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Some remarks on the inverse systems of polynomial modules [1]

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Abstract

This paper is devoted to a generalization of the notion of inverse system of a polynomial ideal as can be found in Macaulay's treatise on Modular Systems. The definition of inverse system given here relates polynomial modules to modules of linear forms on polynomials. The most interesting results obtained by Macaulay on the inverse systems of polynomial ideals are particular cases of propositions proved in this article. ©1998 Elsevier Science B.V. All rights reserved.

1. Introduction

The classical prototype of the dual of a polynomial module is the notion of inverse system of a polynomial ideal which has been given by Macaulay in his treatise on Modular Systems [4]. According to Macaulay the inverse system of a polynomial ideal M is the set of all *negative* formal power series $\sum c_{p_1,\dots,p_n}(x_1^{p_1}\cdots x_n^{p_n})^{-1}$ such that

$$\sum_{p_1,\dots,p_n} a_{p_1,\dots,p_n}c_{p_1,\dots,p_n}=0,$$

for every polynomial $\sum a_{q_1,\dots,q_n}x_1^{q_1}\cdots x_n^{q_n}$ in M. From the point of view of this paper, the most interesting property of an inverse system is the one of being a module over the ring of all polynomials in $x_1,\dots,x_n$ with complex coefficients. Here is Macaulay's

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definition of a scalar product: *If* $E = \sum c_{p_1,\dots,p_n}(x_1^{p_1}\cdots x_n^{p_n})^{-1}$ *is a negative power series (no* p_i *negative), and A any polynomial, the part of the expanded product AE which consists of a negative power series will be denoted by* $A{\cdot}E$ *and called the A-derivate of E* [4, p. 69].

The definition of inverse system may be cast in a linear algebraic setting: if we consider $\mathbb{C}[x_1,\dots,x_n]$ as a $\mathbb{C}$-algebra, the inverse system of an ideal $M \subseteq \mathbb{C}[x_1,\dots,x_n]$ is the subspace of $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[x_1,\dots,x_n],\mathbb{C})$ consisting of all $\mathbb{C}$-linear forms $E\colon \mathbb{C}[x_1,\dots,x_n] \to \mathbb{C}$ whose kernel contains M. In this context, Macaulay's scalar product $A{\cdot}E$ can be defined by $(A{\cdot}E)(B) = E(AB)$, $B \in \mathbb{C}[x_1,\dots,x_n]$.

This trivial observation indicates that Macaulay's scalar product depends only on the linearity of E and on the product in $\mathbb{C}[x_1,\dots,x_n]$. It follows that nothing prevents one from extending Macaulay's definition to a more general context. Thus, in this paper we start by considering any algebra over an arbitrary field $\mathbb{K}$. We then restrict ourselves to polynomial rings and show how certain properties of a Gröbner basis for a polynomial module relate to existing results.

2. Duality

In this section we introduce some notations and elementary definitions. The objects we shall be most interested in are modules over some polynomial algebra; nevertheless it is convenient to give the basic definitions in a more general context.

Let $\mathbb{K}$ be a field and let A be an associative $\mathbb{K}$-algebra with unit element 1_A. Let S be a basis of the vector space underlying A and let us represent each element $P \in A$ in the form $\sum_{\mathbf{s}\in S} a_{\mathbf{s}}\mathbf{s}$ where $a_{\mathbf{s}} = 0$ for all but a finite number of indices. Moreover, $A^* := \mathrm{Hom}_{\mathbb{K}}(A,\mathbb{K})$ will denote the usual linear dual of the $\mathbb{K}$-space A.

A family $(f_\alpha)_{\alpha\in J}$ of elements of A^* will be said to be a *summable family* if, for every $P \in A$, the set of all indices $\alpha \in J$ such that $f_\alpha(P) \neq 0$ is finite. If $(f_\alpha)_{\alpha\in J}$ is a summable family of linear forms we will call the *sum of the family* the linear form $\sum_{\alpha\in J} f_\alpha$ defined by

$$\left(\sum_{\alpha\in J} f_\alpha\right)(P) = \sum_{\alpha\in J} f_\alpha(P).$$

In the following, we will consider the powers A^l and $(A^*)^l$ with their A-module structures given by

$$\begin{aligned} A \times A^l &\to A^l \\ (P, (P_1,\dots,P_l)) &\mapsto (PP_1,\dots,PP_l) \end{aligned}$$

and

$$\begin{aligned} A \times (A^*)^l &\to (A^*)^l \\ (P, (f_1,\dots,f_l)) &\mapsto (P{\cdot}f_1,\dots,P{\cdot}f_l), \end{aligned}$$

respectively, where $(P\cdot f_i)(Q)=f_i(QP)$, $1\le i\le l$. Sometimes, using the canonical embeddings $e_1,\dots,e_l$ of A into A^l, we will write $e_1(P_1)+\cdots+e_l(P_l)$ instead of $(P_1,\dots,P_l)$.

For any submodule H of $(A^*)^l$, we define

$$\mathscr{P}(H)=\{(P_1,\dots,P_l)\in A^l \mid P_1\cdot f_1+\cdots+P_l\cdot f_l=0 \text{ for all } (f_1,\dots,f_l)\in H\}.$$

It is easy to check that $\mathscr{P}(H)$ is a submodule of A^l. Our main goal now is to find a "convenient" answer to the following question: given a submodule M of A^l, is there a submodule H of $(A^*)^l$ such that $\mathscr{P}(H)=M$?

Following old-fashioned terminology which had already been used in the case of polynomial ideals by Macaulay, any submodule $H\subseteq(A^*)^l$ such that $\mathscr{P}(H)=M$ will be called an *inverse system* of M. Let us prove that the answer to the previous question is affirmative (Proposition 2.1), so that we may claim that every submodule $M\subseteq A^l$ has an inverse system. For the purpose of our discussion, we first fix a subset B of $S\times\{1,\dots,l\}$ such that the set $\{e_j(\mathbf{t})+M \mid (\mathbf{t},j)\in B\}$ is a basis of the quotient space A^l/M. For every $1\le i\le l$, let

$$e_i(P)+M=\sum_{(\mathbf{t},j)\in B} f^j_{\mathbf{t},i}(P)(e_j(\mathbf{t})+M),\quad P\in A$$

(i.e. $f^j_{\mathbf{t},i}(P)=$ coeff. of $e_j(\mathbf{t})+M$ occurring in $e_i(P)+M$), so that

$$\begin{aligned}(P_1,\dots,P_l)+M&=\sum_{i=1}^{l}(e_i(P_i)+M)=\sum_{i=1}^{l}\sum_{(\mathbf{t},j)\in B} f^j_{\mathbf{t},i}(P_i)(e_j(\mathbf{t})+M)\\&=\sum_{(\mathbf{t},j)\in B}\sum_{i=1}^{l} f^j_{\mathbf{t},i}(P_i)(e_j(\mathbf{t})+M).\end{aligned}$$

Note that for every $P\in A$, there exist only finitely many pairs $(\mathbf{t},j)\in B$ such that $f^j_{\mathbf{t},i}(P)\neq 0$. It follows that $(f^j_{\mathbf{t},i})_{(\mathbf{t},j)\in B}$ is a summable family of linear forms $f^j_{\mathbf{t},i}: A\to\mathbb{K}$, $P\mapsto f^j_{\mathbf{t},i}(P)$. Moreover, if we denote by $\mathrm{NF}_B(P_1,\dots,P_l)$ the unique element of $\mathrm{Span}_{\mathbb{K}}(B)$ such that $(P_1,\dots,P_l)-\mathrm{NF}_B(P_1,\dots,P_l)\in M$, then

$$\mathrm{NF}_B(P_1,\dots,P_l)=\left(\sum_{\mathbf{t}\in T_1}\sum_{i=1}^{l} f^1_{\mathbf{t},i}(P_i)\mathbf{t},\dots,\sum_{\mathbf{t}\in T_l}\sum_{i=1}^{l} f^l_{\mathbf{t},i}(P_i)\mathbf{t}\right), \tag{1}$$

where $T_j=\{\mathbf{t}\in S \mid (\mathbf{t},j)\in B\}$. Hence,

$$\begin{aligned}(P_1,\dots,P_l)\in M &\Leftrightarrow \mathrm{NF}_B(P_1,\dots,P_l)=0\\&\Leftrightarrow P_1\cdot f^j_{\mathbf{t},1}+\cdots+P_l\cdot f^j_{\mathbf{t},l}=0 \text{ for each } (\mathbf{t},j)\in B.\end{aligned}$$

As a straightforward consequence of these remarks we have the following proposition which proves the existence of a submodule H of $(A^*)^l$ such that $\mathscr{P}(H)=M$.

Proposition 2.1. *Let $\mathscr{R}_B(M)$ be the submodule of $(A^*)^l$ generated by the l-tuples $(f^j_{\mathbf{t},1},\dots,f^j_{\mathbf{t},l})_{(\mathbf{t},j)\in B}$. Then $\mathscr{P}(\mathscr{R}_B(M))=M$, which implies the existence of at least one inverse system of M.*

In general, a module M has many non-isomorphic inverse systems. Let $\{H_\alpha(M)\}_{\alpha\in J}$ be the set of all inverse systems of the submodule M. It is easy to check that $H(M) := \sum_{\alpha\in J} H_\alpha(M)$ is also an inverse system of M. Let us prove that $H(M)=\mathscr{S}(M)$, where

$$\mathscr{S}(M) := \{(f_1,\dots,f_l)\in(A^*)^l \mid P_1\cdot f_1+\cdots+P_l\cdot f_l=0 \text{ for all } (P_1,\dots,P_l)\in M\}.$$

Evidently, $\mathscr{S}(M)$ is a submodule of $(A^*)^l$ containing $\mathscr{R}_B(M)$. We now show that $\mathscr{S}(M)$ is the largest inverse system of M.

Proposition 2.2. *Let M be a submodule of A^l. Then $\mathscr{S}(M)$ is an inverse system of M. Moreover, if H is an inverse system of M then $H\subseteq\mathscr{S}(M)$.*

Proof. We have to prove $\mathscr{P}(\mathscr{S}(M))=M$. The inclusion $M\subseteq\mathscr{P}(\mathscr{S}(M))$ is trivial. Conversely, suppose $(P_1,\dots,P_l)\in\mathscr{P}(\mathscr{S}(M))$. Then $\sum_{i=1}^l P_i\cdot f_i=0$ for all $(f_1,\dots,f_l)\in\mathscr{S}(M)$. In particular, $\sum_{i=1}^l P_i\cdot f^j_{\mathbf{t},i}=0$ for each $(\mathbf{t},j)\in B$, i.e., $(P_1,\dots,P_l)\in\mathscr{P}(\mathscr{R}_B(M))=M$.

Finally, let H be any inverse system of M. If $(f_1,\dots,f_l)\in H$ then $P_1\cdot f_1+\cdots+P_l\cdot f_l=0$ for every $(P_1,\dots,P_l)\in M=\mathscr{P}(H)$. Hence $(f_1,\dots,f_l)\in\mathscr{S}(M)$. □

In conclusion, let us prove that the summable families $(f^j_{\mathbf{t},i})_{(\mathbf{t},j)\in B}$, $1\le i\le l$, are a pseudo-basis of the $\mathbb{K}$-linear space $\mathscr{S}(M)$.

Proposition 2.3. *Let M be a submodule of A^l. Then $(f_1,\dots,f_l)\in\mathscr{S}(M)$ if and only if there exists a family $(b_{\mathbf{t},j})_{(\mathbf{t},j)\in B}$ of elements of $\mathbb{K}$ such that*

$$(f_1,\dots,f_l)=\sum_{(\mathbf{t},j)\in B} b_{\mathbf{t},j}(f^j_{\mathbf{t},1},\dots,f^j_{\mathbf{t},l}).$$

Proof. Let $(b_{\mathbf{t},j})_{(\mathbf{t},j)\in B}$ be an arbitrary family of elements of $\mathbb{K}$. Since $(b_{\mathbf{t},j}f^j_{\mathbf{t},i})_{(\mathbf{t},j)\in B}$, $1\le i\le l$, is a summable family of linear forms on A, then for each $(P_1,\dots,P_l)\in M$ and each $Q\in A$ we have

$$\left(P_1\cdot\sum_{(\mathbf{t},j)\in B} b_{\mathbf{t},j}f^j_{\mathbf{t},1}+\cdots+P_l\cdot\sum_{(\mathbf{t},j)\in B} b_{\mathbf{t},j}f^j_{\mathbf{t},l}\right)(Q)$$

$$=\sum_{(\mathbf{t},j)\in B} b_{\mathbf{t},j}\left(\sum_{i=1}^l f^j_{\mathbf{t},i}(QP_i)\right)=0;$$

i.e.,

$$\left(\sum_{(\mathbf{t},j)\in B} b_{\mathbf{t},j} f^j_{\mathbf{t},1},\ldots,\sum_{(\mathbf{t},j)\in B} b_{\mathbf{t},j} f^j_{\mathbf{t},l}\right)\in\mathscr{S}(M).$$

Conversely, suppose $(f_1,\ldots,f_l)\in\mathscr{S}(M)$. Since, for every $P\in A$,

$$e_i(P)-\sum_{(\mathbf{t},j)\in B} f^j_{\mathbf{t},i}(P)e_j(\mathbf{t})\in M,$$

we have

$$\begin{aligned} f_i(P) &= (0\cdot f_1+\cdots+P\cdot f_i+\cdots+0\cdot f_l)(1_A)=\left(\sum_{(\mathbf{t},j)\in B} f^j_{\mathbf{t},i}(P)\mathbf{t}\cdot f_j\right)(1_A)\\ &= \sum_{(\mathbf{t},j)\in B} f_j(\mathbf{t})f^j_{\mathbf{t},i}(P). \end{aligned}$$

Thus $f_i=\sum_{(\mathbf{t},j)\in B} f_j(\mathbf{t})f^j_{\mathbf{t},i}$. $\square$

Note that for many concrete problems the conditions given above allow an effective calculation of the linear forms $f^j_{\mathbf{t},i}$. For instance, for finitely generated modules M over a wide class of algebras (e.g. polynomial algebras, Weyl algebras, enveloping algebras of Lie algebras) there exists an effective procedure for computing $f^j_{\mathbf{t},i}(P)$ using Buchberger's algorithm (cf. [6, 7]). In these cases Proposition 2.3 turns out to be a handy computational tool.

3. Inverse systems of polynomial modules

This section is devoted to proving the existence of finitely generated inverse systems of polynomial modules. With regard to the polynomial ideals this fact was first stated by Macaulay in [4, p. 91]. We will denote by $\mathbb{K}[\mathbf{x}]$ the polynomial algebra in $x_1,\ldots,x_n$ and will make use of the notations

$$\mathbf{i}:=(i_1,\ldots,i_n)\in\mathbb{N}^n \quad\text{and}\quad \mathbf{x}^{\mathbf{i}}:=x_1^{i_1}\cdots x_n^{i_n}.$$

Let M be a submodule of $\mathbb{K}[\mathbf{x}]^l$. Fix a monomial order $\preceq$ on $\mathbb{K}[\mathbf{x}]^l$, that is a total order $\preceq$ on the set

$$\{e_i(\mathbf{x}^{\mathbf{p}})\mid \mathbf{p}\in\mathbb{N}^n, 1\le i\le l\}$$

such that

$$e_i(\mathbf{x}^{\mathbf{p}})\preceq e_j(\mathbf{x}^{\mathbf{q}}) \Rightarrow e_i(\mathbf{x}^{\mathbf{p}})\preceq e_i(\mathbf{x}^{\mathbf{p}+\mathbf{h}})\preceq e_j(\mathbf{x}^{\mathbf{q}+\mathbf{h}}).$$

Under these assumptions we set

$$B := \{(\mathbf{p},j)\in\mathbb{N}^n\times\{1,\dots,l\}\mid e_j(\mathbf{x}^{\mathbf{p}})\notin \mathrm{LT}(M)\}$$

where $\mathrm{LT}(M)$ is the $\mathbb{K}[\mathbf{x}]$-module generated by the leading terms of the elements of M with respect to the order $\preceq$. It is well known that the set $\{e_j(\mathbf{x}^{\mathbf{p}})+M\mid(\mathbf{p},j)\in B\}$ forms a basis for the vector $\mathbb{K}$-space $\mathbb{K}[\mathbf{x}]^l/M$. Hence, by (1), it follows

$$(P_1,\dots,P_l)\overset{(\mathrm{mod}\ M)}{\equiv}\left(\sum_{\mathbf{p}\in B_1}\sum_{i=1}^{l} f^1_{\mathbf{p},i}(P_i)\mathbf{x}^{\mathbf{p}},\dots,\sum_{\mathbf{p}\in B_l}\sum_{i=1}^{l} f^l_{\mathbf{p},i}(P_i)\mathbf{x}^{\mathbf{p}}\right) \tag{2}$$

where $B_j=\{\mathbf{p}\in\mathbb{N}^n\mid(\mathbf{p},j)\in B\}$ and we have written $f^j_{\mathbf{p},i}(P_i)$ instead of $f^j_{\mathbf{x}^{\mathbf{p}},i}(P_i)$. Our proof of the existence of finitely generated inverse systems of M relies on a characterization of the set B as a disjoint union of "nice" subsets. Since the module $\mathrm{LT}(M)$ may be written as

$$\mathrm{LT}(M)=\bigoplus_{j=1}^{l} I_j$$

where I_j is the monomial ideal generated by the set $\{\mathbf{x}^{\mathbf{p}}\mid e_j(\mathbf{x}^{\mathbf{p}})\in\mathrm{LT}(M)\}$, we have

$$(\mathbf{p},j)\in B\Leftrightarrow \mathbf{p}\in B_j\Leftrightarrow \mathbf{x}^{\mathbf{p}}\notin I_j.$$

It follows that we can describe the set B through the complement of the monomial ideals I_j, $1\le j\le l$. The description of the complement of a monomial ideal we will give in this section is due to Janet (see [1,2]). The following notion is central to Janet's theory.

Let X be a finite set of monomials of $\mathbb{K}[\mathbf{x}]$; we shall say that the indeterminate x_k is *multiplicative* for $x_1^{p_1}\cdots x_{k-1}^{p_{k-1}}x_k^{p_k}\cdots x_n^{p_n}\in X$ if $p_k\ge q_k$, for each $x_1^{p_1}\cdots x_{k-1}^{p_{k-1}}x_k^{q_k}\cdots x_n^{q_n}\in X$. Moreover, for each $1\le k\le n$ we define

$$\begin{aligned}C^{(k)} := \{x_1^{p_1}\cdots x_k^{p_k}\mid p_k<q_k \text{ for some } & x_1^{p_1}\cdots x_{k-1}^{p_{k-1}}x_k^{q_k}\cdots x_n^{q_n}\in X\\ \text{and } & x_1^{p_1}\cdots x_{k-1}^{p_{k-1}}x_k^{p_k}\cdots x_n^{q_n}\notin X\}.\end{aligned}$$

The indeterminates $x_{k+1},\dots,x_n$ as well as the multiplicative indeterminates of $x_1^{p_1}\cdots x_{k-1}^{p_{k-1}}$, as elements of $\{x_1^{p_1}\cdots x_{k-1}^{p_{k-1}}\}\cup X$, will be called *multiplicative indeterminates* of the monomial $x_1^{p_1}\cdots x_k^{p_k}\in C^{(k)}$.

Note that, given the finite set X, the sets $C^{(k)}$ are uniquely determined and pairwise disjoint; thus, we can set

$$\mu(\mathbf{p},X)=\{k\mid x_k \text{ multiplicative indeterminate for } \mathbf{x}^{\mathbf{p}}\in X\cup C^{(1)}\cup\cdots\cup C^{(n)}\}.$$

By the definition of a multiplicative indeterminate of a monomial belonging to $C^{(k)}$ it follows that

$$\mu(\mathbf{p},X)=\emptyset\Rightarrow \mathbf{x}^{\mathbf{p}}\in C^{(n)},$$

while

$$\mathbf{x}^{\mathbf{p}} \in C^{(k)}, \quad 1 \le k \le n-1, \Longrightarrow \mu(\mathbf{p}, X) \neq \emptyset.$$

Example. Suppose $X = \{x^4y^3, x^2y^5\}$. In the following tables we list the multiplicative indeterminates for the elements of X:

Monomials	Multiplicative indeterminates
x^4y^3	$x \quad y$
x^2y^5	$\cdot \quad y$

and those for the elements of $C^{(1)}$ and $C^{(2)}$:

Sets	Monomials	Multiplicative indeterminates
$C^{(1)}$	$x^3, \quad x, \quad 1$	$\cdot \quad y$
$C^{(2)}$	$x^4y^2, \quad x^4y, \quad x^4$	$x \quad \cdot$
	$x^2y^4, \quad x^2y^3, \quad x^2y^2, \quad x^2y, \quad x^2$	$\cdot \quad \cdot$

Let I be a monomial ideal of $\mathbb{K}[\mathbf{x}]$. We shall say that a finite set X of generators of I is *complete* if it is possible to obtain any monomial of I multiplying an element of $\mathbf{x}^{\mathbf{p}} \in X$ by a power product involving only multiplicative indeterminates of $\mathbf{x}^{\mathbf{p}}$. Given a finite set of generators of a monomial ideal I, there exists an algorithm for computing a complete set of generators of I (see [1, p. 80]) as well as the sets $C^{(k)}$.

Example. The set $X = \{x^4y^3, x^2y^5\}$ is not a complete set of generators of the ideal $I = \langle X \rangle$: it is not possible to obtain the monomial $x^3y^5 \in I$ as a product of an element of X by a power product involving only its multiplicative indeterminates. A complete set of generators of I is $\{x^4y^3, x^3y^5, x^2y^5\}$. In this case we have:

Monomials	Multiplicative indeterminates
x^4y^3	$x \quad y$
x^3y^5	$\cdot \quad y$
x^2y^5	$\cdot \quad y$

Sets	Monomials	Multiplicative indeterminates	
$C^{(1)}$	$x,\quad 1$	$\cdot$	y
$C^{(2)}$	$x^4y^2,\quad x^4y,\quad x^4$	x	$\cdot$
	$x^3y^4,\quad x^3y^3,\quad x^3y^2,\quad x^3y,\quad x^3$	$\cdot$	$\cdot$
	$x^2y^4,\quad x^2y^3,\quad x^2y^2,\quad x^2y,\quad x^2$	$\cdot$	$\cdot$

Finally, for each $\mathbf{x}^{\mathbf{h}} \in C^{(1)} \cup \cdots \cup C^{(n)}$, we set

$$C(\mathbf{h}) = \begin{cases} \left\{ \mathbf{x}^{\mathbf{h}+\mathbf{k}} \mid \mathbf{x}^k = \prod_{s\in\mu(\mathbf{h},X)} x_s^{k_s},\ k_s \in \mathbb{N} \right\}, & \text{if } \mu(\mathbf{h},X) \neq \emptyset \\ \{\mathbf{x}^{\mathbf{h}}\}, & \text{otherwise.} \end{cases}$$

Proposition 3.1. *If X is a complete set of generators of a monomial ideal I, then the complement of I can be written as the finite disjoint union of the sets $C(\mathbf{h})$.*

Proof. See [1, p. 90]. □

Example. The complement of the ideal $I = \langle x^4y^3, x^3y^5, x^2y^5 \rangle$ is

$$C(0,0) \cup C(1,0) \cup C(4,2) \cup C(4,1) \cup C(4,0) \cup$$
$$\{x^3y^4, x^3y^3, x^3y^2, x^3y, x^3, x^2y^4, x^2y^3, x^2y^2, x^2y, x^2\}.$$

Turning to the description of the set B_j, $1 \leq j \leq l$, let us denote by X_j the complete set of generators of I_j, computed by Janet's algorithm, starting from the set $\{\mathbf{x}^{\mathbf{p}} \mid e_j(\mathbf{x}^{\mathbf{p}}) \in \mathrm{LT}(M)\}$. Thus, we set

$$C_j^{(k)} = \{x_1^{p_1} \cdots x_k^{p_k} \mid p_k < q_k \text{ for some } x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} x_k^{q_k} \cdots x_n^{q_n} \in X_j$$
$$\text{and } x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} x_k^{p_k} \cdots x_n^{q_n} \notin X_j\}$$

and

$$C_j(\mathbf{h}) = \begin{cases} \left\{ \mathbf{x}^{\mathbf{h}+\mathbf{k}} \;\middle|\; \mathbf{x}^{\mathbf{k}} = \prod_{s\in\mu(\mathbf{h},X_j)} x_s^{k_s},\ k_s \in \mathbb{N} \right\}, & \text{if } \mu(\mathbf{h},X_j) \neq \emptyset, \\ \{\mathbf{x}^{\mathbf{h}}\}, & \text{otherwise.} \end{cases}$$

Under these assumptions we consider the sets

$$B_j^{(0)} = \{\mathbf{h} \in \mathbb{N}^n \mid \mathbf{x}^{\mathbf{h}} \in C_j^{(n)},\ \mu(\mathbf{h},X_j) = \emptyset\},$$
$$B_j^{(1)} = \{\mathbf{h} \in \mathbb{N}^n \mid \mathbf{x}^{\mathbf{h}} \in C_j^{(1)} \cup \cdots \cup C_j^{(n)},\ \mu(\mathbf{h},X_j) \neq \emptyset\},$$
$$B_j^{(\mathbf{h})} = \{\mathbf{k} \in \mathbb{N}^n \mid \mathbf{x}^{\mathbf{h}+\mathbf{k}} \in C_j(\mathbf{h}),\ \mu(\mathbf{h},X_j) \neq \emptyset\}.$$

Then, as a consequence of Proposition. 3.1, each component of (2) can be written in the form

$$\sum_{\mathbf{p}\in B_j}\sum_{i=1}^{l} f^j_{\mathbf{p},i}(P_i)\mathbf{x}^{\mathbf{p}} = \sum_{\mathbf{p}\in B_j^{(0)}}\sum_{i=1}^{l} f^j_{\mathbf{p},i}(P_i)\mathbf{x}^{\mathbf{p}} + \sum_{\mathbf{h}\in B_j^{(1)}}\sum_{\mathbf{k}\in B_j^{(\mathbf{h})}}\sum_{i=1}^{l} f^j_{\mathbf{h}+\mathbf{k},i}(P_i)\mathbf{x}^{\mathbf{h}+\mathbf{k}}$$

$$= \sum_{\mathbf{p}\in B_j^{(0)}}\sum_{i=1}^{l} f^j_{\mathbf{p},i}(P_i)\mathbf{x}^{\mathbf{p}} + \sum_{\mathbf{h}\in B_j^{(1)}}\sum_{i=1}^{l} \Phi^j_{\mathbf{h},i}(P_i)\mathbf{x}^{\mathbf{h}}, \tag{3}$$

where

$$\Phi^j_{\mathbf{h},i}(P_i) = \sum_{\mathbf{k}\in B_j^{(\mathbf{h})}} f^j_{\mathbf{h}+\mathbf{k},i}(P_i)\mathbf{x}^{\mathbf{k}}$$

is an element of the polynomial ring $\Upsilon_{\mathbf{h}}$ in the indeterminates $x_k, k\in\mu(\mathbf{h},X_j)$, with coefficients in $\mathbb{K}$.

Example. Let M be the submodule of $\mathbb{K}[x,y]^2$ generated by the set $\{(x^4y^3,y),(x^2y^5,x)\}$. Let us denote by $\succ$ the lexicographic ordering on $\mathbb{K}[x,y]$, $x\succ y$, and by $\succ_2$ the monomial order on $\mathbb{K}[x,y]^2$ defined by

$$e_i(x^hy^k)\succ_2 e_j(x^ry^s) \Leftrightarrow x^hy^k\succ x^ry^k \text{ or, if } x^hy^k=x^ry^s, i<j.$$

Then

$$\mathrm{Lt}(M)=\langle e_1(x^4y^3), e_1(x^2y^5), e_2(x^3)\rangle = I_1\oplus I_2$$

with $I_1=\langle x^4y^3,x^2y^5\rangle$ and $I_2=\langle x^3\rangle$. Hence,

$$B_1^{(0)}=\{(3,4),(3,3),(3,2),(3,1),(3,0),(2,4),(2,3),(2,2),(2,1),(2,0)\},$$

$$B_1^{(1)}=\{(0,0),(1,0),(4,2),(4,1),(4,0)\},$$

$$B_1^{(0,0)}=B_1^{(1,0)}=\{(0,k_2)\mid k_2\in\mathbb{N}\},$$

$$B_1^{(4,2)}=B_1^{(4,1)}=B_1^{(4,0)}=\{(k_1,0)\mid k_1\in\mathbb{N}\}.$$

Thus,

$$\Phi^1_{(0,0),i}(P_i)=\sum_{k\in\mathbb{N}} f^1_{(0,k),i}(P_i)y^k, \qquad \Phi^1_{(1,0),i}(P_i)=\sum_{k\in\mathbb{N}} f^1_{(1,k),i}(P_i)y^k,$$

$$\Phi^1_{(4,2),i}(P_i)=\sum_{k\in\mathbb{N}} f^1_{(4+k,2),i}(P_i)x^k, \qquad \Phi^1_{(4,1),i}(P_i)=\sum_{k\in\mathbb{N}} f^1_{(4+k,1),i}(P_i)x^k,$$

$$\Phi^1_{(4,0),i}(P_i)=\sum_{k\in\mathbb{N}} f^1_{(4+k,0),i}(P_i)x^k.$$

In a similar way,

$$B_2^{(0)} = \emptyset,$$

$$B_2^{(1)} = \{(0,0),(1,0),(2,0)\},$$

$$B_2^{(0,0)} = B_2^{(1,0)} = B_2^{(2,0)} = \{(0,k_2) \mid k_2 \in \mathbb{N}\}$$

and

$$\Phi^2_{(0,0),i}(P_i) = \sum_{k\in\mathbb{N}} f^2_{(0,k),i}(P_i)y^k, \quad \Phi^2_{(1,0),i}(P_i) = \sum_{k\in\mathbb{N}} f^2_{(1,k),i}(P_i)y^k,$$

$$\Phi^2_{(2,0),i}(P_i) = \sum_{k\in\mathbb{N}} f^2_{(2,k),i}(P_i)y^k.$$

Turning to the general case, from formula (3) we deduce that there exists a finite set of $\Upsilon_{\mathbf{h}}$-linear maps $\Phi^j_{\mathbf{h},i}, \mathbf{h} \in B_j^{(1)}, 1 \le j \le l, 1 \le i \le l$, from $\mathbb{K}[\mathbf{x}]$ into $\Upsilon_{\mathbf{h}}$ defined by the correspondences $P \mapsto \Phi^j_{\mathbf{h},i}(P)$. The existence of such a finite set of linear forms and the fact that in $\mathbb{K}[\mathbf{x}]^*$, as a $\mathbb{K}[\mathbf{x}]$-module, there exist linearly independent elements, imply the existence of finitely generated inverse systems of M. In order to make this paper self-contained we give an example of a $\mathbb{K}[\mathbf{x}]$-linearly independent element of $\mathbb{K}[\mathbf{x}]^*$:

$$\alpha : \mathbb{K}[\mathbf{x}] \to \mathbb{K},$$

$$\mathbf{x}^{\mathbf{j}} \mapsto \begin{cases} 1, & \text{if } j_1 = \cdots = j_n = p(p+3)/2, \quad p = 0,1,2\ldots \\ 0, & \text{otherwise.} \end{cases}$$

To prove that α is $\mathbb{K}[\mathbf{x}]$-linearly independent, consider any polynomial $P = \sum_{\mathbf{i}} a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}, i_1 \le d_1, \ldots, i_n \le d_n$ and $a_{d_1,\ldots,d_n} \neq 0$. Let $d = \max\{d_1,\ldots,d_n\}$. If $k = d(d+3)/2$, a straightforward calculation shows that

$$(P\cdot\alpha)(x_1^{k-d_1}\cdots x_n^{k-d_n}) = a_{d_1,\ldots,d_n} \neq 0.$$

We now have all the ingredients for proving the main result.

Proposition 3.2. *Let M be a submodule of $\mathbb{K}[\mathbf{x}]^l$ and let H the submodule of $(\mathbb{K}[\mathbf{x}]^*)^l$ generated by the l-tuples of linear forms $(f^j_{\mathbf{p},1},\ldots,f^j_{\mathbf{p},l})$, $\mathbf{p} \in B_j^{(0)}, 1 \le j \le l$, and by $(\alpha_{\mathbf{h}} \circ \Phi^j_{\mathbf{h},1},\ldots,\alpha_{\mathbf{h}} \circ \Phi^j_{\mathbf{h},l})$, $\mathbf{h} \in B_j^{(1)}, 1 \le j \le l$, where $\alpha_{\mathbf{h}}$ is a $\Upsilon_{\mathbf{h}}$-linearly independent element of $\Upsilon^*_{\mathbf{h}}$. Then H is an inverse system of M.*

Proof. Because of formula (2), if $(P_1,\ldots,P_l) \in M$, then

$$P_1 \cdot f^j_{\mathbf{p},1} + \cdots + P_l \cdot f^j_{\mathbf{p},l} = 0, \quad \mathbf{p} \in B_j, \ 1 \le j \le l.$$

Thus,

$$P_1\cdot f^j_{\mathbf{p},1} + \cdots + P_l\cdot f^j_{\mathbf{p},l} = 0, \quad \mathbf{p} \in B_j^{(0)}, \ 1 \le j \le l.$$

Moreover, by (3), for every $Q \in \mathbb{K}[\mathbf{x}]$, we have

$$\begin{aligned}&(P_1 \cdot (\alpha_{\mathbf{h}} \circ \Phi^j_{\mathbf{h},1}) + \cdots + P_l \cdot (\alpha_{\mathbf{h}} \circ \Phi^j_{\mathbf{h},l}))(Q)\\ &\quad = \alpha_{\mathbf{h}}(\Phi^j_{\mathbf{h},1}(P_1 Q) + \cdots + \Phi^j_{\mathbf{h},l}(P_l Q)) = 0, \quad \mathbf{h} \in B^{(1)}_j, \ 1 \le j \le l.\end{aligned}$$

Hence,

$$P_1 \cdot (\alpha_{\mathbf{h}} \circ \Phi^j_{\mathbf{h},1}) + \cdots + P_l \cdot (\alpha_{\mathbf{h}} \circ \Phi^j_{\mathbf{h},l}) = 0, \quad \mathbf{h} \in B^{(1)}_j, \ 1 \le j \le l.$$

It follows that $(P_1, \ldots, P_l) \in \mathscr{P}(H)$. Conversely, if $(P_1, \ldots, P_l) \in \mathscr{P}(H)$ then

$$\begin{aligned}&\left(\sum_{i=1}^{l} \Phi^j_{\mathbf{h},i}(P_i)\right) \cdot \alpha_{\mathbf{h}}(1) = \alpha_{\mathbf{h}}\left(\sum_{i=1}^{l} \Phi^j_{\mathbf{h},i}(P_i)\right)\\ &\quad = P_1 \cdot (\alpha_{\mathbf{h}} \circ \Phi^j_{\mathbf{h},1}) + \cdots + P_l \cdot (\alpha_{\mathbf{h}} \circ \Phi^j_{\mathbf{h},l}) = 0, \quad \mathbf{h} \in B^{(1)}_j, \ 1 \le j \le l.\end{aligned}$$

Thus, keeping in mind that the maps $\Phi^j_{\mathbf{h},i}$ are $\Upsilon_{\mathbf{h}}$-linear, we have

$$\left(\sum_{i=1}^{l} \Phi^j_{\mathbf{h},i}(P_i)\right) \cdot \alpha_{\mathbf{h}}(\mathbf{x}^{\mathbf{r}}) = \left(\sum_{i=1}^{l} \Phi^j_{\mathbf{h},i}(\mathbf{x}^{\mathbf{r}} P_i)\right) \cdot \alpha_{\mathbf{h}}(1) = 0, \quad \mathbf{h} \in B^{(1)}_j, \ 1 \le j \le l,$$

for all $\mathbf{x}^{\mathbf{r}} \in \Upsilon_{\mathbf{h}}$. Since $\alpha_{\mathbf{h}}$ is $\Upsilon_{\mathbf{h}}$-linearly independent, the polynomials

$$\sum_{i=1}^{l} \Phi^j_{\mathbf{h},i}(P_i) = \sum_{i=1}^{l} \sum_{\mathbf{k} \in B^{(\mathbf{h})}_j} f^j_{\mathbf{h}+\mathbf{k},i}(P_i)\mathbf{x}^{\mathbf{k}}, \quad \mathbf{h} \in B^{(1)}_j, \ 1 \le j \le l,$$

coincide with the zero polynomial. Hence, for each $\mathbf{x}^{\mathbf{q}} \in \mathbb{K}[\mathbf{x}]$,

$$\begin{aligned}&(P_1 \cdot f^j_{\mathbf{h}+\mathbf{k},1} + \cdots + P_l \cdot f^j_{\mathbf{h}+\mathbf{k},l})(\mathbf{x}^{\mathbf{q}}) = \mathbf{x}^{\mathbf{q}} \cdot \sum_{i=1}^{l} f^j_{\mathbf{h}+\mathbf{k},i}(P_i) = 0,\\ &\quad \mathbf{h} \in B^{(1)}_j, \ \mathbf{k} \in B^{(\mathbf{h})}_j, 1 \le j \le l.\end{aligned}$$

It follows that

$$P_1 \cdot f^j_{\mathbf{h}+\mathbf{k},1} + \cdots + P_l \cdot f^j_{\mathbf{h}+\mathbf{k},l} = 0, \quad \mathbf{h} \in B^{(1)}_j, \ \mathbf{k} \in B^{(\mathbf{h})}_j, \ 1 \le j \le l.$$

On the other hand,

$$P_1 \cdot f^j_{\mathbf{p},1} + \cdots + P_l \cdot f^j_{\mathbf{p},l} = 0, \quad \mathbf{p} \in B^{(0)}_j, \ 1 \le j \le l,$$

so, by (2), we can conclude that $(P_1, \ldots, P_l) \in M$. □

4. Inverse systems and polynomial ideals

If we assume $l = 1$, every submodule M of $\mathbb{K}[\mathbf{x}]$ is an ideal of $\mathbb{K}[\mathbf{x}]$ and an inverse system of M is any submodule H of $\mathbb{K}[\mathbf{x}]^*$ such that $\mathrm{Ann}(H) = M$. Thus we can state the following proposition.

Proposition 4.1 (Macaulay [4]). *For every ideal I of $\mathbb{K}[\mathbf{x}]$ there exists a finitely generated inverse system.*

An equivalent proposition can be found in [8, pp. 32, 52]

Proposition 4.2 (Oberst [8]). *The $\mathbb{K}[\mathbf{x}]$-module $\mathbb{K}[\mathbf{x}]^*$ is a large injective cogenerator.*

Let $\mathbb{K}[\mathbf{x}]_0^*$ be the submodule of $\mathbb{K}[\mathbf{x}]^*$ of the linear forms f such that $f(\mathbf{x}^{\mathbf{i}})=0$ for all but a finite number of monomials $\mathbf{x}^{\mathbf{i}}$ of $\mathbb{K}[\mathbf{x}]$. The ideals I of $\mathbb{K}[\mathbf{x}]$ which are contained in the ideal $(x_1,\dots,x_n)$ are characterized by $\mathscr{S}(I)\cap\mathbb{K}[\mathbf{x}]_0^*\neq\{0\}$.

Proposition 4.3. *$I\subseteq(x_1,\dots,x_n)$ if and only if $\mathscr{S}(I)\cap\mathbb{K}[\mathbf{x}]_0^*\neq\{0\}$.*

Proof. If $I\subseteq(x_1,\dots,x_n)$ then the linear form

$$\begin{aligned}&\mathbb{K}[\mathbf{x}]\to\mathbb{K};\\ &\mathbf{x}^{\mathbf{i}}\mapsto\begin{cases}1, & \text{if } \mathbf{i}=0,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

is an element of $\mathscr{S}(I)\cap\mathbb{K}[\mathbf{x}]_0^*$. Conversely, suppose that $\mathscr{S}(I)\cap\mathbb{K}[\mathbf{x}]_0^*\neq\{0\}$. If f is any nonzero element of $\mathscr{S}(I)\cap\mathbb{K}[\mathbf{x}]_0^*$ then $P\cdot f=0$ for every $P\in I$. Since $f\neq 0$, there exists $\mathbf{p}\in\mathbb{N}^n$ such that $f(\mathbf{x}^{\mathbf{p}})\neq 0$ and $f(\mathbf{x}^{\mathbf{q}})=0$ for every $\mathbf{q}\in\mathbb{N}^n$ with $q_1+\cdots+q_n>p_1+\cdots+p_n$. Therefore, if $\sum_{\mathbf{i}}a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}\in I$ we have

$$\left(\mathbf{x}^{\mathbf{p}}\sum_{\mathbf{i}}a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}\cdot f\right)(1)=a_0 f(\mathbf{x}^{\mathbf{p}})=0.$$

This implies that no polynomial $P\in I$ has a constant term, that is $I\subseteq(x_1,\dots,x_n)$. □

The last result can be sharpened as follows (see [4, p. 75]).

Proposition 4.4. $\operatorname{Ann}(\mathscr{S}(I)\cap\mathbb{K}[\mathbf{x}]_0^*)=I\mathbb{K}[[\mathbf{x}]]\cap\mathbb{K}[\mathbf{x}]$.

Proof. Put $O=(x_1,\dots,x_n)$ and note that $\mathbb{K}[\mathbf{x}]_0^*=\sum_{t\geq 1}\mathscr{S}(O^t)$. Then we have

$$\begin{aligned}\operatorname{Ann}(\mathscr{S}(I)\cap\mathbb{K}[\mathbf{x}]_0^*)&=\operatorname{Ann}\left(\mathscr{S}(I)\cap\left(\sum_{t\geq 1}\mathscr{S}(O^t)\right)\right)\\ &=\operatorname{Ann}\left(\sum_{t\geq 1}(\mathscr{S}(I)\cap\mathscr{S}(O^t))\right)\\ &=\bigcap_{t\geq 1}\operatorname{Ann}(\mathscr{S}(I)\cap\mathscr{S}(O^t))=\bigcap_{t\geq 1}(I+O^t).\end{aligned}$$

Therefore $\mathrm{Ann}(\mathscr{S}(I) \cap \mathbb{K}[\mathbf{x}]_0^*)$ is the closure of I in $\mathbb{K}[\mathbf{x}]$, in the O-adic topology. On the other hand, $I\mathbb{K}[[\mathbf{x}]]$ is the closure of I in $\mathbb{K}[[\mathbf{x}]] = \widehat{\mathbb{K}[\mathbf{x}]}$; thus $\mathrm{Ann}(\mathscr{S}(I) \cap \mathbb{K}[\mathbf{x}]_0^*) = I\mathbb{K}[[\mathbf{x}]] \cap \mathbb{K}[\mathbf{x}]$. $\square$

Corollary 1. *If $I = \bigcap_i Q_i$ is a minimal primary decomposition of I where $Q_i \subset (x_1, \ldots, x_n)$ for every i, then one of the inverse systems of I is contained in $\mathbb{K}[\mathbf{x}]_0^*$.*

Proof. From the hypothesis we see that $I = I\mathbb{K}[[\mathbf{x}]] \cap \mathbb{K}[\mathbf{x}]$, so the corollary follows from Proposition. 4.4. $\square$

Corollary 2 (Macaulay [4]). *If I is a homogeneous ideal of $\mathbb{K}[\mathbf{x}]$ then one of its inverse systems is contained in $\mathbb{K}[\mathbf{x}]_0^*$.*

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