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Smooth structures on certain moduli spaces for bundles on a surface

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Abstract

Let Σ be a closed surface, G a compact Lie group, not necessarily connected, with Lie algebra $\mathfrak{g}$, $\xi : P \to \Sigma$ a principal G-bundle, let $N(\xi)$ denote the moduli space of central Yang–Mills connections on ξ, with reference to suitably chosen additional data, and let $\mathrm{Rep}_\xi(\Gamma, G)$ be the space of representations of the universal central extension Γ of the fundamental group of Σ in G that corresponds to ξ. We construct smooth structures on $N(\xi)$ and $\mathrm{Rep}_\xi(\Gamma, G)$, that is, algebras of continuous functions which restrict to smooth functions on the strata of certain associated stratifications; by means of a detailed investigation of the derivative of the holonomy we show thereafter that, with reference to these smooth structures, the assignment to a smooth connection A of its holonomies with reference to suitable closed paths yields a diffeomorphism from $N(\xi)$ onto $\mathrm{Rep}_\xi(\Gamma, G)$; moreover, we show that the derivative of the latter at the non-singular points of $N(\xi)$ amounts to a certain twisted integration mapping relating a suitable de Rham theory with group cohomology with appropriate coefficients. Finally, we examine the infinitesimal geometry of these moduli spaces by means of the smooth structures and, for illustration, we show that, on the moduli space of flat SU(2)-connections for a surface of genus two which, as a space, is just complex projective 3-space, our smooth structure looks rather different from the standard structure. © 1998 Elsevier Science B.V.

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0. Introduction

Let X be a decomposed topological space, e ach piece of the decomposition being a smooth manifold. A *smooth structure* on X is an algebra $C^\infty(X)$ of continuous

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functions on X which, on each piece, restrict to smooth functions. We shall refer to such a space as a *smooth* space. In the present paper, we endow certain moduli spaces with a smooth structure and thereafter analyze their singular structure and infinitesimal geometry by means of it. It belongs to a series of papers about a program revealing the structure of these moduli spaces by means of the symplectic or more generally Poisson geometry of certain related classical constrained systems but its results are of interest in their own right. In [12] we construct the searched for Poisson structures on the moduli spaces, thereby obtaining structures of a *stratified symplectic space* in the sense of [30]; such a structure encapsulates the *mutual positions* of symplectic structures on the strata. It is known that some of these moduli spaces carry the additional structure of a (complex) projective variety which, however, does not shed too much light on the singular behavior of the symplectic or Poisson structures in general; in fact, it may happen that the symplectic structure is singular whereas the complex analytic one is not. An example will be mentioned shortly. On the other hand, the singular behavior of the symplectic or more generally Poisson structures can entirely be understood in the framework of the real algebraic geometry of appropriate smooth structures on these moduli spaces, to which the present paper is devoted. We shall relate the smooth structures with appropriate complex analytic structures elsewhere by means of a suitable notion of polarization for Poisson structures; this will generalize the classical description of a Kähler structure in terms of a holomorphic polarization and in particular will provide the necessary means to talk about *mutual positions* of Kähler structures on the strata.

We explain at first briefly the moduli spaces. Let Σ be a closed surface, G a compact Lie group, not necessarily connected, with Lie algebra $\mathfrak{g}$, and $\xi : P \to \Sigma$ a principal G-bundle, having a connected total space P. Further, pick a Riemannian metric on Σ and an *orthogonal structure* on $\mathfrak{g}$, that is, an adjoint action invariant scalar product. These data then determine a Yang–Mills theory studied for connected G extensively by Atiyah–Bott in [4] to which we refer for background and notation. We only mention that a connection is said to be *Yang–Mills* provided it satisfies the corresponding Yang–Mills equations and *central* when its curvature is a 2-form on Σ with values in the Lie algebra of the center of G. The *moduli space* $N(\xi)$ of central Yang–Mills connections is then that of gauge equivalence classes of central Yang–Mills connections; it is a compact space, including as special cases certain moduli spaces of flat connections and the Narasimhan–Seshadri-moduli spaces [26] of semistable holomorphic vector bundles. For example, as a space, the moduli space of flat SU(2)-connections for a surface of genus 2 is just complex projective 3-space [25]; as a complex analytic space, it is non-singular but the symplectic or more general stratified symplectic structure degenerates on a Kummer surface; see [16]. For a general bundle ξ and structure group G, we shall assume throughout that the space $N(\xi)$ is non-empty, that is, that Yang–Mills connections exist. For example, this will be the case for a connected structure group, cf. [4].

In [10] we have shown that the assignment to a connection of its holonomies, with reference to suitably chosen closed paths, induces a homeomorphism, referred to as

Wilson loop mapping for a reason given in Section 2 below, from $N(\xi)$ onto a certain representation space $\mathrm{Rep}_\xi(\Gamma, G)$ for the universal central extension Γ of the fundamental group π of Σ. While the space $N(\xi)$ *depends* on the choice of Riemannian metric on Σ the space $\mathrm{Rep}_\xi(\Gamma, G)$ does not. One of our aims is to show that, with reference to appropriate additional structure, the Wilson loop mapping is in fact a diffeomorphism.

We now give a brief overview of the paper. Section 1 is preliminary in character. In Section 2 we determine the derivative of the holonomy, viewed as a map from the space of connections to the structure group, once the appropriate additional requisite data have been chosen. In Section 3 we introduce our algebras of smooth functions and spell out the *first chief result* of the paper, Theorem 3.8; it will say that, the spaces $N(\xi)$ and $\mathrm{Rep}_\xi(\Gamma, G)$ being decomposed into connected components of orbit types in the appropriate sense, the Wilson loop mapping is fact a diffeomorphism. In Section 4 we give a description of the twisted integration mapping tailored to our purposes. In Section 5 we rework and extend the classical relationship between the infinitesimal structure of representation spaces and group cohomology which goes back at least to Weil [31, 32], cf. [27]. In Section 6 we reduce the smooth structures of $N(\xi)$ and $\mathrm{Rep}_\xi(\Gamma, G)$ near any of its points to that of local models of a kind introduced in an earlier paper [11], endowed with suitable smooth structures. This will be our *second chief result*. Our *third chief result*, Theorem 6.15 below, will be the existence of suitable partitions of unity; this will then enable us to complete the proof of Theorem 3.8 mentioned above. In Section 7 we examine the infinitesimal structure of our spaces of interest. In particular, we shall establish the fact that the space $N(\xi)$ is locally semialgebraic. Finally, in Section 8 we examine the moduli space of flat SU(2)-connections for a surface of genus 2 which, cf. what was said above, as a space is just complex projective 3-space. We shall see that, as a smooth space with the appropriate smooth structure, it looks rather different; for example, at 16 isolated points, the Zariski tangent space has (real) dimension 10.

Abstracting the structure of the spaces $N(\xi)$ and $\mathrm{Rep}_\xi(\Gamma, G)$ isolated in the present paper we are led to spaces with an algebra of functions which, locally, look like the reduced space of a momentum mapping for a representation of a compact Lie group which varies over the space, with the obvious smooth structure on the reduced space. This class of spaces may well be worth an independent investigation.

1. Preliminaries

Let M be a finite-dimensional smooth connected manifold, not necessarily compact, G a (real) Lie group, not necessarily compact, $\mathfrak{g}$ its (real) Lie algebra, and $\xi: P \to M$ a principal G-bundle over M, with G acting on the right of P. We denote the action of $x \in G$ by $\mathrm{R}_x : p \mapsto px$, where $p \in P$. The affine space $\mathscr{A}(\xi)$ of smooth connections on ξ inherits an obvious action of the group $\mathscr{G}(\xi)$ of gauge transformations and so does the graded vector space $\Omega^*(M, \mathrm{ad}(\xi))$. We pick a *base point* $Q \in M$ and a pre-image $\hat{Q} \in P$; then assignment to a gauge transformation γ on ξ of $x_\gamma \in G$ defined by

$\gamma(\hat{Q}) = \hat{Q}x_\gamma$ furnishes a surjective homomorphism

$$\mathscr{G}(\xi) \to G \tag{1.1}$$

whose kernel is the *group* $\mathscr{G}^Q(\xi)$ of (at Q) *based gauge transformations*. The *adjoint bundle* $\text{ad}(\xi)$ is the Lie algebra bundle over M associated with ξ and the adjoint action of G on $\mathfrak{g}$. With the obvious bracket, its space of sections $\Omega^0(M, \text{ad}(\xi))$ is the Lie algebra of $\mathscr{G}(\xi)$ in a natural fashion. The tangent bundle of a smooth manifold X will be written $\tau_X : \text{T}X \to X$.

We shall not distinguish in notation between the naive objects and their Sobolev completions [4, 6, 23, 24].

2. The derivative of the holonomy

Let $I = [0, b]$ be an interval and $u : I \to M$ a smooth path in M having starting point Q. For a connection A, we denote by $u_{A,\hat{Q}} : I \to P$ the *horizontal* lift of u, having starting point $\hat{Q}$. For $t \in I$, let $u_{A,\hat{Q},t} : [0, t] \to P$ be the restriction of $u_{A,\hat{Q}}$ to $[0, t]$.

Among the various descriptions of the space $\Omega^j(M, \text{ad}(\xi))$ of j-forms with values in the adjoint bundle $\text{ad}(\xi)$ we shall take here that in terms of G-invariant horizontal $\mathfrak{g}$-valued forms on P. The following will be crucial.

Theorem 2.1. *With reference to a suitable Sobolev topology on $\mathscr{A}(\xi)$, the assignment to $(A,t) \in \mathscr{A}(\xi) \times I$ of the horizontal lift $u_{A,\hat{Q}}(t)$ furnishes a continuous map U from $\mathscr{A}(\xi) \times I$ to P whose restriction to any smooth finite-dimensional submanifold of $\mathscr{A}(\xi) \times I$ is smooth. Given a connection A on ξ and a 1-form $\vartheta \in \Omega^1(M, \text{ad}(\xi)) = \text{T}_A\mathscr{A}(\xi)$, an explicit formula for the partial derivative $(\partial U/\partial \vartheta)(A,t) = \text{d}U(A,t)(\vartheta, 0)$ is given by*

$$\frac{\partial U}{\partial \vartheta}(A,t) = u_{A,\hat{Q}}(t) \int_{u_{A,\hat{Q},t}} \vartheta \in \text{T}_{U(A,t)}P. \tag{2.2}$$

Remark 2.3. Some comment about the interpretation of the formula (2.2) might be in order: The 1-form ϑ being viewed as a G-invariant $\mathfrak{g}$-valued one on P which vanishes on the vertical vectors, the integral $\int_{u_{A,\hat{Q},t}} \vartheta$ is well defined as an element of the Lie algebra $\mathfrak{g}$. Moreover, by construction, $u_{A,\hat{Q}}(t) \in P$, and the expression

$$u_{A,\hat{Q}}(t) \int_{u_{A,\hat{Q},t}} \vartheta \in \text{T}_{u_{A,\hat{Q}}(t)}P$$

refers to the element which is obtained when the canonical injection from $P \times \mathfrak{g}$ into the total space $\text{T}P$ is applied to the pair $(u_{A,\hat{Q}}(t), \int_{u_{A,\hat{Q},t}} \vartheta)$.

Remark 2.4. The existence of the derivative of U, restricted to an arbitrary smooth finite-dimensional submanifold, and that of corresponding derivatives of arbitrarily high order, follows from standard facts about analytical dependence of the solution of a differential equation on suitable parameters.

Proof of Theorem 2.1. In the good range $k > \dim M/2$, convergence in the Sobolev topology H^k implies uniform convergence, cf. [7, Section 6]. This implies readily that U is continuous.

The smooth tangent space $\mathrm{T}_A\mathscr{A}(\xi)$ is naturally identified with the vector space of 1-forms $\Omega^1(M, \mathrm{ad}(\xi))$ with values in the adjoint bundle, and, for a fixed value of $t \in I$, we look for the derivative $\mathrm{T}_A U_t : \mathrm{T}_A\mathscr{A}(\xi) \to \mathrm{T}_{U_t}P$ of the map $U_t : \mathscr{A}(\xi) \to P$ which is given by the assignment to a connection A of the value $U_t(A) = u_{A,\hat{Q}}(t) \in P$. Thus, given $\vartheta \in \Omega^1(M, \mathrm{ad}(\xi))$, all we need is an expression for the partial derivative

$$\frac{\partial U_t}{\partial \vartheta}(A) = \mathrm{T}_A U_t(\vartheta) \in \mathrm{T}_{u_{A,\hat{Q}}(t)}P.$$

To obtain such an expression, given $s \in \mathbb{R}$ and $\vartheta \in \Omega^1(M, \mathrm{ad}(\xi))$, we consider the horizontal lift $u_{A+s\vartheta,\hat{Q}} : I \to P$ of u. It is clear that the assignment to $(s,t) \in I \times I$ of $u_{A+s\vartheta,\hat{Q}}(t)$ yields a smooth map $\hat{u} : I \times I \to P$, and what we are looking for is an expression for the partial derivative of this map at $s = 0$, whatever $t \in I$. To simplify notation, write $v = u_{A,\hat{Q}} : I \to P$ for the horizontal lift of u. It is obvious that there is a unique map $a : I \times I \to G$ such that, for every $(s,t) \in I \times I$,

$$\hat{u}(s,t) = v(t)a(s,t).$$

When we fix s and differentiate this identity with respect to the parameter t we obtain the identity

$$\hat{u}_t' = v_t' a_t + v_t a_t';$$

here we have written $a_t = a(s,t) \in G$, $\hat{u}_t = \hat{u}(s,t) \in P$, $v_t = v(t) \in P$; furthermore, with a notation used e.g. on p. 69 of [18], $\hat{u}_t'$ is the tangent vector to the curve $(s,t) \mapsto \hat{u}(s,t)$ (s fixed) at the point $u(s,t)$, and v_t' and a_t' refer to the corresponding tangent vectors of the other curves coming into play. Let $\omega : \mathrm{T}P \to \mathfrak{g}$ be the connection form of A; then $\omega - s\vartheta$ is the connection form of $A + s\vartheta$. Exploitation of the fact that $\hat{u}_t'$ is horizontal for the connection $A + s\vartheta$ yields

$$\begin{aligned}
0 &= (\omega - s\vartheta)(\hat{u}_t') \\
&= \omega(u_t') - s\vartheta(\hat{u}_t') \\
&= \omega(v_t' a_t + v_t a_t') - s\vartheta(v_t' a_t + v_t a_t') \\
&= \omega(v_t' a_t) + \omega(v_t a_t') - s\vartheta(v_t' a_t) - s\vartheta(v_t a_t') \\
&= \omega((\mathrm{R}_{a_t})_* v_t') + \omega(v_t a_t') - s\vartheta((\mathrm{R}_{a_t})_* v_t'),
\end{aligned}$$

since $v_t a_t'$ is vertical and since ϑ is zero on vertical vectors; we remind the reader that $R_{a_t} : P \to P$ refers to the action of G on P. Moreover, since the curve v_t is horizontal with respect to A, G-invariance of ω implies that $\omega((\mathrm{R}_{a_t})_* v_t')$ equals $\mathrm{ad}_{a_t^{-1}}\omega(v_t')$ which is zero; likewise, G-invariance of ϑ implies $\vartheta((\mathrm{R}_{a_t})_* v_t') = \mathrm{ad}_{a_t^{-1}}\vartheta(v_t')$. Further, by construction, $\omega(v_t a_t')$ equals $a_t^{-1}a_t' \in \mathfrak{g} = \mathrm{T}_e G$. Consequently, the fact that $\hat{u}_t'$ is horizontal for the connection $A + s\vartheta$ entails that a_t satisfies the differential equation

$$0 = a_t^{-1}a_t' - s\,\mathrm{ad}_{a_t^{-1}}\vartheta(v_t') \in \mathfrak{g}$$

in the Lie algebra $\mathfrak{g}$ of G or, equivalently, the differential equation

$$0 = a_t' a_t^{-1} - s\,\vartheta(v_t') \in \mathfrak{g}.$$

When we differentiate this equation with respect to s we obtain

$$0 = \frac{\partial}{\partial s}(a_t' a_t^{-1}) - \vartheta(v_t') \in \mathfrak{g},$$

that is,

$$(*) \qquad 0 = \left(\frac{\partial}{\partial s}a_t'\right) a_t^{-1} + a_t'\left(\frac{\partial}{\partial s}a_t^{-1}\right) - \vartheta(v_t') \in \mathfrak{g}.$$

Finally, we observe that by construction the map a is subject to the conditions $a(0,t) = e = a(s,0)$. In particular, for $s = 0$, do we have $a_t' = 0$, and hence, for $s = 0$, the differential equation $(*)$ simplifies to

$$(**) \qquad 0 = \frac{\partial}{\partial s}a_t' - \vartheta(v_t') \in \mathfrak{g}.$$

However, since a is defined on the product of two intervals, we may interchange partial derivatives and obtain, for $s = 0$, the differential equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial a_t}{\partial s} - \vartheta(v_t') \in \mathfrak{g}.$$

From this we conclude that

$$\frac{\partial a}{\partial s}(0,t) = \int_0^t \vartheta(v_\tau')\,\mathrm{d}\tau = \int_{u_{A,\hat{Q},t}} \vartheta \in \mathfrak{g}. \qquad \square$$

By a *smooth* map h on $\mathscr{A}(\xi)$ with values in a smooth finite-dimensional manifold we mean henceforth a continuous map h whose restriction to an arbitrary smooth finite-dimensional submanifold of $\mathscr{A}(\xi)$ is smooth in the ordinary sense. We can then still talk about the derivative of h: for a point A of $\mathscr{A}(\xi)$, by the *differential* or *derivative* $\mathrm{d}h(A)$, *evaluated at a* 1*-form* $\vartheta \in \Omega^1(M, \mathrm{ad}(\xi))$, we mean the corresponding partial derivative.

For a smooth closed path $w : [0,b] \to M$, with starting point $Q \in M$, the *holonomy* $\mathrm{Hol}_{w,\hat{Q}}(A) \in G$ *of* A *along* w *with reference to* $\hat{Q}$ is defined by

$$w_{A,\hat{Q}}(b) = \hat{Q}\,\mathrm{Hol}_{w,\hat{Q}}(A) \in P.$$

For $y \in G$ we denote by L_y the operation of left translation from $\mathfrak{g}$ to $\mathrm{T}_y G$.

Corollary 2.5. *When u is closed the holonomy along u furnishes a smooth map $\mathrm{Hol}_{u,\hat{Q}}$ from $\mathscr{A}(\xi)$ to G. Moreover, at a connection A, with $y = \mathrm{Hol}_{u,\hat{Q}}(A) \in G$, the differential $d\mathrm{Hol}_{u,\hat{Q}}(A) : \mathrm{T}_A\mathscr{A}(\xi) \to \mathrm{T}_yG$ assigns to a smooth 1-form $\vartheta \in \mathrm{T}_A\mathscr{A}(\xi) = \Omega^1(M, \mathrm{ad}(\xi))$ the value $L_y \int_{u_{A,\hat{Q}}} \vartheta \in \mathrm{T}_yG$. Finally, this map is invariant in the sense that, given a gauge transformation γ, whatever smooth connection A, $\mathrm{Hol}_{u,\hat{Q}}(\gamma A) = x_\gamma \mathrm{Hol}_{u,\hat{Q}}(A) x_\gamma^{-1}$. (See Section 1 for the notation x_γ.)*

Proof. Let $\vartheta \in \Omega^1(M, \mathrm{ad}(\xi)) = \mathrm{T}_A\mathscr{A}(\xi)$ and $Y = \int_{u_{A,\hat{Q}}} \vartheta \in \mathfrak{g}$. By Theorem 2.1, an explicit formula for the partial derivative $(\partial U_b/\partial\vartheta)(A) = \mathrm{d}U_b(A)(\vartheta)$ of the map U_b from $\mathscr{A}(\xi)$ to P which assigns $u_{A,\hat{Q}}(b) \in P$ to $A \in \mathscr{A}(\xi)$ is given by

$$\frac{\partial U_b}{\partial\vartheta}(A) = \frac{\mathrm{d}}{\mathrm{d}t}(u_{A,\hat{Q}}(b)\exp tY)|_{t=0} \in \mathrm{T}_{u_{A,\hat{Q}}(b)}P.$$

The derivative $\mathrm{T}_aG \to \mathrm{T}_{\hat{Q}a}P$ at $a \in G$ of the smooth map from G to P which assigns to $a \in G$ the point $\hat{Q}a \in P$ may be described by the assignment to L_aZ of $\hat{Q}L_aZ$, for $Z \in \mathfrak{g}$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_{A,\hat{Q}}(b)\exp tY)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\hat{Q}\,\mathrm{Hol}_{w,\hat{Q}}(A)\exp tY)|_{t=0} = \hat{Q}L_y \int_{u_{A,\hat{Q}}} \vartheta. \qquad \square$$

Now we pick smooth closed curves $w_1, \ldots, w_n$ in M starting at Q and representing a set of generators $x_1, \ldots, x_n$ of the fundamental group $\pi = \pi_1(M, Q)$; we write $\mathbf{w} = (w_1, \ldots, w_n)$ and denote by F the free group on $x_1, \ldots, x_n$. The assignment to a connection A of $(\mathrm{Hol}_{w_1,\hat{Q}}(A), \ldots, \mathrm{Hol}_{w_n,\hat{Q}}(A)) \in G^n$ yields a map

$$\rho = \mathrm{Hol}_{\mathbf{w},\hat{Q}} : \mathscr{A}(\xi) \to G^n \tag{2.6}$$

which, in view of Theorem 2.1, is smooth in the sense that its restriction to an arbitrary smooth finite-dimensional submanifold of $\mathscr{A}(\xi)$ is smooth. We refer to ρ as *Wilson loop mapping* since, for G compact, its composite with a smooth G-invariant function on $\mathrm{Hom}(F, G)$ yields a smooth $\mathscr{G}(\xi)$-invariant function on $\mathscr{A}(\xi)$ generalizing what is called a (classical) *Wilson loop observable* in the physics literature. Here is an immediate consequence of Corollary 2.5.

Theorem 2.7. *At a connection A, with*

$$\rho(A) = (\mathrm{Hol}_{w_1,\hat{Q}}(A), \ldots, \mathrm{Hol}_{w_n,\hat{Q}}(A)) = (y_1, \ldots, y_n) \in G^n,$$

the differential $\mathrm{d}\rho(A) : \mathrm{T}_A\mathscr{A}(\xi) \to \mathrm{T}_{\rho(A)}G^n = \mathrm{T}_{y_1}G \times \cdots \times \mathrm{T}_{y_n}G$ of (2.6) is given by the assignment to $\vartheta \in \Omega^1(M, \mathrm{ad}(\xi)) = \mathrm{T}_A\mathscr{A}(\xi)$ of

$$I_{\mathbf{w},A,\hat{Q}}(\vartheta) = \left(L_{y_1} \int_{\hat{w}_1} \vartheta, \ldots, L_{y_n} \int_{\hat{w}_n} \vartheta\right) \in \mathrm{T}_{y_1}G \times \cdots \times \mathrm{T}_{y_n}G,$$

where, with an abuse of notation, for $1 \le j \le n$, $\hat{w}_j$ denotes the horizontal lift of w_j with reference to A and $\hat{Q}$.

3. The first main result

In this section we introduce our algebras of smooth functions and spell out the first main result of the paper. We return to the circumstances of the Introduction. Thus, Σ denotes a closed surface, G a Lie group which we now assume compact but not necessarily connected, with Lie algebra $\mathfrak{g}$, $\xi : P \to \Sigma$ a principal G-bundle, having a connected total space P, and $Q \in \Sigma$ a chosen base point. Consider the standard presentation

$$\mathscr{P} = \langle x_1, y_1, \ldots, x_\ell, y_\ell; r \rangle, \quad r = [x_1, y_1] \cdot \cdots \cdot [x_\ell, y_\ell], \tag{3.1}$$

of the fundamental group $\pi = \pi_1(\Sigma, Q)$, the number ℓ being the genus of Σ; we denote by F the free group on $x_1, y_1, \ldots, x_\ell, y_\ell$ and by N the normal closure of r in F. The quotient group $\Gamma = F/[F, N]$ yields the *universal central extension*

$$0 \to \mathbf{Z} \to \Gamma \to \pi \to 1 \tag{3.2}$$

of π; cf. [4, Section 6; and 10, Section 2]. The topology of the bundle ξ determines an element X_ξ of the Lie algebra z of the centre Z of G which is a topological characteristic class of ξ; see [4] for the case of a connected structure group G and Section 1 of our paper [10] for the general case. The evaluation map which assigns $(\phi(x_1), \phi(y_1), \ldots, \phi(x_\ell), \phi(y_\ell)) \in G^{2\ell}$ to $\phi \in \operatorname{Hom}(F, G)$ identifies $\operatorname{Hom}(F, G)$ with $G^{2\ell}$. Let $\mathrm{H}_\xi(\Gamma, G)$ be the subspace of $\operatorname{Hom}(F, G)$ consisting of homomorphisms $\chi \in \operatorname{Hom}(F, G)$ such that

$$[\chi(x_1), \chi(y_1)] \cdot \cdots \cdot [\chi(x_\ell), \chi(y_\ell)] = \exp(X_\xi) \in Z. \tag{3.3}$$

This space is manifestly compact and hence has only finitely many connected components; furthermore, it is a finite union of real algebraic sets, which, in turn, also implies that it has only finitely many connected components since this is true of any real algebraic set, cf. [35].

The values of the restriction of the Wilson loop mapping (2.6) to the subspace $\mathscr{N}(\xi)$ of central Yang–Mills connections lie in $\mathrm{H}_\xi(\Gamma, G)$; we denote by $\operatorname{Hom}_\xi(\Gamma, G)$ its image in $\mathrm{H}_\xi(\Gamma, G)$; it is a space of homomorphisms from Γ to G. A more intrinsic description of the resulting surjection from $\mathscr{N}(\xi)$ onto $\operatorname{Hom}_\xi(\Gamma, G)$ may be found in (3.8) of our paper [10]. The connected components of $\operatorname{Hom}_\xi(\Gamma, G)$ are parametrized by the points of the corresponding π_0-orbit in $\operatorname{Hom}(\pi, \pi_0)$, where π_0 refers to the group of connected components of G; in particular, when G is connected, $\operatorname{Hom}_\xi(\Gamma, G)$ is connected. Let I_ξ denote the ideal in the algebra $C^\infty(\operatorname{Hom}(F, G))$ of smooth functions on $\operatorname{Hom}(F, G)$ that vanish on the subspace $\operatorname{Hom}_\xi(\Gamma, G)$ of $\operatorname{Hom}(F, G)$, and *define* an algebra $C^\infty(\operatorname{Hom}_\xi(\Gamma, G))$ of continuous functions on $\operatorname{Hom}_\xi(\Gamma, G)$ by

$$C^\infty(\operatorname{Hom}_\xi(\Gamma, G)) = C^\infty(\operatorname{Hom}(F, G))/I_\xi. \tag{3.4}$$

This algebra is often called that of *Whitney smooth functions* on $\operatorname{Hom}_\xi(\Gamma, G)$, cf. [34]. We note that here and henceforth spaces may arise which are not necessarily connected.

When we talk about an algebra of continuous functions on such a space we always mean an algebra of continuous functions on a connected component. We do not indicate this explicitly, to avoid an orgy of notation.

Let $\mathrm{Rep}_\xi(\Gamma, G) = \mathrm{Hom}_\xi(\Gamma, G)/G$. We *define* an algebra $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ of continuous functions on $\mathrm{Rep}_\xi(\Gamma, G)$ by

$$C^\infty(\mathrm{Rep}_\xi(\Gamma, G)) = (C^\infty(\mathrm{Hom}(F, G)))^G / I_\xi^G, \tag{3.5}$$

that is, we take that of smooth G-invariant functions $(C^\infty(\mathrm{Hom}(F, G)))^G$ on $\mathrm{Hom}(F, G)$ modulo its ideal I_ξ^G of functions that vanish on $\mathrm{Hom}_\xi(\Gamma, G)$. By construction this is an algebra of functions on $\mathrm{Rep}_\xi(\Gamma, G)$ in an obvious fashion. Since G is compact, the canonical map from $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ to $(C^\infty(\mathrm{Hom}_\xi(\Gamma, G)))^G$ is a bijection whence $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ may as well be described as the algebra of G-invariant *Whitney smooth functions* on $\mathrm{Hom}_\xi(\Gamma, G)$. Since we shall not need this fact we refrain from giving the details here.

In the same vein, denote by $C^\infty(\mathscr{A}(\xi))$ the algebra of smooth functions on $\mathscr{A}(\xi)$ in the sense explained in Section 2 above; we then *define* the algebra $C^\infty(\mathscr{N}(\xi))$ on $\mathscr{N}(\xi)$ as the quotient algebra $C^\infty(\mathscr{A}(\xi))/J_\xi$, where J_ξ refers to the ideal of functions in $C^\infty(\mathscr{A}(\xi))$ that vanish on the subspace $\mathscr{N}(\xi)$ of $\mathscr{A}(\xi)$, and we *define* an algebra $C^\infty(N(\xi))$ of continuous functions on the moduli space $N(\xi) = \mathscr{N}(\xi)/\mathscr{G}(\xi)$ of central Yang–Mills connections by

$$C^\infty(N(\xi)) = (C^\infty(\mathscr{A}(\xi)))^{\mathscr{G}(\xi)} / I_\xi^{\mathscr{G}(\xi)}, \tag{3.6}$$

that is, we take the algebra of smooth $\mathscr{G}(\xi)$-invariant functions $(C^\infty(\mathscr{A}(\xi)))^{\mathscr{G}(\xi)}$ on $\mathscr{A}(\xi)$ modulo its ideal $I_\xi^{\mathscr{G}(\xi)}$ of functions that vanish on $\mathscr{N}(\xi)$. By construction, this is an algebra of functions on $N(\xi)$, in an obvious fashion.

The decomposition of $N(\xi)$ into connected components of orbit types of classes of central Yang–Mills connections endows $N(\xi)$ with a structure of a *decomposed space*, in fact, see [11, (1.2)], with that of a *stratified space*. The *pieces* are smooth manifolds, parametrized by conjugacy classes (K) of subgroups K of G; the piece $N_{(K)}(\xi)$ corresponding to (K) consists of classes $[A]$ of central Yang–Mills connections A having stabilizer $Z_A \subseteq \mathscr{G}(\xi)$ whose image in G under (1.1) is conjugate to K.

We now pick smooth closed paths $u_1, v_1, \ldots, u_\ell, v_\ell$ in Σ representing the generators $x_1, y_1, \ldots, x_\ell, y_\ell$, so that the standard cell decomposition of Σ with a single 2-cell e corresponding to r results, and, furthermore, a base point $\hat{Q} \in P$ so that $\xi(\hat{Q}) = Q \in \Sigma$. Then the Wilson loop mapping ρ from $\mathscr{A}(\xi)$ to $\mathrm{Hom}(F, G)$ with reference to these data, cf. (2.6), induces a homeomorphism

$$\rho_\flat : N(\xi) \to \mathrm{Rep}_\xi(\Gamma, G); \tag{3.7}$$

it coincides with the map given in (3.8.2) of our paper [10]. By an abuse of language, we refer to $\rho_\flat$ as *Wilson loop mapping* as well. It is independent of the choices made to define ρ.

The decomposition of $\mathrm{Rep}_\xi(\Gamma, G)$ into connected components of orbit types of representations has as well pieces parametrized by conjugacy classes (K) of subgroups of G; the piece $\mathrm{R}_{(K)}(\xi)$ corresponding to (K) consists of classes $[\phi]$ of homomorphisms ϕ from Γ to G having stabilizer $Z_\phi \subseteq G$ conjugate to K. The Wilson loop mapping $\rho_\flat$ is manifestly compatible with the decompositions since (1.1) identifies the stabilizer Z_A of a connection A with the stabilizer $Z_{\rho(A)}$ of $\rho(A) \in \mathrm{Hom}(F, G)$, cf. e.g. [14, (2.4)]. Consequently, the Wilson loop mapping, restricted to a piece $N_{(K)}(\xi)$ of $N(\xi)$, is a homeomorphism onto the corresponding piece $\mathrm{R}_{(K)}(\xi)$ of the decomposition of $\mathrm{Rep}_\xi(\Gamma, G)$. In particular, each connected component of a piece $\mathrm{R}_{(K)}(\xi)$ of the decomposition of $\mathrm{Rep}_\xi(\Gamma, G)$ into G-orbit types inherits a structure of a smooth manifold from the corresponding stratum of $N(\xi)$ in such a way that this decomposition of $\mathrm{Rep}_\xi(\Gamma, G)$ is as well a stratification.

Given smooth spaces $(X, C^\infty(X))$ and $(Y, C^\infty(Y))$, a map $\phi : X \to Y$ is said to be *smooth* provided for every $f \in C^\infty(Y)$ the composite $f \circ \phi$ is a smooth function on X, that is, lies in $C^\infty(X)$. The usual notion of diffeomorphism carries over as well: A smooth homeomorphism is a *diffeomorphism* provided its inverse map is also smooth. Here is the *first main result* of the paper.

Theorem 3.8. *With reference to the decompositions into connected components of orbit types, the algebras $C^\infty(N(\xi))$ and $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ yield smooth structures on $N(\xi)$ and $\mathrm{Rep}_\xi(\Gamma, G)$, respectively, and the Wilson loop mapping $\rho_\flat$ from $N(\xi)$ to $\mathrm{Rep}_\xi(\Gamma, G)$ is a diffeomorphism of smooth spaces.*

Remarks about the proof. The restriction of a function in $C^\infty(N(\xi))$ to a piece is a smooth function in the ordinary sense, and the same is true of $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$. This is a consequence of the fact that the restriction of a smooth function to a smooth submanifold is a smooth function on the submanifold. A more formal proof will be given in Section 6 below. Hence, the algebras $C^\infty(N(\xi))$ and $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ furnish smooth structures as asserted. Smoothness of the map $\rho_\flat$ follows at once from the facts that the Wilson loop mapping ρ from $\mathscr{A}(\xi)$ to $\mathrm{Hom}(F, G)$ is smooth and $\mathscr{G}(\xi)$-invariant, cf. Theorem 2.7, where $\mathscr{G}(\xi)$ acts on $\mathrm{Hom}(F, G)$ through the projection (1.1). Moreover ρ^* is manifestly injective since $\rho_\flat$ is a homeomorphism and hence identifies the algebras of continuous functions on these spaces. The surjectivity of ρ^* will be established in Section 6 below by a partition of unity argument. □

Notice that a priori the smooth structure $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ depends on the choice of presentation of π but *not* on the chosen Riemannian metric on Σ while the space $N(\xi)$ and hence a fortiori its smooth structure $C^\infty(N(\xi))$ depend on the chosen Riemannian metric on Σ but *not* on the choice of presentation of π. Theorem 3.8 implies that the smooth structure on $\mathrm{Rep}_\xi(\Gamma, G)$ does *not* depend on the choice of presentation. Furthermore, a diffeomorphism ϕ of Σ preserving ξ will induce a commutative diagram

$$\begin{array}{ccc} N(\xi) & \longrightarrow & \mathrm{Rep}_\xi(\Gamma, G) \\ {\scriptstyle \phi^\sharp}\big\downarrow & & \big\downarrow{\scriptstyle \phi^\flat} \\ \widetilde{N}(\xi) & \longrightarrow & \mathrm{Rep}_\xi(\Gamma, G) \end{array}$$

of diffeomorphisms of smooth spaces, where $\widetilde{N}(\xi)$ denotes the moduli space of central Yang–Mills connections for the image under ϕ of the chosen Riemannian metric on Σ. We hope to return to this issue at another occasion.

4. The twisted integration mapping in de Rham theory

In the present section we work out a precise description of the twisted integration mapping tailored to our purposes.

Let G be a Lie group, not necessarily compact, and consider a principal G-bundle $\xi : P \to M$ over an *arbitrary* smooth connected finite-dimensional manifold M having connected total space P. As before we pick a base point Q of M and a pre-image $\hat{Q} \in P$ of Q. Given a flat connection A on ξ, the holonomy representation $\phi = \rho(A)$ of $\pi = \pi_1(M, Q)$ in G induces a structure of a π-module on $\mathfrak{g}$ through the adjoint action, and it is folk lore that the cohomology $\mathrm{H}^*_A(M, \mathrm{ad}(\xi))$ is isomorphic to the cohomology of M with the appropriate local coefficients, cf. e.g. [27, VII.7.3, p. 107]. We need a more precise description of a somewhat more general result, to be spelled out below.

Consider the universal covering $\widetilde{M} \to M$ of M; we suppose that things have been set up in such a way that π acts on the *right* of $\widetilde{M}$, and we pick a pre-image $\widetilde{Q} \in \widetilde{M}$ of Q.

Proposition 4.1. *Every smooth flat connection A on ξ determines a unique smooth map $\sigma = \sigma_{A,\hat{Q},\widetilde{Q}}$ from $\widetilde{M}$ to P which, with respect to the corresponding holonomy representation $\rho(A)$ of π in G, furnishes a morphism of (right) principal bundles over M.*

Proof. This is established by an argument of the kind for the *Reduction theorem* in [18, II.7.1]; for later reference we sketch the construction of σ: Given $T \in \widetilde{M}$, let $\widetilde{w}$ be a smooth path in $\widetilde{M}$, necessarily horizontal, joining $\widetilde{Q}$ and T, let w be the path in M obtained by projecting $\widetilde{w}$ into M, and let $\hat{w}$ be the unique lift of w that is horizontal for A and has starting point $\hat{Q}$; then the value $\sigma(T)$ is defined as the end point of $\hat{w}$. Since A is flat, the value $\sigma(T)$ does not depend on the choice of $\widetilde{w}$. □

Let $\zeta : E \to M$ be a smooth vector bundle associated to ξ and the finite-dimensional real representation V of G. Then $\Omega^*(M, \zeta)$ amounts to the G-invariant horizontal forms in $\Omega^*(P, V)$ and the operator d_A of covariant derivative of a *flat* connection A is a differential on $\Omega^*(M, \zeta)$. The following is immediate.

Corollary 4.2. *For every flat connection A, the map from $\Omega^*(P,V)$ to $\Omega^*(\widetilde{M},V)$ induced by $\sigma_{A,\hat{Q},\widetilde{Q}}$, cf. Proposition 4.1, passes to an isomorphism $\sigma^*_{A,\hat{Q},\widetilde{Q}}$ of chain complexes from $(\Omega^*(M,\zeta),d_A)$ onto the subcomplex $(\Omega^*(\widetilde{M},V),d)^\pi$ of π-invariant V-valued forms on $\widetilde{M}$, the necessary π-module structure on V coming from the holonomy $\pi \to G$ of A combined with the G-action on V.*

Given a homomorphism ϕ from π to G and a representation V of G, we write $(C^*(M,V_\phi),d)$ for the subcomplex of π-invariant V-valued cellular cochains on $\widetilde{M}$ and we denote by $\mathrm{H}^*(M,V_\phi)$ the resulting *π-equivariant* cohomology of $\widetilde{M}$ with values in V. It is naturally isomorphic to the cohomology of M with *local coefficients* determined by ϕ and the representation of G on V. The usual integration mapping $(\Omega^*(\widetilde{M},V),d) \to (C^*(\widetilde{M},V),d)$ from the de Rham complex to that of usual cellular cochains is compatible with the π-actions. Taking invariants and combining it with $\sigma^*_{A,\hat{Q},\widetilde{Q}}$, for a given flat connection A, we obtain the chain mapping

$$(\Omega^*(M,\zeta),d_A) \to (\Omega^*(\widetilde{M},V),d)^\pi \to (C^*(M,V_{\rho(A)}),d). \tag{4.3}$$

Henceforth, we refer to it as the *twisted integration mapping* in de Rham theory; it induces an isomorphism from $\mathrm{H}^*_A(M,\zeta)$ onto $\mathrm{H}^*(M,V_{\rho(A)})$ a special case of which is the folk lore isomorphism mentioned earlier.

Under our circumstances, twisted integration furnishes such an isomorphism even for a central connection which is not necessarily flat, in the following way: Recall [10] that a smooth connection A on ξ is said to be *central* provided its curvature K_A is a 2-form on M with values in the Lie algebra z of the centre Z of G. To apply what is said above to a central connection, write Z_e for the connected component of the identity of Z, let $G^\sharp = G/Z_e$, $P^\sharp = P/Z_e$, and consider the induced principal $G^\sharp$-bundle $\xi^\sharp : P^\sharp \to M$; since the adjoint representation of G on $\mathfrak{g}$ factors through a representation of $G^\sharp$ the bundle $\xi^\sharp$ is still a principal one for $\mathrm{ad}(\xi)$. Consequently, a central connection A on ξ induces a flat connection $A^\sharp$ on $\xi^\sharp$; the operator d_A of covariant derivative is then a differential on $\Omega^*(M,\mathrm{ad}(\xi))$, and we can apply what is said above to the vector bundle $\zeta = \mathrm{ad}(\xi)$ and corresponding principal bundle $\xi^\sharp$. Maintaining the notation established in Section 2, we suppose that the smooth closed curves $w_1,\ldots,w_n$ are the 1-cells of a cell decomposition of M with the single zero cell Q, and we thus in particular identify the fundamental group $\pi_1(M^1,Q)$ of the 1-skeleton M^1 of M with the free group F. Let, then, A be a central connection on ξ, and let $\phi = \rho(A) : F \to G$. With reference to the image of $\hat{Q}$ in $P^\sharp$, the homomorphism ϕ manifestly passes to the standard holonomy homomorphism from π to $G^\sharp$ for the resulting flat connection $A^\sharp$ on $\xi^\sharp$. Abusing notation somewhat, we write $\mathfrak{g}_\phi$ for the Lie algebra $\mathfrak{g}$ together with the π-module structure induced by ϕ and hence by A; the resulting twisted integration mapping, with target the corresponding *cellular* cochains, then looks like

$$(\Omega^*(M,\mathrm{ad}(\xi)),d_A) \to (C^*_{\mathrm{cell}}(M,\mathfrak{g}_\phi),d) \tag{4.4}$$

and induces, in particular, an isomorphism Int_A from $\mathrm{H}^*_A(M,\mathrm{ad}(\xi))$ onto $\mathrm{H}^*(M,\mathfrak{g}_\phi)$. When M is aspherical, the complex of cellular chains $\mathbf{C}^{\mathrm{cell}}(\widetilde{M})$ of the universal cover $\widetilde{M}$ with its right π-module structure is a free resolution of the ground ring in the category of right π-modules; when M is not aspherical, a free resolution $\mathbf{P}$ is obtained by adding to $\mathbf{C}^{\mathrm{cell}}(\widetilde{M})$ more generators in degrees ≥ 2. Consequently, whatever right π-module U, the canonical map from $\mathrm{H}^*(\pi,U)$ to $\mathrm{H}^*(M,U)$ is an isomorphism in degree 1 and we shall take it to be the identity, the first cohomology of π being computed from $\mathbf{P}$. Thus the isomorphism induced by the twisted integration mapping furnishes, in degree 1, an isomorphism Int_A from $\mathrm{H}^1_A(M,\mathrm{ad}(\xi))$ onto $\mathrm{H}^1(\pi,\mathfrak{g}_\phi)$ while, for aspherical M, in arbitrary degree, it yields an isomorphism Int_A from $\mathrm{H}^*_A(M,\mathrm{ad}(\xi))$ onto $\mathrm{H}^*(\pi,\mathfrak{g}_\phi)$.

5. Representation spaces

It remains to rework and extend the classical relationship between the infinitesimal structure of representation spaces and group cohomology, cf. [27, 31, 32]. Some care is necessary here since central connections which are not necessarily flat will come into play later.

Let

$$\mathscr{P} = \langle x_1,\dots,x_n; r_1,\dots,r_m\rangle \tag{5.1}$$

be a presentation of a finitely presented group π, and write F for the free group on $x_1,\dots,x_n$, so that $\pi = F/N$, where N refers to the normal closure of $r_1,\dots,r_m$. Recall that, given an element $w \in F$, over any ground ring R, the *right* Fox derivative $\partial w/\partial x_j \in RF$ with respect to the variable x_j, $1 \leq j \leq n$, is given by the equation

$$1 - w = \sum_{j=1}^{n} (1 - x_j)\frac{\partial w}{\partial x_j} \in IF.$$

Here as usual $IK = \ker(\varepsilon : RK \longrightarrow R)$ refers to the *augmentation ideal* of a group K. The usual description of a principal bundle with structure group acting on the *right* forces us to use here *right* Fox derivatives which are less common than *left* Fox derivatives. The Fox calculus, applies to the presentation $\mathscr{P}$, yields the sequence

$$\widehat{\mathrm{R}(\mathscr{P})} : RF \xleftarrow{\partial_1^F} RF[x_1,\dots,x_n] \xleftarrow{\partial_2^F} RF[r_1,\dots,r_m] \tag{5.2}$$

involving the free right RF-modules having $r_1,\dots,r_m$ and $x_1,\dots,x_n$ as bases, respectively; further, the operators ∂_*^F are given by certain explicit formulas; we reproduce them only for the case $m = 1$, which is our primary case of interest, and we write r instead of r_1:

$$\partial_2^F = \left[\frac{\partial r}{\partial x_1},\dots,\frac{\partial r}{\partial x_n}\right]^{\mathrm{t}} : RF[r] \to RF[x_1,\dots,x_n]$$

$$\partial_1^F = [1 - x_1,\dots,1 - x_n] : RF[x_1,\dots,x_n] \to RF,$$

where t refers to the transpose of a vector. Modulo N, (5.2) yields the beginning $\mathrm{R}(\mathscr{P})$ of a free resolution of the ground ring R, viewed as a trivial $R\pi$-module, in the category of right $R\pi$-modules; the distinction between $\mathrm{R}(\mathscr{P})$ and $\widehat{\mathrm{R}(\mathscr{P})}$ will be important in [12].

Given a right RF-module U, with structure map χ from F to $\mathrm{Aut}(U)$, application of the functor $\mathrm{Hom}_{RF}(-,U)$ to (5.2) yields the sequence $\mathrm{Hom}_{RF}(\widehat{\mathrm{R}(\mathscr{P})},U)$ which, in view of the obvious identifications $\mathrm{Hom}_{RF}(\widehat{\mathrm{R}_0(\mathscr{P})},U) = U$, $\mathrm{Hom}_{RF}(\widehat{\mathrm{R}_1(\mathscr{P})},U) = U^n$, $\mathrm{Hom}_{RF}(\widehat{\mathrm{R}_2(\mathscr{P})},U) = U^m$, looks like

$$\mathrm{Hom}_{RF}(\widehat{\mathrm{R}(\mathscr{P})},U) : U \xrightarrow{\delta^0_\chi} U^n \xrightarrow{\delta^1_\chi} U^m. \tag{5.3}$$

Here the operators δ_χ depend on the RF-module structure on U while the modules U^m, U^n, U depend only on the presentation whence the notation. When χ factors through a right $R\pi$-module structure on U, (5.3) is a cochain complex $(C^*(\mathscr{P},U),\delta^*_\chi)$ computing low-dimensional cohomology groups of π with coefficients in U. Further, the subgroup of 1-cocycles $Z^1(\mathscr{P},U) = \ker(\delta^1_\chi)$ then depends only on π, $\mathfrak{g}$, and χ, and not on a choice of presentation (5.1), and we shall therefore write $Z^1(\pi,U)$ instead of $Z^1(\mathscr{P},U)$.

Henceforth, we take $R = \mathbb{R}$, the reals, and $U = \mathfrak{g}$, the Lie algebra of G, viewed as a *right* G-module in the usual way. The assignment to $(\chi(x_1),\dots,\chi(x_n))$ of $\chi \in \mathrm{Hom}(F,G)$ identifies $\mathrm{Hom}(F,G)$ with G^n, and that of the m-tuple

$$(r_1(\chi x_1,\dots,\chi x_n),\dots,r_m(\chi x_1,\dots,\chi x_n))$$

to $\chi \in \mathrm{Hom}(F,G)$ yields a smooth map Φ from $\mathrm{Hom}(F,G)$ to G^m. Moreover, for every $\chi \in \mathrm{Hom}(F,G)$, we denote by ω_χ the smooth map from G to $\mathrm{Hom}(F,G)$ which assigns $x^{-1}\chi x \in \mathrm{Hom}(F,G)$ to $x \in G$. For later reference we reproduce the tangent behavior of these maps:

Let χ be a homomorphism from F to G; we write $\mathfrak{g}_\chi$ for the Lie algebra $\mathfrak{g}$, viewed as a right F-module via χ and the adjoint representation. The homomorphism χ being viewed as the point $\mathbf{y} = (y_1,\dots,y_n) = (\chi(x_1),\dots,\chi(x_n))$ of G^n, its operation of *left translation* L_χ from $\mathfrak{g}^n$ to $\mathrm{T}_\chi\mathrm{Hom}(F,G)$ amounts to $\mathrm{L}_{y_1} \times \cdots \times \mathrm{L}_{y_n}$ from $\mathfrak{g}^n$ to $\mathrm{T}_{y_1}G \times \cdots \times \mathrm{T}_{y_n}G$. Accordingly, we write $\mathrm{L}_{\Phi(\chi)}$ for the corresponding operation of left translation from $\mathfrak{g}^m$ to $\mathrm{T}_{\Phi(\chi)}G^m = \mathrm{T}_{r_1(\mathbf{y})}G \times \cdots \times \mathrm{T}_{r_m(\mathbf{y})}G$. The following is well known, cf. [8, 27, 31, 32].

Proposition 5.4. *The tangent maps* $\mathrm{T}_e\omega_\chi$ *and* $\mathrm{T}_\chi\Phi$ *and the operations of left translation make commutative the diagram*:

$$\begin{array}{ccccc}
\mathrm{T}_eG & \xrightarrow{\mathrm{T}_e\omega_\chi} & \mathrm{T}_\chi\mathrm{Hom}(F,G) & \xrightarrow{\mathrm{T}_\chi\Phi} & \mathrm{T}_{\Phi(\chi)}G^m \\
{\scriptstyle \mathrm{Id}}\uparrow & & {\scriptstyle \mathrm{L}_\chi}\uparrow & & {\scriptstyle \mathrm{L}_{\Phi(\chi)}}\uparrow \\
\mathfrak{g} & \xrightarrow[\delta^0_\chi]{} & \mathfrak{g}^n & \xrightarrow[\delta^1_\chi]{} & \mathfrak{g}^m
\end{array}$$

where δ^0_χ *and* δ^1_χ *refer to the corresponding operators in* (5.3), *for* $U = \mathfrak{g}_\chi$.

For a homomorphism χ from F to G having the property that each $\chi(r_j)$ lies in the centre of G, the Lie algebra $\mathfrak{g}$ inherits a structure of a right π-module which we still denote by $\mathfrak{g}_\chi$.

Corollary 5.5. *At a homomorphism χ from F to G having the property that each $\chi(r_j)$ lies in the centre of G, left translation L_χ from $C^1(\mathscr{P}, \mathfrak{g}_\chi) = \mathfrak{g}^n$ to $\mathrm{T}_\chi \mathrm{Hom}(F, G)$ identifies the subspace $Z^1(\pi, \mathfrak{g}_\chi)$ of 1-cocycles with the kernel of the tangent map $\mathrm{T}_\chi \Phi$ from $\mathrm{T}_\chi \mathrm{Hom}(F, G)$ to $\mathrm{T}_{\Phi(\chi)} G^m$ and, moreover, the subspace $B^1(\pi, \mathfrak{g}_\chi)$ of 1-coboundaries with the tangent space $\mathrm{T}_\chi(G\chi) \subseteq \mathrm{T}_\chi \mathrm{Hom}(F, G)$ to the G-orbit $G\chi$ of χ in $\mathrm{Hom}(F, G)$.*

Proposition 5.6. *For every $\chi \in \mathrm{Hom}(F, G)$ having the property that each $\chi(r_j)$ lies in the centre of G, for each $x \in G$, the vector space automorphism $\mathrm{Ad}(x)$ of $\mathfrak{g}$ is an isomorphism of right $\mathbb{R}\pi$-modules from $\mathfrak{g}_\chi$ to $\mathfrak{g}_{x\chi}$ and hence induces an isomorphism $\mathrm{Ad}_\flat(x)$ from $\mathrm{H}^1(\pi, \mathfrak{g}_\chi)$ onto $\mathrm{H}^1(\pi, \mathfrak{g}_{x\chi})$.*

Proof. This is left to the reader. □

We now have the machinery in place to relate the derivative of the Wilson loop mapping (2.6) with twisted 1-cochains and integration. We suppose that (5.1) is the presentation $\mathscr{P}$ of the fundamental group $\pi = \pi_1(M, Q)$ having generators and relations represented by the smooth closed curves $w_1, \ldots, w_n$, cf. Sections 2 and 4 above, and attaching maps of the 2-cells of the cell decomposition of M, respectively.

Let A be a central connection on ξ, let $\phi = \rho(A) : F \to G$ and, as before, write $\mathfrak{g}_\phi$ for the Lie algebra $\mathfrak{g}$, with right π-module structure induced by ϕ. Notice the cellular 1-cochains $C^1_{\mathrm{cell}}(M, \mathfrak{g}_\phi)$ coincide with the 1-cochains $C^1(\mathscr{P}, \mathfrak{g}_\phi)$ with reference to $\mathscr{P}$, cf. (5.3).

Theorem 5.7. *The differential $d\rho(A) : \mathrm{T}_A \mathscr{A}(\xi) \to \mathrm{T}_\phi \mathrm{Hom}(F, G)$ of the Wilson loop mapping ρ from $\mathscr{A}(\xi)$ to $\mathrm{Hom}(F, G)$ amounts to the composite of the twisted integration mapping from $\mathrm{T}_A \mathscr{A}(\xi) = \Omega^1(M, \mathrm{ad}(\xi))$ to $C^1(\mathscr{P}, \mathfrak{g}_\phi)$ with left translation L_ϕ from $C^1(\mathscr{P}, \mathfrak{g}_\phi) = \mathfrak{g}^n$ to $\mathrm{T}_\phi \mathrm{Hom}(F, G)$.*

Proof. In view of what was said about the map from $\widetilde{M}$ to P in the proof of Proposition 4.1 and, furthermore, in view of the description (4.4) of the twisted integration mapping, the statement follows at once from Theorem 2.7 and the fact that the cellular 1-cochains $C^1_{\mathrm{cell}}(M, \mathfrak{g}_\phi)$ coincide with the 1-cochains $C^1(\mathscr{P}, \mathfrak{g}_\phi)$. □

6. Reduction of the smooth structures to the local models

We return to the situation of the Introduction. For intelligibility we assemble at first a number of facts established in our papers [10, 11, 14].

The Lie group G is now assumed compact, and its Lie algebra $\mathfrak{g}$ is assumed endowed with an invariant inner product, referred to henceforth as an *orthogonal structure*. The orthogonal structure on $\mathfrak{g}$ combined with the usual wedge product of forms $\wedge$ and integration induces a non-degenerate bilinear pairing $(\cdot,\cdot)$ between $\Omega^*(\Sigma, \mathrm{ad}(\xi))$ and $\Omega^{2-*}(\Sigma, \mathrm{ad}(\xi))$ given by $(\zeta, \lambda) = \int_\Sigma \zeta \wedge \lambda$. In particular, this furnishes a *weakly* symplectic structure σ on $\Omega^1(\Sigma, \mathrm{ad}(\xi))$ and hence one on $\mathscr{A}(\xi)$, cf. [4; 11, (1.1)]. Furthermore, the space $\Omega^2(\Sigma, \mathrm{ad}(\xi))$ of 2-forms being identified with the dual of $\Omega^0(\Sigma, \mathrm{ad}(\xi))$ via $(\cdot,\cdot)$, the assignment to a connection A of its curvature K_A yields a momentum mapping J from $\mathscr{A}(\xi)$ to $\Omega^2(\Sigma, \mathrm{ad}(\xi))$, for the action of the group $\mathscr{G}(\xi)$ of gauge transformations on $\mathscr{A}(\xi)$, cf. [4].

Let A be a central Yang–Mills connection, fixed until further notice. Its operator $d_A : \Omega^*(\Sigma, \mathrm{ad}(\xi)) \to \Omega^{*+1}(\Sigma, \mathrm{ad}(\xi))$ of covariant derivative is a differential. Hence, the cohomology $\mathrm{H}_A^* = \mathrm{H}_A^*(\Sigma, \mathrm{ad}(\xi))$ is defined. The Lie bracket on $\mathfrak{g}$ induces a graded Lie algebra structure $[\cdot,\cdot]_A$ on H_A^* and the orthogonal structure on $\mathfrak{g}$ together with $(\cdot,\cdot)$ a non-degenerate graded bilinear pairing $(\cdot,\cdot)_A$ between H_A^* and H_A^{2-*}. In particular, the latter identifies H_A^2 with the dual of the Lie algebra $\mathrm{H}_A^0 = z_A$ of the stabilizer $Z_A \subseteq \mathscr{G}(\xi)$ of A, and the constituent of $(\cdot,\cdot)_A$ in degree 1 is a symplectic structure σ_A on H_A^1. Moreover, the assignment to $\eta \in \mathrm{H}_A^1$ of $\Theta_A(\eta) = \frac{1}{2}[\eta,\eta]_A$ yields a momentum mapping Θ_A from H_A^1 to H_A^2 for the Z_A-action on H_A^1, cf. [11, (1.2.5)], in fact, the *unique* one with $\Theta_A(0) = 0$. Write H_A for its reduced space. By [11, (2.32)], the reduced space H_A is a local model for $N(\xi)$ near $[A]$ in the sense that the data induce a homeomorphism of a neighborhood of $[0] \in \mathrm{H}_A$ onto a neighborhood of $[A]$ in $N(\xi)$. Our aim is to show that H_A is a local model near $[A]$ for *all* the structure of interest to us. To this end we observe first that H_A inherits an obvious smooth structure which we explain under more general circumstances:

Let M be a (finite-dimensional) symplectic manifold, with a hamiltonian action of a compact Lie group K and momentum mapping μ from M to k^*, and let $V = \mu^{-1}(0)$ denote its zero locus, so that the reduced space looks like $M_{\mathrm{red}} = V/K$. With respect to the decomposition into connected components of orbit types, the algebra of Whitney smooth functions

$$C^\infty(V) = C^\infty(M)/I_V, \tag{6.1.1}$$

where I_V refers to the ideal of functions that vanish on V, endows V with a smooth structure; likewise, the algebra

$$C^\infty(M_{\mathrm{red}}) = C^\infty(M)^K/(I_V \cap (C^\infty(M)^K)) \tag{6.1.2}$$

yields a smooth structure on the reduced space in an obvious fashion, where $C^\infty(M)^K$ refers to the subalgebra of K-invariant functions. By construction, $C^\infty(M_{\mathrm{red}})$ is an algebra of continuos functions on M_{red}. In particular, this construction, applied to $M = \mathrm{H}_A^1$, $\mu = \Theta_A$, and $K = Z_A$, yields the smooth space $(\mathrm{H}_A, C^\infty(\mathrm{H}_A))$. We mention in passing that it inherits a structure of stratified symplectic space [30]. Our present aim is to show that the latter is a local model for $(N(\xi), C^\infty(N(\xi)))$ near $[A] \in N(\xi)$.

Let $(X, C^\infty(X))$ be a smooth space, and let Y be an open subset of X. In order to avoid to have to talk about *sheaves* of germs of smooth functions, we *define* a notion of *induced* smooth structure on Y in the following way: We shall say that a continuous function f on Y is *smooth* if every point y of Y has an open neighborhood U so that the restriction of f to U coincides with the restriction to U of a smooth function on X, that is, a member of $C^\infty(X)$. These smooth functions on Y constitute an algebra $C^\infty(Y)$ of continuous functions on Y which we refer to as its *induced smooth structure.* Notice the restriction map from $C^\infty(X)$ to $C^\infty(Y)$ is *not* in general surjective. When X is a smooth manifold, with its standard smooth structure, and Y an open subset of X, the algebra $C^\infty(Y)$ is that of smooth functions on Y in the ordinary sense.

Theorem 6.2. *Near $[A] \in N(\xi)$, the smooth space $(\mathrm{H}_A, C^\infty(\mathrm{H}_A))$ is a local model for $(N(\xi), C^\infty(N(\xi)))$. More precisely, the choice of A (in its class $[A]$) induces a diffeomorphism of an open neighborhood W_A of $[0] \in \mathrm{H}_A$ onto an open neighborhood U_A of $[A] \in N(\xi)$, where W_A and U_A are endowed with the induced smooth structures $C^\infty(W_A)$ and $C^\infty(U_A)$, respectively.*

To spell out the representation space version of Theorem 6.2, let $\phi = \rho(A) : \Gamma \to G$. Every $\psi \in \mathrm{Hom}_\xi(\Gamma, G)$ is manifestly of this form and, given such a ψ, a central Yang–Mills connection on ξ which is mapped to ψ under ρ is unique up to based gauge transformations; see [10]. The same kind of structure as that denoted above by $(\cdot,\cdot)_A$, Θ_A, and $[\cdot,\cdot]_A$, is available on $\mathrm{H}^*_\phi = \mathrm{H}^*(\pi, \mathfrak{g}_\phi)$ and the twisted integration mapping from H^*_A to H^*_ϕ identifies the respective structures. In particular, the Lie bracket on $\mathfrak{g}$ induces a graded Lie algebra structure $[\cdot,\cdot]_\phi$ on H^*_ϕ. Further, the orthogonal structure on $\mathfrak{g}$ induces a graded non-degenerate bilinear pairing on H^*_ϕ which in degree 1 amounts to a symplectic structure σ_ϕ on H^1_ϕ, and the assignment to $\eta \in \mathrm{H}^1_\phi$ of $\Theta_\phi(\eta) = \frac{1}{2}[\eta,\eta]_\phi$ yields a momentum mapping Θ_ϕ from H^1_ϕ to H^2_ϕ, for the action of the stabilizer $Z_\phi \subseteq G$ of $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$ on H^1_ϕ; notice that the surjection (1.1) passes to an isomorphism from Z_A to Z_ϕ identifying the stabilizers. Moreover, the construction (6.1.2), applied to $M = \mathrm{H}^1_\phi$, $\mu = \Theta_\phi$, and $K = Z_\phi$, yields the smooth space $(\mathrm{H}_\phi, C^\infty(\mathrm{H}_\phi))$. It also inherits a structure of stratified symplectic space.

Theorem 6.3. *Near $[\phi] \in \mathrm{Rep}_\xi(\Gamma, G)$, the smooth space $(\mathrm{H}_\phi, C^\infty(\mathrm{H}_\phi))$ is a local model for the smooth space $(\mathrm{Rep}_\xi(\Gamma, G), C^\infty(\mathrm{Rep}_\xi(\Gamma, G)))$. More precisely, the choice of ϕ (in its class $[\phi]$) induces a diffeomorphism of an open neighborhood W_ϕ of $[0] \in \mathrm{H}_\phi$ onto an open neighborhood U_ϕ of $[\phi] \in \mathrm{Rep}_\xi(\Gamma, G)$, where W_ϕ and U_ϕ are endowed with the induced smooth structures $C^\infty(W_\phi)$ and $C^\infty(U_\phi)$, respectively.*

Addendum. *Under the circumstances of Theorems 6.2 and 6.3, for suitable choices of the data, twisted integration identifies the local models. More precisely, for a suitable choice of the data, the twisted integration mapping $\mathrm{Int_A}$ from H^*_A to H^*_ϕ identifies the symplectic structures σ_A on H^1_A and σ_ϕ on H^1_ϕ, the stabilizers Z_A and Z_ϕ, and momentum mappings Θ_A and Θ_ϕ, and hence the stratified symplectic spaces H_A and*

H_ϕ. *Consequently, the twisted integration mapping and the Wilson loop mapping* $\rho_\flat$ *from* $N(\xi)$ *to* $\mathrm{Rep}_\xi(\Gamma, G)$ *yield a commutative diagram*

$$\begin{array}{ccc} W_A & \longrightarrow & U_A \\ {\scriptstyle \mathrm{Int}_{A\sharp}} \big\downarrow & & \big\downarrow {\scriptstyle \rho_\flat|} \\ W_\phi & \longrightarrow & U_\phi \end{array}$$

of diffeomorphisms between smooth spaces, the four spaces being endowed with the smooth structures mentioned earlier. Here $\mathrm{Int}_{A\sharp}$ *denotes the map induced by twisted integration and* $\rho_\flat|$ *the restriction of the Wilson loop mapping to* U_A, *and the unlabelled horizontal arrows are the maps coming into play in Theorems* 6.2 *and* 6.3.

The proofs of Theorems 6.2 and 6.3 require some preparation. Near A, the pre-image $\mathscr{A}_A = J^{-1}(\mathscr{H}^2_A(\Sigma, \mathrm{ad}(\xi)))$ of the space $\mathscr{H}^2_A(\Sigma, \mathrm{ad}(\xi))$ of harmonic 2-forms is a smooth Z_A-invariant submanifold of $\mathscr{A}(\xi)$, cf. [11], and the operator d_A gives rise to the exact sequence

$$0 \to \mathrm{T}_A\mathscr{A}_A \to \mathrm{T}_A\mathscr{A}(\xi) \xrightarrow{d_A} \Omega^2(\Sigma, \mathrm{ad}(\xi)) \to \mathrm{H}^2_A(\Sigma, \mathrm{ad}(\xi)) \to 0 \tag{6.4}$$

of real vector spaces whence, in particular, $\mathrm{T}_A\mathscr{A}_A = Z^1_A(\Sigma, \mathrm{ad}(\xi))$, the corresponding space of 1-cocycles; here the tangent space $\mathrm{T}_A\mathscr{A}(\xi)$ is identified with $\Omega^1(\Sigma, \mathrm{ad}(\xi))$ as usual. Let $\mathscr{M}_A$ be a smooth finite-dimensional Z_A-invariant submanifold of $\mathscr{A}_A$ containing A, of the kind coming into play in the proofs of [11, (2.32)] and [14, (1.2)]; in particular, $\mathrm{T}_A\mathscr{M}_A = \mathscr{H}^1_A(\Sigma, \mathrm{ad}(\xi))$, the subspace of harmonic 1-forms in $\Omega^1(\Sigma, \mathrm{ad}(\xi))$; in Proposition 6.11 below we shall pick $\mathscr{M}_A$ suitably. We remind the reader that $\mathscr{N}(\xi) \subseteq \mathscr{A}(\xi)$ denotes the subspace of central Yang–Mills connections. It is clear that the assignment to a pair (γ, A) in $\mathscr{G}(\xi) \times \mathscr{A}(\xi)$ of $\gamma(A)$ induces an injective $\mathscr{G}(\xi)$-invariant immersion

$$\mathscr{G}(\xi) \times_{Z_A} \mathscr{M}_A \to \mathscr{A}(\xi) \tag{6.5}$$

identifying $\mathscr{G}(\xi) \times_{Z_A} \mathscr{M}_A$ with a smooth $\mathscr{G}(\xi)$-invariant codimension 0 submanifold of $\mathscr{A}_A$ containing a $\mathscr{G}(\xi)$-invariant neighborhood of A in $\mathscr{N}(\xi)$. In particular, the derivative of this immersion at A amounts to the inclusion of $Z^1_A(\Sigma, \mathrm{ad}(\xi))$ into $\Omega^1(\Sigma, \mathrm{ad}(\xi))$.

By [11, (2.18)], the 2-form σ on $\mathscr{A}(\xi)$ passes to a symplectic structure ω_A on the smooth manifold $\mathscr{M}_A$, and J induces a momentum mapping ϑ_A from $\mathscr{M}_A$ to $\mathrm{H}^2_A(\Sigma, \mathrm{ad}(\xi))$ for the Z_A-action, with $\vartheta(A) = 0$; here $\mathrm{H}^2_A(\Sigma, \mathrm{ad}(\xi))$ is identified with the dual z^*_A of the Lie algebra $z_A = \mathrm{H}^0_A(\Sigma, \mathrm{ad}(\xi))$ as explained above; see (2.21) in [11] for details. We now consider the Marsden–Weinstein reduced space $\mathscr{W}_A = \vartheta_A^{-1}(0)/Z_A$. It is obvious that (6.5) induces an injection

$$\mathscr{W}_A \to N(\xi) \tag{6.6}$$

of $\mathscr{W}_A$ into $N(\xi)$ identifying $\mathscr{W}_A$ with an open neighborhood U_A of $[A]$ in $N(\xi)$, and in this way (6.6) furnishes a *model* of a neighborhood of $[A]$ in $N(\xi)$. Likewise, the composite of (6.6) with the Wilson loop mapping $\rho_\flat$ from $N(\xi)$ onto $\mathrm{Rep}_\xi(\Gamma, G)$ is an injection

$$\mathscr{W}_A \to \mathrm{Rep}_\xi(\Gamma, G) \tag{6.7}$$

of $\mathscr{W}_A$ into $\mathrm{Rep}_\xi(\Gamma, G)$ identifying $\mathscr{W}_A$ with an open neighborhood U_ϕ of $[\phi]$ in $\mathrm{Rep}_\xi(\Gamma, G)$ whence (6.7) furnishes a *model* of a neighborhood of $[\phi]$ in $\mathrm{Rep}_\xi(\Gamma, G)$. With respect to the decompositions into connected components of orbit types, the embeddings (6.6) and (6.7) are decomposition preserving. The construction (6.1.2) applied to $M = \mathscr{W}_A$, $\mu = \vartheta_A$, and $K = Z_A$, yields a smooth structure $C^\infty(\mathscr{W}_A)$ on $\mathscr{W}_A$, and the embeddings (6.6) and (6.7) are smooth since they preserve the decompositions into orbit types. Let $C^\infty(U_A)$ and $C^\infty(U_\phi)$ be the induced smooth structures on U_A and U_ϕ, respectively; it is obvious that (6.6) and (6.7) induce smooth maps

$$(\mathscr{W}_A, C^\infty(\mathscr{W}_A) \to (U_A, C^\infty(U_A)) \tag{6.8}$$

and

$$(\mathscr{W}_A, C^\infty(\mathscr{W}_A)) \to (U_\phi, C^\infty(U_\phi)). \tag{6.9}$$

Moreover, the Wilson loop mapping from $N(\xi)$ to $\mathrm{Rep}_\xi(\Gamma, G)$ passes to a smooth map

$$(U_A, C^\infty(U_A)) \to (U_\phi, C^\infty(U_\phi)) \tag{6.10}$$

in such a way that (6.9) is the composite of (6.8) and (6.10). Since each of (6.8)–(6.10) are homeomorphisms between the underlying spaces, the induced maps $C^\infty(U_A) \to C^\infty(\mathscr{W}_A)$, etc. between the algebras of smooth functions are injective. We now show that they are surjective, for a suitable choice of the data. This will almost establish the statements of Theorems 6.2 and 6.3, except that $\mathscr{W}_A$ comes into play rather than an open neighborhood W_A of $[0] \in \mathrm{H}_A$. We proceed as follows:

The composite

$$\mathscr{G}(\xi) \times_{Z_A} \mathscr{M}_A \to \mathrm{Hom}(F, G). \tag{6.11.1}$$

of (6.5) with the Wilson loop mapping from $\mathscr{A}(\xi)$ to $\mathrm{Hom}(F, G)$ is $\mathscr{G}(\xi)$-invariant, with respect to the $\mathscr{G}(\xi)$-action on $\mathrm{Hom}(F, G)$ induced by (1.1) and, furthermore, factors through the obvious surjection

$$\mathscr{G}(\xi) \times_{Z_A} \mathscr{M}_A \to G \times_{Z_A} \mathscr{M}_A \tag{6.11.2}$$

and hence passes to a smooth G-invariant map

$$G \times_{Z_A} \mathscr{M}_A \to \mathrm{Hom}(F, G). \tag{6.11.3}$$

Proposition 6.11. *For a suitable choice of $\mathscr{M}_A$, the map* (6.11.3) *is a smooth injective G-invariant immersion identifying $G \times_{Z_A} \mathscr{M}_A$ with a smooth G-submanifold of* $\mathrm{Hom}(F, G)$ *containing a G-invariant neighborhood of ϕ in* $\mathrm{Hom}_\xi(\Gamma, G)$.

To prepare for the proof, we recall that the tangent space $\mathrm{T}_A\mathscr{M}_A$ equals the space $\mathscr{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ of harmonic 1-forms and the tangent space $\mathrm{T}_{(e,A)}(\mathscr{G}(\xi) \times_{Z_A} \mathscr{M}_A)$ equals the space $Z^1_A(\Sigma, \mathrm{ad}(\xi))$ of 1-cocycles; the latter, in turn, decomposes into the direct sum of $B^1_A(\Sigma, \mathrm{ad}(\xi))$ and $\mathscr{H}^1_A(\Sigma, \mathrm{ad}(\xi))$. At the point (e, A), the tangent space of $G \times_{Z_A} \mathscr{M}_A$ equals likewise the direct sum of $B^1(\pi, \mathfrak{g}_\phi)$ and $\mathscr{H}^1_A(\Sigma, \mathrm{ad}(\xi))$, and the smooth map (6.11.2) has tangent map

$$B^1_A(\Sigma, \mathrm{ad}(\xi)) \oplus \mathscr{H}^1_A(\Sigma, \mathrm{ad}(\xi)) \xrightarrow{(\mathrm{Int}_A|, \mathrm{Id})} B^1(\pi, \mathfrak{g}_\phi) \oplus \mathscr{H}^1_A(\Sigma, \mathrm{ad}(\xi)), \qquad (6.11.4)$$

where $\mathrm{Int}_A|$ refers to the restriction of the twisted integration mapping Int_A from $\Omega^*(\Sigma, \mathrm{ad}(\xi))$ to $C^*(\mathscr{P}, \mathfrak{g}_\phi)$, cf. (4.4), to the 1-coboundaries. However, the restriction of the twisted integration mapping to the subspace of 1-cocycles $Z^1_A(\Sigma, \mathrm{ad}(\xi))$ amounts to a surjection of $Z^1_A(\Sigma, \mathrm{ad}(\xi))$ onto $Z^1(\pi, \mathfrak{g}_\phi)$, as inspection of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^0_A & \longrightarrow & \Omega^0 & \longrightarrow & Z^1_A & \longrightarrow & \mathrm{H}^1_A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^0_\phi & \longrightarrow & C^0 & \longrightarrow & Z^1_\phi & \longrightarrow & \mathrm{H}^1_\phi & \longrightarrow & 0 \end{array}$$

with the obvious unlabelled arrows reveals, where we have written $\mathrm{H}^*_A = \mathrm{H}^*_A(\Sigma, \mathrm{ad}(\xi))$, $\Omega^0 = \Omega^0(\Sigma, \mathrm{ad}(\xi))$, $Z^1_A = Z^1_A(\Sigma, \mathrm{ad}(\xi))$, $\mathrm{H}^*_\phi = \mathrm{H}^*(\pi, \mathfrak{g}_\phi)$, $C^0 = C^0(\mathscr{P}, \mathfrak{g}_\phi)$, $Z^1_\phi = Z^1(\pi, \mathfrak{g}_\phi)$ for short. The diagram has exact rows; its outermost columns are isomorphisms; and the arrow from Ω^0 to C^0 is manifestly surjective. This implies that (6.11.4) is surjective. In fact, write $\mathscr{H}^*(\pi, \mathfrak{g}_\phi)$ for the isomorphic image in $Z^*(\pi, \mathfrak{g}_\phi)$ of the subspace of harmonic forms $\mathscr{H}^*_A(\Sigma, \mathrm{ad}(\xi))$ in $\Omega^*(\Sigma, \mathrm{ad}(\xi))$ under the twisted integration mapping Int_A so that the canonical epimorphism from $Z^*(\pi, \mathfrak{g}_\phi)$ onto $\mathrm{H}^*(\pi, \mathfrak{g}_\phi)$ passes to an isomorphism from $\mathscr{H}^*(\pi, \mathfrak{g}_\phi)$ onto $\mathrm{H}^*(\pi, \mathfrak{g}_\phi)$. The direct sum of $B^1(\pi, \mathfrak{g}_\phi)$ and $\mathscr{H}^1(\pi, \mathfrak{g}_\phi)$ equals the space $Z^1(\pi, \mathfrak{g}_\phi)$ of 1-cocycles, and the surjection of $Z^1_A(\Sigma, \mathrm{ad}(\xi))$ onto $Z^1(\pi, \mathfrak{g}_\phi)$ factors through the induced isomorphism $(\mathrm{Id}, \mathrm{Int}_A)$ from $B^1(\pi, \mathfrak{g}_\phi) \oplus \mathscr{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ onto $B^1(\pi, \mathfrak{g}_\phi) \oplus \mathscr{H}^1(\pi, \mathfrak{g}_\phi)$, whence (6.11.4) is surjective. Consequently, (6.11.2) a submersion near the point (e, A).

Proof of Proposition 6.11. The tangent map of (6.11.3) at the point (e, A) is the composite of:

(i) the isomorphism $(\mathrm{Id}, \mathrm{Int}_A)$ from $B^1(\pi, \mathfrak{g}_\phi) \oplus \mathscr{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ onto $B^1(\pi, \mathfrak{g}_\phi) \oplus \mathscr{H}^1(\pi, \mathfrak{g}_\phi)$,

(ii) the inclusion of $B^1(\pi, \mathfrak{g}_\phi) \oplus \mathscr{H}^1(\pi, \mathfrak{g}_\phi) = Z^1(\pi, \mathfrak{g}_\phi)$ into $C^1(\mathscr{P}, \mathfrak{g}_\phi)$ and, finally,

(iii) left translation L_ϕ from $C^1(\mathscr{P}, \mathfrak{g}_\phi)$ to $\mathrm{T}_\phi \mathrm{Hom}(F, G)$.

In fact, in view of Theorem 5.7 the derivative of (6.11.1) at A amounts to the twisted integration mapping Int_A from $\Omega^1(\Sigma, \mathrm{ad}(\xi))$ to $C^1(\mathscr{P}, \mathfrak{g}_\phi)$, restricted to the tangent space $\mathrm{T}_A \mathscr{A}_A = Z^1_A(\Sigma, \mathrm{ad}(\xi)) \subseteq \Omega^1(\Sigma, \mathrm{ad}(\xi))$, combined with left translation L_ϕ from $C^1(\mathscr{P}, \mathfrak{g}_\phi)$ to $\mathrm{T}_\phi \mathrm{Hom}(F, G)$. However, it is manifest that this tangent map factors through the map from $Z^1_A(\Sigma, \mathrm{ad}(\xi))$ onto $Z^1(\pi, \mathfrak{g}_\phi) = B^1(\pi, \mathfrak{g}_\phi) \oplus \mathscr{H}^1(\pi, \mathfrak{g}_\phi)$ induced by the twisted integration mapping and hence through (6.11.4). Hence the tangent map of (6.11.3) at the point (e, A) decomposes into the three pieces (i)–(iii) and is therefore injective since so is the inclusion of $Z^1(\pi, \mathfrak{g}_\phi)$ into $C^1(\mathscr{P}, \mathfrak{g}_\phi)$. This implies that the smooth map (6.11.3) is an immersion near (e, A); hence, for a suitable choice of $\mathscr{M}_A$, it is injective.

Finally, since $\mathscr{G}(\xi) \times_{Z_A} \mathscr{M}_A$, viewed as a smooth $\mathscr{G}(\xi)$-invariant codimension 0 submanifold of $\mathscr{A}_A$ via (6.5), contains a $\mathscr{G}(\xi)$-invariant neighborhood of A in $\mathscr{N}(\xi)$, and, furthermore, since (6.11.2) is a submersion, the image of $G \times_{Z_A} \mathscr{M}_A$ under (6.11.3) contains a G-invariant neighborhood of ϕ in $\mathrm{Hom}_\xi(\Gamma, G)$ as asserted. □

Henceforth, we assume that the smooth manifold $\mathscr{M}_A$ has been chosen in such a way that (6.11.3) is injective. This enables us to relate the smooth structures of $N(\xi)$ near $[A]$ and of $\mathrm{Rep}_\xi(\Gamma, G)$ near $[\phi]$ with that of $\mathscr{W}_A$ near A by means of (6.11.3).

To verify surjectivity of the induced map from $C^\infty(U_\phi)$ to $C^\infty(\mathscr{W}_A)$, let $h : \mathscr{W}_A \to \mathbb{R}$ be a function in $C^\infty(\mathscr{W}_A)$. Then there is a unique continuous function f on U_ϕ whose composite with (6.9) equals h. We must show that f lies in $C^\infty(U_\phi)$. In order to see this, let H be a smooth Z_A-invariant function on $\mathscr{M}_A$ representing h. Abusing notation, we denote its canonical extension to a G-invariant function on $G \times_{Z_A} \mathscr{M}_A$ by H as well. The space $G \times_{Z_A} \mathscr{M}_A$ being identified with a smooth G-invariant submanifold of $\mathrm{Hom}(F, G)$ via (6.11.3), we must show that H extends locally to a G-invariant function on $\mathrm{Hom}(F, G)$. However, given a homomorphism ψ from Γ to G in the image of (6.11.3), there is an open G-invariant neighborhood U of ψ in the image of (6.11.3) and a smooth G-invariant function $\widetilde{H}$ on $\mathrm{Hom}(F, G)$ whose restriction to U coincides with the restriction of H to U. By construction, $\widetilde{H}$ represents a function in $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ and hence one in $C^\infty(U_\phi)$ which, on a neighborhood of $[\psi]$ in U_ϕ, coincides with f. Since ψ is arbitrary, this shows that f is smooth as asserted, that is, lies in $C^\infty(U_\phi)$. Consequently (6.9), and hence (6.8) and (6.10), are diffeomorphisms of smooth spaces.

To complete the proofs of Theorems 6.2 and 6.3 we recall that, by [11, (2.31)], a suitable Kuranishi map furnishes a Z_A-equivariant symplectomorphism Φ_A from $\mathscr{M}_A$ onto a Z_A-invariant ball B_A in $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ about the origin, cf. [11, (2.29)], and this map preserves the momentum mappings Θ_A and ϑ_A. Marsden–Weinstein reduction applied to B_A and Θ_A, restricted to B_A, then yields the open subspace W_A of H_A we are looking for, and the Kuranishi map induces a homeomorphism of a neighborhood of $[A]$ in $N(\xi)$ onto W_A. See [11, (2.32)] for details. Moreover, the construction (6.1.2) applied to $M = B_A$, $\mu = \Theta_A$, restricted to B_A, and $K = Z_A$, yields a smooth structure $C^\infty(W_A)$ in such a way that Φ_A induces a diffeomorphism from $(\mathscr{W}_A, C^\infty(\mathscr{W}_A))$ onto $(W_A, C^\infty(W_A))$. Hence the data induce a diffeomorphism of $(W_A, C^\infty(W_A))$ onto $(U_A, C^\infty(U_A))$. This completes the proof of Theorem 6.2. The same construction applies to the image B_ϕ of B_A in H^1_ϕ under the twisted integration mapping Int_A from H^1_A to H^1_ϕ, the momentum mapping Θ_ϕ, and the stabilizer Z_ϕ of ϕ; it yields the open subspace W_ϕ of H_ϕ we are looking for and a smooth structure $C^\infty(W_\phi)$, together with a diffeomorphism of $(W_\phi, C^\infty(W_\phi))$ onto $(U_\phi, C^\infty(U_\phi))$. This completes the proof of Theorem 6.3.

Moreover, the constructions have been carried out in such a way that the statement of the Addendum is immediate. □

We now proceed towards the proof of Theorem 3.8. Henceforth, A will denote a central Yang–Mills connection which is no longer fixed. At first, we must show that the restriction of a smooth function on $N(\xi)$ and likewise on $\mathrm{Rep}_\xi(\Gamma, G)$ to a stratum is a smooth function on the stratum in the ordinary sense. In view of Theorems 6.2 and 6.3, it suffices to prove that, under the circumstances of the construction (6.1.2), the restriction of a smooth function in $C^\infty(M_{\mathrm{red}})$ to a piece of M_{red} is smooth in the ordinary sense. However, this amounts to the fact that the restriction to a smooth submanifold of a smooth function defined on a smooth manifold is smooth on the submanifold.

As immediate consequence of the Addendum to Theorem 6.3 we see that the Wilson loop mapping from $N(\xi)$ to $\mathrm{Rep}_\xi(\Gamma, G)$ is *locally* a diffeomorphism. To see that this is *globally* so, we establish the existence of suitable partitions of unity. We begin with the following, the proof of which is routine and therefore left to the reader.

Lemma 6.12. *Let W be a finite-dimensional complex representation of a compact Lie group K, and let B be an open K-invariant neighborhood of the origin. Then there are open K-invariant neighborhoods Q and R of the origin with $\overline{Q} \subseteq R$ and $\overline{R} \subseteq B$, together with a smooth K-invariant real-valued function H on B with*

$$H|\overline{Q} = 1, \qquad H|B \setminus R = 0.$$

Under the circumstances of Lemma 6.12, suppose the K-representation is unitary, let μ denote its unique momentum mapping from W to k^* having the value zero at the origin, let W_{red} be its reduced space, and let $C^\infty(W_{\mathrm{red}})$ be the corresponding smooth structure (6.1.2). Here is an immediate consequence of Lemma 6.12.

Corollary 6.13. *Let P be an open neighborhood in W_{red} of the class $[0]$ of the origin, with its induced smooth structure $C^\infty(P)$. Then there are open neighborhoods Q and R in W_{red} of $[0]$ with $\overline{Q} \subseteq R$ and $\overline{R} \subseteq P$, together with a smooth function $h \in C^\infty(P)$, with*

$$h|\overline{Q} = 1, \qquad h|P \setminus R = 0.$$

Corollary 6.14. *Given an arbitrary open neighborhood $U_{[A]}$ of the point $[A]$ of $N(\xi)$, there are open neighborhoods $Q_{[A]}$ and $R_{[A]}$ of $[A]$ in $N(\xi)$, with $\overline{Q}_{[A]} \subseteq R_{[A]}$ and $\overline{R}_{[A]} \subseteq U_{[A]}$, together with a smooth function $h_{[A]}$ on $N(\xi)$ with*

$$h_{[A]}|\overline{Q}_{[A]} = 1, \qquad h_{[A]}|N(\xi) \setminus R_{[A]} = 0.$$

Likewise, given an arbitrary open neighborhood $U_{[\phi]}$ of the point $[\phi]$ of $\mathrm{Rep}_\xi(\Gamma, G)$, there are open neighborhoods $Q_{[\phi]}$ and $R_{[\phi]}$ of $[\phi]$ in $\mathrm{Rep}_\xi(\Gamma, G)$, with $\overline{Q}_{[\phi]} \subseteq R_{[\phi]}$ and $\overline{R}_{[\phi]} \subseteq U_{[\phi]}$, together with a smooth function $h_{[\phi]}$ on $\mathrm{Rep}_\xi(\Gamma, G)$ with

$$h_{[\phi]}|\overline{Q}_{[\phi]} = 1, \qquad h_{[\phi]}|\mathrm{Rep}_\xi(\Gamma, G) \setminus R_{[\phi]} = 0.$$

When $[\phi] = \rho_\flat[A]$ *and* $U_{[\phi]} = \rho_\flat(U_{[A]})$, *under the Wilson loop mapping* $\rho_\flat$ *from* $N(\xi)$ *to* $\mathrm{Rep}_\xi(\Gamma, G)$, *things may be arranged in such a way that* $\rho_\flat$ *identifies* $Q_{[A]}$, $R_{[A]}$, *and* $h_{[A]}$ *with, respectively,* $Q_{[\phi]}$, $R_{[\phi]}$, *and* $h_{[\phi]}$.

Proof. This is a consequence of Theorems 6.2 and 6.3, its Addendum, and Corollary 6.13. □

For each point $[A]$ of $N(\xi)$, pick an injection of W_A into $N(\xi)$ of the kind coming into play in Theorem 6.2 above, and write $U_A \subseteq N(\xi)$ for the image of W_A in $N(\xi)$, so that U_A is an open neighborhood of $[A]$ in $N(\xi)$, as in Theorem 6.2; we then write $\phi = \rho(A)$ and $U_\phi \subseteq \mathrm{Rep}_\xi(\Gamma, G)$ for the image of U_A under the Wilson loop mapping, as in Theorem 6.3. Here is our *third main result.*

Theorem 6.15. *There is a finite open cover of* $N(\xi)$ *by open sets of the kind* U_A *together with a smooth partition of unity subordinate to this cover. Moreover, there is a finite open cover of* $\mathrm{Rep}_\xi(\Gamma, G)$ *by open sets of the kind* U_ϕ *together with a smooth partition of unity subordinate to this cover in such a way that the Wilson loop mapping identifies the covers and partitions of unity.*

Proof. By Corollary 6.14, for every central Yang–Mills connection A, there are open neighborhoods $Q_{[A]}$ and $R_{[A]}$ of $[A]$ in $N(\xi)$, with $\overline{Q}_{[A]} \subseteq R_{[A]}$ and $\overline{R}_{[A]} \subseteq U_{[A]}$, together with a smooth function $h_{[A]}$ on $N(\xi)$ with

$$h_{[A]}|\overline{Q}_{[A]} = 1, \qquad h_{[A]}|N(\xi)\backslash R_{[A]} = 0.$$

The subsets $Q_{[A]}$ constitute an open cover of $N(\xi)$. Since $N(\xi)$ is compact, there is a finite subcover $\{Q_1, \dots, Q_m\}$. Each Q_λ lies in some U_λ; the corresponding family $\{U_\lambda\}$ is the open cover of $N(\xi)$ we are aiming at. Moreover, for each λ, there is a function $h_\lambda \in C^\infty(N(\xi))$ so that h_λ has the constant value 1 on $\overline{Q}_\lambda$ and is zero outside an open neighborhood of $\overline{Q}_\lambda$ in U_λ. Let $h = \sum h_\lambda$; then $h \in C^\infty(N(\xi))$ and $h[A] \geq 1$, whatever $[A] \in N(\xi)$. The family $\{e_\lambda\}$, where $e_\lambda = h_\lambda/h$, then furnishes the desired partition of unity.

The same kind of construction yields the asserted open cover and smooth partition of unity for $\mathrm{Rep}_\xi(\Gamma, G)$, and the Wilson loop mapping identifies the covers and partitions of unity. □

We can now complete the proof of Theorem 3.8: Let $\{h_1, \dots, h_m\}$ be the partition of unity subordinate to the open cover $\{U_1, \dots, U_m\}$ in Theorem 6.15. Given $f \in C^\infty(N(\xi))$, let $f_\lambda = f h_\lambda$; this is a smooth function, that is, $f_\lambda \in C^\infty(N(\xi))$. By construction, each f_λ has a pre-image in $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$. Consequently, f has a pre-image in $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ whence the map from $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ to $C^\infty(N(\xi))$ induced by the Wilson loop mapping is surjective. This completes the proof of Theorem 3.8.

7. Cohomology, Zariski tangent spaces, and local semi-algebraicity

In this section we study the infinitesimal structure of our spaces of interest.

Given a smooth space $(X, C^\infty(X))$, for each point $x \in X$, the *ideal* $\mathfrak{m}_x$ of x consists of all functions in $C^\infty(X)$ vanishing at x; as usual, the space of *differentials* $\Omega_x(X)$ at x is the vector space $\Omega_x(X) = \mathfrak{m}_x/\mathfrak{m}_x^2$, and the *Zariski tangent space* T_xX is the dual space $\mathrm{T}_xX = \Omega_x(X)^* = (\mathfrak{m}_x/m_x^2)^*$. When X is a smooth manifold near a point x in the usual sense, with standard smooth structure near x, the Zariski tangent space boils down to the usual smooth tangent space T_xX whence there is no risk of confusion in notation. Here is another well-known description of the Zariski tangent space: Let $x \in X$ and view $\mathbb{R}$ as a $C^\infty(X)$-module, written $\mathbb{R}_x$, by means of the evaluation mapping from $C^\infty(X)$ to $\mathbb{R}$ which assigns to a function f its value $f(x)$ at $x \in X$; now a *derivation* at $x \in X$ is a linear map d from $C^\infty(X)$ to $\mathbb{R}$ satisfying the usual *Leibniz* rule

$$\mathrm{d}(fh) = (\mathrm{d}f)h(x) + f(x)\,\mathrm{d}h.$$

We denote the real vector space of all derivations of $C^\infty(X)$ in $\mathbb{R}_x$ by $\mathrm{Der}(C^\infty(X), \mathbb{R}_x)$. For $x \in X$, the assignment to $\phi \in \mathrm{T}_xX$ of the derivation d_ϕ at x given by $\mathrm{d}_\phi(f) = \phi(f - f_x)$ identifies T_xX with $\mathrm{Der}(C^\infty(X), \mathbb{R}_x)$; here $f \in C^\infty(X)$ and f_x denotes the function having constant value $f(x)$.

Given smooth spaces $(X, C^\infty(X))$, $(Y, C^\infty(Y))$, and a smooth map ϕ from X to Y, the *derivative* at a point $x \in X$ is the dual $\mathrm{d}\phi_x : \mathrm{T}_xX \to \mathrm{T}_{\phi x}Y$ of the linear map from $\mathfrak{m}_{\phi(x)}/\mathfrak{m}_{\phi(x)}^2$ to $\mathfrak{m}_x/\mathfrak{m}_x^2$ induced by ϕ.

Let $(X, C^\infty(X))$ be a smooth space, and let U be an open subset of X. We shall say that a smooth function h on X is a *bump* function *with support in* U if there are open subsets Q and R of X with $\overline{Q} \subseteq R$ and $\overline{R} \subseteq U$, so that

$$h|\overline{Q} = 1, \qquad h|X \backslash R = 0.$$

Given a point x of X, we shall say that X *has smooth bump functions arbitrarily close to* x if for every open neighborhood U of x in X there is a smooth bump function h having the value 1 near x, with support in U. From Corollary 6.14 above we deduce at once the following.

Proposition 7.1. *The spaces* $N(\xi)$ *and* $\mathrm{Rep}_\xi(\Gamma, G)$ *have smooth bump functions arbitrarily close to every point.*

Let $(X, C^\infty(X))$ be a smooth space having smooth bump functions arbitrarily close to every point. We recall the following well-known fact and reproduce a proof for completeness.

Proposition 7.2. *For every connected open subset* Y, *with induced smooth structure* $C^\infty(Y)$, *the inclusion* j *from* Y *to* X *induces an isomorphism of Zariski tangent spaces for every* $x \in Y$.

Proof. If f is a smooth function which is constant on a neighborhood U of $x \in X$, then $\mathrm{d}f$ is zero for every derivation d from $C^\infty(X)$ to $\mathbb{R}_x$. In fact, the differential of a constant function (on X) is zero, and hence we may assume that f has the value zero on U. Given a bump function h with support in U, we then have

$$0 = \mathrm{d}(fh) = \mathrm{d}fh(x) + f(x)\mathrm{d}h = \mathrm{d}f$$

since $h(x) = 1$ and $f(x) = 0$.

In particular, for every derivation d from $C^\infty(X)$ to $\mathbb{R}_x$, the value $\mathrm{d}h$ is zero for every bump function h near $x \in X$. Hence, given an arbitrary function $f \in C^\infty(X)$ and a bump function h near x, for every derivation d from $C^\infty(X)$ to $\mathbb{R}_x$, we have

$$\mathrm{d}(fh) = (\mathrm{d}f)h(x) + f(x)\mathrm{d}h = \mathrm{d}f.$$

Let $x \in Y$, and let h be bump function on X with $h(y) = 1$ near x having support in Y. Given a derivation d from $C^\infty(X)$ to $\mathbb{R}_x$ and $f \in C^\infty(Y)$, the function fh is defined on X, and $\mathrm{d}f = \mathrm{d}(fh)$ extends d to a derivation from $C^\infty(Y)$ to $\mathbb{R}_x$. This shows the induced map from $\mathrm{Der}(C^\infty(Y), \mathbb{R}_x)$ to $\mathrm{Der}(C^\infty(X), \mathbb{R}_x)$ is surjective. Moreover, if a derivation d from $C^\infty(Y)$ to $\mathbb{R}_x$ goes to zero in $\mathrm{Der}(C^\infty(X), \mathbb{R}_x)$, it must itself be zero since $\mathrm{d}f = \mathrm{d}(fh)$ for every f and every bump function h. □

In view of Proposition 7.2, there is no need for us to talk about *sheaves* of germs of smooth functions in order to define Zariski tangent spaces, etc. In fact, in view of Theorem 6.2, 6.3 and Proposition 7.1, 7.2 entails at once the following:

Theorem 7.3. *For every central Yang–Mills connection A, the inclusion of an open subspace of the kind U_A into $N(\xi)$ induces an isomorphism of Zariski tangent spaces from $\mathrm{T}_{[A]}U_A$ onto $\mathrm{T}_{[A]}N(\xi)$. Likewise, for every $\phi \in \mathrm{Hom}_\xi(F, G)$, the inclusion of an open subspace of the kind U_ϕ into $\mathrm{Rep}_\xi(\Gamma, G)$ induces an isomorphism of Zariski tangent spaces from $\mathrm{T}_{[\phi]}U_\phi$ onto $\mathrm{T}_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$. Consequently, a choice of representative A (in its class $[A]$) induces an isomorphism of Zariski tangent spaces from $\mathrm{T}_{[0]}\mathrm{H}_A$ onto $\mathrm{T}_{[A]}N(\xi)$, and a choice of representative ϕ (in its class $[\phi]$) induces an isomorphism of Zariski tangent spaces from $\mathrm{T}_{[0]}\mathrm{H}_\phi$ onto $\mathrm{T}_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$.*

This reduces the study of the Zariski tangent spaces to our local models, to which we now turn. Let W be a finite-dimensional unitary representation of a compact Lie group K, and let Θ denote its unique momentum mapping from W to k^* having the value zero at the origin; further, let $V = \Theta^{-1}(0)$, with smooth structure $C^\infty(V)$ given by (6.1.1) and $W_{\mathrm{red}} = V/K$, its reduced space, with smooth structure $C^\infty(W_{\mathrm{red}})$ given by (6.1.2). By Corollary 6.13, $(W_{\mathrm{red}}, C^\infty(W_{\mathrm{red}}))$ has smooth bump functions arbitrarily close to every point. Hence, by Proposition 7.1, the inclusion of an arbitrary open connected subset of W_{red} containing the class [0] of the origin, with its induced smooth structure, induces an isomorphism of Zariski tangent spaces at [0]. The Zariski tangent space T_0V of V at the origin equals the linear span $\mathrm{Vect}(V)$ of V in W, and projection from V to W_{red} induces a linear map λ from T_0V to $\mathrm{T}_{[0]}W_{\mathrm{red}}$. To deduce information about the Zariski tangent space $T_{[0]}W_{\text{red}}$, we denote the space of K-invariants by W^K and its counter part, that is the space arising from dividing out the K-action, by W_K. The kernel $J_K(W)$ of the canonical projection from W to W_K is the linear span of the elements $xw - w$, $x \in K$, $w \in W$. A little thought reveals that the orthogonal complement of W^K in W equals the subspace $J_K(W)$, that is, as a K-representation, $W = W^K \oplus J_K(W)$. Moreover, the zero locus V contains the subspace W^K of K-invariants, and the projection from V to W_{red}, restricted to W^K, is a homeomorphism identifying the latter with the (smooth) stratum S in which the class [0] of the origin lies. The (smooth) tangent space $T_{[0]}S$ of S at [0] is thus just a copy of W^K, and the inclusion of S into W_{red} induces an injection of $T_{[0]}S \cong W^K$ into $T_{[0]}W_{\text{red}}$. Furthermore, with respect to the decomposition into connected components of orbit types, the algebra of invariants $(C^\infty(W))^K$ endows the orbit space W/K with a smooth structure $C^\infty(W/K)$, and the inclusion of W_{red} into W/K is smooth. Since the induced map from $C^\infty(W/K)$ to $C^\infty(W_{\text{red}})$ is surjective, the derivative $T_{[0]}W_{\text{red}} \to T_{[0]}(W/K)$ of this inclusion is in fact injective.

Lemma 7.4. *Suppose the zero locus V of Θ spans W so that the Zariski tangent space T_0V equals W whence the linear map λ then goes from W to $T_{[0]}W_{\text{red}}$. Then λ has kernel $J_K(W)$ and image equal to the (smooth) tangent space T_0S, viewed as a subspace of $T_{[0]}W_{\text{red}}$. In particular, λ is injective and hence an isomorphism if and only if W is a trivial K-representation.*

In the language of [2, p. 71], the condition says that, V being viewed as a *constraint set*, the "spanning condition" is satisfied at $0 \in V$.

Proof. View W as a real vector space, consider the algebra $\mathbb{R}[W]$ of real polynomials on W, and pick a finite set of homogeneous generators $(\kappa_1, \ldots, \kappa_k)$ of the subalgebra $\mathbb{R}[W]^K$ of K-invariant polynomials. Then the Hilbert map κ from W to $\mathbb{R}^k$ which assigns $\kappa(w) = (\kappa_1(w), \ldots, \kappa_k(w))$ to a vector $w \in W$ descends to an injective map $\tilde{\kappa}$ from W/K to $\mathbb{R}^k$. In view of a result of [28], with reference to the smooth structure $C^\infty(W/K)$, the map $\tilde{\kappa}$ is proper, that is, the induced map from $C^\infty(\mathbb{R}^k)$ to $C^\infty(W/K)$ is surjective, and hence the derivative of $\tilde{\kappa}$ at the orbit $[0] = 0 \cdot K$ is injective; further, when the number k is minimal, by a result of [22], this derivative is even an isomorphism from $T_{[0]}(W/K)$ onto $\mathbb{R}^k$. Thus, for k minimal, the canonical map from W to $T_{[0]}(W/K)$ comes down to the derivative $d\kappa(0) : W \to \mathbb{R}^k$ of the Hilbert map at the origin, and the latter decomposes into the linear map λ from W to $T_{[0]}W_{\text{red}}$ and the injection from $T_{[0]}W_{\text{red}}$ into $T_{[0]}(W/K)$ which embeds $T_{[0]}W_{\text{red}}$ into a k-dimensional vector space. However, $W = W^K \oplus J_K(W)$, and $d\kappa(0)$ vanishes on $J_K(W)$ and identifies W^K with a subspace of $\mathbb{R}^k$, in fact, with what corresponds to the tangent space T_0S. In particular, λ to be injective means that W^K equals W, that is to say, that K acts trivially on W. □

Next we recall the following well-known fact.

Proposition 7.5. *As a smooth space, W_{red} is semi-algebraic.*

We reproduce a proof, for reference in the next section.

Proof. After a choice of invariant polynomials $(\kappa_1, \ldots, \kappa_k)$ has been made, by the Tarski–Seidenberg theorem, the resulting injective map $\widetilde{\kappa}$ from W/K to $\mathbb{R}^k$ realizes W/K as a semi-algebraic subset of $\mathbb{R}^k$, in fact, of the real affine categorical quotient $W//K$, that is, of the real affine variety determined by a finite set of relations for the algebra of invariants $\mathbb{R}[W]^K$. The composite of $\widetilde{\kappa}$ with the canonical injection of W_{red} into W/K embeds W_{red} into $\mathbb{R}^k$. To see this embedding is semi-algebraic, write I_V for the ideal of V in $\mathbb{R}[W]$ and consider the real affine coordinate ring $A[V] = \mathbb{R}[W]/I_V$ of V. Since K is compact, the canonical map from $\mathbb{R}[W]^K/I_V^K$ to the K-invariants $A[V]^K$ is an isomorphism. Let $\phi_1, \ldots, \phi_\ell$ be a finite set of generators of I_V^K; when we write them out in the generators $(\kappa_1, \ldots, \kappa_k)$, we obtain a polynomial map Φ from $\mathbb{R}^k$ to $\mathbb{R}^\ell$ so that W_{red} amounts to the intersection of W/K with the real affine set $\Phi^{-1}(0)$ whence W_{red} is semi-algebraic in $\mathbb{R}^k$. □

Remark. We have seen above that the inclusion of W_{red} into W/K induces an embedding of the Zariski tangent space $\mathrm{T}_{[0]}W_{\mathrm{red}}$ into the Zariski tangent space $\mathrm{T}_{[0]}(W/K)$. The above embedding of W_{red} into $\mathrm{T}_{[0]}(W/K)$ passes to an embedding into $\mathrm{T}_{[0]}W_{\mathrm{red}}$. In fact, the embedding of W/K into its Zariski tangent space is induced by the canonical embedding of W into its tangent space $\mathrm{T}_0 W$ which assigns to a vector $w \in W$ its directional derivative at the origin on smooth functions on W. It is obvious that this association passes to one which assigns to a vector $w \in V$ an element in the Zariski tangent space $\mathrm{T}_0 V$, viewed as a linear subspace of $\mathrm{T}_0 W$, and hence, by K-invariance, to an embedding of W_{red} into its Zariski tangent space $\mathrm{T}_{[0]}W_{\mathrm{red}}$ as a semi-algebraic set. An example will be examined in the next section.

We now apply the above to moduli spaces. For a central Yang–Mills connection A, we shall denote by V_A the zero locus of the quadratic mapping Θ_A from H^1_A to H^2_A, cf. Section 6, and likewise, for ϕ in $\mathrm{Hom}_\xi(\Gamma, G)$, we shall denote by V_ϕ the zero locus of the quadratic mapping Θ_ϕ from H^1_ϕ to H^2_ϕ.

Lemma 7.6. *For every central Yang–Mills connection A, the zero locus V_A spans* $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$. *Likewise, for every ϕ in* $\mathrm{Hom}_\xi(\Gamma, G)$, *the zero locus V_ϕ spans* $\mathrm{H}^1(\pi, \mathfrak{g}_\phi)$.

The proof of this lemma requires some preparation. We shall denote by $\mathcal{N}(\xi)^-$ the subspace of central Yang–Mills connections A having the property that the Lie bracket $[\cdot,\cdot]_A$ is zero on H^1_A. Recall that a description of the space $\mathcal{A}_A(\xi)$ for a central Yang–Mills connection A has been reproduced in Section 6 above. It is proved in [11] (2.8) that, near a central Yang–Mills connection A, the space $\mathcal{N}(\xi)$ coincides with $\mathcal{A}_A(\xi)$ and hence is smooth near A, with tangent space $\mathrm{T}_A\mathcal{N}(\xi)$ equal to the space $Z^1_A(\Sigma, \mathrm{ad}(\xi))$ of 1-cocycles if and only if A lies in $\mathcal{N}(\xi)^-$. Thus, the subspace $\mathcal{N}(\xi)^-$ is a smooth submanifold of $\mathcal{A}(\xi)$, and from the exactness of (6.4) we deduce that, for every point

A of $\mathcal{N}(\xi)^-$, the operator of covariant derivative d_A gives rise to the exact sequence

$$0 \to \mathrm{T}_A\mathcal{N}(\xi) \to \mathrm{T}_A\mathscr{A}(\xi) \xrightarrow{\mathrm{d}_A} \Omega^2(\Sigma, \mathrm{ad}(\xi)) \to \mathrm{H}^2_A(\Sigma, \mathrm{ad}(\xi)) \to 0 \tag{7.7}$$

of real vector spaces. In fact, the points of $\mathcal{N}(\xi)^-$ are exactly the weakly regular points [1, p. 300] for the momentum mapping J from $\mathscr{A}(\xi)$ to $\Omega^2(\Sigma, \mathrm{ad}(\xi))$, cf. Section 6.

Denote by $\mathcal{N}^{\mathrm{top}}(\xi)$ the subspace of $\mathcal{N}(\xi)$ which consists of central Yang–Mills connections A having the property that Z_A acts trivially on $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$, so that the top stratum $N^{\mathrm{top}}(\xi)$ equals $\mathcal{N}^{\mathrm{top}}(\xi)/\mathscr{G}(\xi)$, see our paper [14]. By [14, (1.5)], there is a certain subgroup Z^{top} of G, unique up to conjugacy, such that under (1.1) the image of the stabilizer Z_A of every central Yang–Mills connection A in $\mathcal{N}^{\mathrm{top}}(\xi)$ is conjugate to Z^{top}. Since Θ_A is a momentum mapping for every central Yang–Mills connection A, $\mathcal{N}^{\mathrm{top}}(\xi)$ is a subspace of $\mathcal{N}^-(\xi)$, in fact, a smooth codimension zero submanifold since for every $A \in \mathcal{N}^{\mathrm{top}}(\xi)$ the tangent map of the inclusion $\mathcal{N}^{\mathrm{top}}(\xi) \subseteq \mathcal{N}^-(\xi)$ amounts to the identity mapping of $Z^1_A(\Sigma, \mathrm{ad}(\xi))$.

In what follows, by the *dimension* $\dim V_A$ of V_A we mean the dimension of its nonsingular part $V_A^- \subseteq V_A$.

Proof of Lemma 7.6. Since the top stratum $N^{\mathrm{top}}(\xi)$ is dense in $N(\xi)$, cf. [14, (1.4)], arbitrarily close to $[A]$ there is a point $[\widetilde{A}]$ in the top stratum, and we may assume that the group Z^{top} is the stabilizer $Z_{\widetilde{A}}$ of $\widetilde{A}$. Then a neighborhood of the point x of V_A corresponding to $\widetilde{A}$ is the total space of a Z_A-fiber bundle, having as base space a neighborhood of the class $[x]$ in V_A/Z_A and as fiber the homogeneous space Z_A/Z^{top}. Consequently,

$$\begin{aligned}\dim V_A = \dim \mathrm{T}_x V_A &= \dim N^{\mathrm{top}}(\xi) + \dim Z_A - \dim Z^{\mathrm{top}}\\ &= \dim \mathrm{H}^1_{\widetilde{A}} + \dim Z_A - \dim Z^{\mathrm{top}}.\end{aligned}$$

However, for every central Yang–Mills connection $\widetilde{A}$, the twisted integration mapping yields an isomorphism from $\mathrm{H}^*_{\widetilde{A}}(\Sigma, \mathrm{ad}(\xi))$ onto $\mathrm{H}^*(\pi, \mathfrak{g}_{\rho_{\widetilde{A}}})$. Now an Euler characteristic argument in the chain complex calculating the corresponding group cohomologies establishes equality between the two alternating sums $\dim \mathrm{H}^0_A - \dim \mathrm{H}^1_A + \dim \mathrm{H}^2_A$ and $\dim \mathrm{H}^0_{\widetilde{A}} - \dim \mathrm{H}^1_{\widetilde{A}} + \dim \mathrm{H}^2_{\widetilde{A}}$. Since $\dim \mathrm{H}^2_A = \dim \mathrm{H}^0_A = \dim Z_A$ and $\dim \mathrm{H}^2_{\widetilde{A}} = \dim \mathrm{H}^0_{\widetilde{A}} = \dim Z^{\mathrm{top}}$, we conclude

$$\dim \mathrm{H}^1_{\widetilde{A}} - 2\dim Z^{\mathrm{top}} = \dim \mathrm{H}^1_A - 2\dim Z_A, \tag{7.6.1}$$

and thence

$$\dim V_A = \dim \mathrm{H}^1_A - \dim Z_A + \dim Z^{\mathrm{top}}. \tag{7.6.2}$$

Next we assert that, at the image x of $[\widetilde{A}]$ in $V_A \subseteq \mathrm{H}^1_A$, the derivative $\mathrm{d}\Theta_A(x) : \mathrm{H}^1_A \to \mathrm{H}^2_A$ of Θ_A has rank

$$\mathrm{rank}(\mathrm{d}\Theta_A(x)) = \dim \mathrm{H}^2_A - \dim Z^{\mathrm{top}} = \dim Z_A - \dim Z^{\mathrm{top}}. \tag{7.6.3}$$

Now, at a point $\bar{A} \in \mathscr{A}_A$, the smooth submanifold $\mathscr{A}_A$ of $\mathscr{A}(\xi)$ has tangent space

$$\mathrm{T}_{\bar{A}}\mathscr{A}_A = \{\phi;\ \mathrm{d}_{\bar{A}}\phi \in \mathscr{H}^2_A(\Sigma, \mathrm{ad}(\xi))\} \subseteq \Omega^1(\Sigma, \mathrm{ad}(\xi)).$$

In other words, the right-hand unlabelled arrow being the inclusion, the square

$$\begin{array}{ccc} \mathrm{T}_{\bar{A}}\mathscr{A}_A & \xrightarrow{\mathrm{d}_{\bar{A}}|} & \mathscr{H}^2_A(\Sigma, \mathrm{ad}(\xi)) \\ \downarrow & & \downarrow \\ \mathrm{T}_{\bar{A}}\mathscr{A}(\xi) & \xrightarrow{\mathrm{d}_{\bar{A}}} & \Omega^2(\Sigma, \mathrm{ad}(\xi)) \end{array} \tag{7.6.4}$$

is a pull back diagram. By construction, $\mathscr{N}(\xi) = \{\hat{A} \in \mathscr{A}_A;\ K_{\hat{A}} = K_\xi\}$; here K_ξ refers to the element of $\mathscr{H}^2_A(\Sigma, \mathrm{ad}(\xi))$ determined by the topology of ξ, see [11, Section 2]. Since (7.6.4) is a pull back diagram, by standard principles, at a point $\hat{A}$ of $\mathscr{N}^-(\xi)$ the sequence (7.7) induces an exact sequence of real vector spaces

$$0 \to \mathrm{T}_{\hat{A}}\mathscr{N}(\xi) \to \mathrm{T}_{\hat{A}}\mathscr{A}_A \xrightarrow{\mathrm{d}_{\hat{A}}} \mathscr{H}^2_A(\Sigma, \mathrm{ad}(\xi)) \to \mathrm{H}^2_{\hat{A}}(\Sigma, \mathrm{ad}(\xi)). \tag{7.6.5}$$

Notice at present we cannot assert that the last arrow in (7.6.5) is surjective.

Next we recall that, for $\hat{A}$ in $\mathscr{N}^-(\xi)$ and close to A, the smooth submanifold $\mathscr{M}_A$ of $\mathscr{A}_A$, cf. [11, (2.16)] and Section 6 above, has tangent space $\mathrm{T}_{\hat{A}}\mathscr{M}_A$ equal to $\mathrm{T}_{\hat{A}}\mathscr{A}_A \cap \ker(\mathrm{d}^*_A)$; hence such a point $\hat{A}$ gives rise to the exact sequence

$$0 \to (\mathrm{T}_{\hat{A}}\mathscr{N}(\xi) \cap \ker(\mathrm{d}^*_A)) \to \mathrm{T}_{\hat{A}}\mathscr{M}_A \to \mathrm{d}_{\hat{A}}(\mathrm{T}_{\hat{A}}\mathscr{M}_A) \to 0$$

which, cf. [10, Section 2], with $\mathscr{N}_A = \mathscr{N}(\xi) \cap \mathscr{M}_A$, looks like

$$0 \to \mathrm{T}_{\hat{A}}\mathscr{N}_A \to \mathrm{T}_{\hat{A}}\mathscr{M}_A \to \mathrm{d}_{\hat{A}}(\mathrm{T}_{\hat{A}}\mathscr{M}_A) \to 0. \tag{7.6.6}$$

We note that, near A, $\mathscr{N}_A$ also equals the intersection $\mathscr{N}(\xi) \cap (A + \ker(\mathrm{d}^*_A))$.

Let now $\widetilde{A}$ be a point close to A representing a point of N^{top}; then $\widetilde{A}$ lies in particular in $\mathscr{N}^-(\xi)$, and near $\widetilde{A}$, the restriction to $\mathscr{N}_A$ of the projection map from $\mathscr{N}(\xi)$ onto $N(\xi)$ is a fiber bundle map onto its image, having fiber the homogeneous space $Z_A/Z_{\widetilde{A}}$. Consequently, in view of (7.6.1),

$$\begin{aligned} \dim \mathscr{N}_A &= \dim N(\xi) + \dim Z_A - \dim Z_{\widetilde{A}} \\ &= \dim \mathrm{H}^1_{\widetilde{A}} + \dim Z_A - \dim Z_{\widetilde{A}} \\ &= \dim \mathrm{H}^1_A + \dim Z_{\widetilde{A}} - \dim Z_A. \end{aligned}$$

However, $\dim \mathscr{M}_A = \dim \mathrm{H}^1_A$. Consequently,

$$\begin{aligned} \dim \mathrm{d}_{\widetilde{A}}(\mathrm{T}_{\widetilde{A}}\mathscr{M}_A) &= \dim \mathrm{H}^1_A - \dim \mathscr{N}_A \\ &= \dim Z_A - \dim Z_{\widetilde{A}} \\ &= \dim \mathrm{H}^2_A - \dim \mathrm{H}^2_{\widetilde{A}}, \end{aligned}$$

whence the exact sequence (7.6.5) furnishes the exact sequence

$$0 \to \mathrm{T}_{\tilde{A}}\mathscr{N}_A \to \mathrm{T}_{\tilde{A}}\mathscr{M}_A \xrightarrow{\mathrm{d}_{\tilde{A}}} \mathscr{H}_A^2(\Sigma, \mathrm{ad}(\xi)) \to \mathrm{H}^2_{\tilde{A}}(\Sigma, \mathrm{ad}(\xi)) \to 0 \tag{7.6.7}$$

of finite-dimensional real vector spaces; notice its exactness at $\mathrm{T}_{\tilde{A}}\mathscr{M}_A$ is implied by that of (7.6.6). By construction, the Kuranishi map identifies (7.6.7) with the sequence

$$0 \to \mathrm{T}_x V_A \to \mathrm{T}_x \mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi)) \xrightarrow{\mathrm{d}\Theta_A(x)} \mathrm{H}^2_A(\Sigma, \mathrm{ad}(\xi)) \to \mathrm{H}^2_{\tilde{A}}(\Sigma, \mathrm{ad}(\xi)) \to 0 \tag{7.6.8}$$

which is therefore exact. In particular, the point $x \in V_A$ is weakly regular for Θ_A, and hence $\mathrm{d}\Theta_A(x)$ has rank asserted in (7.6.3).

Finally, we show that the latter implies that the *real* linear span $\mathrm{Vect}(V_A)$ of V_A in $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ equals the whole space $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$. In fact, the cone V_A is obviously stable under Z_A. Moreover, in view of [11, (2.27)], for every $\eta \in \mathscr{H}^1_A(\Sigma, \mathrm{ad}(\xi))$, the value $[\eta, \eta] \in \mathscr{H}^2_A(\Sigma, \mathrm{ad}(\xi))$ is zero if and only if $[*\eta, *\eta] = 0$; here $*$ refers to the corresponding duality operator, cf. [11, (1.1.5)]. Consequently, the cone V_A is stable under the duality operator $*$. However, this d·ality operator induces the *complex* structure on $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$. Hence the *real* linear span $\mathrm{Vect}(V_A)$ of V_A in $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ equals its *complex* linear span in $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$; the complex vector space $\mathrm{Vect}(V_A)$ thus inherits a structure of a unitary Z_A-representation, and as a unitary Z_A-representation, the space $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ decomposes into the direct sum of $\mathrm{Vect}(V_A)$ and its orthogonal complement $\mathrm{Vect}(V_A)^\perp$. Moreover, the restrictions Θ^1_A and Θ^2_A of Θ_A to $\mathrm{Vect}(V_A)$ and $\mathrm{Vect}(V_A)^\perp$, respectively, are the unique momentum mappings for these unitary Z_A-representations having the value zero at the origin. By construction, the cone V_A lies in the summand $\mathrm{Vect}(V_A)$, whence the zero locus $(\Theta^2_A)^{-1}(0) \subseteq \mathrm{Vect}(V_A)^\perp$ consists merely of the origin. Hence, whatever weakly regular point x of V_A, the rank of the derivative $\mathrm{d}\Theta_A(x)$ coincides with the rank of the restriction $\mathrm{d}\Theta^1_A(x)$ to $\mathrm{T}_x \mathrm{Vect}(V_A) = \mathrm{Vect}(V_A)$. Consequently, $\dim V_A = \dim \mathrm{Vect}(V_A) - \dim Z_A + \dim Z^{\mathrm{top}}$. However, in view of (7.6.2), this can only happen if $\dim \mathrm{Vect}(V_A) = \dim \mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$, whence $\mathrm{Vect}(V_A) = \mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ as asserted. □

Remark 7.8. Let K be a compact Lie group, with Lie algebra k, let W be an n-dimensional unitary representation of K, and let μ be the unique momentum mapping from W to k^* having the value zero at the origin. Its derivative at the origin is zero, the kernel of $\mathrm{d}\mu(0)$ in fact equals the whole space W, and the Zariski tangent space $\mathrm{T}_0(\mu^{-1}(0))$ at the origin of the zero locus $\mu^{-1}(0)$ is obviously a subspace of the kernel of $\mathrm{d}\mu(0)$. However, in general, the Zariski tangent space does *not* coincide with the kernel of $\mathrm{d}\mu(0)$. To see this, suppose that the irreducible representations in W are all non-trivial, that K is a subgroup of the unitary group $\mathrm{U}(n)$, and that K contains the central circle subgroup S^1 of $\mathrm{U}(n)$. Since the momentum mapping for the S^1-action on $\mathbb{C}^n$ is given by the assignment to $\mathbf{z} \in \mathbb{C}^n$ of $\|\mathbf{z}\|^2$, the zero level set $\mu^{-1}(0)$ will then consist of the origin only, the Zariski tangent space of which is of course trivial. Thus, Lemma 7.6 is *non-trivial*.

The decompositions of $N(\xi)$ and $\mathrm{Rep}_\xi(\Gamma, G)$ into connected components of orbit types have been shown to be a stratification in [14]. If $[A]$ lies in the stratum $N_{(K)}$, the inclusion of $N_{(K)}$ into $N(\xi)$ induces an injection $\mathrm{T}_{[A]}(N_{(K)}) \to \mathrm{T}_{[A]}N(\xi)$ of Zariski tangent spaces, and $\mathrm{T}_{[A]}(N_{(K)})$ will in this way be viewed as a linear subspace of $\mathrm{T}_{[A]}N(\xi)$; this is e.g. a consequence of Theorem 6.2 combined with Lemma 7.4. Notice that $\mathrm{T}_{[A]}N_{(K)}$ amounts to the usual smooth tangent space of the smooth manifold $N_{(K)}$. It is clear that the same kind of remarks can be made for an arbitrary point $[\phi]$ of $\mathrm{Rep}_\xi(\Gamma, G)$ and the stratum $\mathrm{Rep}_\xi(\Gamma, G)_{(K)}$ in which it lies. A point in the top stratum $N^{\mathrm{top}}(\xi)$ will be referred to as a *non-singular* point of $N(\xi)$, cf. [14]. Accordingly, the representation space $\mathrm{Rep}_\xi(\Gamma, G)$ has a *non-singular* part or *top* stratum $\mathrm{Rep}_\xi^{\mathrm{top}}(\Gamma, G)$, and a point in $\mathrm{Rep}_\xi^{\mathrm{top}}(\Gamma, G)$ will be said to be a *non-singular* point of $\mathrm{Rep}_\xi(\Gamma, G)$. We now collect a number of consequences of the above results.

7.9. Let $[A]$ be a point of $N(\xi)$. In view of Theorem 6.2 and Lemma 7.6, a choice of representative A in its class $[A]$ determines a linear map λ_A from $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ to $\mathrm{T}_{[A]}N(\xi)$. In fact, this map is the composite of the linear map λ from $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ to $\mathrm{T}_{[0]}W_A$, cf. Lemma 7.4, with the derivative of the injection of W_A into $N(\xi)$ given in Theorem 6.2, where W_A refers to an open neighborhood of the class of zero in H_A of the kind coming into play in Theorem 6.2. By construction, λ_A depends on the Kuranishi map; however the latter, in turn, depends merely on the data coming into play in the definition of $N(\xi)$. It is in this sense that a choice of representative A of $[A]$ in fact *determines* λ_A. The map λ_A has the following properties:

(1) It is independent of the choice of A in the sense that, for every gauge transformation $\gamma \in \mathscr{G}(\xi)$, the composite

$$\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi)) \xrightarrow{\gamma_\sharp} \mathrm{H}^1_{\gamma(A)}(\Sigma, \mathrm{ad}(\xi)) \xrightarrow{\lambda_{\gamma(A)}} \mathrm{T}_{[A]}N(\xi)$$

of the induced linear isomorphism $\gamma_\sharp$ with $\lambda_{\gamma(A)}$ coincides with λ_A.

(2) Its kernel equals the subspace $J_K(\mathrm{H}^1_A)$ of $\mathrm{H}^1_A = \mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$, where $K = Z_A$, the stabilizer of A.

(3) Its image equals the (smooth) tangent space $\mathrm{T}_{[A]}(N_{(K)})$, viewed as a subspace of $\mathrm{T}_{[A]}N(\xi)$ in a sense explained above, where $N_{(K)}$ denotes the stratum in which $[A]$ lies.

(4) It is an isomorphism if and only if $[A]$ is a non-singular point of $N(\xi)$.

These follow at once from Lemma 7.4 except statement (1) the proof of which we leave to the reader.

7.10. Let $[\phi]$ be a point of $\mathrm{Rep}_\xi(\Gamma, G)$. In view of Theorem 6.3 and Lemma 7.6, a choice of representative ϕ in $\mathrm{Hom}_\xi(\Gamma, G)$ in its class $[\phi]$ determines a linear map λ_ϕ from $\mathrm{H}^1(\pi, \mathfrak{g}_\phi)$ to $\mathrm{T}_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$. In fact, this map is the composite of the linear map λ from $\mathrm{H}^1(\pi, \mathfrak{g}_\phi)$ to $\mathrm{T}_{[0]}W_\phi$, cf. Lemma 7.4, with the derivative of the injection of W_ϕ into $\mathrm{Rep}_\xi(\Gamma, G)$ given in Theorem 6.3, where W_ϕ refers to an open neighborhood of the class of zero in H_ϕ of the kind coming into play in Theorem 6.3. By construction, the injection of W_ϕ into $\operatorname{Rep}_\xi(\Gamma, G)$ depends a priori on the Kuranishi map and in particular on the choice of Riemannian metric on Σ. However, λ_ϕ does *not* depend on this choice. In fact, by Theorem 5.7, the derivative of the Wilson loop mapping ρ from $\mathscr{A}(\xi)$ to $\operatorname{Hom}(F, G)$ at a central Yang–Mills connection A, restricted to the subspace $Z_A^1(\Sigma, \operatorname{ad}(\xi))$ of 1-cocycles in $\Omega^1(\Sigma, \operatorname{ad}(\xi)) = \mathrm{T}_A\mathscr{A}(\xi)$, amounts to the composite

$$Z_A^1(\Sigma, \operatorname{ad}(\xi)) \xrightarrow{\operatorname{Int}_A|} Z^1(\pi, \mathfrak{g}_\phi) \xrightarrow{\mathrm{L}_\phi} \mathrm{T}_\phi \operatorname{Hom}_\xi(\Gamma, G)$$

of the restriction $\operatorname{Int}_A|$ of the twisted integration mapping with left translation L_ϕ from $Z^1(\pi, \mathfrak{g}_\phi)$ to $\mathrm{T}_\phi \operatorname{Hom}_\xi(\Gamma, G)$, whatever Riemannian metric on Σ; here $\phi = \rho(A) \in \operatorname{Hom}_\xi(\Gamma, G)$. Since every $\phi \in \operatorname{Hom}_\xi(\Gamma, G)$ arises in this way, for every such ϕ, the diagram

$$\begin{array}{ccc} Z^1(\pi, \mathfrak{g}_\phi) & \xrightarrow{\mathrm{L}_\phi} & \mathrm{T}_\phi \operatorname{Hom}_\xi(\Gamma, G) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(\pi, \mathfrak{g}_\phi) & \xrightarrow{\lambda_\phi} & \mathrm{T}_\phi \operatorname{Rep}_\xi(\Gamma, G) \end{array}$$

is commutative, the unlabelled vertical maps being the obvious ones. Hence, a choice of representative ϕ in its class $[\phi]$ indeed determines a linear map λ_ϕ as asserted which does *not* depend on a choice of Riemannian metric on Σ. The map λ_ϕ has the following properties:

(1) It is independent of the choice of ϕ in the sense that, for every $x \in G$, the composite

$$\mathrm{H}^1(\pi, \mathfrak{g}_\phi) \xrightarrow{\operatorname{Ad}_\flat(x)} \mathrm{H}^1(\pi, \mathfrak{g}_{x\phi}) \xrightarrow{\lambda_{x\phi}} \mathrm{T}_{[\phi]} \operatorname{Rep}_\xi(\Gamma, G)$$

of the induced linear isomorphism $\operatorname{Ad}_\flat(x)$ with $\lambda_{x\phi}$ coincides with λ_ϕ.

(2) Its kernel equals the subspace $J_K(\mathrm{H}^1(\pi, \mathfrak{g}_\phi))$ of $\mathrm{H}^1(\pi, \mathfrak{g}_\phi)$, where $K = Z_\phi$, the stabilizer of ϕ.

(3) Its image equals the (smooth) tangent space $\mathrm{T}_{[\phi]}(\operatorname{Rep}_\xi(\Gamma, G)_{(K)})$, viewed as a subspace of $\mathrm{T}_{[\phi]} \operatorname{Rep}_\xi(\Gamma, G)$ in a sense explained above, where $\operatorname{Rep}_\xi(\Gamma, G)_{(K)}$ denotes the stratum in which $[\phi]$ lies.

(4) It is an isomorphism if and only if $[\phi]$ is a non-singular point of $\operatorname{Rep}_\xi(\Gamma, G)$.

These follow again at once from Lemma 7.4 except statement (1) the proof of which is formally the same as that of (7.9(1)).

The statements of (7.9) and (7.10) are related by the fact that, for every central Yang–Mills connection A, the diagram

$$\begin{array}{ccc} \mathrm{H}_A^1(\Sigma, \operatorname{ad}(\xi)) & \xrightarrow{\lambda_A} & \mathrm{T}_{[A]}N(\xi) \\ {\scriptstyle \operatorname{Int}_A}\downarrow & & \downarrow{\scriptstyle d\rho_\flat[A]} \\ \mathrm{H}^1(\pi, \mathfrak{g}_{\rho(A)}) & \xrightarrow[\lambda_{\rho(A)}]{} & \mathrm{T}_{[\rho(A)]} \operatorname{Rep}_\xi(\Gamma, G) \end{array} \tag{7.11}$$

is commutative. Thus at a non-singular point $[A]$ of $N(\xi)$, the derivative of the Wilson loop mapping comes down to the twisted integration mapping Int_A from $\mathrm{H}_A^1(\Sigma, \operatorname{ad}(\xi))$ to $\mathrm{H}^1(\pi, \mathfrak{g}_{\rho(A)})$.

Remark 7.12. At a singular point $[\phi]$ of $\mathrm{Rep}_\xi(\Gamma, G)$, the Zariski tangent space $\mathrm{T}_{[\phi]}\mathrm{Rep}_\xi(\Gamma, G)$ with respect to the smooth structure $C^\infty(\mathrm{Rep}_\xi(\Gamma, G))$ does *not* boil down to $\mathrm{H}^1(\pi, \mathfrak{g}_\phi)$, cf. what is said on p. 205 of [8]. An example where this phenomenon really occurs will be given in the next section.

Here is an immediate consequence of Theorems 6.2 and 6.3 and Proposition 7.5.

Theorem 7.13. *As smooth spaces, $N(\xi)$ and its diffeomorphe $\mathrm{Rep}_\xi(\Gamma, G)$ are locally semi-algebraic.*

Next we spell out our *fifth main result*. For every $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$, the kernel of the derivative dr_ϕ from $\mathrm{T}_\phi \mathrm{Hom}(F, G)$ to $\mathrm{T}_{\exp(X_\xi)}G$, with reference to the word map r from $\mathrm{Hom}(F, G)$ to G, yields a notion of *not necessarily reduced* Zariski tangent space, and it is clear that the Zariski tangent space $\mathrm{T}_\phi \mathrm{Hom}_\xi(\Gamma, G)$ with reference to the smooth structure $C^\infty(\mathrm{Hom}_\xi(\Gamma, G))$ (introduced in Section 3) is a subspace thereof; however, a priori the two spaces should *not* be confused.

Theorem 7.14. *For every point $\phi \in \mathrm{Hom}_\xi(\Gamma, G)$, the Zariski tangent space with reference to $C^\infty\left(\mathrm{Hom}_\xi(\Gamma, G)\right)$ coincides with the kernel of the derivative $\mathrm{d}r_\phi$.*

Thus our *reduced* Zariski tangent space coincides with the other notion of Zariski tangent space. However, we do not know whether the ideal in $C^\infty(\mathrm{Hom}(F, G))$ corresponding to the word map r coincides with its real radical.

Proof. Let A be a central Yang–Mills connection so that $\rho(A) = \phi$. The smooth G-invariant immersion (6.11.3) identifies the subspace $G \times_{Z_A} \vartheta_A^{-1}(0)$ with a G-invariant neighborhood of ϕ in $\mathrm{Hom}_\xi(\Gamma, G)$; here ϑ_A refers to the momentum mapping coming into play in Section 6. However the Kuranishi map Φ_A, cf. [11, (2.29)] and what is said in Section 6, identifies the inclusion of $G \times_{Z_A} \vartheta_A^{-1}(0)$ into $G \times_{Z_A} \mathscr{M}_A$ with the inclusion of $G \times_{Z_A} V_A$ into $G \times_{Z_A} \mathrm{H}^1_A$, where $V_A \subseteq \mathrm{H}^1_A$ refers to the cone $\Theta_A^{-1}(0)$; see Section 6 for any unexplained notation. Now the tangent space $\mathrm{T}_{[e,0]}\left(G \times_{Z_A} \mathrm{H}^1_A\right)$ decomposes into a direct sum of $B^1(\pi, \mathfrak{g}_\phi)$ and H^1_A, that is, it amounts to the space $Z^1(\pi, \mathfrak{g}_\phi)$ of 1-cocycles, and in suitable coordinates near the point $[e, 0]$, the space $G \times_{Z_A} V_A$ boils down to the zero locus of the composition of the projection from $Z^1(\pi, \mathfrak{g}_\phi)$ to $\mathrm{H}^1(\pi, \mathfrak{g}_\phi)$ with the momentum mapping Θ_A from H^1_A to H^2_A. In view of Lemma 7.6, this implies that the Zariski tangent space of $G \times_{Z_A} V_A$ at the point $[e, 0]$ equals the tangent space $\mathrm{T}_{[e,0]}\left(G \times_{Z_A} \mathrm{H}^1_A\right)$ whence the assertion. □

8. An example

Consider the moduli space N of flat SU(2)-connections for a surface Σ of genus 2. This example is already sufficiently general to visualize the global picture which emerges. As a space, N is just complex projective 3-space, by a result of Narasimhan and Ramanan [25]. However, we shall see that, as a smooth space, with smooth structure (3.6), it looks rather different.

Write $G = \mathrm{SU}(2)$, and let $Z = \{\pm 1\}$ denote the centre of G and $T = S^1 \subseteq G$ the standard circle subgroup inside G; it is a maximal torus. The decomposition of N according to orbit types of flat connections has the three pieces N_G, $N_{(T)}$, and N_Z, where the subscript refers to the conjugacy class of stabilizer; we recall that N_G consists of 16 isolated points and that $N_{(T)}$ is connected.

In view of what is said in Section 6 of our paper [16] near a point of the middle stratum $N_{(T)}$, as a smooth space, N looks like a product of a standard $\mathbb{R}^4$ with a copy of the half cone $C = \{(u,v,r); u^2+v^2=r^2; r \geq 0\}$, with smooth structure induced by the embedding of C in 3-space with coordinates (u,v,r). In fact, the latter arises as the reduced space for the diagonal $\mathrm{SO}(2,\mathbb{R})$-action on $W = \mathbb{R}^2 \times \mathbb{R}^2$ with its obvious symplectic structure, in the following way: Let $K = \mathrm{SO}(2,\mathbb{R})$, and write elements of W in the form $w = (q,p) \in \mathbb{R}^2 \times \mathbb{R}^2$. The algebra $\mathbb{R}[W]^K$ of invariants is generated by qq, pp, qp, $|qp|$, and the momentum mapping μ is given by $\mu(q,p) = |qp|$. However, μ generates the ideal I_V of polynomials in $\mathbb{R}[W]$ vanishing on the zero locus $V = \mu^{-1}(0)$; since μ is K-invariant it also generates the ideal I_V^K of K-invariant polynomials vanishing on V. Thus the coordinate ring $A[V]$ has four generators while the subalgebra of K-invariants $A[V]^K$ is generated by $u = qq - pp$, $v = 2qp$, $r = qq + pp$, subject to the relation $r^2 = u^2 + v^2$. The real affine categorical quotient $W//K$ is the double cone given by this equation while the reduced space W_{red} amounts to the positive cone C, with the cone point included. Moreover, it is manifest that the Zariski tangent space T_0C at the cone point 0 has dimension 3. In fact, the invariants u,v,r induce a map λ from $\mathbb{R}^4$ to $\mathbb{R}^3$ passing through a map $\tilde{\lambda}$ from $\mathbb{R}^4/K$ to $\mathbb{R}^3$; now λ has derivative zero at the origin while the derivative of $\tilde{\lambda}$ induces an isomorphism from $\mathrm{T}_{[0]}$ onto $\mathbb{R}^3$. Hence for a point $[A]$ of the middle stratum $N_{(T)}$, the Zariski tangent space $\mathrm{T}_{[A]}N$ has dimension $4+3=7$. On the other hand, the dimension of $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ equals 8, and the linear map λ_A from $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ to $\mathrm{T}_{[A]}N$ has rank four since the derivative of λ at the origin has rank zero. Thus the Zariski tangent space $\mathrm{T}_{[A]}N$ can in no way be identified with the cohomology group $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$, cf. Remark 7.12.

Likewise, in view of what is said in Section 7 of our paper [16], near any of the 16 points of N_G, as a smooth space, N looks like the reduced space for the momentum mapping μ from $W = (\mathbb{R}^3)^4$ to the dual of $\mathrm{so}(3,\mathbb{R})$, for the diagonal $\mathrm{SO}(3,\mathbb{R})$-action on W with its obvious symplectic structure, the $\mathrm{SO}(3,\mathbb{R})$-action on $\mathbb{R}^3$ being the obvious one. With the notation $(q_1,p_1,q_2,p_2) \in (\mathbb{R}^3)^4$ for the elements of W, the momentum mapping μ is given by the assignment to (q_1,p_1,q_2,p_2) of $q_1 \wedge p_1 + q_2 \wedge p_2$. Moreover, by invariant theory, cf. [33, 15], the ten distinct invariants

$$q_iq_j,\ q_ip_j,\ p_ip_j, \quad 1 \leq i,j \leq 2, \tag{8.1}$$

among the scalar products, together with the four determinants

$$|q_1p_1q_2|,\ |q_1p_1p_2|,\ |q_1q_2p_2|,\ |p_1q_2p_2|, \tag{8.2}$$

constitute a complete set of invariants for the $\mathrm{SO}(3,\mathbb{R})$-action on W. However, for $(q_1, p_1, q_2, p_2) \in V = \mu^{-1}(0)$, that is, when $q_1 \wedge p_1 + q_2 \wedge p_2 = 0$, any three of (q_1, p_1, q_2, p_2) are linearly dependent, that is, (q_1, p_1, q_2, p_2) lie in a plane in $\mathbb{R}^3$, whence the four determinants (8.2) vanish on V, and the algebra of invariants $A[V]^{\mathrm{SO}(3,\mathbb{R})}$ in the coordinate ring $A[V] = \mathbb{R}[W]/I_V$ is in fact generated by the ten scalar products; these induce the quadratic $\mathrm{SO}(3,\mathbb{R})$-invariant map

$$\lambda : W \longrightarrow \mathrm{S}^2(\mathbb{R}^4), \quad \lambda(q_1, p_1, q_2, p_2) = \begin{bmatrix} q_1q_1 & q_1q_2 & q_1p_1 & q_1p_2 \\ q_2q_1 & q_2q_2 & q_2p_1 & q_2p_2 \\ p_1q_1 & p_1q_2 & p_1p_1 & p_1p_2 \\ p_2q_1 & p_2q_2 & p_2p_1 & p_2p_2 \end{bmatrix} \tag{8.3}$$

into the 10-dimensional real vector space $\mathrm{S}^2(\mathbb{R}^4)$ of symmetric 4 by 4 matrices which, in turn, passes to an embedding

$$\tilde{\lambda} : W_{\mathrm{red}} \longrightarrow \mathrm{S}^2(\mathbb{R}^4) \tag{8.4}$$

of W_{red} into $\mathrm{S}^2(\mathbb{R}^4)$ as a real semi-algebraic set S; more details about this semi-algebraic realization will be given below. We assert at first that the Zariski tangent space T_0S at the origin equals the whole ambient space, that is, has dimension 10. In fact, S is a cone since for $(q_1, p_1, q_2, p_2) \in V$ and $t \in \mathbb{R}$,

$$\tilde{\lambda}[t(q_1, p_1, q_2, p_2)] = t^2\tilde{\lambda}[q_1, p_1, q_2, p_2] \in S.$$

Hence for $x \in S$, the half line $\{tx; t \geq 0\}$ lies in S. Let v be an arbitrary vector in $\mathbb{R}^3$ of length one. Then the vectors

$$\begin{matrix} (v,0,0,0), & (0,v,0,0), & (0,0,v,0), & (0,0,0,v), & (v,v,0,0), \\ (v,0,v,0), & (v,0,0,v), & (0,v,v,0), & (0,v,0,v), & (0,0,v,v) \end{matrix} \tag{8.5}$$

all lie in V, and inspection shows that their images in S under λ are linearly independent in the ambient vector space $\mathrm{S}^2(\mathbb{R}^4)$ and hence constitute a basis. In fact,

$$\lambda(v,0,0,0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \lambda(v,v,0,0) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

etc. Consequently, the linear span of the cone S equals the whole ambient space $\mathrm{S}^2(\mathbb{R}^4)$, and hence the latter coincides with the Zariski tangent space T_0S at the origin as asserted. In particular, the minimal number of generators of the algebra $A[V]^{\mathrm{SO}(3,\mathbb{R})}$ is ten, and this is also the minimal number of generators of $C^\infty(W_{\mathrm{red}})$ since if fewer generators did suffice the dimension of the Zariski tangent space would be smaller.

These observations translate to the moduli space N in the following way: Let $[A]$ be a point in N_G. Then the Zariski tangent space $\mathrm{T}_{[A]}N$ has dimension 10 and hence the minimal number of generators of $C^\infty(N)$ near $[A]$ or rather that of its germ at $[A]$ is 10. Moreover, a closer look reveals that the Zariski tangent space $\mathrm{T}_{[A]}N$ equals that of $\mathrm{T}_{[A]}N_{(T)}$, with reference to the induced smooth structure $C^\infty(N_{(T)})$. In fact, in the language of constrained systems, $N_{(T)}$ corresponds to reduced states where each of the two particles individually has angular momentum zero, cf. what is said in our paper [16], and hence the images of the ten vectors (8.5) under λ already lie in the part of S which corresponds to $N_{(T)}$. In particular, the minimal number of generators of the induced smooth structure $C^\infty(N_{(T)})$ near $[A]$ or rather that of its germ at $[A]$ is still 10. Finally, the linear map λ_A from $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ to $\mathrm{T}_{[A]}N$ is zero since the derivative of λ at the origin is zero. Thus, the Zariski tangent space $\mathrm{T}_{[A]}N$ can in no way be identified with the cohomology group $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$, cf. Remark 7.12. It seems also worthwhile pointing out that, cf. [16], as a complex variety, near a point $[A]$ in N_G, the stratum $N_{(T)}$ looks like the quadric $Y^2 = XZ$ in complex 3-space and hence at a point $[A]$ in N_G the complex Zariski tangent space of $N_{(T)}$ has dimension 3. Thus, we see once more that, as a smooth space, the moduli space N of flat SU(2)-connections for a surface Σ of genus 2 looks rather different from complex projective 3-space with its standard smooth structure.

More information about the geometry of N near a point $[A]$ in N_G can be obtained in the following way: The cone V in W may be defined as the zero locus of the single homogeneous real quartic function Ψ on W given by the formula

$$\Psi(q_1, p_1, q_2, p_2) = (q_1 \wedge p_1 + q_2 \wedge p_2)(q_1 \wedge p_1 + q_2 \wedge p_2).$$

However, this function looks like

$$\Psi(q_1, p_1, q_2, p_2) = \begin{vmatrix} q_1q_1 & q_1p_1 \\ p_1q_1 & p_1p_1 \end{vmatrix} + 2\begin{vmatrix} q_1q_2 & q_1p_2 \\ p_1q_2 & p_1p_2 \end{vmatrix} + \begin{vmatrix} q_2q_2 & q_2p_2 \\ p_2q_2 & p_2p_2 \end{vmatrix}$$

and hence passes to a quadratic function ψ on $\mathrm{S}^2(\mathbb{R}^4)$. Next we observe that the reduced space W_{red} with respect to the $\mathrm{SO}(3,\mathbb{R})$-action coincides with the reduced space with respect to the action of the larger group $\mathrm{O}(3,\mathbb{R})$ since the four determinants (8.2) which distinguish between the two reduced spaces vanish on V; this is a special phenomenon due to the fact that we are considering angular momentum of two particles in $\mathbb{R}^3$. Now W_{red} appears as the zero locus of the single function ψ on $W/\mathrm{O}(3,\mathbb{R})$. However, by invariant theory, the ten distinct inner products (8.1) constitute a complete set of invariants for the $\mathrm{O}(3,\mathbb{R})$-action on W subject to the single defining relation

$$\begin{vmatrix} q_1q_1 & q_1q_2 & q_1p_1 & q_1p_2 \\ q_2q_1 & q_2q_2 & q_2p_1 & q_2p_2 \\ p_1q_1 & p_1q_2 & p_1p_1 & p_1p_2 \\ p_2q_1 & p_2q_2 & p_2p_1 & p_2p_2 \end{vmatrix} = 0. \tag{8.6}$$

Consequently, the affine categorical quotient $W//\mathrm{O}(3,\mathbb{R})$ amounts to the space of singular symmetric 4×4 matrices, and $W/\mathrm{O}(3,\mathbb{R})$ is realized as its semi-algebraic subset which consists of non-negative semi-definite matrices. Thus, the reduced space W_{red} and hence the space N near a point $[A]$ in N_G appear as the zero locus of the single function ψ on the subspace of singular non-negative semi-definite matrices. The determinant and ψ clearly yield two $\mathrm{SO}(3,\mathbb{R})$-invariant polynomials vanishing on V, that is, elements of the ideal $I_V^{\mathrm{SO}(3,\mathbb{R})}$ but these two will *not* generate $I_V^{\mathrm{O}(3,\mathbb{R})}$. In fact, we can at once write down the following six $\mathrm{O}(3,\mathbb{R})$-invariant polynomials which vanish on V and are quadratic in the generators (8.1) of $A[W]^{\mathrm{O}(3,\mathbb{R})}$:

$$(q_1 \wedge q_2)\mu, \quad (q_1 \wedge p_1)\mu, \quad (q_1 \wedge p_2)\mu, \quad (q_2 \wedge p_1)\mu, \quad (q_2 \wedge p_2)\mu, \quad (p_1 \wedge p_2)\mu \tag{8.7}$$

More explicitly, (a,b) denoting any of the six couples (q_1,q_2), etc., we have

$$((a \wedge b)\mu)(q_1,p_1,q_2,p_2) = \begin{vmatrix} aq_1 & ap_1 \\ bq_1 & bp_1 \end{vmatrix} + \begin{vmatrix} aq_2 & ap_2 \\ bq_2 & bp_2 \end{vmatrix}.$$

Moreover, from the six relations

$$|u_{j_1}u_{j_2}u_{j_3}||v_{j_1}v_{j_2}v_{j_3}| = \begin{vmatrix} u_{j_1}v_{j_1} & u_{j_1}v_{j_2} & u_{j_1}v_{j_3} \\ u_{j_2}v_{j_1} & u_{j_2}v_{j_2} & u_{j_2}v_{j_3} \\ u_{j_3}v_{j_1} & u_{j_3}v_{j_2} & u_{j_3}v_{j_3} \end{vmatrix}$$

among the $\mathrm{SO}(3,\mathbb{R})$-invariants (8.1) and (8.2), cf. [33], where $|u_{j_1}u_{j_2}u_{j_3}|$ and $|v_{j_1}v_{j_2}v_{j_3}|$ refer to any of the four determinants (8.2), we conclude that on V all 3×3 minors of $\lambda(q_1,p_1,q_2,p_2)$ vanish; these 3×3 minors yield six additional $\mathrm{O}(3,\mathbb{R})$-invariant polynomials vanishing on V, of degree three in the generators (8.1) of $A[W]^{\mathrm{O}(3,\mathbb{R})}$. In particular, the image $\tilde{\lambda}(W_{\mathrm{red}})$ lies in the subspace of symmetric 4×4 matrices having rank at most 2. We conjecture that the six quadratic polynomials (8.7) and the six cubic ones arising from the 3×3 minors constitute a complete set of generators of the ideal $I_V^{\mathrm{O}(3,\mathbb{R})} = I_V^{\mathrm{SO}(3,\mathbb{R})}$.

The methods of Lerman et al. [20] yield a geometric description of W_{red}, viewed as a subspace of that of symmetric 4×4 matrices: Let J be the symplectic operator on $\mathbb{R}^4$: $J^2 = -1$, $J^{\mathrm{t}}J = \mathrm{Id}$, $\sigma(v,w) = vJw$. The assignment $S \mapsto JS$ identifies $\mathrm{S}^2(\mathbb{R}^4)$ with the Lie algebra $\mathrm{sp}(2,\mathbb{R})$, and a result in [20] implies that $\tilde{\lambda}$ identifies W_{red} with the closure of the nilpotent orbit in $\mathrm{sp}(2,\mathbb{R})$ which corresponds to positive symmetric 4×4 matrices of rank at most 2 having kernel a coisotropic subspace. The Lie algebra $\mathrm{sp}(2,\mathbb{R})$ has rank two – in fact it is the split real form of C_2 which coincides with B_2, though – and its algebra of $\mathrm{Sp}(2,\mathbb{R})$-invariants under the adjoint representation is a polynomial algebra, generated by the Killing form and the determinant. Hence, the nilvariety $\mathrm{Nil}(\mathrm{sp}(2,\mathbb{R}))$ is of real dimension 8; it consists of singular matrices in $\mathrm{sp}(2,\mathbb{R})$ having vanishing Killing form, and its subspace $\mathrm{Nil}^+(\mathrm{sp}(2,\mathbb{R}))$ of non-negative semi-definite matrices is a union $\mathbf{n}_0 \cup \mathbf{n}_1 \cup \mathbf{n}_2 \cup \mathbf{n}_3$ of four nilpotent adjoint orbits, $\mathbf{n}_j$ being the subspace of non-negative semi-definite rank j matrices. The reduced space W_{red} now appears as the union $\mathbf{n}_0 \cup \mathbf{n}_1 \cup \mathbf{n}_2$. It may be described as a zero locus in $\mathrm{Nil}^+(\mathrm{sp}(2,\mathbb{R}))$ in various ways, that is,

— of the function ψ or what corresponds to it, restricted to $\mathrm{Nil}^+(\mathrm{sp}(2,\mathbb{R}))$,

— of the functions (8.7) or what corresponds to them, restricted to $\mathrm{Nil}^+(\mathrm{sp}(2,\mathbb{R}))$; in fact, the two functions $(q_1 \wedge p_1)\mu$ and $(q_2 \wedge p_2)\mu$ already suffice;

— of the six 3×3 minors, restricted to $\mathrm{Nil}^+(\mathrm{sp}(2,\mathbb{R}))$.

Somewhat amazingly, since, as a space, W_{red} is smooth in the ordinary sense, in fact a copy of real affine 6-dimensional space, the union $\mathbf{n}_0 \cup \mathbf{n}_1 \cup \mathbf{n}_2$ is just a real affine 6-dimensional space.

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