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Some classes of linearizable polynomial maps

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Abstract

It is known that for a polynomial automorphism F with strongly nilpotent Jacobian matrix the automorphism sF is linearizable for some scalar s. This paper gives a slight generalization of this property and describes a class of linearizable polynomial maps of finite order which includes triangular automorphisms. Some conditions of proven theorems enable us to give a counterexample to the Linearization problem and the Fixed point problem in finite characteristics. © 1998 Elsevier Science B.V.

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1. Introduction

Let $\Bbbk$ be a field, not necessarily algebraically closed. By $\mathbb{A}^n$ we denote the affine space of dimension n over $\Bbbk$. The problem which we are studying is the following:

How can an algebraic group G act algebraically on affine space $\mathbb{A}^n$?

As an example of such an action one can take the action of $GL_n(\Bbbk)$ on $\Bbbk^n \cong \mathbb{A}^n$ in the natural way. This example can be generalized as follows. Denote by $\mathcal{G}_n$ the group of algebraic automorphisms of affine space $\mathbb{A}^n$. An element F of this group can be given by an n-tuple $(f_1,\dots,f_n)$ of polynomials $f_i \in \Bbbk[x_1,\dots,x_n]$ with the condition that they generate the same polynomial ring, i.e. $\Bbbk[f_1,\dots,f_n] = \Bbbk[x_1,\dots,x_n]$. In terms of algebraic maps it means that F gives an invertible algebraic morphism of $\mathbb{A}^n$ to itself. We call this group the affine Cremona group.

This group has a structure of an infinite-dimensional algebraic group as it was shown by Shafarevich [11] and the action of subgroups of this group can be viewed in the

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context of our question. Any element of $\mathcal{G}_n$ generates a subgroup and we can ask how this element acts on affine space.

The group $\mathcal{G}$ contains two important subgroups defined in the following way:

$$Aff_n(\mathbb{k}) = \{F = (f_1,\dots,f_n) \in \mathcal{G}_n | f_i \text{ is linear for all } 1 \le i \le n\},$$
$$\mathcal{J}_n = \{F = (f_1,\dots,f_n) \in \mathcal{G}_n | f_i \in \mathbb{k}[x_1,\dots,x_i]\}.$$

The first one is the group of affine transformations and the second is called the Jonquière group and contains the triangular transformations. If $F \in \mathcal{J}_n$ and $F = (f_1,\dots,f_n)$ then

$$f_i(x_1,\dots,x_i) = s_i x_i + g_i(x_1,\dots,x_{i-1})$$

and $s_i \in \mathbb{k}^*$.

Definition 1.1. We say that a polynomial map is linearizable if it is conjugated to some element of $Aff_n(\mathbb{k})$ and it is triangularizable if it is an element of $\mathcal{J}_n$ again up to conjugation by an element of $\mathcal{G}_n$. We give the same names to the action of a subgroup of $\mathcal{G}_n$ if any element of this subgroup is conjugated to an element of $Aff_n(\mathbb{k})$ or $\mathcal{J}_n$, respectively.

The following results were proved for actions on affine space over the field of complex numbers but the questions for other ground fields are also valid and interesting.

For $n = 2$ the Cremona group is generated by $Aff_2(\mathbb{k})$ and $\mathcal{J}_2$ and it follows from a result of van der Kulk [9] that any action on affine space is either linearizable or triangularizable. In higher dimensions the structure of $\mathcal{G}_n$ is not so clear. The Cremona group for $n \ge 3$ probably is not generated by Jonquière group and affine subgroup as an abstract group but it is known that it is generated by these subgroups as an algebraic group. To find an example of an element which is not in the subgroup generated by $Aff_n(\mathbb{k})$ and $\mathcal{J}_n$ but in its closure is an open problem and one candidate was conjectured by Nagata in [10].

There exists also an example of an action of algebraic group on affine space of dimension 3 which is neither triangularizable nor linearizable. Such an action was first constructed by Bass [2].

One can consider a smaller class of subgroups acting on $\mathbb{A}^n$. Studying reductive subgroups of $\mathcal{G}_n$ we can expect more properties for this action. There was stated:

Linearization problem. *Is any action of a reductive algebraic group G on affine space linearizable?*

This problem was conjectured to be true by Kambayashi [7] but recently several counterexamples were found by Schwarz [12]. If a group G acts on affine space one can consider a set $(\mathbb{A}^n)^G$ of points fixed under this action. If action of the group is linearizable then it must be isomorphic to some linear subspace of $\mathbb{A}^n$. This leads to another problem.

Fixed point problem. *Given an action of a reductive algebraic group G on $\mathbb{A}^n$. Is the fixed point set $(\mathbb{A}^n)^G$ isomorphic to $\mathbb{A}^d$ for some d or at least not empty?*

In the case of non-reductive groups acting on affine space the action has no fixed points but if the group is reductive there is no general answer.

Another interesting problem which is closely related to these two is:

Cancelation problem. *Given a variety Y and an isomorphism $Y \times \mathbb{A}^m \cong \mathbb{A}^n$ for some m and n. Does it imply that Y is isomorphic to $\mathbb{A}^{n-m}$?*

A positive solution of the Linearization problem or even the Fixed point problem would imply the positive answer to the Cancelation problem. Consider the action of $\mathbb{Z}/2\mathbb{Z}$ on $Y \times \mathbb{A}^m$ given by $(x, y) \mapsto (x, -y)$. Then Y is the fixed point set, hence isomorphic to some $\mathbb{A}^d$. The same trick can be done with any other reductive group acting on affine space, but this particular case is interesting in the context of statements proved below.

We can view the contents of this paper from another angle. There is the well-known and still open Jacobian conjecture. Let us denote JF the Jacobian matrix of the transformation $F = (f_1, \dots, f_n)$, i.e.,

$$JF = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

Jacobian conjecture. *Let $\mathbf{k}$ be a field of characteristic zero. If $F : \mathbb{A}^n \to \mathbb{A}^n$, $F = (f_1, \dots, f_n)$ is a polynomial mapping with $JF \in \mathbf{k}^*$, then F is invertible, i.e., $F \in \mathcal{G}_n$.*

It is proved by Bass, et al. [3] that it is enough to show this for maps of the form $F = X + H$, where $X = (x_1, \dots, x_n)$, $H = (h_1, \dots, h_n)$ and h_i are cubic homogeneous polynomials such that JH is nilpotent.

If we replace the condition that JH is nilpotent by the requirement that it is strongly nilpotent then this statement follows from the work of van den Essen and Hubbers [5]. This work will be discussed in Section 3.

Studying this case further leads to the conjecture of Meisters which says:

Conjecture (Meisters). *Let $F = X + H$ be a cubic homogeneous polynomial map with* $\det JF = 1$. *Then for almost all $s \in \mathbf{k}^*$ (except for a finite set of roots of unity) the polynomial map sF is linearizable.*

This conjecture turns out to be false as it was shown by van den Essen [4], but it becomes true after replacing the requirement of nilpotence of JF by strong nilpotence [5]. This paper gives a slight generalization of the last property and also gives a class of linearizable polynomial maps of finite order. Some conditions of proven theorems enable us to give a counterexample to the Linearization problem and the Fixed point problem in finite characteristics. This example turns out to be much easier than the one which was given recently by Asanuma [1].

A more detailed overview can be found in the paper by Kraft [8].

2. Linearization of triangular maps

From now on we denote by $\Bbbk$ a field and $\mathbb{A}^n$ affine space over $\Bbbk$ of dimension n. We consider the subgroup $\mathcal{J}_n$ in $\operatorname{Aut}\mathbb{A}^n$ containing (lower) triangular transformations

$$F = [s_1x_1 + a_1, s_2x_2 + a_2(x_1), \dots, s_nx_n + a_n(x_1, \dots, x_{n-1})],$$

where all $s_i \neq 0$ and group law is given by $(F \cdot G)(x_1, \dots, x_n) = G(F(x_1, \dots, x_n))$.

If $A \in GL_n(\Bbbk)$ is a matrix then it can be considered as an element of $\operatorname{Aut}\mathbb{A}^n$,

$$A = [a_{11}x_1 + \cdots + a_{1n}x_n, \cdots, a_{n1}x_1 + \cdots + a_{nn}x_n],$$

in particular, if $A \in GL_n(\Bbbk)$ is a (lower) triangular matrix then

$$A = [a_{11}x_1, a_{22}x_2 + a_{21}x_1, \dots, a_{nn}x_n + \cdots + a_{n1}x_1]$$

and it acts on $\mathcal{J}_n$ by multiplication from the left. We denote the subgroup of such matrices $T_n(\Bbbk)$. In the same way we can consider affine transformations as elements of $\operatorname{Aut}\mathbb{A}^n$.

The main result of this section is given by the following theorem.

Theorem 2.1. *Let $\Bbbk$ be a field and $F : \mathbb{A}^n \to \mathbb{A}^n$ a triangular polynomial map of the form*

$$F = [s_1x_1 + a_1, s_2x_2 + a_2(x_1), \dots, s_nx_n + a_n(x_1, \dots, x_{n-1})].$$

Then, if $F^m = I$ for some $m \in \mathbb{N}$ and characteristic of $\Bbbk$ does not divide m, there exists a triangular automorphism $\varphi \in \operatorname{Aut}\mathbb{A}^n$ such that $\varphi^{-1}F\varphi \in \mathit{Aff}_n(\Bbbk)$.

Before proving this we need some other results which in some sense generalize Theorem 3.2 from [5].

Theorem 2.2. *Let $\Bbbk$ be a field and $F : \mathbb{A}^n \to \mathbb{A}^n$ a triangular polynomial map of the form*

$$F = [x_1 + a_1, x_2 + a_2(x_1), \dots, x_n + a_n(x_1, \dots, x_{n-1})].$$

For almost all $A \in T_n(\Bbbk)$ there exists a triangular automorphism $\varphi \in \operatorname{Aut}\mathbb{A}^n$ such that $\varphi^{-1}AF\varphi \in \operatorname{Aff}_n(\Bbbk)$. More precisely, consider a triangular matrix

$$A = \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ a_{21} & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & s_n \end{pmatrix} \tag{1}$$

as a point of affine space $\mathbb{A}^d$ of dimension $d = \frac{1}{2}(n+1)n$. Then the matrices, for which the automorphism AF can be linearized, form a Zariski open subset in the space $\mathbb{A}^d$. The complement of this subset is defined by equations in the variables s_i.

The proof of this theorem is nearly analogous to that of Theorem 3.2 from [5]. First we need a definition and some lemmas.

Definition 2.1. We introduce the reverse lexicographical order on monomials in n variables $x_1,\dots,x_n$. Namely, we say $x_1^{i_1}\cdots x_n^{i_n} > x_1^{i'_1}\cdots x_n^{i'_n}$ if $i_k > i'_k$ for some $k \in \{1,2,\dots,n\}$ and $i_j = i'_j$ for all $j \in \{k+1,\dots,n\}$.

Lemma 2.3. *For each $1 \le j \le n-1$, let $\ell_j(x_1,\dots,x_{j-1})$ be linear in the variables $x_1,\dots,x_{j-1}$ and let $\mu \in \Bbbk^*$. Then the leading monomial with respect to the introduced order in expansion of*

$$\mu \prod_{j=1}^{n-1} (s_j x_j + \ell_j(x_1,\dots,x_{j-1}))^{i_j}$$

is

$$\mu s_1^{i_1}\cdots s_{n-1}^{i_{n-1}} x_1^{i_1}\cdots x_{n-1}^{i_{n-1}}.$$

Proof. The result follows from the fact that the leading monomial of the product is the product of leading monomials. □

Lemma 2.4. *If $F,G \in \mathscr{J}_n$ and for some $1 \le k \le n$ they are linear up to $(k-1)$th coordinate, then the highest-order term of the product $(F\cdot G)_{|_k}$ has order less then or equal to the maximum of orders of such terms of $F_{|_k}$ and $G_{|_k}$.*

Proof. The proof follows from Lemmas 2 and 3 in [6]. □

Lemma 2.5. *Let F be a polynomial map of the form*

$$[s_1x_1 + \ell_1, s_2x_2 + \ell_2(x_1), \dots, s_{n-1}x_{n-1} + \ell_{n-1}(x_1,\dots,x_{n-2}),$$
$$s_nx_n + a(x_1,\dots,x_{n-1}) + \ell_n(x_1,\dots,x_{n-1})],$$

where $a(x_1,\dots,x_{n-1})$ is a polynomial with leading monomial $\lambda x_1^{i_1}\dots x_{n-1}^{i_{n-1}}$ and $s_1^{i_1}\dots s_{n-1}^{i_{n-1}} s_n^{-1} \neq 1$, $\ell_i(x_1,\dots,x_{i-1})$ are linear. Then there exists a polynomial map φ of triangular form such that

$$\varphi^{-1}F\varphi = [s_1x_1+\ell_1, s_2x_2+\ell_2(x_1),\dots,s_{n-1}x_{n-1}+\ell_{n-1}(x_1,\dots,x_{n-2}),\\ s_nx_n+\tilde{a}(x_1,\dots,x_{n-1})+\ell_n(x_1,\dots,x_{n-1})],$$

where the leading monomial of $\tilde{a}(x_1,\dots,x_{n-1})$, say $\tilde{\lambda}x_1^{j_1}\dots x_{n-1}^{j_{n-1}}$, is of a strictly lower order than the leading monomial of $a(x_1,\dots,x_{n-1})$.

Proof. Take φ of the special form

$$\varphi = [x_1,\dots,x_{n-1},x_n+\mu x_1^{i_1}\dots x_{n-1}^{i_{n-1}}]$$

for some $\mu \in \mathbf{k}$. We will try to find μ such that

$$F\varphi = \varphi([s_1x_1+\ell_1, s_2x_2+\ell_2(x_1),\dots,s_{n-1}x_{n-1}+\ell_{n-1}(x_1,\dots,x_{n-2}),\\ s_nx_n+\tilde{a}(x_1,\dots,x_{n-1})+\ell_n(x_1,\dots,x_{n-1})]). \tag{2}$$

Automorphism φ does not change first $(n-1)$ coordinates, so we will look at the last one:

$$F\varphi_{|_n} = s_nx_n+\lambda x_1^{i_1}\dots x_{n-1}^{i_{n-1}}+\hat{a}(x_1,\dots,x_{n-1})+\ell_n(x_1,\dots,x_{n-1})\\ +\mu\prod_{j=1}^{n-1}(s_jx_j+\ell_j(x_1,\dots,x_{j-1}))^{i_j},$$

where $\hat{a}(x_1,\dots,x_{n-1}) = a(x_1,\dots,x_{n-1})-\lambda x_1^{i_1}\cdots x_{n-1}^{i_{n-1}}$, and on the right-hand side,

$$\varphi([s_1x_1+\ell_1,\dots,s_nx_n+\tilde{a}(x_1,\dots,x_{n-1})+\ell_n(x_1,\dots,x_{n-1})])_{|_n}\\ = s_nx_n+s_n\mu x_1^{i_1}\dots x_{n-1}^{i_{n-1}}+\tilde{a}(x_1,\dots,x_{n-1})+\ell_n(x_1,\dots,x_{n-1}).$$

By subtracting first equation from the second, under assumption (2), we get

$$(s_n\mu-\lambda)x_1^{i_1}\dots x_{n-1}^{i_{n-1}}+\hat{\hat{a}}(x_1,\dots,x_{n-1}) = \mu\prod_{j=1}^{n-1}(s_jx_j+\ell_j(x_1,\dots,x_{j-1}))^{i_j},$$

where $\hat{\hat{a}} = \tilde{a}-\hat{a}$. Now we can focus our attention to the coefficients of $x_1^{i_1}\dots x_{n-1}^{i_{n-1}}$, because all other terms have strictly lower order. From Lemma 2.3 the right-hand side coefficient of $x_1^{i_1}\dots x_{n-1}^{i_{n-1}}$ is $\mu s_1^{i_1}\dots s_{n-1}^{i_{n-1}}$. We have

$$s_n\mu-\lambda = \mu s_1^{i_1}\dots s_{n-1}^{i_{n-1}}$$

or

$$\mu = \frac{\lambda}{s_n(1-s_1^{i_1}\dots s_{n-1}^{i_{n-1}}s_n^{-1})}.$$

By assumption, $s_1^{i_1}\dots s_{n-1}^{i_{n-1}}s_n^{-1} \neq 1$ and $s_n \neq 0$, so such μ exists. $\square$

Now we can prove Theorem 2.2.

Proof. First note that AF has lower triangular form, so we will look at such transformations. Let

$$F = [s_1x_1 + a_1, s_2x_2 + a_2(x_1), \dots, s_nx_n + a_n(x_1, \dots, x_{n-1})].$$

Then exactly in the same way as in [5, Theorem 3.2] applying previous lemma finite number of times one can get triangular automorphism with desired property.

The complement of the Zariski open set where the linearization is possible is defined by equations of two types:

$$s_i = 0 \quad \text{and} \quad s_1^{i_1} \dots s_{k-1}^{i_{k-1}} = s_k$$

also coming from the lemma. □

Finally, we come to the proof of Theorem 2.1.

Proof. First of all note that if $F^m = I$ then $s_i^m = 1$ for all i. Now we are going to follow the proof of Theorem 2.2 constructing φ with the desired property. There φ was constructed as the composition of isomorphisms lowering the order of coordinates of F. The first coordinate is obviously linear. We are going to prove the theorem using induction on coordinates and inverse induction on the highest order of terms of coordinates. By Lemma 2.4 this order cannot become higher than the highest order of the original transformation. By induction hypothesis there exists φ_k-triangular automorphism such that $\varphi_k^{-1}F\varphi_k$ is linear up to $(k-1)$th coordinate. Let

$$G = \varphi_k^{-1}F\varphi_k = [s_1x_1 + \ell_1, s_2x_2 + \ell_2(x_1), \\ \dots, s_{k-1}x_{k-1} + \ell_{k-1}(x_1, \dots, x_{k-2}), s_kx_k + g(x_1, \dots, x_{k-1}), \dots],$$

where $\ell_i(x_1, \dots, x_{i-1})$ are linear and

$$G^m = (\varphi^{-1}F\varphi)^m = \varphi^{-1}F^m\varphi = I.$$

It remains to show that there exists an automorphism ψ such that

$$\psi^{-1}G\psi = [s_1x_1 + \ell_1, s_2x_2 + \ell_2(x_1), \\ \dots, s_{k-1}x_{k-1} + \ell_{k-1}(x_1, \dots, x_{k-2}), s_kx_k + \tilde{g}(x_1, \dots, x_{k-1}), \dots]$$

and $\tilde{g}$ has strictly lower order than g.

By Lemma 2.5 we can do this if $s_1^{i_1} \dots s_{k-1}^{i_{k-1}} s_k^{-1} \neq 1$ where $\lambda x_1^{i_1} \dots x_{k-1}^{i_{k-1}}$ is the highest order term of g. Suppose $s_1^{i_1} \dots s_{k-1}^{i_{k-1}} s_k^{-1} = 1$. We are going to look at the coefficient μ of highest-order term of kth coordinate of G^m. By Lemma 2.4 it has the same order as the highest-order term of kth coordinate of G or lower (then $\mu = 0$):

$$G = [s_1x_1 + \ell_1, \dots, s_{k-1}x_{k-1} + \ell_{k-1}(x_1, \dots, x_{k-2}), \\ s_kx_k + \lambda x_1^{i_1} \dots x_{k-1}^{i_{k-1}} + \hat{g}(x_1, \dots, x_{k-1}), \dots],$$

using notation from the proof of Lemma 2.5,

$$G^2 = [\ldots, s_k^2 x_k + \lambda(s_k + s_1^{i_1} \cdots s_{k-1}^{i_{k-1}}) x_1^{i_1} \cdots x_{k-1}^{i_{k-1}} + \ldots, \ldots],$$

$$G^3 = [\ldots, s_k^3 x_k + \lambda(s_k^2 + s_1^{i_1} \cdots s_{k-1}^{i_{k-1}}(s_k + s_1^{i_1} \cdots s_{k-1}^{i_{k-1}}))$$
$$\times x_1^{i_1} \cdots x_{k-1}^{i_{k-1}} + \ldots, \ldots],$$

and finally

$$\mu = \lambda(s_k^{m-1} + s_1^{i_1} \cdots s_{k-1}^{i_{k-1}}(s_k^{m-2}$$
$$+ \ldots + s_1^{i_1} \cdots s_{k-1}^{i_{k-1}}(s_k^2 + s_1^{i_1} \cdots s_{k-1}^{i_{k-1}}(s_k + s_1^{i_1} \cdots x_{k-1}^{i_{k-1}}))))$$
$$= \lambda(s_k^{m-1} + s_1^{i_1} \cdots s_{k-1}^{i_{k-1}} s_k^{m-2} + \ldots + (s_1^{i_1} \cdots s_{k-1}^{i_{k-1}})^{m-2} s_k + (s_1^{i_1} \cdots s_{k-1}^{i_{k-1}})^{m-1})$$
$$= \lambda m s_k^{-1}$$

by the relation $s_1^{i_1} \cdots s_{k-1}^{i_{k-1}} s_k^{-1} = 1$. Thus, the kth coordinate of G^m is not equal to x_k but $G^m = I$. This contradiction shows that $s_1^{i_1} \cdots s_{k-1}^{i_{k-1}} s_k^{-1} \neq 1$ on every step of the construction of the automorphism and F is linearizable. □

3. Linearizable maps with scalar diagonal

Let F be a polynomial automorphism. We can write it in the form $F = X + H$ where $X = (x_1, \ldots, x_n)$ and $H = (H_1(x_1, \ldots, x_n), \ldots, H_n(x_1, \ldots, x_n))$. By JH we denote the Jacobian matrix of H.

Definition 3.1. The Jacobian matrix JH is called strongly nilpotent if the matrix $JH(Y_{(1)}) \cdots JH(Y_{(n)}) = 0$, where $Y_{(i)}$ are n sets of n new variables.

In [5] it is proved that if F is a polynomial map with strongly nilpotent Jacobian matrix then there exists an automorphism $T \in GL_n(\Bbbk)$ such that $T^{-1}FT$ is triangular. By $C(F)$ we denote the centralizer of T in $GL_n(\Bbbk)$.

Theorem 3.1. *Let $\Bbbk$ be a field and $F : \mathbb{A}^n \to \mathbb{A}^n$ a polynomial map of the form $F = X + H$ with JH strongly nilpotent. Then for all $A \in C(F) \cap T_n(\Bbbk)$ of the form (1) for which $s_1^{i_1} \cdots s_n^{i_n} \neq 1$ for some finite set of vectors $(i_1, \cdots, i_n) \in \mathbb{Z}^n$ there exists a linearly triangularizable automorphism $\varphi \in \operatorname{Aut} \mathbb{A}^n$ such that $\varphi^{-1} A F \varphi \in Aff_n(\Bbbk)$.*

Remark. $C(F) \cap T_n(\Bbbk) \neq \emptyset$ (it contains scalar matrices). After applying the conditions it is again nonempty.

Proof. By the Theorem 1.6 from [5] there exists $T \in GL_n(\Bbbk)$ such that $T^{-1}FT$ has triangular form. We have

$$G = T^{-1}AFT = AT^{-1}FT \in \mathscr{J}_n$$

and by Theorem 2.2 there exists a triangular automorphism $\psi \in \mathrm{Aut}\mathbb{A}^n$ such that $\psi^{-1}G\psi$ is linear. Defining $\varphi = T\psi T^{-1}$ we get that $\varphi^{-1}AF\varphi$ is linear. □

Corollary 3.2 (Van den Essen and Hubbers [5, Theorem 3.2]). *Let $\Bbbk$ be a field, $F : \mathbb{A}^n \to \mathbb{A}^n$ a polynomial map of the form $F = X + H$ with JH strongly nilpotent. Then for all s except for a finite set of roots of unity, there exists a $\Bbbk$-linearly triangularizable automorphism φ such that $\varphi^{-1}sF\varphi \in \mathit{Aff}_n(\Bbbk)$.*

Proof. Scalar matrix commute with any matrix from $GL_n(\Bbbk)$. □

Theorem 3.3. *Let $\Bbbk$ be a field and $F : \mathbb{A}^n \to \mathbb{A}^n$ a polynomial map of the form $F = X + H$ with JH strongly nilpotent. Then, if $(sF)^m = I$ for some $s \in \Bbbk$ and $m \in \mathbb{N}$ and the characteristic of $\Bbbk$ does not divide m, there exists a linearly triangularizable automorphism $\varphi \in \mathrm{Aut}\mathbb{A}^n$ such that $\varphi^{-1}sF\varphi \in \mathit{Aff}_n(\Bbbk)$.*

Proof. By the Theorem 1.6 from [5], F is triangularizable so there exists $T \in GL_n(\Bbbk)$ such that $T^{-1}FT \in \mathscr{J}_n$. Then

$$(T^{-1}sFT)^m = (sT^{-1}FT)^m = T^{-1}(sF)^mT = I$$

and by Theorem 2.1 there exists $\psi \in \mathrm{Aut}\,\mathbb{A}^n$ such that $\psi^{-1}T^{-1}sFT\psi$ is triangular. Letting $\varphi = T\psi T^{-1}$, we get the desired result. □

4. Non-linearizable reductive group actions

Theorems 2.1 and 2.2 provide some conditions to check whether the triangular map is linearizable. What happens if these conditions are not satisfied?

Theorem 2.1 is true if the characteristic of the field does not divide the order of the transformation. It is easy to give an example of a non-linearizable map if this condition is not satisfied. Take $\Bbbk = \bar{\mathbb{F}}_p$ for some prime p and the map $F = [x, y + x(x-1)]$. It is clear that $F^p = I$ but it is not linearizable because its fixed point set is $\{0\} \times \mathbb{A}^1 \cup \{1\} \times \mathbb{A}^1 \subset \mathbb{A}^2$ and it is not isomorphic to any linear subspace of $\mathbb{A}^2$. On the other hand, if a polynomial map is linearizable then its fixed point set should be isomorphic to some affine space. This is also a counterexample to the fixed point problem in prime characteristic.

Now for the Theorem 2.2. If the conditions are not satisfied it does not mean that the map is not linearizable. Take, for example, $F = [x + 1, y - 3x^2 - 2x - 1]$. Here $s_1 = s_2 = 1$ and for any integer vector (i_1, i_2) we get $s_1^{i_1}s_2^{i_2} = 1$ but F is linearizable by conjugation with element $\varphi = [x, y + x^3]$ and $\varphi^{-1}F\varphi = [x + 1, y + x]$.

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