Journal of Pure and Applied Algebra 38 (1985) 277-283
North-Holland

SOME CURIOSITIES OF RINGS OF ANALYTIC FUNCTIONS

C.U. JENSEN

Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark

Communicated by C. Löfwall
Received 8 March 1985

Dedicated to Jan-Erik Roos on his 50-th birthday

1. It is the purpose of this paper to exhibit some properties of certain rings of analytic functions which may be a little unexpected.

Let E be the ring of all entire functions in one complex variable, i.e. the subring of $\mathbb{C}[[X]]$ consisting of all formal power series with infinite convergence radius. More generally, for a subfield K of $\mathbb{C}$ let $E(K)$ be the subring of $K[[X]]$ formed by all power series with infinite convergence radius. If ϱ is a positive real number, let $E(\varrho, K)$, resp. $\bar{E}(\varrho, K)$ be the subring of $K[[X]]$ consisting of all power series with convergence radius $>\varrho$, resp. $\geq\varrho$.

It is well known that $E(K)$ is a non-Noetherian domain and that $E(K)$ is a Bezout domain, i.e. any finitely generated ideal is principal. If $K=\mathbb{C}$ this was proved by Wedderburn [13] and in the general case by Helmer [4], (who apparently was unaware of [13]). As for the Krull dimension of E the first 'result' appeared in [10] stating that K-dim $E=1$. An error in the proof was noticed by Kaplansky [5] and K-dim E is actually infinite. We shall give more precise results concerning the length of chains of prime ideals of E.

As shown in [7] the global dimension of E is ≥ 3, while the exact value of gl.dim E cannot be determined from the usual axioms of set theory (ZFC): For any t, $3\leq t\leq\infty$, the statement gl.dim $E=t$ is consistent with ZFC, in fact, even consistent with ZFC + MA, (MA denoting Martin's axiom).

The corresponding results hold true if E is replaced by $E(K)$ or $\bar{E}(\varrho, K)$. The proofs only require minor modifications. For $E(\varrho, K)$, however, the situation is completely different. For any positive ϱ and any field $K\subseteq\mathbb{C}$ the ring $E(\varrho, K)$ is Euclidean, in particular a PID. Since for instance $\bar{E}(1, \mathbb{C})=\bigcap_{n=1}^{\infty} E(1-1/n, \mathbb{C})$ we obtain a decreasing sequence of PID's whose intersection is a Bezout domain of undecidable global dimension and of uncountable Krull dimension.

The stable range (in the sense of Bass [1]) of the above rings depends on K. If $K\subseteq\mathbb{R}$ the stable range of each of the rings $E(\varrho, K)$, $\bar{E}(\varrho, K)$ and $E(K)$ is 2, otherwise, when $K\nsubseteq\mathbb{R}$ the stable range is 1.

0022-4049/85/$3.30

Finally, we consider the rings E_r of entire functions of order $<r$, where $0<r<\infty$, cf. [11]. We recall that a function $f\in E$ belongs to E_r if there exist real numbers c and a, $a<r$, such that $|f(x)|\leq c\exp(|x|^a)$ for all $x\in\mathbb{C}$. From an analytic point of view the functions of finite order are, after the polynomials, the simplest entire functions [11]. However, the ring-theoretic structure of E_r is much more complicated than that of E. Just as E the rings E_r are non-Noetherian, but unlike E none of the rings E_r is a Bezout domain. The stable range of E_r is >1, but the precise value is unknown.

From hadamard's factorization theorem [11] it follows that each E_r is completely integrally closed in its quotient field. Concerning the global dimension of E_r we are only able to show that $\text{gl.dim}\, E_r\geq 3$ and that the statement $\text{gl.dim}\, E_r=\infty$ is consistent with ZFC + MA. The Krull dimension of E_r is uncountable and behaves to a large extent like that of E.

The above methods also allow us to determine the Krull dimension of the ring of all infinitely often differentiable real functions and a class of subrings hereof.

2. In this section we prove those results mentioned in Section 1 which do not invoke logic or set theory.

Theorem 2.1. *For any subfield K of $\mathbb{C}$ any $\varrho>0$ the ring $R=E(\varrho,K)$ is Euclidean; in particular, R is a PID.*

Proof. For $f\in R\setminus 0$ let $N(f)$ be the number of zeros (counted with multiplicities) in the closed disc $|z|\leq\varrho$. Obviously $N(fg)=N(f)+N(g)$ for all $f,g\in R\setminus 0$ and $N(f)=0$ if and only if f is a unit in R. We shall show that for any two elements $f,g\in R$, $g\neq 0$ there exist elements q and $r\in R$ such that

$$f=gq+r \quad \text{where } r=0 \text{ or } N(r)<N(g).$$

Here, we may, of course, assume that $g\nmid f$.

The functions f and g can be written in the form

$$f=\bar{f}u, \qquad g=\bar{g}v,$$

where

(1) $\bar{f}$ (resp. $\bar{g}$) is a polynomial in $\mathbb{C}[X]$ whose roots are exactly the zeros of f (resp. g) in the disc $|z|\leq\varrho$, counted with multiplicities, and $\bar{f}$ (resp. $\bar{g}$) has real coefficients when $K\subseteq\mathbb{R}$.

(2) u (resp. v) is a unit in $E(\varrho,\mathbb{C})$ and u (resp. v) belongs to $E(\varrho,\mathbb{R})$ when $K\subseteq\mathbb{R}$.

By the usual algorithm for polynomials there exist elements $\bar{q}$ and $\bar{r}\in\mathbb{C}[X]$ such that $\bar{f}=\bar{g}\bar{q}+\bar{r}$ and $\bar{r}$ has degree smaller than $N(g)=$ degree of $\bar{g}$. If $K\subseteq\mathbb{R}$ the polynomials $\bar{q}$ and $\bar{r}$ have real coefficients. Hence

$$f=g(v^{-1}u\bar{q})+u\bar{r}$$

where the number of zeros of $u\bar{r}$ is smaller than $N(g)$. Here $u\bar{r}$ and $v^{-1}u\bar{q}$ may not have coefficients in K, but if $K \subseteq \mathbb{R}$ they belong to $\mathbb{R}$.

If $g = b_n x^n + b_{n+1}x^{n+1} + \dots, b_n \neq 0$, then $u\bar{r} = \sum_{i=0}^{\infty} a_i x^i$ where $a_0, a_1, \dots, a_{n-1} \in K$.

For any $h \in E(\varrho, \mathbb{C})$, resp. $h \in E(\varrho, \mathbb{R})$ when $K \subseteq \mathbb{R}$, we can write

$$f = g(v^{-1}u\bar{q} - h) + (hg + u\bar{r}).$$

Since K is dense in $\mathbb{R}$ when $K \subseteq \mathbb{R}$ and K is dense in $\mathbb{C}$ when $K \nsubseteq \mathbb{R}$, we can define successively the coefficients in

$$h = h_0 + h_1 x + h_2 x^2 + \dots$$

such that

$$hg + u\bar{r} = (h_0 + h_1 x + h_2 x^2 + \dots)(b_n x^n + b_{n+1}x^{n+1} + \dots) + \sum_{i=0}^{\infty} a_i x^i$$

have coefficients in K, the convergence radius of h is $> \varrho$, and (by continuity arguments) $hg + u\bar{r}$ has at most $N(\bar{r}) = N(u\bar{r})$ zeros in the disc $|z| \leq \varrho$. Hence

$$f = gq + r,$$

$$q = v^{-1}u\bar{q} - h,$$

$$r = hg + u\bar{r},$$

yields the desired decomposition of f.

Theorem 2.2. *Let K be a subfield of $\mathbb{C}$ and ϱ a positive number. If $K \subseteq \mathbb{R}$ each of the rings $E(K)$, $\bar{E}(\varrho, K)$ and $E(\varrho, K)$ has stable range 2. If $K \nsubseteq \mathbb{R}$ each of the above rings has stable range 1.*

Proof. The proof of Proposition 1.1 of [8], (cf. the addendum of that paper) also works for the rings $\bar{E}(\varrho, K)$ and $E(\varrho, K)$, $K \nsubseteq \mathbb{R}$, so that the rings $E(K)$, $\bar{E}(\varrho, K)$ and $E(\varrho, K)$ have stable range 1 when $K \nsubseteq \mathbb{R}$. (For $E(\mathbb{C})$ cf. also [9].)

Next we consider the case where $K \subseteq \mathbb{R}$. Let R denote one of the rings $E(K)$, $\bar{E}(\varrho, K)$ or $E(\varrho, K)$. For $f \in R$ let $Z(f)$ be the set of zeros α of f where we require $|\alpha| \leq \varrho$ if $R = E(\varrho, K)$ and $|\alpha| < \varrho$ if $R = \bar{E}(\varrho, K)$.

We first prove that the stable range of R is ≤ 2. For any three functions f, g and $h \in R$ such that $Rf + Rg + Rh = R$ we have to find elements $\lambda, \mu \in R$ for which $R(f + \lambda h) + R(g + \mu h) = R$, cf. [12]. The condition $Rf + Rg + Rh = R$ implies $Z(f) \cap Z(g) \cap Z(h) = \emptyset$. By interpolation we can find $\lambda \in R$ such that $\lambda(\alpha) \neq -f(\alpha)/h(\alpha)$ for all $\alpha \in Z(g), \alpha \notin Z(h)$. Hence $Z(f + \lambda h) \cap Z(g) = \emptyset$ and thus $R(f + \lambda h) + Rg = R$.

To show that the stable range of R is $\neq 1$ we consider the functions $f = x$ and $g = 4x^2 - \varrho^2$, where ϱ may be any positive number in the case $R = E(K)$. Obviously, $Rf + Rg = R$ while $R(f + \lambda g) \neq R$ for every $\lambda \in R$, since $f + \lambda g$ is a real-valued continuous function and $(f + \lambda g)(\varrho/2) > 0$ and $(f + \lambda g)(-\varrho/2) < 0$.

Remark 1. That the stable range of $E(\varrho, K)$ is ≤ 2 could also be seen from Theorem 2.1 since any PID has stable range ≤ 2.

Remark 2. From Theorems 2.1 and 2.2 it follows that any matrix in $\mathrm{SL}(n, E(\varrho, K))$ is a product of elementary matrices. If $K \subsetneqq \mathbb{R}$, any matrix in $\mathrm{SL}(n, E(\varrho, K))$ is a product of at most $2n(n-1)$ elementary matrices. (The bound is probably not best possible.) The corresponding result is not true if $K \subseteqq \mathbb{R}$. For instance, for $n=2$ there is no number f such that any matrix in $\mathrm{SL}(2, E(\varrho, K))$ is a product of at most f elementary matrices. In fact, no such bound exists for the matrices

$$\begin{pmatrix} \cos(tx) & -\sin(tx) \\ \sin(tx) & \cos(tx) \end{pmatrix}, \quad t \in \mathbb{N}.$$

3. In this section we deal with the remaining assertions in Section 1. For the proofs we need some general results about ultrapowers of $\mathbb{Z}$ over a countable index set I. Let $\mathscr{F}$ be a non-principal ultrafilter on I and $\hat{\mathbb{Z}} = \mathbb{Z}^I/\mathscr{F}$ the corresponding ultrapower of $\mathbb{Z}$. $\hat{\mathbb{Z}}$ has a natural structure as a totally ordered group. Let $\mathscr{C}$ be the family of convex subgroups ('isolated subgroups' in the terminology of [14]). By set-theoretical inclusion $\mathscr{C}$ is a totally ordered Dedekind complete set. Further, let $\mathscr{P}$ be the family of 'principal' convex subgroups of $\hat{\mathbb{Z}}$. If $a>0$, the principal convex subgroup generated by a is

$$\langle a \rangle = \{x \in \hat{\mathbb{Z}} \mid -na < x < na \text{ for some } n \in \mathbb{N}\}.$$

$\mathscr{P}$ is totally ordered by set-theoretical inclusion and forms as such an η_1-set. This means that for any two countable families $\{\langle a \rangle\}$ and $\{\langle b \rangle\}$ of $\mathscr{P}$ such that any $\langle a \rangle \subsetneqq$ any $\langle b \rangle$ there exists a y for which

$$\langle a \rangle \subsetneqq \langle y \rangle \subsetneqq \langle b \rangle \tag{$*$}$$

for all $\langle a \rangle$ and $\langle b \rangle$.

We may assume that all a and b are positive and $(*)$ can be written

$$\left.\begin{matrix} na < y \\ my < b \end{matrix}\right\} \quad \text{for all } a \text{ and } b \text{ and all } n, m \in \mathbb{N}.$$

Since any finite subsystem of the above family of inequalities is solvable in $\hat{\mathbb{Z}}$ and $\hat{\mathbb{Z}}$ is $\aleph_1$-saturated, the above countable system of inequalities has a solution $y \in \hat{\mathbb{Z}}$. Consequently $\mathscr{P}$ is an η_1-set. If we assume MA, there exists a non-principal ultrafilter $\mathscr{F}$ on I such that $\hat{\mathbb{Z}} = \mathbb{Z}^I/\mathscr{F}$ is $2^{\aleph_0}$-saturated [2]. The above construction shows that in this case $\mathscr{P}$ is an η_α-set, where $2^{\aleph_0} = \aleph_\alpha$, ($\alpha$ being an ordinal).

By [3] it follows that $\mathscr{C}$ has at least $2^{\aleph_1}$ elements. If we assume MA and $2^{\aleph_0} = \aleph_\alpha$, then $2^{\aleph_0}$ is regular [6], and the proof in [3, pp. 185–188] shows that $\mathscr{C}$ has cardinality $2^{2^{\aleph_0}}$.

Moreover, if we have convex subgroups of $\hat{\mathbb{Z}}$

$$\mathfrak{a} \subsetneqq \mathfrak{b} \subsetneqq \mathfrak{c},$$

then there exist elements b and $c \in \hat{\mathbb{Z}}$ such that

$$\mathfrak{a} \subsetneqq \langle b \rangle \subsetneqq \mathfrak{b} \subsetneqq \langle c \rangle \subsetneqq \mathfrak{c}$$

and the fact that $\mathscr{P}$ is an η_1-set (resp. η_α-set if we assume MA and choose $\mathscr{F}$ suitably) implies that there are $\geq 2^{\aleph_1}$ (resp. $2^{2^{\aleph_0}}$) convex subgroups between $\mathfrak{a}$ and $\mathfrak{c}$.

After these preliminary remarks we return to the results in Section 1. It presents no difficulties to modify the proof in [7] to obtain

Theorem 3.1. *Let K be a subfield of $\mathbb{C}$ and ϱ a positive number. Then the rings $E(K)$ and $\bar{E}(\varrho, K)$ have global dimension ≥ 3. Moreover, for any t, $3 \leq t \leq \infty$, the statement* "gl.dim $E(K) = t$" *(resp.* "gl.dim $\bar{E}(\varrho, K) = t$"*) is consistent with* ZFC + MA.

Remark. It is an open question whether the statement "gl.dim$E(K) = 3$" resp. "gl.dim $\bar{E}(\varrho, K) = 3$" is consistent with ZFC + $\overline{\text{CH}}$, where $\overline{\text{CH}}$ denotes the negation of the continuum hypothesis.

Theorem 3.2. *Let $R = E(K)$ or $\bar{E}(\varrho, K)$ as above. Then* K-dim $R \geq 2^{\aleph_1}$. *Moreover,* MA *implies* K-dim $R = 2^{2^{\aleph_0}}$. *If $\mathfrak{p} \subsetneqq \mathfrak{q} \subsetneqq \mathfrak{r}$ are prime ideals of R, then there exists a chain of $2^{\aleph_1}$ prime ideals between $\mathfrak{p}$ and $\mathfrak{r}$.*

Proof. Let I be a countably infinite set of zeros of a function in R and let $\mathscr{F}$ be a non-principal ultrafilter on $\mathfrak{i}$. Then $\mathfrak{m} = \{f \in R \mid Z(f) \cap I \in \mathscr{F}\}$ is a maximal ideal of R and the localization $R_\mathfrak{m}$ is a valuation ring with $\hat{\mathbb{Z}}$ as value group. The statements of the theorem are now just formal consequences of the results in the beginning of this section since there is a (1–1) correspondence between the prime ideals contained in $\mathfrak{m}$ and the convex subgroups of $\hat{\mathbb{Z}}$, and any non-principal maximal ideal of R can be obtained as above.

Remark. The exact value (cardinality) of the Krull dimension of the above rings is probably undecidable in ZFC.

Theorem 3.3. *Let $R = E_r$ be the ring of all entire functions of order $< r$. Then* gl.dim $E_r \geq 3$, *and the statement* "gl.dim $R = \infty$" *is consistent with* ZFC + MA. *Moreover,* K-dim $R \geq 2^{\aleph_1}$ *and* MA *implies* K-dim $R = 2^{2^{\aleph_0}}$.

Proof. Let $I = \{2^n \mid n \in \mathbb{N}\}$. If a_n, $n \in \mathbb{N}$, is a sequence of natural numbers for which a_n/n is bounded there exists a function $f \in R$, for instance

$$f = \prod_{n \in \mathbb{N}} (1 - x/2^n)^a n,$$

for which $Z(f) = I$ and a_n is the multiplicity of 2^n as a zero of f.

For a non-principal ultrafilter $\mathscr{F}$ on I the functions g in R for which $Z(g)\cap I\in\mathscr{F}$ form a prime ideal $\mathfrak{p}$ of R.

For $h\in R$ and $n\in\mathbb{N}$ let $v_n(h)$ be the multiplicity of 2^n as a zero of h and $\hat{v}(h)$ the element of $\hat{\mathbb{Z}}$ determined by the sequence $v_n(h)$, $n\in\mathbb{N}$. Here $\hat{v}$ defines a valuation of the quotient field Q_r of $E_r=R$ with values in $\hat{\mathbb{Z}}$. Since any element in Q_r – by virtue of Hadamard's factorization theorem – can be written as a quotient of two functions in R with disjoint zero sets, it follows that the valuation ring $\hat{V}$ corresponding to $\hat{v}$ is the localization $R_{\mathfrak{p}}$.

For $t\in\mathbb{N}$ let t^* be the element in $\hat{\mathbb{Z}}$ defined by the constant sequence $\{t\}$. Just as in [7] the ideal in $\hat{V}$ consisting of all $s\in\hat{V}$ such that $\hat{v}(s)>t^*$ for all $t\in\mathbb{N}$ is not countably generated. Hence as in [7] we get

$$\text{gl.dim}\,R\geq\text{gl.dim}\,R_{\mathfrak{p}}=\text{gl.dim}\,\hat{V}\geq 3.$$

If we assume MA there exists an ultrafilter $\mathscr{F}'$ on I such that $\mathbb{Z}/\mathscr{F}'$ is $s^{\aleph_0}$-saturated [2]. The statement $2^{\aleph_0}=\aleph_{\omega+1}$ is consistent with ZFC + MA (cf. [6]). Consequently, it follows as in [7] that the statement "gl.dim $\hat{V}'=\infty$" – and thus the statement "gl.dim$R=\infty$" – is consistent with ZFC + MA, where $\hat{V}'$ denotes the valuation ring constructed as above from the ultrafilter $\mathscr{F}'$.

Let w be the element in $\hat{\mathbb{Z}}$ defined by the sequence $\{n\}$, $n\in\mathbb{N}$. The principal convex subgroup $\langle w\rangle$ of $\hat{\mathbb{Z}}$ is in the value group of $\hat{v}$. Now the remaining assertions of Theorem 3.3 follow – just as in Theorem 3.2 – from the results in the beginning of this section.

Finally, we consider infinitely often differentiable functions f from $\mathbb{R}$ to $\mathbb{R}$. For $\alpha\in\mathbb{R}$ we define $v_\alpha(f)=n$, $n\geq 0$, if $f(\alpha)=\dots=f^{(n-1)}(\alpha)=0$, $f^{(n)}(\alpha)\neq 0$ and $v_\alpha(f)=\infty$, if $f^{(i)}(\alpha)=0$ for all $i\geq 0$.

If $\mathscr{F}$ is a non-principal ultrafilter on $\mathbb{N}$ we define $\hat{v}(f)$ as the element in $\mathbb{Z}^{\mathbb{N}}/\mathscr{F}\cup\{\infty\}$ determined by the sequence $\{v_n(f)\}$, $n\in\mathbb{N}$.

For any ring R, $E(\mathbb{Q})\subseteq R\subseteq C^\infty(\mathbb{R})$, where $C^\infty(\mathbb{R})$ denotes the ring of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ we obtain by $\hat{v}$ a valuation (for rings with zero-divisors) of R with values in $\mathbb{Z}^{\mathbb{N}}/\mathscr{F}\cup\{\infty\}$ and by arguments similar to the previous we get

Theorem 3.4. *Let R be any ring for which $E(\mathbb{Q})\subseteq R\subseteq C^\infty(\mathbb{R})$. Then R contains a chain of prime ideals of length $2^{\aleph_1}$. Moreover,* MA *implies that* K-dim$\,R=2^{2^{\aleph_0}}$ *and there is a chain of prime ideals of length $2^{2^{\aleph_0}}$.*

References

[1] H. Bass, Algebraic K-Theory (Benjamin, New York, 1968).
[2] E. Ellentuck and R.V.B. Bucker, Martin's axiom and saturated models, Proc. Amer. Math. Soc. 34 (1972) 243–249.
[3] L. Gillman and M. Jerison, Rings of Continuous Functions (Van Nostrand, Princeton, 1960).
[4] O. Helmer, Divisibility properties of integral functions, Duke Math. J. 6 (1940) 345–356.
[5] M. Henriksen, On the prime ideals of the rings of entire functions, Pacific J. Math. 3 (1953) 711–720.
[6] T. Jech, Set Theory (Academic Press, New York, 1978).
[7] C.U. Jensen, La dimension globale de l'anneau des fonctions entières, C.R. Acad. Sci. Paris 294 (1982) 385–386.
[8] C.U. Jensen, Théorie des modèles pour des anneaux de fonctions entières et des corps de fonctions méromorphes, to appear in Mém. Soc. Math. de France.
[9] L.A. Rubel, Solution of problem 6117, Amer. Math. Monthly 85 (1978) 505–506.
[10] O.F.G. Schilling, Ideal theory on open Riemann surfaces, Bull. Amer. Math. Soc. 52 (1946) 945–963.
[11] E.C. Titchmarsh, The Theory of Functions (Oxford Univ. Press, Oxford, 1939).
[12] I.N. Vaserstein, Stable ranges of rings and dimensions of topological spaces, Funk. Anal. Pril. 5 (2) (1971) 17–27.
[13] J.H.M. Wedderburn, On matrices whose coefficients are functions of a single variable, Trans. Amer. Math. Soc. 16 (1915) 328–332.
[14] O. Zariski and P. Samuel, Commutative Algebra II (Van Nostrand, Princeton, 1960).