Spectral Decompositions,
Eisenstein series,
L-functions

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Fourier series *decompose* periodic functions $f(x)$ as sums of simpler periodic functions

$$f(x) \sim \sum_{n \in \mathbf{Z}} c_n \, e^{2\pi i n x}$$

There's a handy formula for the **decomposition coefficients** c_n:

$$c_n = \int_0^1 f(x) \, e^{-2\pi i n x} \, dx$$

This decomposition is **discrete** in the sense that the atomic pieces, the exponential functions, are themselves in the space of functions being decomposed.

For future reference, we note that it may be better to consider periodic functions $f(x+1) = f(x)$ as living on the circle $\mathbf{R}/\mathbf{Z}$ rather than on the unit interval.

By contrast, Fourier *transforms* give a **continuous decomposition**

$$f(x) \sim \int_{-\infty}^{+\infty} \hat{f}(t)\, e^{2\pi i t x}\, dt$$

of $L^2(\mathbf{R})$, with a good formula for $\hat{f}$:

$$\hat{f}(t) = \int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i t x}\, dx$$

This deserves to be called continuous since it is an *integral* (rather than a *sum*), and since the exponential functions are not in $L^2(\mathbf{R})$, so are themselves are *not* in the space being decomposed. (This is a little strange.)

The exponential functions' not being in L^2 creates trouble both in defining $f \to \hat{f}$ in the first place, and in the reconstitution of f from $\hat{f}$, since the integrals generally don't converge.

The Fourier series decomposition of $L^2(0,1)$ lets us write the same thing in two different ways.

As a first example, the Fourier coefficients of $f(x) = x - \frac{1}{2}$ are easily computed

$$c_n = \begin{cases} -1/2\pi i n & (\text{for } n \neq 0) \\ 0 & (\text{for } n = 0) \end{cases}$$

The Plancherel formula

$$\int_0^1 |f(x)|^2 \, dx = \sum_{n \in \mathbf{Z}} |c_n|^2$$

applied to $f(x) = x - \frac{1}{2}$ gives

$$\frac{1}{12} = \frac{1}{(2\pi)^2} \cdot \sum_{n \neq 0} \frac{1}{n^2}$$

giving the popular formula

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Though the Fourier series

$$x - \frac{1}{2} \sim \frac{1}{-2\pi i} \sum_{n\neq 0} \frac{e^{2\pi i n x}}{n}$$

for $f(x) = x - \frac{1}{2}$ doesn't converge absolutely, it does converge for $0 < x < 1$. For example,

$$\frac{1}{2} = \frac{1}{2} + \frac{1}{4} = -f(\frac{1}{4}) + f(\frac{1}{4})$$

$$= \sum_{n\neq 0} \frac{-1}{2\pi i n} \left(-e^{2\pi i n(1/4)} + e^{2\pi i n(3/4)}\right)$$

$$= 2 \cdot \sum_{1\leq n \,\mathrm{odd}} \frac{-1}{2\pi i n} \left(-i^n + i^{-n}\right)$$

which simplifies to the ever-popular

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots$$

A slightly less trivial application of the Fourier series

$$f(x) = x - \frac{1}{2} \sim \frac{1}{-2\pi i} \sum_{n \neq 0} \frac{e^{2\pi i n x}}{n}$$

is the example

$$-\frac{1}{3} = f(\frac{1}{3}) - f(\frac{2}{3}) = 2 \cdot \frac{-1}{2\pi i} \sum_{n=1}^{\infty} \frac{1}{n} \cdot (\omega^n - \omega^{2n})$$

(letting $\omega = e^{2\pi i/3}$ and grouping $\pm n$ terms). This simplifies to

$$\frac{4\pi}{3\sqrt{3}} = \frac{1}{1} - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \ldots$$

Similarly, for $f(x) = \frac{1}{2}x(x-1) + \frac{1}{12}$

$$\frac{1}{2}x(x-1) + \frac{1}{12} \quad \sim \quad \frac{1}{4\pi^2} \sum_{n \neq 0} \frac{e^{2\pi i n x}}{n^2}$$

and as an example of an application, letting $\omega = e^{2\pi i/5}$ and grouping $\pm n$ terms,

$$\frac{2}{25} = f(\frac{1}{5}) - f(\frac{2}{5}) - f(\frac{3}{5}) + f(\frac{4}{5})$$

$$= \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{\omega^n - \omega^{2n} - \omega^{3n} + \omega^{4n}}{n^2}$$

This simplifies (less trivially than before) to

$$\frac{4\pi^2}{25\sqrt{5}} = \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} + \dots$$

Indeed, there are polynomials $B_o(x)$, $B_1(x)$, $B_2(x)$, ... (essentially Bernouilli polynomials) with coefficients in $\mathbf{Q}$ so that

$$(2\pi i)^\ell \cdot B_\ell(x) \quad \sim \quad \sum_{n\neq 0} \frac{e^{2\pi i n x}}{n^\ell}$$

From this, for an N-periodic $\mathbf{Q}$-valued function χ on $\mathbf{Z}$ with a parity condition

$$\chi(a+N) = \chi(a) \quad \text{and} \quad \chi(-n) = (-1)^\ell\,\chi(n)$$

we can conclude that

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^\ell} \quad \in \quad \pi^\ell \cdot \overline{\mathbf{Q}}$$

More precisely, as was secretly the case in the examples above, if χ is a primitive Dirichlet character modulo N, and $\chi(-1) = (-1)^\ell$, then

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^\ell} = \frac{2\pi^\ell}{g(\chi)} \cdot \sum_{a \bmod N} \chi(a) \cdot B_\ell(\frac{a}{N})$$

where $g(\chi)$ is a Gauss sum

$$g(\chi) = \sum_{a \bmod N} \bar{\chi}(a)\, e^{-2\pi i a/N}$$

For both the circle and the line, there are several possible viewpoints about what we decompose with respect to.

Those decompositions do expand functions in terms of eigenfunctions of the (translation invariant) differential operator d^2/dx^2.

Also, the exponential functions are exactly the functions on $\mathbf{R}/\mathbf{Z}$ or on $\mathbf{R}$ which are *homomorphisms* to $\mathbf{C}^\times$.

Also, Fourier series decompose the *representation space* $L^2(\mathbf{R}/\mathbf{Z})$ for the translation action $R_t f(x) = f(x+t)$ into *irreducible* one-dimensional representations $\mathbf{C} \cdot e^{2\pi i n x}$ with $n \in \mathbf{Z}$. (Similarly for Fourier transforms.)

Fourier series expand functions in eigenfunction expansions for the adjoint-closed algebra of *compact* operators

$$R_\eta f(x) = \int_{\mathbf{R}} \eta(t)\, f(x+t)\, dt$$

for *test functions* $\eta \in \mathrm{C}_c^\infty(\mathbf{R})$. (By contrast, on $L^2(\mathbf{R})$ these operators are *not* compact.)

It is more interesting to look at decomposition of functions on more complicated objects than the *circle* (for Fourier series), or the *line* (for Fourier transforms).

For example, consider the linear fractional (Möbius) transformation action of $\Gamma = SL(2,\mathbf{Z})$ on the complex upper half-plane $\mathcal{H}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az+b}{cz+d}$$

The space we want is the quotient

$$X = \Gamma\backslash\mathcal{H} \approx \text{ sphere } \mathbf{P}^1 \text{ with } \infty \text{ removed}$$

The metric on the $\mathbf{P}^1$ is inherited from the $SL(2,\mathbf{R})$-invariant metric $\frac{dx\,dy}{y^2}$ on $\mathcal{H}$. It is not the usual sphere metric, but instead makes the missing point *infinitely far away*.

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2,\mathbf{R})$ there are formulas

$$\frac{d}{dz}\left(\frac{az+b}{cz+d}\right) = \frac{1}{(cz+d)^2}$$

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}\,(z)}{|cz+d|^2}$$

If we want to look at *holomorphic* functions on $\mathcal{H}$ with some sort of nice behavior under the action of Γ, it turns out that actual *invariance* is too restrictive.

A holomorphic **modular form** of weight $2k$ is a holomorphic function f on $\mathcal{H}$

- Which is of *moderate growth*: for some N

$$f(x+iy) = O(y^N) \text{ as } y \to +\infty$$

- Which for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ satisfies

$$f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z)) = (cz+d)^{2k} \cdot f(z)$$

The last condition is equivalent to the *invariance* of the *form*

$$f(z)\, dz^k$$

under the action of Γ. That is, holomorphic modular forms are Γ-invariant sections of a line bundle, rather than being Γ-invariant *functions*.

The simplest examples of modular forms are **holomorphic Eisenstein series**

$$E_{2k}(z) = \sum_{m,n}{}' \frac{1}{(cz+d)^{2k}} \quad (m, n \text{ not both } 0)$$

The Eisenstein series $g_2 = E_4$ and $g_3 = E_6$ occur in the Weierstrasss equation

$$\wp'(u)^2 = 4\wp(u)^3 - 60 g_2 \wp(u) - 140 g_3$$

for the elliptic function

$$\wp(z) = \sum_{m,n}{}' \left(\frac{1}{(u - (mz+n))^2} - \frac{1}{(mz+n)^2} \right)$$

It is *elementary* to verify that Eisenstein series are modular forms.

The other simple kind of holomorphic modular forms (though with a somewhat smaller Γ in place of $SL(2, \mathbf{Z})$) is a **theta series**

$$\theta_Q(z) = \sum_{v \in \mathbf{Z}^n} e^{2\pi i Q[v] \cdot z}$$

where Q is a $\mathbf{Z}$-valued positive-definite quadratic form on $\mathbf{R}^n$. It is *not* entirely elementary to verify that theta series are modular forms.

Note that $\Gamma = sl(2, \mathbf{Z})$ contains the subgroup

$$N_{\mathbf{Z}} = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbf{Z}\}$$

whose elements act on $\mathcal{H}$ by integer translations

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} (x + iy) = (x + n) + iy$$

Also, for such matrices the corresponding $cz+d$ is simply 1. Thus, modular forms $f(z)$ have **Fourier expansions**

$$f(z) = \sum_{n \in \mathbf{Z}} c_n(y)\, e^{2\pi i n x}$$

Since $f(z)$ is holomorphic, necessarily $c_n(y)$ is a constant multiple of $e^{-2\pi i n y}$. Further, the moderate growth condition excludes negative-index Fourier terms, so actually the Fourier expansion looks like

$$f(z) = \sum_{n \geq 0} c_n\, e^{2\pi i n z}$$

That is, we can *decompose* modular forms by the action of translations. Such $f(z)$ is called a **cuspform** if $c_o = 0$, in which case $f(z)$ is actually of *rapid decay* as $y \to +\infty$.

The Fourier expansions of holomorphic Eisenstein series can be directly computed, using residue calculus from complex analysis and using a little Fourier analysis.

$$E_{2k}(z) = 2\cdot\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n\geq 1} \sigma_{2k-1}(n)\, e^{2\pi i n z}$$

where as usual

$$\sigma_{2k-1}(n) = \sum_{1\leq d|n} d^{2k-1} \qquad \zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

The higher Fourier coefficients are elementary, while by contrast the constant term $2\cdot\zeta(2k)$ is not.

An elementary complex analysis argument shows (much like Liouville's theorem) that the only weight-zero modular forms are constants, and for low weights

$$\begin{array}{rcl}
\text{weight 0 modular forms} & = & \mathbf{C} \\
\text{weight 2 modular forms} & = & \{0\} \\
\text{weight 4 modular forms} & = & \mathbf{C} \cdot E_4 \\
\text{weight 6 modular forms} & = & \mathbf{C} \cdot E_6 \\
\text{weight 8 modular forms} & = & \mathbf{C} \cdot E_8 \\
\text{weight 10 modular forms} & = & \mathbf{C} \cdot E_{10} \\
\text{weight 12 modular forms} & = & \mathbf{C} \cdot E_{12} \oplus \mathbf{C} \cdot \Delta
\end{array}$$

where Ramanujan's

$$\Delta = \frac{1}{1728}(\left(\frac{E_4}{2\zeta(4)}\right)^3 - \left(\frac{E_6}{2\zeta(6)}\right)^2)$$

$$= e^{2\pi i z} \cdot \left(\prod_{n=1}^{\infty} (1 - e^{2\pi i n z})\right)^{24}$$

is the lowest-weight cuspform.

Restrictions on existence of modular forms imply relations. Some senseless but possibly slightly charming examples are some relations among the sum-of-powers-of-divisors functions σ_{2k-1} that occur in the higher Fourier coefficients of Eisenstein series:

$$\sigma_7(n) \;=\; \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\,\sigma_3(n-m)$$

$$\begin{aligned} 11\sigma_9(n) \;&=\; 21\sigma_5(n) - 10\sigma_3(n) \\ &\quad+\; 5040 \sum_{m=1}^{n-1} \sigma_3(n)\,\sigma_5(n-m) \end{aligned}$$

which follow from

$$\left(\frac{E_4}{2\zeta(4)}\right)^2 = \frac{E_8}{2\zeta(8)}$$

and

$$\frac{E_4}{2\zeta(4)} \cdot \frac{E_6}{2\zeta(6)} = \frac{E_{10}}{2\zeta(10)}$$

Similarly, constraints on existence of modular forms yield relations between theta series and Eisenstein series. First, at the level of an amusement, one can prove the traditional

$$\begin{aligned} &\text{number of ways to express an odd } n \text{ as} \\ &\quad \text{sum of 8 squares } n = x_1^2 + \ldots + x_8^2 \\ &\qquad\qquad\qquad \text{with } (x_i \in \mathbf{Z}) \\ &= 16\,\sigma_3(n) \end{aligned}$$

More seriously, the *Siegel-Weil* formula asserts that normalized Eisenstein series $E_{4k}/2\zeta(4k)$ are *rational-coefficient* linear combinations of theta series.

Since Fourier coefficients of theta series are rational, and since all parts of these Eisenstein series are explicit except for the $\zeta(2k)$,

$$\zeta(2k) = \text{ rational } \times \pi^{2k}$$

This argument works as well for zeta functions of any *totally-real* algebraic number field, such as $\mathbf{Q}(\cos 2\pi/n)$ for integer n.

The best of Riemann's proofs of the analytic continuation of $\zeta(s) - \sum 1/n^2$ uses an **integral representation**

$$\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s) = \int_0^\infty y^{s/2}\,\frac{\theta(iy)-1}{2}\,\frac{dy}{y}$$

with θ defined on the upper half-plane $\mathcal{H}$ by

$$\theta(z) = \sum_{n\in\mathbf{Z}} e^{\pi i n^2 z}$$

The Poisson summation formula $\sum_{n\in\mathbf{Z}} f(n) = \sum_{n\in\mathbf{Z}} \hat{f}(n)$ applied to the test function $f(n) = e^{-\pi n^2 y}$ gives

$$\theta(iy) = \frac{1}{\sqrt{y}}\,\theta(-1/iy)$$

from which we obtain the analytic continuation:

Proof of analytic continuation of $\zeta(s)$

$$\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s) = \int_1^\infty + \int_0^1$$

$$= \text{ entire } + \int_0^1 y^{s/2}\,\frac{\theta(iy)}{2}\,\frac{dy}{y} - \int_0^1 \frac{y^{s/2}}{2}\,\frac{dy}{y}$$

$$= \text{ entire } + \int_1^\infty y^{-s/2}\,\frac{\theta(-1/iy)}{2}\,\frac{dy}{y} - \frac{1}{s}$$

$$= \text{ entire } + \int_1^\infty y^{-s/2}\,\frac{\theta(iy)}{2\sqrt{y}}\,\frac{dy}{y} - \frac{1}{s}$$

$$= \text{ entire } + \int_1^\infty y^{(1-s)/2}\,\frac{\theta(iy)-1}{2}\,\frac{dy}{y}$$

$$+ \int_1^\infty \frac{y^{(1-s)/2}}{2}\,\frac{dy}{y} - \frac{1}{s}$$

$$= \text{ entire}(s) + \text{ entire}(1-s) \ - \frac{1}{1-s} - \frac{1}{s}$$

Symmetry of the last expressions gives the *functional equation*

$$\pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s) = \pi^{(1-s)/2}\,\Gamma(\frac{1-s}{2})\,\zeta(1-s)$$

Hecke used holomorphic cuspforms $f(z) = \sum_{n>0} c_n\, e^{2\pi i n z}$ to obtain analytic continuations and functional equations of **L-functions**

$$L_f(s) = \sum_{n>0} \frac{c_n}{n^s}$$

by an **integral representation** parallel to Riemann's, but simpler

$$(2\pi)^{-s}\Gamma(s)\, L_f(s) = \int_0^\infty y^s\, f(iy)\, \frac{dy}{y}$$

$$= \int_1^\infty + \int_0^1 = \text{ entire } + \int_1^\infty y^s\, f(iy)\, \frac{dy}{y}$$

$$= \text{ entire } + \int_1^\infty y^{-s}\, f(-1/iy)\, \frac{dy}{y}$$

$$= \text{ entire } + (-1)^k \int_1^\infty y^{2k-s}\, f(iy)\, \frac{dy}{y}$$

$$= \text{ entire}(s) + (-1)^k \text{ entire}(2k-s)$$

A less elementary family of modular forms consists of **Maass' waveforms**, Γ-invariant moderate-growth functions on $\mathcal{H}$ which are eigenfunctions fo the $SL(2,\mathbf{R})$-invariant Laplacian

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

We want a decomposition of $L^2(\Gamma\backslash\mathcal{H})$ *by* Δ.

The simplest (but not square-integrable) waveform is another type of Eisenstein series

$$E_s(z) = \sum_{\gcd(m,n)=1} \frac{y^s}{|mz+n|^{2s}}$$

$$= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}\,(\gamma(z))^s$$

where Γ_∞ consists of elements $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ in Γ. The Δ-eigenvalue of E_s is

$$\lambda = \lambda_s = s(s-1)$$

Waveforms have Fourier expansions

$$f(z) = \sum_n c_n(y)\, e^{2\pi i n z}$$

where the Fourier coefficient satisfies a differential equation

$$u" - (\frac{\lambda}{y^2} + 4\pi^2 n^2)u = 0$$

The Eisenstein series correctly suggests the entirely unobvious fact that an *integral representation* for a solution with eigenvalue $\mu(\mu - 1)$ is

$$u(y) = \int_{-\infty}^{+\infty} \frac{y^\mu}{(x^2 + y^2)^\mu}\, e^{-2\pi i n x}\, dx$$

This integral only converges for $\mathrm{Re}\,(\mu) > \frac{1}{2}$, but can be rearranged to a better form

$$u(y) = \frac{\pi^\mu}{\Gamma(\mu)} \int_0^\infty e^{-\pi n^2/t}\, e^{-\pi y^2 t}\, t^{\mu - \frac{1}{2}}\, \frac{dt}{t}$$

This solution is the *only* rapidly decreasing solution (as $y \to +\infty$).

Maass did for waveforms what Hecke did in the holomorphic case. For cuspidal waveform (zero-th Fourier coefficient 0) with normalized rapid-decay u_n with eigenvalue $\mu(\mu - 1)$

$$f(z) = \sum_{n \neq 0} c_n \cdot u_n(y)\, e^{2\pi i n x}$$

the same integral transform is appropriate

$$\int_0^\infty y^s\, f(iy)\, \frac{dy}{y}$$

$$= \pi^{-s}\, \Gamma(\frac{s+\mu}{2})\, \Gamma(\frac{s+1-\mu}{2}) \sum_{n \geq 1} \frac{c_n}{n^s}$$

The computation of the gamma factors is feasible when the integral representation of the u_n's is known.

The same argument proves analytic continuation and functional equation.

What's in $L^2(\Gamma\backslash\mathcal{H})$ besides cuspidal waveforms?

The orthogonal complement of cuspforms is spanned by functions

$$\theta_\varphi(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(\mathrm{Im}\,(\gamma z))$$

where $\varphi \in \mathrm{C}_c^\infty(0,\infty)$, so decompose *these* into Δ-eigenfunctions. Define TF on $\mathrm{C}_c^\infty(0,\infty)$

$$TF(s) = \int_{-\infty}^{\infty} F(r)\, r^{-s}\, \frac{dr}{r}$$

Theorem:

$$\theta_\varphi(g) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} T\varphi(s)\, E_s(g) \quad (\mathrm{Re}\,(s) > 1)$$

The complement of cuspforms is integrals of Eisenstein series. However, we should push the line of integration to $\mathrm{Re}\,(s) = 1/2$, despite the Eisenstein series only converging for $\mathrm{Re}\,(s) > 1$. *Meromorphically continue the Eisenstein series.*

One can compute separately, much as in the holomorphic case

$$E_s(z) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} + \text{higher-order terms}$$

where ξ is ζ with its proper gamma factor

$$\xi(s) = \pi^{-s/2}\, \Gamma(\frac{s}{2})\, \zeta(s)$$

Though the zeta function was understood prior to the Eisenstein series, Selberg noted that meromorphic continuation of E_s would prove meromorphic continuation of $\zeta(s)$.

This idea was vastly extended by Langlands, who looked at constant terms of more complicated Eisenstein series on bigger groups, like $GL(n)$, obtaining in 1960's the *meromorphic continuation* of a large family of L-functions. Shahidi extended and refined Langlands' idea to obtain *functional equations* as well. This is the *Langlands-Shahidi* method for obtaining integral representations.

The *Rankin-Selberg*-method is another type of integral representation which makes essential use of the meromorphic continuation of Eisenstein series.

For holomorphic cuspforms $f(z) = \sum a_n e^{2\pi i n z}$, $g(z) = \sum b_n e^{2\pi i n z}$ of weight $2k$

$$\int_{\Gamma\backslash\mathcal{H}} f(z)\,\overline{g(z)}\,E_s(z)\,y^{2k}\,\frac{dx\,dy}{y^2}$$

$$= (4\pi)^{-s}\,\Gamma(s)\,\sum_n \frac{a_n\overline{b_n}}{n^s}$$

The rapid decay of the cuspforms combined with the moderate growth (even of the meromorphically continued Eisenstein series) makes the integral converge away from the poles of the Eisenstein series.

Bernstein refined of one of Selberg's methods for meromorphic continuation of Eisenstein series.

Consider the holomorphically parametrized system of equations

$$\begin{cases} \Delta u & = & s(s-1)\cdot u \\ \left(y\frac{\partial}{\partial y} - (1-s)\right) c(u) & = & (2s-1)\, y^s \end{cases}$$

For u a nice function of moderate growth on $\Gamma\backslash\mathcal{H}$. Note that E_s satisfies this system, and the trickier part of the constant term is masked.

Theorem (Bernstein) Suppose that for s in a non-empty open subset of $\mathbf{C}$ the system has the unique solution E_s. Suppose that (locally everywhere) for some n there is a holomorphic family of linear maps

$$h_s : \mathbf{C}^n \to \{\text{moderate growth functions on } \Gamma\backslash\mathcal{H}\}$$

so that all solutions of the system lie inside the image $h_s(\mathbf{C}^n)$. Then E_s has a meromorphic continuation to $s \in \mathbf{C}$.

The first condition is *uniqueness*, the second *finiteness*.

Proof: Fix $s \notin \mathbf{R}$ with $\mathrm{Re}(s) >> 0$. If a function v_s other than E_s were to satisfy the system, then

$$E_s - v_s = c \cdot y^{1-s} + \text{ higher}$$

The *theory of the constant term* assures that the higher-order terms are of rapid decay, so this difference would be square-integrable. But the Laplacian Δ on $\mathcal{H}$ is essentially self-adjoint, so any $L^2(\Gamma\backslash\mathcal{H})$ eigenfunction must have a *real* eigenvalue. But $s(s-1) \notin \mathbf{R}$, contradiction. Thus, for s in a non-empty open set E_s is the unique solution to the system.

For large real T, let

$$y_T^s = \begin{cases} y^s & (\text{for } y \geq T) \\ 0 & (\text{else}) \end{cases}$$

And form special Eisenstein series by

$$E(y_T^s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y_T^s(\gamma z)$$

where y_T^s is viewed as a function on $\mathcal{H}$. These converge for all s.

If v_s is a solution of $\Delta v_x = s(s-1)$, then by the theory of the constant term

$$v_s = aE(y_T^s) + bE(y_T^{1-s}) + \text{ rapid decay}$$

Thus, any solution of the original system is inside the space

$$\mathbf{C} \cdot E(y_T^s) + \mathbf{C} \cdot E(y_T^{1-s}) + \{ \text{ rapid decay } \}$$

In fact, solutions are inside a space

$$\mathbf{C} \cdot E(y_T^s) + \mathbf{C} \cdot E(y_T^{1-s})$$

$$+\{ \text{ constant term compactly supported } \}$$

Lemma (Selberg-Bernstein) Let V be a Banach space. Consider a parametrized family of linear maps $T_s(v) = 0$, with $T_s : V \to W$, where W is a Banach space, and $s \to T_s$ is holomorphic for the uniform-norm Banach-space topology. Suppose that for some fixed s_o the operator T_{s_o} has a left inverse modulo compact operators, that is, that there exists an operator $A : W \to V$ so that

$$A \circ \lambda_{s_o} = 1_V + \text{(compact operator)}$$

Then the finiteness condition holds.

But the theorem of Selberg, Gelfand, Piatetski-Shapiro, Langlands on the *compactness* of integral operators on modular forms with compact-support constant term meets the condition of the lemma!

The same type of argument applies at the very least to Eisenstein series made with cuspidal data on maximal proper parabolics. (The expanded version of my Tel Aviv talk (March 2001) gives details.)

A refinement of Bernstein's criterion allows for discussion of poles in terms of unitariness of associated representations.

Via Langlands-Shahidi and Rankin-Selberg integral representations, knowledge of poles of Eisenstein series gives information about poles of L-functions.
