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Stripping and conjugation in the Steenrod algebra

Judith H. Silverman*

Indiana University – Purdue University at Columbus, 4601 Central Avenue, Columbus, IN 47203, USA

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Abstract

Let $S(k;f) = Sq(2^{k-1}f) \cdot Sq(2^{k-2}f) \cdots Sq(2f) \cdot Sq(f)$ in the mod-2 Steenrod algebra $\mathscr{A}^*$, and let χ denote the canonical antiautomorphism of $\mathscr{A}^*$. Given positive integers k, Λ and j with $1 \le j \le \Lambda$, we prove that

$$\chi S(k; 2^{\Lambda} - j) = S(\Lambda - (j-1); 2^{j-1}(2^k - 1)) \cdot \chi S(k; 2^{j-1} - j),$$

generalizing formulae of Davis and the author. Our proof relies on the "stripping" action of the dual Steenrod algebra $\mathscr{A}_*$ or $\mathscr{A}^*$ itself, which we identify as a special case of a general Hopf algebra phenomenon.

Given a positive integer f, denote by $\mu(f)$ the minimal number of summands in any representation of f in the form $\sum(2^{i_k} - 1)$. The antiautomorphism formula above implies that for $f = 2^{\Lambda} - j$, $1 \le j \le \Lambda + 2$, the excess of $\chi S(k;f)$ satisfies $ex(\chi S(k;f)) = (2^k - 1)\mu(f)$ for all k, confirming the conjecture of the author (Silverman, 1993) for such f. We also prove that $ex(\chi S(k;f)) \le (2^k - 1)\mu(f)$ for all f and k. © 1997 Elsevier Science B.V.

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1. Introduction

1.1. Informal statement of results

The mod-2 Steenrod algebra $\mathscr{A}^*$ is multiplicatively generated by the Steenrod squares $Sq(i)$ in dimension $|Sq(i)| = i$, $0 \le i < \infty$. The product $Sq(a_1) \cdots Sq(a_n)$ is *admissible* if $a_i \ge 2a_{i+1}$ for $i < n$, and $a_n > 0$ if $n > 1$; the admissible elements form an additive basis of $\mathscr{A}^*$. The *excess* of the admissible element $Sq(a_1) \cdots Sq(a_n)$ of

* E-mail: jsilver@indyvax.iupui.edu.

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dimension d is given by $\sum_{i=1}^{n-1}(a_i - 2a_{i+1}) + a_n = 2a - d$ [15]. In general, the excess of a sum of admissibles is the minimum of the excesses of the summands.

We write

$$S(k;f) = Sq(2^{k-1}f) \cdot Sq(2^{k-2}f) \cdots Sq(2f) \cdot Sq(f)$$

and apologize for the change in notation from [10]. The dimension $|S(k;f)| = (2^k - 1)f$ and the excess $ex(S(k;f)) = f$.

The Steenrod algebra is a connected Hopf algebra, and as such has a unique antiautomorphism, commonly denoted by χ [5]. Following [16], we write $\hat{\theta}$ for $\chi\theta$. In particular, $\hat{Sq}(a) = \chi Sq(a)$ and $\hat{S}(k;f) = \chi S(k;f)$.

In Section 6, we prove the following antiautomorphism formula, generalizing results of [2, 10]:

Theorem 1.1. *Let k and Λ be positive integers, and suppose that $1 \le j \le \Lambda$. Then*

$$\hat{S}(k; 2^\Lambda - j) = S(\Lambda - (j-1); 2^{j-1}(2^k - 1)) \cdot \hat{S}(k; 2^{j-1} - j).$$

The Steenrod algebra $\mathscr{A}^*$ acts on $\mathbb{F}_2[x_1, \dots, x_s]$ according to well-known rules. The image of this action, i.e., the set of polynomials which can be written $F = \sum_{i>0} Sq(i)F_i$, is related to various entities of importance in algebraic topology; among these are $Ext^s_{\mathscr{A}^*}(\mathbb{F}_2, \mathbb{F}_2)$ [13] and cobordism classes of closed manifolds [9]. In addition, this image contains information about the simple representations of the general linear group $GL(s, \mathbb{F}_2)$ [17]. In [10], we discuss the connection between this image and the excess of the Steenrod operations $S(k;f)$, and frame a conjecture which would permit the argument of [18] to prove the conjecture of [12] concerning this image. A stronger version of this conjecture appears in [11] and is equivalent to Conjecture 1.2 as stated in Section 1.4. In order to state our present results without introducing further notation, we recall here the original statement of the conjecture:

Conjecture 1.2 (Weak version). *Let f be a positive integer. Then for all positive integers k we have*

$$ex(\hat{S}(k;f)) = (2^k - 1)ex(\hat{Sq}(f)) \; (= (2^k - 1)ex(\hat{S}(1;f))).$$

Theorem 1.1 will imply

Theorem 1.3. *Conjecture 1.2 is true for f satisfying $2^\Lambda - (\Lambda + 2) \le f \le 2^\Lambda - 1$ for some $\Lambda \ge 0$.*

We also prove that

Theorem 1.4. *One of the inequalities of Conjecture 1.2 is true for all f;*

$$ex(\hat{S}(k;f)) \le (2^k - 1)ex(\hat{Sq}(f))$$

for all $k, f \ge 0$.

In the remainder of this section, we introduce enough notation to state Conjecture 1.2 in full.

1.2. E-notation for admissibles

Denote by $\mathscr{S}$ the set of finite sequences of non-negative integers. We define the dimension of an element $S = (s_1, s_2, \ldots, s_n) \in \mathscr{S}$ to be $|S| = \sum_i s_i(2^i - 1)$, its length $l(S) = n$, and its excess $ex(S) = \sum_i s_i$.

For our purposes, it will be convenient to parametrize the admissible basis in terms of the numbers $s_i = a_i - 2a_{i+1}$, the contributions to excess at each stage. That is, given a sequence $S = (s_1, \ldots, s_n) \in \mathscr{S}$, we define the admissible element $E(S) = Sq(a_1) \cdots Sq(a_n)$ where $a_n = s_n$ and $a_i = 2a_{i+1} + s_i$ for $1 \le i \le n-1$. For example, if $S = (0, \ldots, 0, \underset{j}{f})$, then $E(S) = S(j; f)$ as defined in Section 1.1. We have that $ex(E(S)) = ex(S)$ and $|E(S)| = |S|$.

1.3. Elements of minimal excess

We now single out particular basis elements in each degree. Given a positive integer f, we denote by $\mu(f)$ the least excess of all sequences in $\mathscr{S}$ of dimension f. Let $\Lambda(f) = \max\{\lambda : 2^\lambda - 1 \le f\}$. In [14], Singer observes that for any f there exists a unique sequence $R_1(f) = (r_1, \ldots, r_{\Lambda(f)}) \in \mathscr{S}$ of dimension f such that $r_i \le 1$ for all i, except that the first non-trivial r_i is ≤ 2; this sequence has $ex(R_1(f)) = \mu(f)$. The corresponding admissible element $E(R_1(f))$ is thus of minimal excess among all elements of the admissible basis in dimension f.

The sequence $R_1(f)$ may be constructed inductively by increasing the $\Lambda(f)$th entry of $R_1(f - (2^{\Lambda(f)} - 1))$ by 1.

Example. We have

$$R_1(2^\Lambda - 1) = (0, \ldots, 0, \underset{\Lambda}{1}),$$

$$R_1(2^\Lambda - j) = \begin{cases} \left(0, \ldots, 0, \underset{\Lambda-j+1}{2}, 1, 1, \ldots, \underset{\Lambda-1}{1}\right), & 2 \le j \le \Lambda, \\ \left(1, 1, \ldots, \underset{\Lambda-1}{1}\right), & j = \Lambda + 1, \\ \left(0, 1, \ldots, \underset{\Lambda-1}{1}\right), & j = \Lambda + 2. \end{cases}$$

Consequently,

$$\mu(2^\Lambda - j) = \begin{cases} j, & j \le \Lambda, \\ \Lambda - 1, & j = \Lambda + 1, \\ \Lambda - 2, & j = \Lambda + 2. \end{cases}$$

For $k \geq 1$, let $R_k(f) = ((2^k-1)r_1, (2^k-1)r_2, \ldots, (2^k-1)r_{\Lambda(f)})$. Then $R_k(f)$ and the corresponding admissible basis element $E(R_k(f))$ have dimension $(2^k-1)f$ and excess $(2^k-1)\mu(f)$.

1.4. Results

As we shall see in Section 3.3, the admissible element $E(R_1(f))$ appears in $\hat{Sq}(f) = \hat{S}(1;f)$, and so $ex(\hat{S}(1;f)) = \mu(f)$. Conjecture 1.2 below purports to generalize these phenomena to $k > 1$.

Conjecture 1.2 (Silverman [11]). *Let f be a positive integer. Then for all positive integers k we have*
(i) *the element $E(R_k(f))$ is a (non-trivial) summand in the admissible-basis representation of $\hat{S}(k;f)$, and*
(ii) *its excess is minimal among all such summands, so that $ex(\hat{S}(k;f)) = ex(E(R_k(f))) = (2^k-1)\mu(f)$.*

In this paper, we prove the following theorems:

Theorem 1.3. *Conjecture 1.2 is true for numbers of the form $f = 2^\Lambda - j$, $1 \leq j \leq \Lambda + 2$.*

Theorem 1.4. *Part (i) of Conjecture 1.2 holds for all pairs (f,k) of positive integers, and consequently $ex(\hat{S}(k;f)) \leq (2^k-1)\mu(f)$.*

2. Hopf algebras

Recent work on nilpotence in $\mathscr{A}^*$ has exploited the "stripping" action of the dual Steenrod algebra $\mathscr{A}_*$ on $\mathscr{A}^*$ itself [7, 16]. In this section, we identify the stripping action as a general Hopf algebra phenomenon. I thank Bill Schmitt, Grant Walker, and Reg Wood for elucidating this point of view.

Let A^* be a Hopf algebra over a field K with diagonal Δ^*, multiplication ϕ^*, and conjugation χ [6]. We continue to write $\chi\theta$ as $\hat{\theta}$ or $\overline{\theta}$ depending on the typographical complexity of θ. Let A_* be the dual Hopf algebra and write $\langle , \rangle : A_* \otimes A^* \longrightarrow K$ for the inner product. In what follows, $y_i \in A_*$, $\theta_j \in A^*$, $\Delta^*\theta = \sum \theta' \otimes \theta''$ and $\phi_* y = \sum y' \otimes y''$; we write $\theta_1\theta_2$ for $\phi^*(\theta_1 \otimes \theta_2)$ and $y_1 y_2$ for $\Delta_*(y_1 \otimes y_2)$.

There is a natural action of A_* on A^* in which $y \in A_*$ acts via

$$D(y) : A^* \xrightarrow{\Delta^*} A^* \otimes A^* \xrightarrow{1 \otimes \langle y, \rangle} A^*. \tag{1}$$

Following [1, 16], we refer to $D(y)$ as the operation of *stripping by* y. Evidently $D(y_1 + y_2) = D(y_1) + D(y_2)$; coassociativity of A^* gives $D(y_1 y_2) = D(y_1) \circ D(y_2)$.

Let $\mathscr{D} = \{D(y) : y \in A_*\}$. Define the maps

$$\chi : \mathscr{D} \to \mathscr{D}, \qquad D(y) \mapsto D(\hat{y});$$
$$\Delta_* : \mathscr{D} \otimes \mathscr{D} \to \mathscr{D}, \qquad D(y_1) \otimes D(y_2) \mapsto D(\Delta_*(y_1 \otimes y_2));$$
$$\phi_* : \mathscr{D} \to \mathscr{D} \otimes \mathscr{D}, \qquad D(y) \mapsto (D \otimes D)(\phi_* y) = \sum D(y') \otimes D(y'').$$

Henceforth we write $\hat{D}(y)$ for $\chi(D(y))$. If A^* is cocommutative, a determined chase of the defining diagrams (see for example Sections 4 and 8 of [6]) reveals the following three equations:

$$D(y) \circ \phi^* = \phi^* \circ (\phi_* D(y)),$$
$$\text{i.e. } D(y)(\theta_1\theta_2) = \sum D(y')\theta_1 \cdot D(y'')\theta_2. \tag{2}$$

$$\hat{D}(y) = \chi \circ D(y) \circ \chi, \quad \text{i.e. } \hat{D}(y)\theta = \overline{D(y)\hat{\theta}}. \tag{3}$$

$$\hat{D}(y) \circ \phi^* = \phi^* \circ ((\chi \otimes \chi) T \phi_* D(y)),$$
$$\text{i.e. } \hat{D}(y)(\theta_1\theta_2) = \sum \hat{D}(y'')\theta_1 \cdot \hat{D}(y')\theta_2. \tag{4}$$

In Sections 3 and 4 below, we discuss the stripping action in the case where $A^* = \mathscr{A}^*$, the mod-2 Steenrod algebra.

3. The Milnor basis

3.1. Notation

The dual Hopf algebra $\mathscr{A}_*$ of $\mathscr{A}^*$ is a polynomial algebra over $\mathbb{F}_2$ on generators ξ_i in dimension $2^i - 1$, $1 \le i < \infty$ [5]. For $S = (s_1, \ldots, s_n) \in \mathscr{S}$, write $\xi(S)$ for the monomial $\xi_1^{s_1} \cdots \xi_n^{s_n}$; evidently the dimension of this monomial is $\sum_i s_i(2^i - 1) = |S|$. The Milnor basis of $\mathscr{A}^*$ itself is the basis dual to the monomials in the ξ_i; we denote the dual element to $\xi(S)$ by $M(S)$. Then $|M(S)| = |E(S)|$ and $ex(M(S)) = \sum s_i = ex(E(S))$ [3].

In [8], Monks shows that for all $S \in \mathscr{S}$, each of $E(S)$ and $M(S)$ appears in the representation of the other in the appropriate basis. Moreover, the difference $\delta(S) = E(S) - M(S)$ satisfies $ex(\delta(S)) \ge ex(S)$ and also $\delta(S) <_E E(S)$, $\delta(S) <_M M(S)$, where $<_E$ and $<_M$ are the orderings induced on $\mathscr{A}^*$ by the right-lexicographical ordering of $\mathscr{S}$ relative to the admissible and Milnor bases respectively. This justifies the current admissible version of Conjecture 1.2, which was originally stated in terms of the Milnor basis [11].

3.2. Length of a Steenrod operation

Recall from Section 1.2 that the length of a sequence $S = (s_1, \ldots, s_n) \in \mathscr{S}$ is n. Given $\theta \in \mathscr{A}^*$, define its *admissible length* (resp. *Milnor length*) by

$$l_A(\theta) = \max\{l(S) \colon \theta = \sum E(S)\} \quad (\text{resp. } l_M(\theta) = \max\{l(S) \colon \theta = \sum M(S)\}).$$

It follows from Monks's result that $l_A(\theta) = l_M(\theta)$ for all θ. Denote this common value by $l(\theta)$, the *length* of θ.

3.3. Conjugation in the Milnor basis

The canonical antiautomorphism χ has the property that for all positive integers f, the element $\hat{Sq}(f)$ is the sum of all Milnor basis elements $M(S)$ of the appropriate dimension [5], and consequently, as indicated in Section 1.4 above, we have

$$ex(\hat{Sq}(f)) = \mathrm{ex}\left(\sum_{|S|=f} Sq(S)\right) = \mu(f). \qquad (5)$$

3.4. Stripping in the Milnor basis

The diagonal homomorphism Δ^* in $\mathscr{A}^*$ is determined by

$$\Delta^* M(S) = \sum_{S'+S''=S} M(S') \otimes M(S''),$$

where addition in $\mathscr{S}$ is componentwise [5]. It follows from the defining equation of stripping (1) that

$$D(\xi(R))M(S) = M(S-R), \qquad (6)$$

where the right-hand side is understood to be 0 if $s_i < r_i$ for any i [4]. In particular, the stripping operations $D(\xi(R))$ do not increase length, and stripping by a basis element of excess e decreases excess by no more than e.

Since $\mathscr{A}_*$ is commutative, we have $D(y) \circ D(y') = D(y') \circ D(y)$ for all $y, y' \in \mathscr{A}_*$.

4. Stripping in the admissible basis

4.1. General formula

In this section we discuss the action of $D(\xi_k)$ on the (not necessarily admissible) product $Sq(a_1)\cdots Sq(a_n)$ (cf. [1, 16]). For $n \geq k$, define $\mathscr{V}_{n,k}$ to be the set of all sequences $(v_1, \ldots, v_n)$ in which the non-zero elements form exactly the subsequence $(2^{k-1}, \ldots, 2, 1)$. For example, $\mathscr{V}_{3,2}$ consists of (0,2,1), (2,0,1), and (2,1,0). For $n < k$, define $\mathscr{V}_{n,k} = \emptyset$.

It is readily verified, using (2) and induction, that

Proposition 4.1. $D(\xi_k)(Sq(a_1)\cdots Sq(a_n)) = \sum_{V\in\mathscr{V}_{n,k}} Sq(a_1 - v_1)\cdots Sq(a_n - v_n)$.

In the special case $k = n$, we find that

Corollary 4.2. (i) *If* $Sq(a_1)\cdots Sq(a_k)$ *is admissible and has excess* e, *then*

$$D(\xi_k)(Sq(a_1)\cdots Sq(a_k)) = Sq(a_1 - 2^{k-1})\cdots Sq(a_{k-1} - 2)\cdot Sq(a_k - 1),$$

which is admissible and has excess $e - 1$.

(ii) *In particular,* $D(\xi_k)S(k;f) = S(k;f-1)$.

In view of (3) and Part (ii) of Corollary 4.2, we have

$$\hat{D}(\xi_k)\hat{S}(k;f) = \hat{S}(k;f-1), \tag{7}$$

which permits the proof of Theorems 1.3 and 1.4 by downward induction in Sections 6 and 5 respectively.

4.2. The stripping operation $\hat{D}(\xi_k)$

Conjugation in $\mathscr{A}_*$ is determined by

$$\hat{\xi}_k = \sum_{\alpha\in \mathrm{Part}(k)} \prod_{i=1}^{n(\alpha)} \xi_{\alpha_i}^{\sigma_i(\alpha)},$$

where Part(k) is the set of sequences $\alpha = (\alpha_1, \ldots, \alpha_{n(\alpha)})$ of positive integers whose sum is k, and where $\sigma_i(\alpha) = \sum_{j=1}^{i-1} \alpha_j$ [5]. It follows that

$$\hat{D}(\xi_k)(\theta) = D(\hat{\xi}_k)(\theta) = \sum_{\alpha} D\left(\prod \xi_{\alpha_i}^{\sigma_i(\alpha)}\right)(\theta)$$

for all $\theta \in \mathscr{A}^*$.

Consequence 4.3. (i) *Stripping by* $\hat{\xi}_k$ *decreases excess by no more than* $2^k - 1 = ex(\xi_1^{2^k-1})$.

(ii) *Since* $D(\xi_l)(Sq(f)) = 0$ *for* $l > 1$, *we have*

$$\hat{D}(\xi_k)Sq(f) = D(\xi_1^{2^k-1})Sq(f) = Sq(f - (2^k - 1))$$

for all f *and* k.

In Section 4.3, we generalize Part (ii) of Consequence 4.3 with a formula partially describing the effect of $\hat{D}(\xi_k^j)$ on $S(\Lambda;f)$ for $\Lambda, j \geq 2$.

4.3. Stripping $S(\Lambda;f)$ by $\hat{\xi}_k^j$

Lemma 4.4. *For any* $\theta \in \mathscr{A}^*$, *we have* $\hat{D}(\xi_k)[S(2;f)\cdot\theta] = Sq(2f)\cdot[\hat{D}(\xi_k)Sq(f)\cdot\theta]$.

Proof. The comultiplication in $\mathscr{A}_*$ is given by $\phi_*(\xi_k) = \sum_{j=0}^{k} \xi_{k-j}^{2^j} \otimes \xi_j$ [5], and consequently (4) implies

$$\hat{D}(\xi_k)[S(2;f)\cdot\theta] = Sq(2f)\cdot[\hat{D}(\xi_k)Sq(f)\cdot\theta] + \sum_{j=1}^{k} \hat{D}(\xi_j)Sq(2f)\cdot[\hat{D}(\xi_{k-j}^{2^j})Sq(f)\cdot\theta].$$

Since $\hat{D}(\xi_j)Sq(2f) = Sq(2f-(2^j-1)) = \hat{D}(\xi_{j-1}^2)Sq(2f-1)$ by Part (ii) of Consequence 4.3, we have

$$\hat{D}(\xi_k)[S(2;f)\cdot\theta] = Sq(2f)\cdot[\hat{D}(\xi_k)Sq(f)\cdot\theta] + \sum_{j=1}^{k} \hat{D}(\xi_{j-1}^2)Sq(2f-1)\cdot\hat{D}((\xi_{k-1-(j-1)}^{2^{j-1}})^2)[Sq(f)\cdot\theta]$$

$$= Sq(2f)\cdot[\hat{D}(\xi_k)Sq(f)\cdot\theta] + \hat{D}(\xi_{k-1}^2)[Sq(2f-1)\cdot Sq(f)\cdot\theta].$$

But $Sq(2f-1)\cdot Sq(f) = 0$, proving the lemma. □

Proposition 4.5. *For $\Lambda \geq 2$ and for any $\theta \in \mathscr{A}^*$,*

$$\hat{D}(\xi_k)[S(\Lambda;f)\cdot\theta] = S(\Lambda-1;2f)\cdot\hat{D}(\xi_k)[Sq(f)\cdot\theta].$$

Proof. The proof is by induction on Λ. The case $\Lambda = 2$ follows from the proof of Lemma 4.4. Suppose that the result is known for $\Lambda - 1$. Then

$$\hat{D}(\xi_k)[S(\Lambda;f)\cdot\theta]$$

$$= \sum_{j=0}^{k} \hat{D}(\xi_j)S(2;2^{\Lambda-2}f)\cdot\hat{D}(\xi_{k-j}^{2^j})[S(\Lambda-2;f)\cdot\theta] \quad \text{(by (4))}$$

$$= \sum_{j=0}^{k} Sq(2^{\Lambda-1}f)\cdot[\hat{D}(\xi_j)Sq(2^{\Lambda-2}f)]\cdot\hat{D}(\xi_{k-j}^{2^j})[S(\Lambda-2;f)\cdot\theta] \quad \text{(Lemma 4.4)}$$

$$= Sq(2^{\Lambda-1}f)\cdot\hat{D}(\xi_k)[S(\Lambda-1;f)\cdot\theta]$$

$$\overset{\text{ind}}{=} Sq(2^{\Lambda-1}f)\cdot S(\Lambda-2;2f)\cdot\hat{D}(\xi_k)[Sq(f)\cdot\theta]$$

$$= S(\Lambda-1;2f)\cdot\hat{D}(\xi_k)[Sq(f)\cdot\theta]. \quad \square$$

Next we generalize Proposition 4.5 to show that applying $\hat{D}(\xi_k)$ to $S(\Lambda;f)$ a total of j times affects only the right-most j places:

Proposition 4.6. $\hat{D}(\xi_k^j)S(\Lambda;f) = S(\Lambda-j;2^jf)\cdot\hat{D}(\xi_k^j)S(j;f)$.

Proof. Proposition 4.5 gives the case $j=1$. Suppose that the result is known for $j' < j$ and all Λ', and for j and $\Lambda' < \Lambda$. We have

$$\begin{aligned}\hat{D}(\xi_k^j)\ S(\Lambda; f) &= \hat{D}(\xi_k)\ [\hat{D}(\xi_k^{j-1})S(\Lambda; f)] \\ &\overset{\text{ind}}{=} \hat{D}(\xi_k)\ [S(\Lambda-(j-1); 2^{j-1}f)\cdot \hat{D}(\xi_k^{j-1})S(j-1; f)] \\ &= S(\Lambda-j; 2^j f)\cdot \hat{D}(\xi_k)[Sq(2^{j-1}f)\cdot \hat{D}(\xi_k^{j-1})\ S(j-1; f)] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(Proposition 4.5)} \\ &\overset{\text{ind}}{=} S(\Lambda-j; 2^j f)\cdot \hat{D}(\xi_k^j)S(j; f). \qquad \square\end{aligned}$$

4.4. Application

In [10], we prove the following conjugation formula, which grounds the inductive proof of Theorem 1.1:

Theorem 4.7 (Silverman [10]). *For all positive integers k and Λ, we have $\hat{S}(k; 2^\Lambda - 1) = S(\Lambda; 2^k - 1)$.*

Stripping both sides of this equation by $\hat{\xi}_k$, using (7) on the left and Proposition 4.5 on the right, we find that

$$\hat{S}(k; 2^\Lambda - 2) = S(\Lambda - 1; 2(2^k - 1))\cdot \hat{D}(\xi_k)Sq(2^k - 1).$$

Since $\hat{D}(\xi_k)Sq(2^k-1) = Sq(0) = 1$ by (6), we find that

$$\hat{S}(k; 2(2^{\Lambda-1}-1)) = S(\Lambda-1; 2(2^k-1)), \tag{8}$$

a formula conjectured in [10] which is a special case of Theorem 1.1 below. As a consequence, we see that $\hat{S}(k; 2(2^\Lambda - 1))$ and $\hat{S}(k; 2^\Lambda - 1)$ are both of length exactly Λ, where length is as defined in Section 3.2.

Recall now from Section 3.4 that stripping operations do not increase length. The result below, conjectured in [11], follows by a sandwich argument from (7) and the conclusion of the previous paragraph.

Theorem 4.8. *If $2^\Lambda - 1 \le f < 2^{\Lambda+1} - 1$, then the elements $\hat{S}(k; f)$ are of length exactly Λ independently of k.*

5. Proof of Theorem 1.4

We are now ready to prove

Theorem 1.4. *Part* (i) *of Conjecture* 1.2 *is always true, and consequently for all $f \ge 1$ we have $ex(\hat{S}(k; f)) \le (2^k - 1)\mu(f)$ for all $k \ge 1$.*

Proof. The cases $f = 2^\Lambda - 1$ and $f = 2(2^\Lambda - 1)$ follow from Theorem 4.7 and (8) respectively. We assume inductively that the result is known for $f \le 2^\Lambda - 1$ and prove it for $2^\Lambda \le f < 2(2^\Lambda - 1)$. Observe that

$$\begin{aligned} \hat{D}(\xi_k^{2^\Lambda-1})\hat{S}(k;2(2^\Lambda-1)) &= \hat{S}(k;2^\Lambda-1) && (7)\\ &= S(\Lambda;2^k-1) && \text{(Theorem 4.7)}\\ &= D(\xi_\Lambda^{2^k-1})S(\Lambda;2(2^k-1)) && \text{(Corollary 4.2 (ii))}\\ &= D(\xi_\Lambda^{2^k-1})\hat{S}(k;2(2^\Lambda-1)) && \text{(Theorem 4.7)}, \end{aligned}$$

i.e., $\hat{D}(\xi_k^{2^\Lambda-1})$ and $D(\xi_\Lambda^{2^k-1})$ agree on $\hat{S}(k;2(2^\Lambda-1))$. Moreover, since stripping operations commute with each other, these two operations also agree on all elements of the form $D(y)\hat{S}(k;2(2^\Lambda-1))$. In particular, they agree on

$$\hat{S}(k;f) = \hat{D}(\xi_k^{2(2^\Lambda-1)-f})\ \hat{S}(k;2(2^\Lambda-1))$$

for all $f \le 2(2^\Lambda - 1)$. Thus by (7) we have

$$D(\xi_\Lambda^{2^k-1})\hat{S}(k;f) = \hat{D}(\xi_k^{2^\Lambda-1})\hat{S}(k;f) = \hat{S}(k;f-(2^\Lambda-1)). \tag{9}$$

By Theorem 4.8, all admissible elements appearing in $\hat{S}(k;f)$ have length $\le \Lambda$, as those in $\hat{S}(k;f-(2^\Lambda-1))$ have length $\le \Lambda - 1$. By Proposition 4.1, those in $\hat{S}(k;f)$ of length $< \Lambda$ vanish when stripped by $\xi_\Lambda^{2^k-1}$. By Part (i) of Corollary 4.2, those of the form $E(r_1,\dots,r_\Lambda)$ either vanish (if $r_\Lambda < 2^k - 1$) or map to the single admissible element $E(r_1,\dots,r_{\Lambda-1},r_\Lambda-(2^k-1))$. Therefore (9) implies that the map

$$E(s_1,\dots,s_{\Lambda-1}) \mapsto E(s_1,\dots,s_{\Lambda-1},2^k-1)$$

assigns to each admissible summand of $\hat{S}(k;f-(2^\Lambda-1))$ an admissible summand of $\hat{S}(k;f)$ of length Λ with last entry 2^k-1. Since by assumption $E(R_k(f-(2^\Lambda-1)))$ appears in $\hat{S}(k;f-(2^\Lambda-1))$, we find from the inductive construction of $R_1(f)$ and the definition of $R_k(f)$ in Section 1.3 that indeed $E(R_k(f))$ appears in $\hat{S}(k;f)$. As the excess of $E(R_k(f))$ is $(2^k-1)\mu(f)$, this completes the inductive step in our proof. □

2. Proof of Theorems 1.1 and 1.3

We now apply results of Section 4.3 to prove Theorems 1.1 and 1.3, restated below for convenience. We will make use of the following observation, immediate from the definition of excess and the Adem relations governing multiplication in $\mathscr{A}^*$:

Observation 6.1. *If $a \in N$ and $Sq(a)\cdot\theta$ has degree d, then $ex(Sq(a)\cdot\theta) \ge 2a - d$.*

Theorem 1.3 for $f = 2^\Lambda - 1$ is an immediate consequence of Theorem 4.7, which states that $\hat{S}(k;2^\Lambda-1) = S(\Lambda;2^k-1)$ for all k and Λ, generalizing a result of [2] to $k > 1$. Theorem 1.1, from which the rest of Theorem 1.3 will follow, likewise generalizes another result of [2]:

Theorem 1.1. *$\hat{S}(k;2^\Lambda - j) = S(\Lambda - (j-1); 2^{j-1}(2^k - 1)) \cdot \hat{S}(k; 2^{j-1} - j)$ for $1 \le j \le \Lambda$.*

Proof. For such j, we have

$$\begin{aligned}
\hat{S}(k;2^\Lambda - j) &= \overline{D^{j-1}(\xi_k)S(k;2^\Lambda - 1)} \quad \text{(Corollary 4.2 (ii))}\\
&= \hat{D}(\xi_k^{j-1})\hat{S}(k;\, 2^\Lambda - 1) \quad \text{(by (3))}\\
&= \hat{D}(\xi_k^{j-1})S(\Lambda; 2^k - 1) \quad \text{(Theorem 4.7)}\\
&= S(\Lambda - (j-1); 2^{j-1}(2^k - 1)) \cdot \hat{D}(\xi_k^{j-1})S(j-1; 2^k - 1)\\
&\qquad \text{(Proposition 4.6)}\\
&= S(\Lambda - (j-1); 2^{j-1}(2^k - 1)) \cdot \hat{D}(\xi_k^{j-1})\hat{S}(k; 2^{j-1} - 1)\\
&\qquad \text{(Theorem 4.7)}\\
&= S(\Lambda - (j-1); 2^{j-1}(2^k - 1)) \cdot \hat{S}(k; 2^{j-1} - j). \qquad \square
\end{aligned}$$

Before recalling the statement of Theorem 1.3, we recall from Section 1.3 that

$$\mu(2^\Lambda - j) = \begin{cases} j, & j \le \Lambda,\\ \Lambda - 1, & j = \Lambda + 1,\\ \Lambda - 2, & j = \Lambda + 2. \end{cases} \tag{10}$$

We are now ready to prove

Theorem 1.3. *Conjecture 1.2 is true for all f satisfying $2^\Lambda - (\Lambda + 2) \le f \le 2^\Lambda - 1$ for some $\Lambda \ge 0$.*

Proof. By Theorem 1.4, we need only show that for $1 \le j \le \Lambda + 2$, we have

$$ex(\hat{S}(k; 2^\Lambda - j)) \ge (2^k - 1)\mu(2^\Lambda - j)$$

for all $k \ge 0$. If $1 \le j \le \Lambda$, then by Observation 6.1 and Theorem 1.1 we have

$$\begin{aligned}
ex(\hat{S}(k; 2^\Lambda - j)) &\ge 2 \cdot 2^{\Lambda - j} \cdot 2^{j-1}(2^k - 1) - (2^k - 1)(2^\Lambda - j)\\
&= (2^k - 1)[2^\Lambda - (2^\Lambda - j)]\\
&= (2^k - 1) \cdot j\\
&= (2^k - 1)\mu(2^\Lambda - j) \quad \text{(by (10))}.
\end{aligned}$$

The result for $j = \Lambda + 1$ and $j = \Lambda + 2$ follows from the case $j = \Lambda$ and Part (i) of Consequence 4.3. $\square$

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