyaga ega bo`lib, $g(x)$ funksiya $(a,b)$ intervalda differentsiallanuvchi bo`lsa, u holda

$$\int f(x)dx = \int f[g(x)]g'(x)dx = \Phi[g(x)] + c \qquad (2)$$

formula o`rinli.

Bo`laklab integrallash usuli.

Agar $u = u(x)$ va $v = v(x)$ funksiyalar (a,b) intervalda uzluksiz $u'(x)$ va $v'(x)$ hosilalariga ega bo`lsa, u holda shu intervalda ushbu

$$\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x) \qquad (3)$$

**bo`laklab integrallash formulasi** o`rinli bo`ladi.

Amaliyot shuni ko`rsatadiki, bo`laklab integrallash usulini qo`llab hisoblanadigan integrallarni asosan uch guruxga ajratish mumkii.

Birinchi guruxga ko`paytuvchining biri ma'lum funksiyaning hosilasi bo`lgan, ikkinchisi esa ushbu

$\ln(x),\ \arcsin x,\ \arccos x,\ arctgx,\ (arctg)^2,\ (\arccos)^2,\ \ln\varphi(x), ...$

funksiyalardan biriga teng bo`lgan funksiyalarning integrallari kiritiladi. Bu holda $u(x)$ deb shu funksiyalar belgilanadi.

Ikkinchi guruxga $\int (ax+b)^n \cos(cx)dx,\ \int (ax+b)^n \sin(cx)dx,$ va $\int (ax+b)^n \cdot e^{cx}dx$ ko`rinishidagi integrallar kiritiladi. Bu holda $u(x) = (ax+b)^n$ deb olinib, bo`laklab integrallash formulasi n marta qo`llaniladi.

Uchinchi guruxga $\int e^{ax}\cos bxdx,\ \int e^{ax}\sin bxdx,\ \int \sin(\ln x)dx,\ \int \cos(\ln x)dx,...$ ko`rinishidagi integrallar kiritiladi. Bunda integralni $I$ deb belgilab, bo`laklab integrallash formulasini ikki marta qo`llasak, I ga nisbatan chizikli tenglamaga kelamiz.

Bu uchta guruxga kirmagan ba’zi bir integrallarni ham bo`laklab integrallash usuli bilan hisoblash mumkin. Masalan,

$$I_n = \int \frac{dx}{(x^2+a^2)^n},\ (n \in N)$$

integral yukoridagi uchta guruxga kirmaydi, lekin bu integralni xam bo`laklab integrallash usuli bilan **rekkurent formulaga** keltirish yordamida hisoblash mumkin:

$$I_{n+1} = \frac{1}{2na^2}\cdot\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2n}\cdot\frac{1}{a^2}I_n \qquad (5)$$

$I_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a}arctg\frac{x}{a} + c$ Agar (5) – tenglikda $n=1$ desak,

$I_2 = \int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^2}\cdot\frac{x}{x^2+a^2} + \frac{1}{2a^2}arctg\frac{x}{a} + c$ ekanini topamiz.

Izox: Ma’lumki, elementar funksiyaning hosilasi yana elementar funksiya bo`lar edi, lekin integral olish uchun bu tasdik o`rinli bo`lishi shart emas, ya’ni ba’zi bir elementar funksiyalarning integrallari elementar funksiya bo`lmay qolishi mumkin. Masalan, ushbu

1. $\int e^{-x^2}dx,$
2. $\int \cos x^2 dx,$
3. $\int \sin x^2 dx,$
4. $\int \frac{dx}{\ln x}\quad (x \ge 0,\ x \ne 1),$
5. $\int \frac{\cos x}{x}dx\quad (x \ne 0),$
6. $\int \frac{\sin x}{x}dx,$

integrallarning har biri elementar funksiyalar yordamida ifodalanmaydi. Bu funksiyalar amaliyotda ko`p uchraganligi sababli ularning qiymatlarini hisoblash uchun alohida jadvallar tuzilgan va ularning grafiklari yasalgan. Shu yo`l bilan elementar funksiyalarda integrallanmaydigan funksiyalar xam to`la o`rganilgan.

Ba’zi integrallar jadvali:

1. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}arctg\frac{x}{a} + C;\ (a \ne 0)$

2. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C;\ (a \neq 0)$

3. $\int \frac{xdx}{a^2 \pm x^2} = \pm\frac{1}{2}\ln\left|a^2 \pm x^2\right| + C;$

4. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C;\ (a > 0)$

5. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C;\ (a > 0)$

6. $\int \frac{xdx}{\sqrt{a^2 \pm x^2}} = \pm\sqrt{a^2 \pm x^2} + C;$

7. $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C;\ (a > 0)$

8. $\int \sqrt{x^2 \pm a^2}dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{x^2 \pm a^2}\right| + C;$

1–misol. O`**zgaruvchilarni almashtirib integrallash formulasi yordamida aniqmas integral topilsin.**

$$\int \cos\frac{1}{x}\frac{dx}{x^2}$$

◄ $\int \cos\frac{1}{x}\frac{1}{x^2}dx = -\int \cos\frac{1}{x}d\left(\frac{1}{x}\right) = \left(\frac{1}{x} = t\right) = -\int \cos tdt = -\sin t + C = -\sin\frac{1}{x} + C;$

►

2–misol. Bo`laklab integrallash formulasidan foydalanib, aniqmas integralni hisoblang.

$$\int (xarctgx + \sqrt{x})dx$$

◄

$$\int\left(xarctgx + \sqrt{x}\right)dx = \int xarctgxdx + \int \sqrt{x}dx = \int xarctgxdx + \int x^{\frac{3}{2}}dx = I_1 + I_2$$

$$I_1 = \int xarctgxdx = \left\{\begin{matrix} u = arctgx & xdx = dv \\ du = \frac{dx}{1+x^2} & v = \frac{x^2}{2} \end{matrix}\right\} = \frac{x^2}{2}arctgx - \frac{1}{2}\int\frac{x^2}{1+x^2}dx =$$

$$= \frac{x^2}{2}arctgx - \frac{1}{2}x + \frac{1}{2}arctgx + C;$$

$$I_1 + I_2 = \left(\frac{x^2}{2} + \frac{1}{2}\right)arctgx - \frac{1}{2}x + \frac{2x^{\frac{5}{2}}}{5} + C;$$ ►

3–misol. Aniqmas integral hisoblansin.

$$\int \frac{x-5}{6+4x-x^2}dx$$

◀Oldin kasr maxrajida turshgan ko`pxadni quyidagi ko`rinishda yozib olamiz.
$6+4x-x^2=10-4+4x-x^2=10-(x-2)^2$

$$\int \frac{x-5}{10-(x-2)^2}dx=\begin{pmatrix} x-2=t \\ dx=dt \end{pmatrix}=\int \frac{t-3}{10-t^2}dt=\int \frac{tdt}{10-t^2}-\int \frac{3dt}{10-t^2}=$$

$$=\frac{1}{2}\int \frac{d(t^2)}{10-t^2}-3\int \frac{dt}{(\sqrt{10})^2-t^2}=-\frac{1}{2}\ln\left|10-t^2\right|-3\int \frac{dt}{(\sqrt{10}-t)(\sqrt{10}+t)}=$$

$$=-\frac{1}{2}\ln\left|10-t^2\right|-\frac{3}{2\sqrt{10}}\left[\int \frac{dt}{\sqrt{10}-t}+\int \frac{dt}{\sqrt{10}+t}\right]=-\frac{1}{2}\ln\left|10-t^2\right|-$$

$$-\frac{3}{2\sqrt{10}}\left[-\ln\left|\sqrt{10}-t\right|+\ln\left|\sqrt{10}+t\right|\right]=-\frac{1}{2}\ln\left|10-t^2\right|-\frac{3}{2\sqrt{10}}\ln\left|\frac{\sqrt{10}+t}{\sqrt{10}-t}\right|=$$

$$=-\frac{1}{2}\ln\left|6+4x-x^2\right|-\frac{3}{2\sqrt{10}}\ln\left|\frac{\sqrt{10}+x-2}{\sqrt{10}-x+2}\right|+C;$$

Mustaqil yechish uchun misollar

1 – topshiriq. O`zgaruvchilarni almashtirib integrallash formulasi yordamida aniqmas integral topilsin.

1.1 $\int \frac{arctgx}{1+x^2}dx$ **1.2** $\int \frac{x^2dx}{1+x^4}$

1.3 $\int \frac{xdx}{\sqrt{x^2+1}}$ 1.4 $\int \frac{dx}{\sqrt{2-3x}}$

1.5 $\int \frac{x^2dx}{\sqrt{x^3-1}}$ 1.6 $\int \frac{dx}{4-3x^2}$

1.7 $\int \frac{xdx}{\sqrt{x^4+x^2+1}}$ 1.8 $\int \frac{dx}{\sqrt{1-4x^2}}$

1.9 $\int tgx\cdot \cos xdx$ 1.10 $\int \frac{x^5}{x^2+1}dx$

1.11 $\int \frac{\sin x - \cos x}{(\cos x + \sin x)^5} dx$ 1.12 $\int \frac{\sqrt[3]{4 - 4x + x^2}}{2 - x} dx$ 1.13 $\int \frac{x^3 - 2x}{x^4 + 1} dx$

1.14 $\int \frac{xdx}{\sqrt{x^4 - 6x^2 + 9}}$

1.15 $\int \frac{e^x dx}{\sqrt[3]{e^x - 1}}$ 1.16 $\int \frac{\ln^3(x-2) + 2}{x - 2} dx$

1.17 $\int \frac{2x + \ln^2 x}{x} dx$ 1.18 $\int \frac{x^2 + arctgx}{1 + x^2} dx$

1.19 $\int \frac{x^3 + arctg\frac{x}{2} dx}{x^2 + 4}$ 1.20 $\int \frac{arctg\sqrt{x}}{\sqrt{x}(1 + x)} dx$

2- topshiriq. Bo`laklab integrallash formulasidan foydalanib, aniqmas integralni hisoblang.

2.1 $\int x^2 \ln xdx$ 2.2 $\int x^2 \sin \quad xdx$

2.3 $\int (4x + 3) \ln xdx$ 2.4 $\int x \ln^2 xdx$

2.5 $\int (x + 3) \cos 2xdx$ 2.6 $\int (x^2 + 1) \sin 4xdx$

2.7 $\int (x + 5) \cos 5xdx$ 2.8 $\int (x^2 - 4) \sin 3xdx$

2.9 $\int (x^2 - 5) \ln xdx$ 2.10 $\int \sqrt{x} \ln^2 xdx$

2.11 $\int (x^2 - 3) \cos 2xdx$ 2.12 $\int arctg\sqrt{x + 2}dx$

2.13 $\int arctg\sqrt{7x + 1}dx$ 2.14 $\int (x - 2) \cos^2 xdx$

2.15 $\int (2 + 7x) \sin^2 xdx$ 2.16 $\int (5x + 6) \cos^2 xdx$

2.17 $\int e^{2x}(x^2 + 1)dx$ 2.18 $\int e^{-2x}(x^2 - 2)dx$

2.19 $\int (arctg\sqrt{x} + x)dx$ 2.20 $\int x^2 arctgxdx$

3 – topshiriq. Aniqmas integral hisoblansin.

3.1 $\int \frac{2x - 1dx}{\sqrt{x^2 + 4x + 1}}$ 3.2 $\int \frac{x + 1}{\sqrt{4x^2 + 4x + 5}} dx$

3.3 $\int \frac{x + 2}{\sqrt{x^2 - 2x + 1}} dx$ 3.4 $\int \frac{x + 3}{\sqrt{4 + 2x - x^2}} dx$

3.5 $\int \frac{xdx}{\sqrt{x \quad + x^2 + 1}}$ 3.6 $\int \frac{2x - 1}{\sqrt{1 + 4x - x^2}} dx$

3.7 $\int \frac{x+1}{4x^2+12x}dx$

3.8 $\int \frac{x-2}{x^2+2x+2}dx$

3.9 $\int \frac{x+3}{3x^2+6x+1}dx$

3.10 $\int \frac{2x-5}{x^2+4x+7}dx$

311 $\int \frac{x-8}{4-3x+x^2}dx$

3.12 $\int \frac{x-5}{x^2+3x+5}dx$

3.13 $\int \frac{x+2}{\sqrt{x^2+4x-5}}dx$

3.14 $\int \frac{xdx}{\sqrt{2x^2-3x+1}}$

3.15 $\int \frac{(3x-1)dx}{\sqrt{3x^2+4x+1}}$

3.16 $\int \frac{5x+1}{\sqrt{2x^2+4x+6}}dx$

3.17 $\int \frac{3x+2}{\sqrt{4x^2-10x+13}}dx$

3.18 $\int \frac{x+3}{\sqrt{4+2x-x^2}}dx$

3.19 $\int \frac{5x+2}{2x^2+3x-4}dx$

3.20 $\int \frac{dx}{x(1+\sqrt[3]{x})}$

12-mustaqil ish.
Mavzu:Ratsional va ba'zi irratsional funksiyalarni integrallash.Trigonometrik funksiyalarni integrallash.

1. Ratsional funksiyalarni integrallash.

1-ta'rif. Agar $R(x)$ funksiyani ikkita ko`pxadning nisbati ko`rinishida yozish mumkin bo`lsa, u holda $R(x)$ ratstsonal **funksiya (yoki ratsional kasr)** deyiladi, ya'ni

$$R(x)=\frac{P_n(x)}{Q_m(x)} \qquad (1)$$

$P_n(x)-n$ - tartibli, $Q_m(x)-m$ - tartibli ko`pxad.

Agar $n\geq m$ bo`lsa kasr **noto`g'ri kasr**; $n<m$ bo`lsa **to`g'ri kasr** deyiladi. Ixtiyoriy noto`g'ri kasr berilgan bo`lsa, ko`pxadni ko`pxadga bo`lish yordamida xar doim uni ko`pxad va to`g'ri kasrning yig'indisi shaklida ifodalash mumkin. Ixtiyoriy to`g'ri kasrni quyidagi 4 ta ko`rinishdagi sodda kasrlarning yig'indisi kabi ifodalash mumkin.

I. $\frac{A}{x-a}$,

II. $\frac{A}{(x-a)^k}$ $(k=2,3,4...)$,

III. $\frac{Mx+N}{x^2+px+q}$,

IV. $\frac{Mx+N}{(x^2+px+q)^m}$ $(m=2,3,...)$,

III va IV da $x^2+px+q\neq 0$, ya'ni $q-\frac{p^2}{4}>0$

I va II ko`rinishidagi sodda kasrlar to`g'ridan to`g'ri integrallanadi. III va IV ko`rinishdagi integrallar $t=x+\frac{p}{2}$ almashtirish yordamida hisoblanadi.

Ba'zi irratsional ko`rinishidagi funksiyalarni integrallash. Eyler almashtirishlari.

$R(x,y)$ deganda x va y o`zgaruvchiga nisbatan ratsional bo`lgan funksiyani tushunamiz.

a) $\int R\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$ integralni hisoblashda $t=\sqrt[n]{\frac{ax+b}{cx+d}}$ almashtirish bajarilsa, ratsional funksiyani integrallashga kelinadi.

b) $\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$ integralni hisoblashda quyidagi Z ta hol qaraladi.

1 – hol. ax^2+bx+c kvadrat uchhad har xil x_1 va x_2 xaqiqiy ildizlarga ega bo`lcin.$\Rightarrow ax^2+bx+c=a(x-x_1)(x-x_2)$ Bynda

$$\sqrt{a(x-x_1)(x-x_2)}=t(x-x_1) \qquad (2)$$

almashtirish bajaramiz.

2 - hol. $a>0$ bo`lsin. Unda

$$\sqrt{ax^2+bx+c)}=t-\sqrt{a}x \ (\textit{ёки} \sqrt{ax^2+bx+c}=t+\sqrt{a}x) \qquad (3)$$

almashtirish bajaramiz.

3 - hol. $c>0$ bo`lsin. U holda

$$\sqrt{ax^2+bx+c)}=tx+\sqrt{c} \ (\textit{ёки} \sqrt{ax^2+bx+c}=tx-\sqrt{c}) \qquad (4)$$

almashtirishni bajarish yordamida hisoblanishi kerak bo`lgan integral ratsional funksiyani integrallashga keltiriladi.

(2) - (4) almashtirishlarga **Eyler almashtirishlari** deb ataladi.

Binomial differentsiallarni va trigonometrik funksiyalarni integrallash.

a) 2 - ta'rif. Ushbu $x^m(a+bx^n)^p\,dx$ ko`rinishidagi ifodaga **binomial differentsial** deb ataladi. Bu erda m,n,p - lar ratsional sonlar.

$$I=\int x^m(a+bx^n)^p\,dx \qquad (5)$$

integral quyidagi 3 ta holda ratsional funksiyaning integraliga keladi.

1 - hol. p **- butun son.** $x=t^N$ almashtirish bajariladi. Bu erda N soni m va n ratsional sonlar (ya'ni kasrlar) maxrajlarning eng kichik umumiy bo`linuvchisi

2 - hol. $\frac{m+1}{n}$ **- butun son.** Bu holda $a+bx^n=Z^N$, almashtirish bajarish kerak, ($N-p$ ratsional sonning maxraji,)

3 - hol. $\frac{m+1}{n}+p$ **- butun son.** Bunda $\frac{a}{x^n}+b=Z^N$, $N-p$ ning maxraji, almashtirish bajarish etarli.

b) $\int R(\sin x,\cos x)\,dx$ integral berilgan bo`lsin. Bu integralni ushbu

$$tg\frac{x}{2}=t, \ -\pi<x<\pi$$

universal almashtirish yordamida xar doim ratsional funksiyani integrallashga keltirish mumkin:

v) Aytaylik,

$$I=\int \sin^n x\cdot\cos^m x dx, \ (n, \ m\in Z)$$

integral berilgan bo`lsin. Bu integralni hisoblash uchun quyidagi hollar karaladi.

1 - hol. p - tok, m - juft $\Rightarrow \cos x=t$ almashtirish bajariladi.

2 - hol. n - juft, m - tok $\Rightarrow \sin x = t$ almashtirish bajariladi.
3 - hol. n va m - tok. Bunda $\cos x = t$, $\sin x = t$ yoki $tgx = t$ almashtirishlardan biri bajariladi.
4 - hol. n va m - juft. Bu holda

$$\sin 2x = 2\sin x \cdot \cos x \text{ va } \cos 2x = \cos^2 x - \sin^2 x$$

formulalardan foydalanib tartib pasaytiriladi va yukoridagi hollardan biriga keltiriladi

Trigonometrik funksiyalarni integrallash.

$$\int R(\sin x, \cos x)dx \qquad (5)$$

integralni qaraylik. (5) integralda $t = tg\frac{x}{2}(-\pi < x < \pi)$

almashtirish bajarilsa, u holda (5) integral ostidagi ifoda t o`zgaruvchining ratsional funksiyasiga aylanadi.

$$\sin x = \frac{2tg\frac{x}{2}}{1 + tg^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1 - tg^2\frac{x}{2}}{1 + tg^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$$x = 2arctgx, \quad dx = \frac{2dt}{1+t^2}$$

munosabatlarni e'tiborga olsak, (5) integral

$$\int R(\sin x, \cos x)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \cdot \frac{2dt}{1+t^2}$$

ratsional funksiyani integrallashga keltiriladi.Ayrim hollarda trigonometrik funksiyalarni integrallashda

$$t = tgx, \; t = \sin x, \; t = \cos x$$

almashtirishlar qulay bo`ladi.

1– misol. $\int \frac{2x^4 - 8x^2 + 1}{x^2 - 4x + 1}dx$ **aniqmas integral hisoblansin.**

◀Biz bu integralni ratsional funksiyani integrallash usulidan foydalanib hisoblaymiz. Avval noto`g'ri kasrni to`g'ri kasrga keltiramiz, so`ngra uni sodda kasrlarga yoyamiz:

$$R(x) = \frac{2x^4 - 8x^2 + 1}{x^2 - 4x + 1} = \left(2x^2 + 8x + 22\right) + \frac{80x - 21}{x^2 - 4x + 1}$$

$$\int \frac{2x^4-8x^2+1}{x^2-4x+1}dx = \int\left[\left(2x^2+8x+22\right)+\frac{80x-21}{x^2-4x+1}\right]dx = I_1+I_2$$

$$I_1=\int\left(2x^2+8x+22\right)dx=\frac{2x^3}{3}+\frac{8x^2}{2}+22+C;$$

$$I_2=\int\frac{80x-21}{x^2-4x+1}dx=\int\frac{80(x-2)dx}{(x-2)^2-3}+\left[\frac{80x-21}{x^2-4x+1}=\frac{80(x-2)}{x^2-4x+4-3}=\frac{80x-21}{(x-2)^2-3}=\right.$$

$$\left.=\frac{80x-21+160-21}{(x-2)^2-3}=\frac{80(x-2)}{(x-2)^2-3}+\frac{139}{(x-2)^2-3}\right]+\int\frac{139}{(x-2)^2-3}dx=\begin{pmatrix}x-2=t\\ dx=dt\end{pmatrix}=$$

$$=80\int\frac{t}{t^2-3}dt+\int\frac{139}{t^2-\left(\sqrt{3}\right)^2}dt=40\int\frac{d\left(t^2\right)}{t^2-3}+\int\frac{139dt}{\left(t-\sqrt{3}\right)\left(t+\sqrt{3}\right)}=40\ln\left|t^2-3\right|+$$

$$+\frac{139}{2\sqrt{3}}\left[\int\frac{dt}{t-\sqrt{3}}-\int\frac{dt}{t+\sqrt{3}}\right]=40\ln\left|t^2-3\right|+\frac{139}{2\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|=40\ln\left|x^2-4x+1\right|+$$

$$+\frac{139}{2\sqrt{3}}\ln\left|\frac{x-2-\sqrt{3}}{x-2+\sqrt{3}}\right|$$

$$I_1+I_2=\frac{2x^3}{3}+4x^2+22x+40\ln\left|x^2-4x+1\right|+\frac{139}{2\sqrt{3}}\ln\left|\frac{x-2-\sqrt{3}}{x-2+\sqrt{3}}\right|+C;\ \blacktriangleright$$

2–misol. Aniqmas integral hisoblansin.

$$\int\frac{2\sqrt[4]{x}+1}{\sqrt[4]{x^3}\left(\sqrt{x}+1\right)}dx$$

$\blacktriangleleft\ \sqrt[4]{x}=t \qquad \sqrt{x}=t^2 \qquad x=t^4 \qquad dx=4t^3dt$

$$\int\frac{2t+1}{t^3\left(t^2+1\right)}\cdot 4t^3dt=4\int\frac{2t+1}{t^2+1}dt=\int\frac{8t}{t^2+1}dt+4\int\frac{dt}{t^2+1}=4\int\frac{d\left(t^2\right)}{t^2+1}+4\int\frac{dt}{t^2+1}=$$

$$=4\ln\left|t^2+1\right|+4arctg\,t=4\ln\left|\sqrt{x}+1\right|+4arctg\sqrt[4]{x}$$

$\blacktriangleright$

3– misol. Ushbu $\int\frac{dx}{\sqrt[4]{1+x^4}}$ **integral hisoblansin.**

$\blacktriangleleft$Integral ostidagi ifoda uchun $a=b=1,\ m=0,\ n=4,\ p=-\frac{1}{4}$ bo\`lib,

$\frac{m+1}{n}+p=\frac{1}{4}-\frac{1}{4}=0$ bo\`ladi.

Demak, $1+x^{-4}=t^4$ almashtirishni bajarish lozim. Natijada:

$$x=\left(t^4-1\right)^{-\frac{1}{4}},\quad \frac{1}{\sqrt[4]{1+x^4}}=\frac{\left(t^4-1\right)^{\frac{1}{4}}}{t},\quad dx=-t^3\left(t^4-1\right)^{-\frac{5}{4}}dt.$$

$$\int\frac{dx}{\sqrt[4]{1+x^4}}=-\int\frac{t^2dt}{t^4-1}=-\frac{1}{2}\left(\int\frac{dt}{t^2-1}+\int\frac{dt}{t^2+1}\right)=\frac{1}{4}\ln\left|\frac{1+t}{1-t}\right|-\frac{1}{2}arctgt+C=$$

$$\frac{1}{4}\ln\frac{\sqrt[4]{1+x^4}+4}{\sqrt[4]{1+x^4}-4}-\frac{1}{2}arctg\frac{\sqrt[4]{1+x^4}}{x}+C$$

►

4– misol. Aniqmas integralni hisoblang.

$$\int\frac{dx}{\sin x(1-\sin x)}$$

◄ $$\int\frac{dx}{\sin x(1-\sin x)}=\int\frac{(1+\sin x)}{\sin x\left(1-\sin^2 x\right)}dx=\int\frac{1+\sin x}{\sin x\cos^2 x}dx=\int\frac{dx}{\sin x\cos^2 x}+$$

$$+\int\frac{dx}{\cos^2 x}=\int\frac{\sin xdx}{\sin^2 x\cos^2 x}+tgx+C=-\int\frac{d(\cos x)}{\left(1-\cos^2 x\right)\cos^2 x}+tgx+C=(\cos x=t)=$$

$$=-\int\frac{dt}{\left(1-t^2\right)t^2}+tgx+C=-\left[\int\frac{dt}{t^2}+\int\frac{dt}{1-t^2}\right]+tgx+C=+\frac{1}{t}+\frac{1}{2}\ln\left|\frac{1-t}{1+t}\right|+tgx+C=$$

$$=\frac{1}{\cos x}+\frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right|+tgx+C=\left(\int\frac{dt}{(1-t)(1+t)}=\frac{1}{2}\int\frac{dt}{1-t}+\frac{1}{2}\int\frac{dt}{1+t}=\frac{1}{2}\ln\left|\frac{1-t}{1+t}\right|\right)=$$

$$=\frac{1}{\cos x}+\frac{1}{2}\ln\left|tg^2\frac{x}{2}\right|+tgx+C;$$

►

Mustaqil yechish uchun misollar.

1 – topshiriq. Aniqmas integral hisoblansin.

1.1 $\int\frac{2x^3+1}{x^3-2x+1}dx$

1.2 $\int\frac{x-12}{(x-5)(x-3)(3x-2)}dx$

1.3 $\int\frac{2x^4+1}{x^2-x+1}dx$

1.4 $\int\frac{x^3}{(x-4)(x-3)(x-2)}dx$

1.5 $\int\frac{x^3-2x}{x^2-3x+3}dx$

1.6 $\int\frac{4x^2+x+1}{x(x-1)(x-2)}dx$

1.7 $\int\frac{2x^4+5x}{x^2-4x-2}dx$

1.8 $\int\frac{x^3-2x+1}{x^3-x}dx$

1.9 $\int \frac{2x^5-3}{x^2+2x-6}dx$ 1.10 $\int \frac{x^4-3x^2-12}{(x-3)(x-1)x}dx$

1.11 $\int \frac{x^4-2x^3+1}{x^2-3x}dx$ 1.12 $\int \frac{x^5-x^3+1}{x^2-x}dx$

1.13 $\int \frac{x-2}{x^2+2x+2}dx$ 1.14 $\int \frac{x^5+3x^3-1}{x^2+x}dx$

1.15 $\int \frac{3x^3+2x^2+1}{(x+2)(x-2)(x-1)}dx$ 1.16 $\int \frac{xdx}{\sqrt{2x^2-3x+1}}$

1.17 $\int \frac{x^3}{(x-1)(x+1)(x+2)}dx$ 1.18 $\int \frac{x^3-4x^2+2}{(x-1)(x+1)(x-3)}dx$

1.19 $\int \frac{x^4+2x^2+1}{x^2+5x+1}dx$ 1.20 $\int \frac{x^5+5x^3+5}{x^3+2x^2+x}dx$

2– topshiriq.Aniqmas integral hisoblansin.

2.1 $\int \frac{dx}{2+\sqrt{x}}$ 2.2 $\int \frac{dx}{x(1+\sqrt{x})}$

2.3 $\int \frac{dx}{(1+\sqrt[3]{x})}$ 2.4 $\int \frac{dx}{1+\sqrt[4]{x}}$.

2.5 $\int \frac{dx}{x(1+\sqrt[3]{x})}$ 2.6 $\int \frac{x}{1+\sqrt{x}}dx$

2.7 $\int \frac{x^2+1}{1+\sqrt{x}}dx$ 2.8 $\int \frac{x^3}{1+\sqrt[3]{x}}dx$

2.9 $\int \frac{\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}dx$. 2.10 $\int \frac{x\sqrt[3]{1+x}}{1+\sqrt[3]{1+x}}dx$

2.11 $\int \frac{1-\sqrt{x+2}}{1+\sqrt{x+2}}dx$ 2.12 $\int \frac{1-\sqrt{x-1}}{1+\sqrt[3]{x-1}}dx$

2.13 $\int \frac{dx}{1+2\sqrt{x}+\sqrt[3]{x}}dx$ 2.14 $\int \frac{dx}{2+x+\sqrt[3]{x}}$

2.15 $\int \frac{\sqrt[5]{x}-1}{1+\sqrt{x}}dx$ 2.16 $\int \frac{\sqrt{x+1}-1}{\sqrt[3]{x+1}+1}dx$

2.17 $\int \frac{x+1}{\sqrt[3]{2x+1}}dx$ 2.18 $\int \frac{\sqrt{x}dx}{1+\sqrt[3]{x^2}}$

2.19 $\int \frac{\sqrt[4]{x}+1}{(\sqrt{x}+4)\sqrt[4]{x^3}}dx$ 2.20 $\int \frac{dx}{\sqrt{x+3}(1+\sqrt[3]{x+3})}$

3– topshiriq. Ainqmas integralni hisoblang.

3.1 $\int \frac{\sin xdx}{1+\sin x}$

3.2 $\int \cos^5 xdx$

3.3 $\int \cos^6 xdx$

3.4 $\int \sin^2 x\cos^4 xdx$

3.5 $\int \sin^3 x\cos^5 xdx$

3.6. $\int \cos^3 x\sin^6 xdx$

3.7 $\int \frac{dx}{\sin^3 x}$

3.8 $\int \frac{dx}{\cos^3 x}$

3.9 $\int \frac{\cos xdx}{1+\cos x+\sin x}$

3.10 $\int \frac{dx}{\cos x(1+\cos x)}$

3.11 $\int \frac{1+\cos x}{\sin x(1+\sin x)}dx$

3.12 $\int \frac{1-\sin x}{\cos x(1+\cos x)}dx$

3.13 $\int \frac{\sin^3 x}{\cos^4 x}dx$

3.14 $\int \frac{\sin^3 x}{\sqrt{\sin x}}dx$

3.15 $\int \frac{\cos x-\sin x}{(1+\sin x)^2}dx$

3.16 $\int \frac{1+tgx}{\sin x+2\cos x}dx$

3.17 $\int \frac{tgx+1}{\sin^2 x+2\cos^2 x-3}dx$

3.18 $\int \sin x\cdot\sin 2x\cdot\sin 4xdx$

3.19 $\int \cos x\cdot\cos 2x\cdot\cos 4xdx$

3.20 $\int \sin 2x\cdot\cos x\cdot\cos 2xdx$

13 –mustaqil ish.

Mavzu: Aniq integral.

$f(x)$ funksiya $[a;b]$ kesmada aniqlangan va uzluksiz bo`lsin. Bu kesmani $a = x_0 < x_1 < x_2 < ... < x_n = b$ nuqtalar bilan $n$ ta qismga bo`lamiz. Har bir (x_{i-1}, x_i) oraliqda ixtiyoriy ξ_i nuqtani olamiz va ushbu yig'indini tuzamiz: $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ bunda $\Delta x_i = x_i - x_{i-1}$. Ushbu $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ ko`rinishdagi yig'indiga **integral yig'indi** deyiladi.

Agar $\max \Delta x_i \to 0$ da $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ yig'indining limiti mavjud va chekli bo`lsa, u holda bu limitga $f(x)$ funksiyadan $a$ dan $b$ gacha olingan **aniq integral** deyiladi va $\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$ ko`rinishda belgilanadi. Bu holda $f(x)$ funksiya $[a;b]$ kesmada integrallanuvchi funksiya deyiladi. a va b sonlar mos ravishda integrallashning quyi va yuqori chegaralari deyiladi.

1-teorema. Agar $f(x)$ funksiya $[a;b]$ oraliqda uzluksiz bo`lsa, u shu oraliqda integrallanuvchi bo`ladi.

2-teorema. Agar $f(x)$ funksiya $[a;b]$ oraliqda chegaralangan va monoton bo`lsa, u shu oraliqda integrallanuvchi bo`ladi.

Aniq integral xossalari.

1^0. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2^0. $\int_a^a f(x)dx = 0$

3^0. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

4^0. $\int_a^b (f_1(x) \pm f_2(x))dx = \int_a^b f_1(x)dx \pm \int_a^b f_2(x)dx$

5^0. $\int_a^b kf(x)dx = k\int_a^b f(x)dx,$ бунда k - узгармас

6^0. agar $[a;b]$ kesmada $f(x) \geq 0$ bo`lsa, u holda $\int_a^b f(x)dx \geq 0$

7^0. agar $[a;b]$ kesmada $f(x)\geq g(x)$ bo`lsa, u holda $\int\limits_a^b f(x)dx \geq \int\limits_a^b g(x)dx$

8^0. agar m va M mos ravishda $f(x)$ funksiyaning $[a;b]$ kesmadagi eng kichik va eng katta qiymati bo`lsa, u holda $m\cdot(b-a)\leq \int\limits_a^b f(x)dx \leq M(b-a)$ tengsizlik o`rinli. (aniq integralni baholash xaqidagi teorema.)

9^0. $\int\limits_a^b f(x)dx = f(c)(b-a)$ bunda $c\in(a;b)$ (o`rta qiymat xaqidagi teorema.)

Agar $F(x)$ $[a;b]$ kesmada uzluksiz $f(x)$ funksiyaning boshlang'ich funksiyalaridan biri bo`lsa, u holda quyidagi formula o`rinli:

$$\int\limits_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a)$$

bu formula **Nyuton –Leybnits formulasi** deyiladi.

1. O`zgaruvchini almashtirib integrallash formulasi.

Agar $f(x)$ funksiya $[a;b]$ kesmada uzluksiz, (a,b) funksiya esa differentsiallanuvchi bo`lib shu bilan birga $a=\varphi(\alpha)$, $b=\varphi(\beta)$ bo`lsa, u holda ushbu tenglik o`rinli: $\int\limits_a^b f(x)dx = \int\limits_\alpha^\beta f(\varphi(t))\varphi'(t)dt$

2. Bo`laklab integrallash formulasi.

Agar $u=u(x)$, $v=v(x)$funksiyalar va ularning xosilalari $[a;b]$ kesmada uzluksiz bo`lsa, u holda $\int\limits_a^b udv = uv\Big|_a^b - \int\limits_a^b vdu$ tenglik o`rinli.

.

1–misol. Anik integral hisoblansin.

$$\int\limits_0^{\frac{\pi}{4}} \frac{dx}{1+2\sin^2 x}$$

$$◄ \int\limits_0^{\frac{\pi}{4}} \frac{dx}{1+2\sin^2 x} = \int\limits_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{dx}{1+2\sin^2 x} = \int\limits_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{dx}{\sin^2 x\left(\frac{1}{\sin^2 x}+2\right)} = -\int\limits_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{d(ctgx)}{\left(\frac{1}{\sin^2 x}-1+3\right)} =$$

$$=\begin{bmatrix} d(ctgx)=-\dfrac{dx}{\sin^2 x} \\ 1+ctg^2x=\dfrac{1}{\sin^2 x}\end{bmatrix}=-\int_{\frac{\pi}{3}}^{\frac{\pi}{4}}\frac{d(ctgx)}{ctg^2x+3}=\begin{pmatrix} ctgx=t; \\ x=\dfrac{\pi}{3} \quad ctg\dfrac{\pi}{3}=\dfrac{1}{\sqrt{3}} \\ x=\dfrac{\pi}{4} \quad ctg\dfrac{\pi}{4}=1\end{pmatrix}=-\int_{\frac{1}{\sqrt{3}}}^{1}\frac{dt}{t^2+3}=$$

$$=-\frac{1}{3}\int_{\frac{1}{\sqrt{3}}}^{1}\frac{dt}{\left(\left(\frac{t}{\sqrt{3}}\right)^2+1\right)}=-\frac{\sqrt{3}}{3}\int_{\frac{1}{\sqrt{3}}}^{1}\frac{d\left(\frac{t}{\sqrt{3}}\right)}{\left(\frac{t}{3}\right)^2+1}=-\frac{\sqrt{3}}{3}arctg\frac{t}{\sqrt{3}}\Bigg|_{\frac{1}{\sqrt{3}}}^{1}=-\frac{\sqrt{3}}{3}arctg\frac{1}{\sqrt{3}}+$$

$$+\frac{\sqrt{3}}{3}arctg\frac{1}{3}=-\frac{\sqrt{3}}{3}\cdot\frac{\pi}{6}+\frac{\sqrt{3}}{3}arctg\frac{1}{3}=-\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}arctg\frac{1}{3}$$

►

2–misol. Anik integral hisoblansin.

$$\int_1^2\frac{e^{2x}}{e^{2x}-e^{-2x}}dx$$

$$◄\int_1^2\frac{e^{2x}}{e^{2x}-e^{-2x}}dx=\frac{1}{2}\int_1^2\frac{d\left(e^{2x}\right)}{e^{2x}-e^{-2x}}=\begin{pmatrix} e^{2x}=t \\ x=1 \ e^2=t \\ x=2 \ e^4=t\end{pmatrix}=\frac{1}{2}\int_{e^2}^{e^4}\frac{dt}{t-\frac{1}{t}}=\frac{1}{2}\int\frac{tdt}{t^2-1}$$

$$=\frac{1}{4}\int_{e^2}^{e^4}\frac{d\left(t^2\right)}{t^2-1}=\frac{1}{4}\ln\left|t^2-1\right|\Big\|_{e^2}^{e^4}=\frac{1}{4}\ln\left|e^8-1\right|-\frac{1}{4}\ln\left|e^4-1\right|=\frac{1}{4}\ln\left|\frac{e^8-1}{e^4-1}\right|=\frac{1}{4}\ln\left|e^4+1\right|$$

►

3–misol. Anik integral hisoblansin.

$$\int_0^{\frac{\pi}{3}} \frac{tg^2 x dx}{4+3\cos 2x}$$

◄

$tgx = t \quad \sin 2x = \dfrac{2tgx}{1+tg^2 x} \qquad \cos 2x = \dfrac{1-tg^2 x}{1+tg^2 x} \quad \sin x = \dfrac{2t}{1+t^2} \qquad \cos x = \dfrac{1-t^2}{1+t^2}$

$tg\dfrac{\pi}{3} = \dfrac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \sqrt{3} \qquad x = arctg\, t \quad dx = \dfrac{dt}{1+t^2}$

almashtirishlarni bajaramiz.

$$\int_0^{\sqrt{3}} \frac{t^2}{4+3\frac{1-t^2}{1+t^2}} \frac{dt}{1+t^2} = \int_0^{\sqrt{3}} \frac{t^2 dt}{4+4t^2+3-3t^2} = \int_0^{\sqrt{3}} \frac{t^2}{7+t^2} dt = \int_0^{\sqrt{3}} \frac{t^2+7-7}{t^2+7} dt =$$

$$= \int_0^{\sqrt{3}} \left(1 - \frac{7}{t^2+7}\right) dt = \left(t - \frac{7}{\sqrt{7}} arctg \frac{t}{\sqrt{7}}\right)\Bigg|_0^{\sqrt{3}} = \sqrt{3} - \frac{1}{\sqrt{7}} arctg \frac{\sqrt{3}}{\sqrt{7}}$$

►

Mustaqil yechish uchun misollar.

1 –topshiriq. Anik integral hisoblansin.

1.1 $\int_0^1 \left(\sqrt{x} + \sqrt[3]{x^2}\right) dx$

1.2 $\int_0^1 \frac{x+1}{x^2+3x+1} dx$

1.3 $\int_0^1 \frac{4x^2-x}{1+x^2} dx$

1.4 $\int_0^2 \frac{x^3}{x^4+4} dx$

1.5 $\int_1^2 \frac{x+1}{x^2+2x} dx$

1.6 $\int_0^{\frac{\pi}{3}} \frac{\cos x + \sin x}{\sin x - \cos x} dx$

1.7 $\int_1^e \frac{dx}{x\left(1+\ln^2 x\right)}$

1.8 $\int_0^1 \frac{xdx}{x^4+2}$

$$1.9\ \int_1^3 \frac{x+\frac{1}{x}}{\sqrt{x^2+1}}dx \qquad 1.10\ \int_0^{\sqrt{3}} \frac{2arctgx+x}{1+x^2}dx$$

$$1.11\ \int_0^{\sqrt{3}} \frac{x-(arctgx)^4}{1+x^2}dx \qquad 1.12\ \int_0^1 \frac{x^3}{x^2+1}dx$$

$$1.13\ \int_0^{\sin 1} \frac{(\arcsin x)^2+1}{\sqrt{1-x^2}}dx \qquad 1.14\ \int_1^3 \frac{1-\sqrt{x}}{\sqrt{x}\cdot(x+1)}dx$$

$$1.15\ \int_{\sqrt{3}}^{\sqrt{8}} \frac{dx}{x\cdot\sqrt{x^2+1}} \qquad 1.16\ \int_1^e \frac{1+\ln x}{x}dx$$

$$1.17\ \int_1^e \frac{x^2+\ln x^2}{x}dx \qquad 1.18\ \int_0^{\frac{\pi}{4}} tgx\cdot\ln(\cos x)dx$$

$$1.19\ \int_0^1 \frac{x+2}{x^2+1}dx \qquad 1.20\ \int_{\pi}^{2\pi} \frac{1+\cos x}{1-\sin x}dx$$

2 – topshiriq. Anik integral hisoblansin.

$$2.1\ \int_{-1}^{1}(x^2+5x+6)\sin xdx \qquad 2.2\ \int_{-2}^{0}(x^2-2x+4)\cos 5xdx$$

$$2.3\ \int_{-1}^{0}(x^2+x+3)\sin 2xdx \qquad 2.4\ \int_{-1}^{1}(x^2+3x+3)\cos 3xdx$$

$$2.5\ \int_1^e \frac{\sin\ln x}{x}dx \qquad 2.6\ \int_1^e \frac{\cos\ln x}{x}dx$$

$$2.7\ \int_0^{\frac{\pi}{6}} e^{3\sin^2 x}\sin 2xdx \qquad 2.8\ \int_0^{\frac{1}{2}} \frac{2^{3arctg2x}}{1+4x^2}dx$$

$$2.9\ \int_0^{\frac{1}{2}} \frac{3^{arctgx}+1}{1+x^2}dx \qquad 2.10\ \int_0^1 \frac{2^{arctgx}-x}{1+x^2}dx$$

$$2.11\ \int_0^{\frac{1}{2}} \frac{4^{arctg3x}+x}{1+9x^2}dx \qquad 2.12\ \int_0^{\frac{1}{2}} \frac{2^{3\arcsin x}}{\sqrt{1-x^2}}dx$$

$$2.13\ \int_0^{\frac{1}{2}} \frac{2^{\arcsin 2x}}{\sqrt{1-4x^2}}dx \qquad 2.14\ \int_0^{\frac{1}{4}} \frac{4^{\arcsin 2x}-2x}{\sqrt{1-4x^2}}dx$$

2.15 $\int_{0}^{\frac{1}{2}} \frac{2^{3\arcsin x}}{\sqrt{1-x^2}}dx$

2.16 $\int_{0}^{\frac{1}{2}} \frac{5^{\arccos x} - x}{\sqrt{1-x^2}}dx$

2.17 $\int_{0}^{\frac{1}{4}} \frac{5^{\arccos 4x} - x}{\sqrt{1+16x^2}}dx$

2.18 $\int_{0}^{1} \frac{5^x}{\sqrt{1+5^x}}dx$

2.19 $\int_{\frac{1}{2}}^{1} \frac{2^x}{\sqrt{1+4^x}}dx$

2.20 $\int_{0}^{1} \frac{e^x}{e^x + e^{-x}}dx$

3 – topshiriq. Anik integral hisoblansin.

3.1 $\int_{0}^{\frac{\pi}{4}} \frac{\sin x dx}{(1+\sin x+\cos x)^2}$

3.2 $\int_{0}^{arctg\frac{1}{2}} \frac{(1+\sin x)dx}{\cos x(1+\cos x)}$

3.3 $\int_{0}^{\frac{\pi}{2}} \frac{\cos x dx}{4+4\cos x}$

3.4 $\int_{-\frac{\pi}{2}}^{2\pi} \frac{\sin x dx}{(1+\cos x-\sin x)^2}$

3.5 $\int_{0}^{2\pi} \frac{\cos x dx}{1+\cos x}$

3.6 $\int_{0}^{arctg\frac{1}{3}} \frac{2+tgxdx}{12\cos^2 x+3\sin^2 x}$

3.7 $\int_{0}^{\frac{\pi}{4}} \frac{(5tgx+2)dx}{2\sin 2x+4}$

3.8 $\int_{0}^{2arctg\frac{1}{2}} \frac{1+\sin x}{(1-\sin x)^2}dx$

3.9 $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos x dx}{1+\sin x-\cos x}$

3.10 $\int_{0}^{\frac{\pi}{2}} \frac{(1+\cos x)dx}{1+\cos x+\sin x}$

3.11 $\int_{0}^{\frac{\pi}{2}} \frac{\cos x dx}{5+4\cos x}$

3.12 $\int_{0}^{\frac{2\pi}{3}} \frac{1+\sin x}{1+\cos x+\sin x}dx$

3.13 $\int_{2arctg\frac{1}{3}}^{2arctg\frac{1}{2}} \frac{dx}{\sin x\cdot(1-\sin x)}$

3.14 $\int_{2arctg\frac{1}{2}}^{\frac{\pi}{2}} \frac{dx}{(1+\sin x-\cos x)}$

3.15 $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\sin x)^2}\, dx$ 3.16 $\int_{\pi}^{5\pi/4} \frac{dx}{\cos x \cdot (1-\cos x)}$

3.17 $\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^2 x(1+\sin x)}$ 3.18 $\int_0^{\pi} \frac{\sin x}{2+\sin x}\, dx$

3.19 $\int_0^{\frac{\pi}{4}} \frac{\sin x dx}{\cos^2 x(1-\sin x)}$ 3.20 $\int_0^{\frac{\pi}{2}} \frac{\cos x dx}{2+\cos x}$

14–mustaqil ish.

Mavzu: Aniq integral tatbiqklari.

Aniq integral yordamida tekis shaklning yuzasini hisoblash.

1) Dekart koordinatalar sistemasida berilgan shaklning yuzasini hisoblash.

$f(x) \in C[a,b]$ bo`lib, $\forall x \in [a,b]$ uchun $f(x) \geq 0$ tengsizlik bajarilsin va D soha quyidagicha aniqlansin:

$$D = \begin{cases} a \leq x \leq b \\ 0 \leq y \leq f(x) \end{cases}$$ - egri chiziqli trapetsiya.

Unda

$$S = \int_a^b f(x)\, dx \qquad (1)$$

tenglik o`rinli.

Agar $f_1(x) \in C[a,b]$, $f_2(x) \in C[a,b]$ bo`lib, $D = \begin{cases} a \leq x \leq b \\ f_1(x) \leq y \leq f_2(x) \end{cases}$

bo`lsa, u holda

$$S = \int_a^b [f_2(x) - f_1(x)]dx \qquad (2)$$

bo`ladi.

2) Qutb koordinatalar sistemasida berilgan shaklning yuzasini hisoblash.

Agar D soha qutb koordinatalar sistemasida

$$D = \begin{cases} \alpha \leq \varphi \leq \beta \\ 0 \leq r \leq r(\varphi) \end{cases}$$

ko`rinishida berilgan bo`lib, $r(\varphi) \in C[\alpha, \beta]$ bo`lsa,

$$S=\frac{1}{2}\int_{\alpha}^{\beta} r^2(\varphi)\,d\varphi \qquad (3)$$

formula o`rinli bo`ladi.

Anik integral yordamida yoy uzunligini hisoblash.

1) Dekart koordiiatalar sistemasida berilgan yoy uzunligini hisoblash.

$f(x)$ funksiya $[a,b]$ kesmada aniqlangan bo`lsin, uning grafigi quyidagi

$$\{(x,\ f(x)) : x\in[a\,b]\}$$

nuqtalar to`plamidan iborat. Shu grafikdagi $A(a,f(a))$ va $B(b,f(b))$ nuqtalar orasidagi $\overset{\cup}{AB}$ egri chizik yoyi uzunligi l ni topish talab qilinsin.

Agar $f'(x)\in C[a,b]$ bo`lsa, unda

$$l=\int_a^b \sqrt{1+[f'(x)]^2}\,dx \qquad (4)$$

bo`ladi.

Agar (14) da $b=x$ desak, $l(x)=\int_a^x \sqrt{1+[f'(x)]^2}\,dx$ bo`lib,

$$\frac{dl}{dx}=\sqrt{1+[f'(x)]^2} \Rightarrow dl=\sqrt{1+[f'(x)]^2}\,dx$$

Bu ifodaga **yoy differentsiali** deb ataladi.

2) Parametrik ko`rinishda berilgan egri chizik yoyining uzunligini hisoblash.

Agar

$$\overset{\cup}{AB}:\begin{cases} x=\varphi(t) \\ y=\psi(t), \end{cases} \quad \alpha\le t\le\beta$$

bo`lib, $\varphi'(t)\in C[\alpha,\beta]$ va $\psi'(t)\in C[\alpha,\beta]$ bo`lsa,

$$l=\int_{\alpha}^{\beta} \sqrt{\varphi'(t)^2+[\psi'(t)]^2}\,dt \qquad (5)$$

bo`ladi.

3) Qutb koordinatalar sistemasida berilgan egri chizik yoyinnng uzunligini hisoblash.

Agar

$$\overset{\cup}{AB}:\begin{cases} \alpha\le\varphi\le\beta, \\ r=r(\varphi) \end{cases}$$

bo`lib, $r'(\varphi)\in C[\alpha,\beta]$ bo`lsa, unda

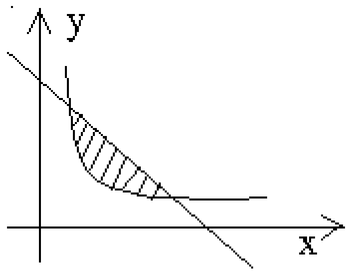
$$l=\int_{\alpha}^{\beta}\sqrt{r^2(\varphi)+[r'(\varphi)]^2}\,d\varphi \qquad (6)$$

formula o`rinli bo`ladi.

1–misol. Quyidagi chiziklar bilan chegaralangan shaklning yuzasi hisoblansin.

$$xy=4;\ \ x+y=5$$

◄ $xy=4,\ \ x+y=5,\quad y=\frac{4}{x},\ \ y=5-x$. Shakl yuzasi: $S=\int_a^b (y_1-y_2)dx$ formula orkali hisoblanadi.



$x(5-x)=4,\ \ 5x-x^2-4=0,\ \ x^2-5x+4=0$

$x_1=\frac{5+3}{2}=4,\ \ x_2=\frac{5-3}{2}=1.$

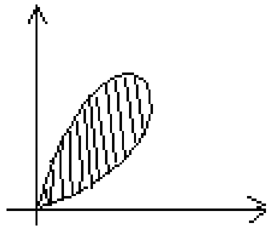
$S=\int_1^4\left(5-x-\frac{4}{x}\right)dx=\left(5x-\frac{x^2}{2}+\frac{4}{x^2}\right)\Big|_1^4=20-8+\frac{1}{4}-5+\frac{1}{2}-4=3+\frac{3}{4}=\frac{15}{4};$►

2–misol. Tenglamalari qutb koordinatalar sistemasida berilgan chiziqlar bilan chegaralangan shaklning yuzasi hisoblansin.

$$r=4\sin 2\varphi;$$

◄Shakl yuzasi: $S=\frac{1}{2}\int_{\alpha}^{\beta} r^2(\varphi)d\varphi$ formula orkali hisoblanadi.

$r\ge 0\Rightarrow 4\sin 2\varphi\ge 0,\ \sin 2\varphi\ge 0,\ 0+2\pi n\le 2\varphi\le \pi+2\pi n,\quad [0,2\pi]$ da $0\le\varphi\le\frac{\pi}{2}$ va $\pi\le\varphi\le\frac{3\pi}{2}$ bo`ladi. $0\le\varphi\le\frac{\pi}{2}$ va $\pi\le\varphi\le\frac{3\pi}{2}$ oraliqlardagi yuzalar teng bo`lganligi uchun faqat $0\le\varphi\le\frac{\pi}{2}$ dagi yuzani hisoblab, so`ngra natija ikkiga ko`paytiriladi.



$$S = 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} r_1^2\,d\varphi = \int_0^{\frac{\pi}{2}} 16\sin^2 2\varphi\,d\varphi = \int_0^{\frac{\pi}{2}} \left(8(1-\cos 4\varphi)\right)d\varphi =$$

$$= 8\left(\varphi - \frac{1}{4}\sin 4\varphi\right)\Bigg|_0^{\frac{\pi}{2}} = 4\pi$$

►

3-misol Parametrik ko`rinishda berilgan egri chiziq yoyining uzunligi hisoblansin

$$\begin{cases} x = 5(t+\sin t) & 0 \le t \le \dfrac{\pi}{2} \\ y = 5(1-\cos t) \end{cases}$$

◄Yoy uzunligi $\ell = \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t)}dt$ bilan hisoblangani uchun

Quyidagi hosilalarni hisoblaymiz: $\begin{cases} x' = 5(1+\cos t) \\ y = 5\sin t \end{cases}$

$$\ell = \int_0^{\frac{\pi}{2}} \sqrt{25(1+\cos t)^2 + 25\sin^2 t}dt = \int_0^{\frac{\pi}{2}} 5\sqrt{2+2\cos t}dt = \int_0^{\frac{\pi}{2}} 5\cdot 2\cos\frac{t}{2}dt =$$

$$= 20\sin\frac{t}{2}\Bigg|_0^{\frac{\pi}{2}} = 20\cdot\frac{\sqrt{2}}{2} = 10\sqrt{2};$$

►

4–misol. Egri chiziq yoyi uzunligini hisoblang.

$$r = 4e^{\frac{5\varphi}{4}} \qquad \frac{\pi}{2} \le \varphi \le \pi$$

◄ Egri chizik yoyi uzunligi $\ell=\int\limits_{\alpha}^{\beta}\sqrt{r'^2(\varphi)+r^2(\varphi)}d\varphi$ formula orkali hisoblanadi.

$$r'=4\cdot\frac{5}{4}e^{\frac{5\varphi}{4}}=5e^{\frac{5\varphi}{4}}$$

$$\ell=\int\limits_{\frac{\pi}{2}}^{\pi}\sqrt{25e^{\frac{5\varphi}{2}}+16e^{\frac{5\varphi}{2}}}\,d\varphi=\int\limits_{\frac{\pi}{2}}^{\pi}e^{\frac{5\varphi}{4}}\sqrt{41}d\varphi=\sqrt{41}\cdot\frac{4}{5}e^{\frac{5\varphi}{4}}\Bigg|_{\frac{\pi}{2}}^{\pi}=\frac{4\sqrt{41}}{5}\left(e^{\frac{5\pi}{4}}-e^{\frac{5\pi}{8}}\right);$$

►

4-misol. Ushbu $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellips yoyining uzunligi hisoblansin.

Ellipsni parametrik ko`rinishida $\begin{cases}x=a\sin t\\ y=b\sin t,\ 0\le t\le 2\pi\end{cases}$ kabi ifodalab olamiz.

Unda

$$l=4l_1=4\cdot\int\limits_0^{\frac{\pi}{2}}\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt=4\int\limits_0^{\frac{\pi}{2}}\sqrt{a^2\cos^2 t+b^2\sin^2 t}\,dt=$$

$$4\int\limits_0^{\frac{\pi}{2}}\sqrt{a^2(1-\sin^2 t)+b^2\sin^2 t}\,dt=4a\int\limits_0^{\frac{\pi}{2}}\sqrt{1-\frac{a^2-b^2}{a^2}\sin^2 t}\,dt=4aE\left(\frac{\sqrt{a^2-b^2}}{a}\right)=4aE(\varepsilon)$$

bu erda $\varepsilon=\frac{\sqrt{a^2-b^2}}{a}$ - ellipsning ekstsentrisiteti.

Mustaqil yechish uchun misollar.

1 – topshiriq. Quyidagi chiziklar bilan chegaralangan shaklning yuzasi hisoblansin.

1.1 $y=(x-2)^3;\ y=-x,\ y=0$ 1.2 $y=x\sqrt{4-x^2};\ y=0,\ (0\le x\le 2)$

1.3 $y=16-x^2;\ y=x^2-x$ 1.4 $y=\sqrt{16-x^2};\ y=0;\ x=0;\ x=2$

1.5 $y=x\sqrt{4-x^2};\ y=0;\ (0\le x\le 2)$ 1.6 $y=\sqrt{e^x-1};\ y=0;\ x=\ln 4$

1.7 $x=(y-2)^3;\ x=4y-8$ 1.8 $y=(x+2)^2;\ y=x+3$

1.9 $y=\arcsin x;\ y=\frac{\pi}{2};\ x=0$ 1.10 $y=x-x^2+1;\ y=x^2-2x+1$

1.11 $y = x\sqrt{36 - x^2}$; $y = 0$; $(0 \le x \le 6)$ 1.12 $x = \arccos y$; $x = 0$; $y = 0$

1.13 $xy = 5; y + x = 6$ 1.14 $y = x^2\sqrt{8 - x^2}$; $y = 0$; $(0 \le x \le 2\sqrt{2})$

1.15 $x = \sqrt{e^x - 1}$; $x = 0$; $y = \ln 2$ 1.16 $y = x\sqrt{4 - x^2}$; $y = 0$; $(0 \le x \le 2)$

1.17 $y = \dfrac{x}{1 + x^2}$; $y = 0$; $x = 1$ 1.18 $x = (y - 3)^3$; $x = 27y - 27$

1.19 $y = \dfrac{x}{(\sqrt{x} + 1)^2}$; $y = 0$; $x = 1$ 1.20 $y = 4 - x^2$; $y = x^2 - 2x$

1.21 $xy = 4$; $x + y = 5$

2 – topshiriq.. **Tenglamalari qutb koordinatalar sistemasida berilgan chiziqlar bilan chegaralangan shaklning yuzasi hisoblansin.**

2.1 $r = 3\cos 3\varphi$; $r \ge 1$ 2.2 $r = 3\sin\varphi$; $r = 4\sin\varphi$

2.3 $r = 6\cos 3\varphi$; $r \ge 3$ 2.4 $r = \cos\varphi; r = \sin\varphi$, $0 \le \varphi \le \dfrac{\pi}{2}$

2.5 $r = 2\cos\varphi + 2\sin\varphi$ 2.6 $r = 2\cos 4\varphi$

2.7 $r = \sin 5\varphi$ 2.8 $r = \cos 2\varphi$

2.9 $r = 3\cos 2\varphi$; $r = 5\cos 2\varphi$ 2.10 $r = 4\cos 6\varphi$

2.11 $r = 1 + \sqrt{2}\sin\varphi$ 2.12 $r = \dfrac{5}{2}\sin\varphi$; $r = \dfrac{3}{2}\sin\varphi$

2.13 $r = 1 + \cos\varphi$ 2.14 $r = \cos\varphi$; $r = 2\cos\varphi$

2.15 $r = \sin\varphi$; $r = 2\sin\varphi$ 2.16 $r = 6\cos 2\varphi$; $r = 2(r \ge 2)$

2.17 $r = 2 + 2\sin\varphi$ 2.18 $r = \cos 3\varphi$; $r = 3(r \ge 3)$

2.19 $r = 1 + 2\cos 3\varphi$ 2.20 $r = 1 - \sin 3\varphi$

3- topshiriq. Egri chiziq yoyi uzunligini hisoblang.

3.1 $y = \sqrt{1 - x^2}$; $0 \le x \le \dfrac{1}{2}$ 3.2 $y = \ln\cos x$; $0 \le x \le \dfrac{\pi}{3}$

3.3 $y = \sqrt{4 - x^2}$; $1 \le x \le 2$ 3.4 $y = x + 2$ $0 \le x \le 2$

3.5 $y = \ln\sin x$; $\dfrac{\pi}{4} \le x \le \dfrac{\pi}{2}$ 3.6 $y = -\ln\cos x$; $0 \le x \le \dfrac{\pi}{4}$

3.7 $y = 1 - \ln(x^2 - 4)$; $3 \le x \le 4$ 3.8 $y = \ln(x^2 - 1)$; $2 \le x \le 4$

3.9 $y = e^x + e^{-x}$; $0 \le x \le 1$ 3.10 $y = e^x + 5$; $\ln\sqrt{2} \le x \le \ln\sqrt{10}$

3.11 $r = \cos\varphi$; $\dfrac{\pi}{3} \le \varphi \le \dfrac{\pi}{2}$ 3.12 $r = \sin\varphi$; $-\dfrac{\pi}{4} \le \varphi \le -\dfrac{\pi}{6}$

3.13 $r = 6\sin\varphi$; $\dfrac{\pi}{3} \le \varphi \le \dfrac{\pi}{2}$ 3.14 $r = 6(1 - \sin\varphi)$; $-\dfrac{\pi}{3} \le \varphi \le \dfrac{\pi}{3}$

3.15 $r = \cos\varphi$; $-\dfrac{\pi}{3} \le \varphi \le -\dfrac{\pi}{6}$ 3.16 $r = \sin\varphi$; $-\dfrac{\pi}{3} \le \varphi \le \dfrac{\pi}{6}$

3.17 $r = 2\sin\varphi;\quad -\frac{\pi}{3} \le \varphi \le -\frac{\pi}{6}$ 3.18 $r = 5\cos\varphi;\quad \frac{\pi}{3} \le \varphi \le \pi$

3.19 $r = e^{\frac{3\varphi}{4}};\quad -\frac{\pi}{3} \le \varphi \le \frac{\pi}{3}$ 3.20 $r = 2e^{\frac{3\varphi}{2}};\quad -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

15-mustaqil ish.
Mavzu: Aniq integralning tadbiqlari.

Aylanma sirtning yuzasi.

$f(x)$ funksiya $[a,b]$ kesmada aniqlangan bo`lsin, uning grafigi quyidagi

$$\{(x, f(x)) : x \in [a b]\}$$

nuqtalar to`plamidan iborat. Shu grafikdagi $A(a, f(a))$ va $B(b, f(b))$ nuqtalar orasidagi $\overset{\cup}{AB}$ egri chizikni qaraymiz.

Aytaylik, $f(x) \in C[a,b]$ bo`lib, $f(x) \geq 0$ bo`lsin. $\overset{\cup}{AB}$ yoyni OX o`qi atrofida aylantiramiz va aylanma sirtni hosil kilamiz. Agar $f'(x) \in C[a,b]$ bo`lsa, unda shu aylanma sirtning yuzasi ushbu

$$S = 2\pi \int_a^b f(x) \cdot \sqrt{1 + [f'(x)]^2}\, dx \qquad (1)$$

formula yordamida hisoblanadi.

$\overset{\cup}{AB}$ egri chiziq yuqori yarim tekislikda joylashgan bo`lib u $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $(\alpha \leq t \leq \beta)$ parametrik tenglama bilan berilgan bo`lsin. Bunda $x = x(t)$ va $y = y(t)$ funksiyalar $[\alpha; \beta]$ da uzluksiz va uzluksiz $x'(t)$, $y'(t)$ hosilalarga ega bo`lsin. Bu egri chiziqni $OX$ o`qi atrofida aylantirishdan hosil bo`lgan aylanma sirtning yuzi

$$S = 2\pi \int_\alpha^\beta y(t) \sqrt{x'^2(t) + y'^2(t)}\, dt \qquad (2)$$

formula yordamida hisoblanadi.

Aniq integral yordamida hajm hisoblash.
Faraz qilaylik, bizga biror T jism berilgan bo`lib, uning $OY$ o`qiga parallel bo`lgan kesimlarining yuzasi ma’lum bo`lsin Bu yuza x o`zgaruvchining funksiyasi bo`ladi, uni $S = S(x)$ deb belgilaylik. Agar $S(x) \in C[a,b]$ bo`lsa, unda T jismning hajmi V ushbu

$$V = \int_a^b S(x)\, dx \qquad (3)$$

formula yordamida hisoblanadi.

Aylanma jismning hajmi. Ushbu

$$D=\begin{cases} a\le x\le b \\ 0\le y\le f(x)\end{cases}$$

egri chiziqli trapetsiyani OX o`qi atrofida aylantirishdan hosil bo`lgan aylanma jismning hajmi

$$V=\pi\int_a^b [f(x)]^2\,dx \qquad (4)$$

formula yordamida hisoblanadi.

Agar shu figuraning o`zi $OY$ o`qi atrofida aylantirilsa, u holda aylanish jismining hajmi $V=2\pi\int_a^b xf(x)dx$ formula bilan hisoblanadi.

O`zgaruvchi kuchning bajargan ishi.

OX o`qida shu o`q bo`ylab biror jism $F=F(x)$ kuch ta'sirida harakat qilayotgan bo`lsin. Agar $F(x)\in C[a,b]$ bo`lsa, $F=F(x)$ kuch ta'sirida jismni $a$ nuqtadan $b$ nuqtaga o`tkazishda bajarilgan ish ushbu

$$A=\int_a^b F(x)\,dx \qquad (5)$$

formula yordamida hisoblanadi

Statik moment. Og'irlik markazi.

Aytaylik, m massaga ega bo`lgan $M(x,y)$ - moddiy nuqta berilgan bo`lsin. my va mx ko`paytmalarga mos ravishda berilgan nuqtaning $OX$ **va** $OY$ **o`qlarga nisbatan statik momentlari** deb ataladi.

Egri chiziqning OX va OY o`qlarga nisbatan **statik momentlari** M_x va M_y lar xam shu kabi aniqlanadi xamda

$$M_x=\int_0^l y\,dl,\ M_y=\int_0^l x\,dl \qquad (6)$$

formulalar yordamida hisoblanadi. Bu erda $dl=\sqrt{(dx)^2+(dy)^2}$ - yoy differentsiali, l esa berilgan egri chiziq uzunligi.

Berilgan egri chizik og'irlik markazining koordinatalari esa ushbu

$$\bar{x}=\frac{M_y}{l},\ \bar{y}=\frac{M_x}{l} \qquad (7)$$

formulalar yordamida hisoblanadi.

Geometrik figuralarning statik momentlari va og'irlik markazi.

Agar geometrik figura

$$D = \begin{cases} a \le x \le b \\ 0 \le y \le f(x) \end{cases}$$

egri chiziqli trapetsiyadan iborat bo`lsa, unda

$$M_x = \frac{1}{2}\int_a^b y^2\,dx, \quad M_y = \frac{1}{2}\int_a^b xy\,dx \tag{8}$$

va

$$\left(\bar{x}, \bar{y}\right) = \left(\frac{M_y}{S}, \frac{M_x}{S}\right) \tag{9}$$

bo`ladi. Bu erda $S = \int_a^b y(x)\,dx$ - trapetsiyaning yuzi.

Elliptik integrallar.

1- Ta'rif. Ushbu

$$F(k,\varphi) = \int_0^{\varphi} \frac{dx}{\sqrt{1-k^2\sin^2 x}} \tag{10}$$

$$F(k,\varphi) = \int_0^{\varphi} \sqrt{1-k^2\sin^2 x}\,dx \tag{11}$$

ko`rinishdagi integrallar I va II - tipdagi **elliptik integrallarning Lejandr formasi** deb ataladi.

(10) va (11) — integral ostidagi funksiyalarning boshlang'ich funksiyalari elementar funksiyalar yordamida ifodalanmaydi. Shuning uchun ham ularning qiymatlarini hisoblash uchun maxsus jadvallar yaratilgan.

Agar (10) va (11) - integrallarda $\varphi = \frac{\pi}{2}$ bo`lsa, u holda

bunday integrallar **to`liq elliptik integrallar** deb ataladi va ular $F(k), E(k)$ kabi belgilanadi.

Demak,

$$F(k) = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-k^2\sin^2 x}} \tag{12}$$

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2 x}\,dx \tag{13}$$

To`lik elliptik integrallarning kiymatlari xam maxsus jadvallar yordamida hisoblanadi.

1–misol. Quyidagi egri chiziqlarni OX o`qi atrofida aylantirishdan hosil bo`lgan aylanish sirtlarining yuzalari topilsin.

$$x^2+(y-1)^2=1$$

$$S=2\pi\int_a^b f^2(x)\sqrt{1+f'^2(x)}dx$$

◀ formulalardan foydalanamiz.

$$S=2\pi\int_a^b y(t)\sqrt{x'^2(t)+y'^2(t)}dt$$

$x^2+(y-1)^2=1$

$$\begin{cases} x=\cos t \\ y-1=\sin t \end{cases} \qquad \begin{cases} x=\cos t \\ y=1+\sin t \end{cases} \quad 0\le t\le 2\pi \qquad \begin{cases} x'=-\sin t \\ y'=\cos t \end{cases}$$

$$S=2\pi\int_0^{2\pi}(1+\sin t)\sqrt{\sin^2 t+\cos^2 t}dt=2\pi\int_0^{2\pi}(1+\sin t)dt=2\pi(t-\cos t)\Big|_0^{2\pi}=4\pi$$

▶

2.21 –misol. Quyidagi sirtlar bilan chegaralangan jismning xajmi topilsin.

$$\frac{x^2}{25}+\frac{y^2}{49}+\frac{z^2}{16}=1;\ z=4;\ z=0$$

◀ $\frac{x^2}{25}+\frac{y^2}{49}+\frac{z^2}{16}=1$ $z=4,\ z=0$ sirtni z o`qiga perpendikular kesim bilan kesamiz. $\dfrac{x^2}{25\left(1-\frac{z^2}{16}\right)}+\dfrac{y^2}{49\left(1-\frac{z^2}{16}\right)}=1$ ellips hosil bo`ladi.

Ellipsning yuzi $S(z)=5\pi\sqrt{1-\frac{z^2}{16}}\cdot 7\sqrt{1-\frac{z^2}{16}}=35\pi\left(1-\frac{z^2}{16}\right)$.

$$V=\int_0^4 S(z)dz=\pi\int_0^4 35\left(1-\frac{z^2}{16}\right)=\pi\left(35z-\frac{35}{16}\cdot\frac{z^3}{3}\right)\Bigg|_0^4=\pi\left(35z-\frac{35}{48}z^3\right)\Bigg|_0^4=$$

$$=\pi\left(35\cdot 4-\frac{35}{48}\cdot 64\right)=\pi\left(140-\frac{35\cdot 4}{3}\right)=140\pi\left(1-\frac{1}{3}\right)=140\pi\cdot\frac{2}{3}=\frac{280}{3}\pi$$

▶

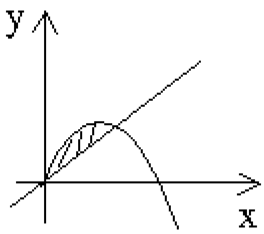
3–misol. Funksiyalar grafiklari bilan chegaralangan figurani OU o`qi atrofida aylantirishdan xosil bo`lgan jismning hajmi topilsin.

$$y = 2x - x^2,\ y = x$$

◀ Figurani Ox o`q atrofida aylantirishdan hosil bo`lgan jism hajmi $V = \pi \int_a^b f^2(x)dx$ dan hisoblanadi. Figurani Oy o`q atrofida aylantirishdan hosil bo`lgan jism hajmi $V = 2\pi \int_a^b xydx$ dan hisoblanadi.

$y = 2x - x^2; \quad y = x$

$2x - x^2 = x \quad x^2 - x = 0 \quad x(x-1) = 0 \quad x = 0 \quad x = 1$



$$V = 2\pi \int_0^1 x(2x - x^2 - x)dx = 2\pi \int_0^1 x(x - x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx = 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4} \right)\Bigg|_0^1 =$$

$$= 2\pi \left(\frac{1}{3} - \frac{1}{4} \right) = 2\pi - \frac{1}{12} = \frac{\pi}{6}$$

▶

4-misol. Quyidagi chiziklar bilan chegaralangan tekis shaklning ogirlik markazi topilsin.

$$x^2 + 2y - 4 = 0;\ y = 0.$$

◀ $x^2 + 2y - 4 = 0,\ \ y = 0$ figura uchun:

$M_x = \frac{1}{2} \int_a^b y^2 dx; \qquad M_y = \int_a^b xydx$

$(\bar{x}, \bar{y}) = \left(\frac{M_y}{S}, \frac{M_x}{S} \right) \qquad S = \int_a^b y(x)dx$ formulalardan foydalanamiz.

$x^2 + 2y - 4 = 0 \quad y = 0 \Rightarrow \quad \frac{4 - x^2}{2} = y \Rightarrow$

$x^2 - 4 = 0 \qquad \Rightarrow \qquad x = \pm 2$

$$M_x=\frac{1}{2}\int_a^b\left(2-\frac{x^2}{2}\right)^2dx=\frac{1}{2}\int_{-2}^{2}\left(4-2x^2+\frac{x^4}{4}\right)dx=\frac{1}{2}\left(4x-\frac{2x^3}{3}+\frac{x^5}{20}\right)\Bigg|_{-2}^{2}=$$

$$=\frac{1}{2}\left(8-\frac{16}{3}+\frac{32}{20}+8-\frac{16}{3}+\frac{32}{20}\right)=\frac{1}{2}\left(8-\frac{32}{3}+\frac{8}{5}\right)=\frac{1}{2}\left(8-\frac{160-24}{15}\right)=4-\frac{136}{30}=$$

$$=4-\frac{68}{15}=\frac{60-68}{15}=\frac{-8}{15}$$

$$M_y=\int_{-2}^{2}x\left(2-\frac{x^2}{2}\right)dx=\int_{-2}^{2}\left(2x-\frac{x^3}{2}\right)dx=\left(2\frac{x^3}{3}-\frac{x^4}{8}\right)\Bigg|_{-2}^{2}=\frac{16}{3}-\frac{16}{8}+\frac{16}{3}+\frac{16}{8}=\frac{32}{3}$$

$$S=\int_{-2}^{2}\left(2-\frac{x^2}{2}\right)dx==\left(2x-\frac{x^3}{6}\right)\Bigg|_{-2}^{2}=4-\frac{8}{6}+4-\frac{8}{6}=\frac{16}{3};$$

$$\bar{x}=\frac{M_y}{S}=\frac{\frac{32}{3}}{\frac{16}{3}}=2,\quad \bar{y}=\frac{M_x}{S}=\frac{-\frac{8}{15}}{\frac{16}{3}}=-\frac{8}{15}\cdot\frac{3}{16}=-\frac{1}{10}$$

$\left(2;-\frac{1}{10}\right)$ nuqta og’irlik markazi.▶

Mustaqil yechish uchun misollar.

1-topshiriq. Quyidagi egri chiziqlarni OX o`qi atrofida aylantirishdan hosil bo`lgan aylanish sirtlarining yuzalari topilsin.

1.1 $y=\frac{x^3}{3};\quad -2\le x\le 2$

1.2 $y=\frac{x^2}{2};\ -1\le x\le 1$

1.3 $y=x^2+2;\ 0\le x\le 2$

1.4 $y=x^3-x;\ 0\le x\le 1$

1.5 $y=\frac{4}{x};\ 1\le x\le 4$

1.6 $y=\frac{2}{x};\ 2\le x\le 4$

1.7 $x=2\sqrt{3}\cos t,\ y=\sin 2t$

1.8 $x=\frac{t^3}{3},\ y=4-\frac{t^2}{2}\quad -1\le t\le 1$

1.9 $y=\frac{x^2}{2}-x;\quad 0\le x\le 2$

1.10 $y=\frac{x^2}{3}-1;\ 1\le x\le 3$

1.11 $y=\cos x;\ 0\le x\le\frac{\pi}{4}$

1.12 $y=\sin x;\ 0\le x\le\frac{\pi}{3}$

1.13 $y=2\cos 2x;\ 0\le x\le\frac{\pi}{6}$

1.14 $y=2\sin 2x;\ 0\le x\le\frac{\pi}{2}$

1.15 $y = tgx;\ \ 0 \le x \le \frac{\pi}{4}$ 1.16 $y = tg2x;\ 0 \le x \le \frac{\pi}{6}$

1.17 $y = e^{-x};\ 0 \le x \le 2$ 1.18 $y = e^{2x};\ -1 \le x \le 1$

1.19 $x^2 + y^2 = 4$ 1.20 $(x-2)^2 + y^2 = 4$

2 – topshiriq. Quyidagi sirtlar bilan chegaralangan jismning xajmi topilsin.

2.1. $\frac{x^2}{4} + y^2 = 1;\ z = 2;\ z = 0.$ 2.2. $z = 2x^2 + 2y;\ z = 2$

2.3. $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{2} = 1;\ z = 0;\ z = 4$ 2.4

$.\frac{x^2}{16} + \frac{y^2}{4} + \frac{z^2}{36} = 1;\ z = 0; z = 4$

2.5. $\frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{4} = 1;\ z = 1;\ z = 2$ 2.6.

$x^2 + 9y^2 = 9;\ z = y;\ z = 0.(y \ge 0)$

2.7. $z = x^2 - 4y;\ z = 2$ 2.8. $\frac{x^2}{4} + \frac{y^2}{16} - z^2 = 1;\ z = 0;\ z = 1$

2.9. $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{25} = 1;\ z = 5$ 2.10.

$\frac{x^2}{25} + \frac{y^2}{9} + \frac{z^2}{16} = 1;\ z = 5;\ z = 0$

2.11. $z = 2x^2 + y^2;\ z = 2$ 2.12.

$\frac{x^2}{81} + y^2 - z^2 = 1;\ z = 0;\ z = 3$

2.13. $\frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{36} = 1;\ z = 0; z = 6$ 2.14.

$4x^2 + y^2 = 4;\ z = y;\ z = 0.(y \ge 0)$

2.15. $z = x^2 + 5y^2;\ z = 5$ 2.16. $x^2 + \frac{y^2}{4} - \frac{z^2}{9} = 1;\ z = 0;\ z = 2$

2.17. $\frac{x^2}{9} + \frac{y^2}{25} - \frac{z^2}{36} = -1;\ z = 12$ 2.18.

$\frac{x^2}{4} + y^2 = 1;\ z = x;\ z = 0.(x \ge 0)$

2.19. $z = x^2 + 16y^2;\ z = 4$ 2.20. $z = 2x^2 + 9y^2;\ z = 3$

3 – topshiriq. Funksiyalar grafiklari bilan chegaralangan figurani OX o`qi atrofida (3.1 –3.11) va OU o`qi atrofida (3.12 –3.21) aylantirishdan xosil bo`lgan jismning hajmi topilsin.

3.1 $y = 2x^2 + 4x - 6;\ y = 0$

3.2 $y = x^2 + 1;\ y = x,\ \ x = 0,\ \ x = 1$

3.3 $y = e^{1-x};\ y = 0,\ x = 0,\ x = 1$

3.4 $y^2 = 4x,\ x^2 = 4y$

3.5 $y = 2\sin^2 x;\ x = \frac{\pi}{2};\ y = 0$

3.6 $y = 3\sin x,\ y = \sin x;\ 0 \le x \le \pi$

3.7 $y = e^{x+1},\ y = 0,\ x = 1$

3.8 $y = x^2;\ y^2 - x = 0$

3.9 $y = x - x^2;\ y = -x$

3.10 $y = e^{3-x},\ y = 0,\ x = 0,\ x = 1$

3.11 $y = x^2,\ y^2 - 2x = 0$

3.12 $x^2 + (y-1)^2 = 1$

3.13 $y = -x^2,\ x = \sqrt{y-1},\ x = 1$

3.14 $y = x^2 + 2x,\ y = -1,\ x = 1$

3.15 $y = 2x^2,\ y = \sqrt{x^2 + 1}$

3.16 $y = \cos\frac{\pi x}{2},\ y = x^2$

3.17 $y = x^3 + 1;\ x = 2;\ y = 0$

3.18 $y = (x-2)^2,\ y = 4$

3.19 $y = 2 - \frac{x^2}{2},\ x + y = 2$

3.20 $y = x^2 - 4x + 4,\ x = 1,\ y = 0$

4 – misol. Quyidagi chiziklarning koordinata o`qlariga nisbatan statik momentlari topilsin.

4.1 $y = \sin x (0 \le y \le \pi)$ egri chiziq yoyining OX o`qiga nisbatan statik momentlari topilsin.

4.2 $y^2 = 4 + x\ \ (y > 0,\ 4 \le x \le 6)$ parabola yoyining OX va OY uklarga nisbatan statik momentlari topilsin.

4.3 $y = \sin 2x\left(0 \le y \le \frac{\pi}{2}\right)$ egri chizik yoyining OX ukiga nisbatan statik momentlari topilsin.

4.4 $y = 1 + x^2\ \ (x > 0,\ 1 \le x \le 2)$ parabola yoyining OX va OY uklarga nisbatan statik momentlari topilsin.

4.5 $y^3 = x\ \ (y > 0,\ 1 \le x \le 8)$ giperbola yoyining OX va OY uklarga nisbatan statik momentlari topilsin.

4.6 $y = 2x^3\ \ (y > 0,\ 1 \le x \le 2)$ giperbola yoyining OX va OY uklarga nisbatan statik momentlari topilsin.

4.7 $y^2 = -x\ \ (y > 0,\ -2 \le x \le -1)$ parabola yoyining OX va OY uklarga nisbatan statik momentlari topilsin.

4.8 $y = x^2 - 2x,\ \ (y > 0,\ 0 \le x \le 2)$ parabola yoyining OX va OY uklarga nisbatan statik momentlari topilsin.

4.9 $\frac{x^2}{4}+\frac{y^2}{9}=1$ ellipsning OX ukidan yukorida joylashgan bulagining koordinata uklariga nisbatan statik momentlari topilsin.

4.10 $2x+y=1,\ x=0,\ y=0$ chiziklar bilan chegaralangan uchburchakning OX va OY uklarga nisbatan statik momentlari topilsin.

4.11 $y^2=2x,\ (y>0,\ 0\le x\le 4)$ parobola yoyining OX va OY uklarga nisbatan statik momentlari topilsin.

4.12 $y=\cos x\left(-\frac{\pi}{2}\le y\le\frac{\pi}{2}\right)$ egri chizik yoyining OX ukiga nisbatan statik momentlari topilsin.

4.13 $\frac{x}{4}+\frac{y}{3}=1$ to`g’ri chizikning koordinata uklari orasida joylashgan kesmasining koordinata uklariga nisbatan statik momentlari topilsin.

4.14 $y=\frac{2}{1+x^2}$ va $y=x^2$ chiziklar bilan chegaralangan shaklning OX ukiga nisbatan statik momentlari topilsin.

4.15 $x^2+y^2=16,\ y\ge 0$ - yarim aylananing ogirlik markazi topilsin.

4.16 $x^{2\backslash 3}+y^{2\backslash 3}=2^{2\backslash 3},\ x\ge 0,\ y\ge 0$ - astroida yoyining ogirlik markazi topilsin.

4.17 $\frac{x^2}{16}+\frac{y^2}{25}\le 1\ (x\ge 0,\ y\ge 0)$ ning ogirlik markazi topilsin.

Quyidagi chiziklar bilan chegaralangan tekis shaklning ogirlik markazi topilsin.

4.18 $y=\cos x,\ y=\sin x;\ 0\le x\le\frac{\pi}{2}$

4.19 $x=y^2,\ y=x^2\ \ (x>0)$.

4.20 $y=x,\ y=-x^2+1\ \ (x\ge 0)$.

4.21 $x^2+2y-4=0;\ y=0$.

Asosiy formulalar.

1. Qisqa ko'paytirish formulalar va Nyuton binomi.

1. $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

.2. $(a \pm b)^3 = a^3 \pm 3a^2 b + 3ab^3 \pm b^3$.

3. $(a \pm b)^4 = a^4 \pm 4a^3 b + 6a^2 b^2 \pm 4ab^3 + b^4$.

4. $a^2 - b^2 = (a - b)(a + b)$.

5. $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

6. $(a + b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$, bu erda

$C_n^k = \frac{n!}{k!(n-k)!}$, $n! = 1 \cdot 2 \cdot \ldots \cdot n$ va $0! = 1$.

7. $a^n - b^n = (a - b)\sum_{k=0}^{n-1} a^{n-1-k} b^k = (a - b)\sum_{k=0}^{n-1} a^k b^{n-1-k} =$

$= (a - b) \cdot \left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}\right)$,

bu erda $n \in N$, $n > 1$

2.Trigonometrik funksiyalar va trigonometriya formulalari.

1) Trigonometrik funksiyalarning ishoralari.

	$\sin x$	$\cos x$	tgx	$ctgx$
$0 < x < \frac{\pi}{2}$	+	+	+	+
$\frac{\pi}{2} < x < \pi$	+	-	-	-
$\pi < x < \frac{3\pi}{2}$	-	-	+	+
$\frac{3\pi}{2} < x < 2\pi$	-	+	-	-

2) Trigonometrik funksiyalarning ba'zi bir burchaklardagi qiymatlari.

Radianlar	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Graduslar	0^0	30^0	45^0	60^0	90^0	180^0	270^0	360^0
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
tgx	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	-	0	-	0
$ctgx$	-	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	-	0	-

3) Asosiy trigonometrik ayniyatlar.

1. $\sin^2 x + \cos^2 x = 1.$ 2. $tgx = \frac{\sin x}{\cos x}, \ \left(x \neq \frac{\pi}{2} + \pi n\right).$

3. $ctgx = \frac{\cos x}{\sin x}, \ x \neq \pi n.$ 4. $tgx \cdot ctgx = 1, \ \left(x \neq \frac{\pi n}{2}\right).$

5. $1 + tg^2 x = \frac{1}{\cos^2 x}, \left(x \neq \frac{\pi}{2} + \pi n\right)$

6. $1 + ctg^2 x = \frac{1}{\sin^2 x}, \ (x \neq \pi n), \ (n \in Z)$

4) Keltirish formulalari.

$y \rightarrow$	$\frac{\pi}{2} \pm x$	$\pi \pm x$	$\frac{3\pi}{2} \pm x$	$2\pi \pm x$
$\sin y$	$\cos x$	$\mp \sin x$	$-\cos x$	$\pm \sin x$
$\cos y$	$\mp \sin x$	$-\cos x$	$\pm \sin x$	$\cos x$
tgy	$\mp ctgx$	$\pm tgx$	$ctgx$	$\pm tgx$
$ctgy$	$\mp tgx$	$\pm ctgx$	tgx	$\pm ctgx$

5) Burchak yig`indisi va ayirmasi uchun formulalar.

1. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

2. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

3. $tg(x \pm y) = \dfrac{tgx \pm tgy}{1 \mp tgxtgy}$

4. $ctg(x \pm y) = \dfrac{ctgx \cdot ctgy \mp 1}{ctgx \pm ctgy}$

6) Ikkilangan va karrali burchak uchun formulalar.

1. $\sin 2x = 2\sin x \cdot \cos x = \dfrac{2tgx}{1 + tg^2 x}$

2. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1 - tg^2 x}{1 + tg^2}$

3. $tg 2x = \dfrac{2tgx}{1 - tg^2 x} = \dfrac{2}{ctgx - tgx}$

4. $ctg 2x = \dfrac{ctg^2 x - 1}{2ctgx} = \dfrac{ctgx - tgx}{2}$

5. $\sin 3x = 3\sin x - 4\sin^3 x$

6. $\cos 3x = 4\cos^3 x - 3\cos x$

7. $tg 3x = \dfrac{3tgx - tg^3 x}{1 - 3tg^2 x}$

8. $ctg 3x = \dfrac{ctg^3 x - 3ctgx}{3ctg^2 x - 1}$

7) YArim burchak uchun formulalar.

1. $\sin \dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{2}}$

2. $\cos \dfrac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{2}}$

3. $tg \dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{1 + \cos x}} = \dfrac{\sin x}{1 + \cos x} = \dfrac{1 - \cos x}{\sin x}$

4. $ctg \dfrac{x}{2} = \pm\sqrt{\dfrac{1 + \cos x}{1 - \cos x}} = \dfrac{\sin x}{1 - \cos x} = \dfrac{1 + \cos x}{\sin x}$

Izox: Tengliklardagi «+» yoki «–» ishora $\dfrac{x}{2}$ burchakning qaysi chorakda joylashganligiga qarab tanlanadi.

8) Trigonometrik funksiyalarning darajalari uchun formulalar.

1. $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

2. $\cos^2 x = \dfrac{1+\cos 2x}{2}$

3. $\sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$

4. $\cos^3 x = \dfrac{3\cos x - \cos 3x}{4}$

9) Trigonometrik funksiyalarning yig`indi va ayirmalari uchun formulalar.

1. $\sin x + \sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

2. $\sin x - \sin y = 2\sin\dfrac{x-y}{2}\cos\dfrac{x+y}{2}$

3. $\cos x + \cos y = 2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

4. $\cos x - \cos y = -2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

5. $\cos x \pm \sin x = \sqrt{2}\sin\left(\dfrac{\pi}{4} \pm x\right) = \sqrt{2}\cos\left(\dfrac{\pi}{4} \pm x\right)$

6. $A\cos x \pm B\sin x = \sqrt{A^2+B^2}\sin(x+y)$, bu erda $A^2+B^2 \neq 0$

$\sin y = \dfrac{A}{\sqrt{A^2+B^2}}, \quad \cos y = \dfrac{B}{\sqrt{A^2+B^2}}$

7. $tgx \pm tgy = \dfrac{\sin(x \pm y)}{\cos x \cos y}$

8. $ctgx \pm ctgy = \dfrac{\sin(x \pm y)}{\sin x \sin y}$

9. $tgx + ctgy = \dfrac{\cos(x-y)}{\cos x \sin y}$

10. $ctgx - tgy = \dfrac{\cos(x+y)}{\sin x \cos y}$

10) Trigonometrik funksiyalarning ko’paytmalari uchun formulalar.

1. $\sin x \sin y = \dfrac{1}{2}[\cos(x-y) - \cos(x+y)]$

2. $\cos x \cos y = \dfrac{1}{2}[\cos(x-y) + \cos(x+y)]$

3. $\sin x \cos y = \dfrac{1}{2}[\sin(x-y) + \sin(x+y)]$

4. $\cos x \sin y = [\sin(x+y) - \sin(x-y)]$

5. $tgxtgy = \dfrac{tgx + tgy}{ctgx + ctgy}$

6. $ctgxctgy = \frac{ctgx + ctgy}{tgx + tgy}$

7. $\sin(x+y)\sin(x-y) = \cos^2 y - \cos^2 x$

8. $\cos(x-y)\cos(x+y) = \cos^2 y - \sin^2 x$

Izox: YUqorida keltirilgan ayniyatlar va formulalar tenglikning har ikkala tomoni ma'noga ega bo'lgan qiymatlarida o'rinli bo'ladi.

3. Teskari trigonometrik funksiyalar.

1. $y = \arcsin x$. $D(y) = [-1;1]$, $E(y) = \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$, $f(-x) = -f(x)$

2. $y = \arccos x$. $D(y) = [-1;1]$, $E(y) = [0;\pi]$, $\arccos(-x) = \pi - \arccos x$

3. $y = arctgx$. $D(y) = (-\infty;\infty)$, $E(y) = \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$, $f(-x) = -f(x)$

4. $y = arcctgx$. $D(y) = (-\infty;\infty)$, $E(y) = (0,\pi)$, $arcctg(-x) = \pi - arcctgx$

4. Trigonometrik tenglamalar.

1. $\sin x = a \Rightarrow \begin{cases} |a| > 1 \Rightarrow x \in \varnothing \\ |a| \le 1 \Rightarrow x = (-1)^k \arcsin a + \pi k \end{cases}$ bu erda $k \in Z$ va

$-\frac{\pi}{2} \le \arcsin a \le \frac{\pi}{2}$

2. $\cos x = a \Rightarrow \begin{cases} |a| > 1 \Rightarrow x \in \varnothing \\ |a| \le 1 \Rightarrow x = \pm \arccos a + 2\pi k \end{cases}$ bu erda $0 \le \arccos a \le \pi$

3. $tgx = a \Rightarrow x = arctga + \pi k$, bu erda $arctga \in \left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ va $a \in R$

4. $tgx = a \Rightarrow x = arctga + \pi k$, bu erda $arcctga \in (0;\pi)$ va $a \in R$

5. Eng sodda trigonometrik tenglamalar yechimlari jadvali $(k \in Z)$.

A	$\sin x = a$	$\cos x = a$
0	$x = \pi k$	$x = \frac{\pi}{2} + 2\pi k$
1	$x = \frac{\pi}{2} + 2\pi k$	$x = 2\pi k$
-1	$x = -\frac{\pi}{2} + 2\pi k$	$x = \pi + 2\pi k$
$\frac{1}{2}$	$x = (-1)^k \frac{\pi}{6} + \pi k$	$x = \pm\frac{\pi}{3} + 2\pi k$

$-\frac{1}{2}$	$x=(-1)^{k+1}\frac{\pi}{6}+\pi k$	$x=\pm\frac{2\pi}{3}+2\pi k$
$\frac{\sqrt{3}}{2}$	$x=(-1)^{k}\frac{\pi}{3}+\pi k$	$x=\pm\frac{\pi}{6}+2\pi k$
$-\frac{\sqrt{3}}{2}$	$x=(-1)^{k+1}\frac{\pi}{3}+\pi k$	$x=\pm\frac{5\pi}{6}+2\pi k$
$\frac{\sqrt{2}}{2}$	$x=(-1)^{k}\frac{\pi}{4}+\pi k$	$x=\pm\frac{\pi}{4}+2\pi k$
	$x=(-1)^{k+1}\frac{\pi}{4}+\pi k$	$x=\pm\frac{3\pi}{4}+2\pi k$

6. Giperbolik tenglamalar.

a	$tgx=a$	$ctgx=a$
0	$x=\pi k$	$x=\frac{\pi}{2}+2\pi k$
1	$x=\frac{\pi}{4}+\pi k$	$x=\frac{\pi}{4}+\pi k$
-1	$x=-\frac{\pi}{4}+\pi k$	$x=\frac{3\pi}{4}+\pi k$
$\sqrt{3}$	$x=\frac{\pi}{3}+\pi k$	$x=\frac{\pi}{6}+\pi k$
$-\sqrt{3}$	$x=-\frac{\pi}{3}+\pi k$	$x=\frac{5\pi}{6}+\pi k$
$\frac{\sqrt{3}}{3}$	$x=\frac{\pi}{6}+\pi k$	$x=\frac{\pi}{3}+\pi k$
$-\frac{\sqrt{3}}{3}$	$x=-\frac{\pi}{6}+\pi k$	$x=\frac{2\pi}{3}+\pi k$

1. $shx = \frac{e^x - e^{-x}}{2}$.   2. $chx = \frac{e^x + e^{-x}}{2}$.

3. $thx = \frac{shx}{chx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.   4. $cthx = \frac{chx}{shx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

5. $ch^2 x - sh^2 x = 1$.   6. $sh2x = 2shx \cdot chx$.

7. $ch2x = ch^2 x + sh^2 x$.   8. $thx \cdot cthx = 1$.

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