SMOOTH SPACES VERSUS CONTINUOUS SPACES IN MODELS FOR SYNTHETIC DIFFERENTIAL GEOMETRY

Ieke MOERDIJK

Universiteit van Amsterdam, Roetersstraat 15, 1018 WB Amsterdam, The Netherlands

Gonzalo E. REYES

Université de Montréal, Montréal, Canada and Rijksuniversiteit Utrecht, Utrecht, The Netherlands

In topos models for synthetic differential geometry we study connections between smooth spaces (which interpret synthetic calculus) and continuous spaces (which interpret intuitionistic analysis). Our main tools are adjoint retractions of toposes and the standard map from the smooth reals to the continuous reals.

This paper deals with models of Synthetic Differential Geometry (SDG). Acquaintance with that theory is not really presupposed, although some familiarity with SDG would put our results into a proper perspective. The interested reader is referred to [10] for information about SDG.

We will study connections between 'smooth' spaces built up from the smooth reals R and 'continuous' (or 'set-like') spaces constructed from the Dedekind reals $\mathbb{R}$ in the topos $\mathcal{G}$ introduced by E. Dubuc (see [3]) as a model for SDG (for the definition of $\mathcal{G}$, see Section 4 of this paper). The properties that are valid for the first exhibit $\mathcal{G}$ as a model of *Synthetic (smooth) Calculus*, whereas those valid for the second (the continuous spaces) tell us to which extent $\mathcal{G}$ is a model of *Intuitionistic Analysis*.

In particular, we have a comparison map, the so-called standard map $\text{st} : R \to \mathbb{R}$ which sends a smooth real $x \in R$ into the obvious Dedekind cut $(\{q \in \mathbb{Q} \mid q < x\}, \{q \in \mathbb{Q} \mid x < q\})$. (This map has been studied in the synthetic context by Barbara Veit, in an unpublished manuscript [18].)

We approach the problem of the connection between these types of spaces by establishing a comparison between the topos $\mathcal{G}$ and a topological topos, the Eucli-

dean topos $\mathscr{E}$ introduced in Section 1, a variant of which was considered in [1]. In fact, we construct an adjoint retraction

$$\mathscr{G} \underset{\rightleftarrows}{\rightarrow} \mathscr{E}$$

with $\varrho\gamma \simeq 1$ and $\gamma^* \dashv \gamma_* = \varrho^* \dashv \varrho_*$. Thus ϱ^* is faithful, preserves exponentials, and moreover preserves continuous spaces. To the casual reader this may seem rather pointless. But is not, since we have a retraction of pointed toposes, and this allows us to reduce some problems in $\mathscr{G}$ to the corresponding ones in $\mathscr{E}$ (Section 2).

As an application of this adjoint retraction, we give explicit descriptions of the continuous spaces in $\mathscr{G}$ mentioned before (Section 5). Indeed, contrary to the smooth spaces, continuous spaces are spaces of points of T_1-locales (equivalently, spaces of models of T_1-propositional theories), and our descriptions follow from the known characterizations of such locales in the topological topos of Section 1.

In Section 4 we show that $\mathscr{G}$ is the generic model of a certain theory of loci, or formal smooth varieties over an Archimedean local ring, and ϱ is essentially the functor which associates with each locus its set of points, considered as a topological space in the site of $\mathscr{E}$. Alternatively, ϱ may be viewed as the extension to the level of toposes (considered as 'generalized spaces') of the standard map st : $R \rightarrow \mathbb{R}$, where R is replaced by the 'generalized smooth space' $\mathscr{G}$, and $\mathbb{R}$, or rather the image of st as a subspace of the continuous space $\mathbb{R}$, by the 'generalized space' $\mathscr{E}$ which is a quotient of a 'generalized continuous space'.

Once that the topos $\mathscr{G}$ has been introduced, we can apply the general machinery of the first two sections: Bar Induction (i.e. separable locales have enough points), Brouwer's theorem on continuity of all functions from the Dedekind reals to itself, and several other continuity results involving both smooth and continuous spaces are obtained in $\mathscr{G}$ (Section 5). The standard map is constantly used, as well as some techniques of van der Hoeven & Moerdijk [5].

In a final section we point out that continuity properties of continuous spaces may fail if we do not assume the smooth reals to be Archimedean, by constructing a topos $\mathscr{G}_{\text{fin}}$ in which the smooth analysis remains essentially the same, whereas for example Brouwer's theorem fails for the Dedekind reals.

As a general remark, we should point out that the continuous spaces in our toposes do not model 'full' Intuitionistic Analysis, as opposed, for example, to the models of van der Hoeven & Moerdijk [5]. For instance, principles of countable choice and dependent choices (DC) fail because of the connectedness properties of the euclidean topos. Indeed, it is an open problem whether SDG is consistent with such choice principles.

On the other hand, some principles of local continuous choice for smooth reals hold in our models, which have no analog in full (i.e. with DC) Intuitionistic Analysis.

Although, as said above, the reader is not presupposed to have detailed knowledge of SDG, we do assume that he or she is familiar with the general theory of Grothendieck toposes, in particular sheaf semantics (see e.g. Reyes [15]). We also

assume the reader to lack some ignorance of the theory of locales; in particular, of how locales classify models of T_1-propositional theories (see Joyal & Tierney [9], Fourman & Grayson [4]).

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1. Topological toposes, the euclidean topos

A site $(\mathbb{C}, J)$ is called *topological* if $\mathbb{C}$ is a category of topological spaces and continuous maps, which is closed under open inclusions (i.e. if X is an object of $\mathbb{C}$ and $U \in \mathscr{O}(X)$, then the inclusion $U \hookrightarrow X$ is an arrow of $\mathbb{C}$) and J is the open cover topology, i.e. J is generated by families of the form

$$\{U_i \hookrightarrow X\}_i$$

where $U_i \in \mathscr{O}(X)$ and $\{U_i\}$ covers X. A topos of the form 'sheaves on a topological site' is called a *topological topos*.

It is well-known that the Dedekind reals, for example, are interpreted in topological toposes by the sheaf of continuous $\mathbb{R}$-valued functions. An elegant way of explaining this, which is due to M. Fourman, is based on the following lemma.

1.1. Lemma. *Let* $(\mathbb{C}, J)$ *be a topological site,* $X \in \mathbb{C}$. *Then* X *is an adjoint retract of the locale* $\Omega(X)$ *of J-closed cribles on X, i.e. there are continuous maps*

$$X \underset{r}{\overset{i}{\rightleftarrows}} \Omega(X)$$

with $ri = 1_X$, $1_{\Omega(X)} \le ir$.

Proof. If $U \in \mathscr{O}(X)$, let $r^*(U)$ be the closed crible generated by $U \hookrightarrow X$, i.e. $r^*(U) = \{Y \xrightarrow{f} X \mid f(Y) \subseteq U\}$. And if $K \in \Omega(X)$, let $i^*(K)$ be the largest open $U \subseteq X$ such that the inclusion $U \hookrightarrow X$ is in K, i.e. $i^*(K) = \bigcup\{V \text{ open} \subseteq X \mid V \hookrightarrow X \in K\}$. i^* and r^* are $\wedge\vee$-maps, and $i^*r^* = 1$ clearly. Also since $i^*(K) \hookrightarrow X \in K$, $r^*i^*(K) \le K$, i.e. $1_{\Omega(X)} \le ir$. $\square$

Let $\mathscr{E} = \mathrm{Sh}(\mathbb{C}, J)$, where $(\mathbb{C}, J)$ is an arbitrary site. Since $\mathrm{Sh}(\mathbb{R})$, sheaves on the (external) space $\mathbb{R}$ of Dedekind reals, classifies the notion of a Dedekind real, the sheaf $\mathbb{R}_{\mathscr{E}}$ of Dedekind reals has as sections at $C \in \mathbb{C}$ the locale maps $\Omega(C) \to \mathbb{R}$. In the case of topological sites there is a simpler representation which uses the fact that $\mathbb{R}$ is a T_1-locale:

1.2. Definition. A locale X is called T_1 if for all pairs $Y \underset{g}{\overset{f}{\rightrightarrows}} X$ into X, $f \le g$ implies $f = g$. (Recall that $f \le g$ iff for the corresponding framemaps $f^*, g^*: \mathscr{O}(X) \rightrightarrows \mathscr{O}(Y)$ it holds that $g^* \le f^*$, i.e. $\forall V \in \mathscr{O}(Y)$ $g^*(V) \le f^*(V)$.)

1.3. Remark. Note that for topological (sober) spaces, a space X is T_1 in the usual topological sense iff for all maps $Y \underset{g}{\overset{f}{\rightrightarrows}} X$, Y a (sober) space, $f \leq g$ implies $f = g$. In particular, if X is a sober space which is T_1 in the above localic sense, X is a T_1-space in the usual sense. The converse need not hold, but all regular (Hausdorff) spaces are T_1 as locales.

1.4. Lemma. *Let* $A \underset{g}{\overset{f}{\rightrightarrows}} B$ *be a pair of continuous maps of locales such that* f^* *is adjoint to* g^*, *say* $f^* \dashv g^*$, *and let* X *be a* T_1-*locale. Then* f *and* g *induce maps* $\mathrm{Cts}(A, X) \rightleftarrows \mathrm{Cts}(B, X)$ *by composition, which are inverse to each other, i.e.*

$$\mathrm{Cts}(A, X) \cong \mathrm{Cts}(B, X).$$

Proof. Since $1_{r(B)} \leq g^*f^*$, $f^*g^* \leq 1_{r(A)}$, this is immediate from definition 1.2. $\square$

1.5. Corollary. *Let* A *be a* T_1-*propositional theory, and let* A_0 *be the corresponding locale in* Sets *(so* $\mathrm{Sh}(A_0)$ *classifies A-models in Grothendieck toposes). Let* $\delta = \mathrm{Sh}(\mathbb{C}, J)$ *be a topological topos. Then the object of A-models in* δ *is given by the sheaf*

$$\mathrm{Cts}(-, A_0) : \mathbb{C}^{\mathrm{op}} \to \mathrm{Sets}.$$

1.6. Examples. The sheaf $\mathbb{R}_\delta$ of Dedekind reals in δ is (isomorphic to) the sheaf $\mathbb{R}_\delta(X) = \mathrm{Cts}(X, \mathbb{R})$, $X \in \mathbb{C}$. Similarly for Bairespace $\mathbb{N}^{\mathbb{N}}$, Cantorspace $2^{\mathbb{N}}$, and the functionspace $\mathbb{R}^{\mathbb{R}}$. Note that $\mathrm{Sh}(\mathbb{R}^{\mathbb{R}})$ classifies continuous maps from the *formal* reals to itself. In topological toposes, however, the locale of formal reals coincides with the space of Dedekind reals, by the following proposition.

1.7. Proposition. *Let* $\delta = \mathrm{Sh}(\mathbb{C}, J)$ *be a topological topos. Then the Fan Theorem holds in* δ. *In particular, the Cantorspace* $2^{\mathbb{N}}$ *and the Dedekind unit interval* $[0, 1] \subseteq \mathbb{R}$ *are compact in* δ.

Proof. Suppose S is a subsheaf of the sheaf $2^{<\mathbb{N}}$ of finite sequences at $X \in \mathbb{C}$, such that

$$X \Vdash \text{"}S \text{ is a monotone inductive bar"}.$$

For each $\alpha \in 2^{\mathbb{N}}$, the corresponding constant function $\alpha : X \to 2^{\mathbb{N}}$ is an element of $2^{\mathbb{N}}(X)$ (an element of the internal Cantorset $2^{\mathbb{N}}$ at stage X, by Corollary 1.5), so $X \Vdash \exists n\, \bar{\alpha}(n) \in S$, where $\bar{\alpha}(n)$ is the initial segment $\{\alpha(i)\}_{i<n}$ of α. Thus, if for a point $x \in X$ we let

$$S_x = \{v \in 2^{<\mathbb{N}} \mid \exists \mathrm{nbd}\ U_x \text{ of } x \text{ with } U_x \Vdash v \in S\},$$

we find that each S_x is a cover of $2^{\mathbb{N}}$ externally. Clearly, S_x is monotone and inductive, so by the external Fan Theorem, $\langle\ \rangle \in S_x$ for all $x \in X$. Thus $X \Vdash \langle\ \rangle \in S$. $\square$

In the topological sites that we will meet later on, all spaces are locally compact, and we can improve a bit on the preceding proposition, by replacing the Fan Theorem by the principle of Bar Induction. The importance of the validity of Bar Induction in our context is that it implies that each 'countably presented' (separable) locale has enough points (cf. Fourman & Grayson [4]).

1.8. Proposition. *Let $\mathscr{E} = \mathrm{Sh}(\mathbb{C}, J)$ be a topological topos, and suppose all $X \in \mathbb{C}$ are locally compact. Then Bar Induction holds in $\mathscr{E}$.*

Proof. Let S be a subsheaf of $\mathbb{N}^{<\mathbb{N}}$ at X, such that

$$X \Vdash \text{``}S \text{ is a monotone inductive bar''}.$$

For each $x \in X$ we may choose a relatively compact neighbourhood V_x. Now if $\alpha \in \mathbb{N}^\mathbb{N}$, the constant function $\alpha : X \to \mathbb{N}^\mathbb{N}$ is an element of $\mathbb{N}^\mathbb{N}(X)$ (Corollary 1.5), and hence $X \Vdash \exists n \, \bar{\alpha}(n) \in S$, i.e. there is a cover $\{U_n^\alpha\}_n$ of X such that $U_n^\alpha \Vdash \bar{\alpha}(n) \in S$. A finite set of these U_n^α's cover V_x, and since S is forced to be monotone,

$$V_x \Vdash \text{``}\bar{\alpha}(m) \in S\text{''}, \quad \text{for some } m.$$

Thus, if we let $S_x = \{u \mid V_x \Vdash u \in S\}$, S_x is an external monotone inductive bar. Hence $\langle \, \rangle \in S_x$ by external Bar Induction, i.e. $V_x \Vdash \langle \, \rangle \in S$. Since the V_x cover X, also $X \Vdash \langle \, \rangle \in S$. $\square$

1.9. Definition. A topological topos that will be of much use to us in the sequel is the *euclidean topos*. Let $\mathbb{E}$ be the topological site consisting of locally closed (=locally compact) subspaces of some $\mathbb{R}^n$, $n \in \mathbb{N}$, and C^∞-maps between them. Recall that if $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ are objects of $\mathbb{E}$, a C^∞-map $X \to Y$ is a function $f : X \to Y$ such that for all $x \in X$ there exists an open nbd U_x of x (in $\mathbb{R}^n$) and C^∞-map $g : U_x \to \mathbb{R}^m$ such that $g \mid (U_x \cap X) = f \mid U_x$. By a partitions of unity argument, $f : X \to Y$ is a C^∞-map iff there exists an open $U \subseteq X$ and a C^∞-map $g : U \to \mathbb{R}^m$ (C^∞ in the usual sense of having all continuous partial derivatives) such that $g \mid X = f$. If $X \subseteq \mathbb{R}^n$ is *closed*, we may take $U = \mathbb{R}^n$ ('smooth Tietze'). Therefore, closed objects of $\mathbb{E}$ are convenient to deal with. Fortunately, we may restrict our attention to closed objects of $\mathbb{E}$, since if $U \subseteq \mathbb{R}^n$ is open, U is isomorphic in $\mathbb{E}$ to a closed set $\hat{U} \subseteq \mathbb{R}^{n+1}$: let $f : \mathbb{R}^n \to [0, 1]$ be a C^∞-characteristic function for U ($f(x) > 0$ iff $x \in U$) and let $\hat{U} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \cdot y = 1\}$. The *euclidean topos* is the topos of sheaves on the site $\mathbb{E}$ with the open cover topology, and is denoted by $\mathscr{E} = \mathrm{Sh}(\mathbb{E})$.

1.10. Proposition. *In the euclidean topos $\mathscr{E}$, all functions $\mathbb{R} \to \mathbb{R}$ are continuous (hence uniformly continuous on closed intervals, by Proposition 1.6).*

Proof. By Corollary 1.5, continuous functions from the formal reals to itself at stage $X \in \mathbb{E}$ correspond to continuous maps $X \to \mathbb{R}^\mathbb{R}$, or $X \times \mathbb{R} \to \mathbb{R}$ in the real world. By Proposition 1.6, the formal reals have enough points in $\mathscr{E}$, i.e. they coincide with

the space of Dedekind reals. Such a map $\varphi: X \times \mathbb{R} \to X$ acts internally, as a natural transformation

$$\tau: \mathbb{R} \to \mathbb{R} \quad \text{over } \mathbb{E}/X$$

by composition: given $Y \xrightarrow{f} X$ in $\mathbb{E}$ and $\alpha \in \mathbb{R}(Y)$, i.e. (Corollary 1.5) $Y \xrightarrow{\alpha} \mathbb{R}$, $\tau_f(\alpha) = \varphi \circ \langle f, \alpha \rangle$. Hence, to show that all internal functions $\mathbb{R} \to \mathbb{R}$ are continuous, we need to prove that any natural transformation $\tau: \mathbb{R} \to \mathbb{R}$ over $\mathbb{E}/X$ comes from such a continuous $\varphi: X \times \mathbb{R} \to \mathbb{R}$.

To this end, pick such a τ, and let $\varphi = \tau_{p_1}(p_2)$, where p_1, p_2 are the projections

$$X \times \mathbb{R} \xrightarrow{\quad p_2 \quad} \mathbb{R}$$

$$\downarrow{p_1}$$

$$X$$

We claim that for any $Y \xrightarrow{f} X$ in $\mathbb{E}$ any continuous $\alpha: Y \to \mathbb{R}$, $\tau_f(\alpha) = \varphi \circ \langle f, \alpha \rangle$. To see this, choose such f and α. α is not a morphism of $\mathbb{E}$, but its restriction to points of Y is, and now we can apply naturality of τ: choose any point $y \in Y$, and apply naturality of τ to the diagram

in $\mathbb{E}/X$. Then one gets

$$\tau_f(\alpha)(y) = \tau_{f \circ y}(\alpha(y)) = \tau_{p_1 \circ f(y)}(\alpha(y)) = \tau_{p_1}(\langle \alpha, f \rangle)(y). \quad \Box$$

1.11. Remark. If M is any space which is locally homeomorphic to a space in $\mathbb{E}$ (e.g. M is a manifold), and N is any space, M and N have interpretations $M_{\mathcal{E}}$ and $N_{\mathcal{E}}$ in $\mathcal{E}$ (as in 1.6: $M_{\mathcal{E}} = \text{Cts}(-, M)$, $N_{\mathcal{E}} = \text{Cts}(-, N)$), and the same proof gives that $\mathcal{E} \models$ "all functions $M_{\mathcal{E}} \to N_{\mathcal{E}}$ are continuous".

2. Adjoint retractions of toposes

As stated in the introduction, one of the themes of this paper is a comparison of the smooth topos $\mathcal{G}$ which will be defined later on, and the euclidean topos $\mathcal{E}$ which we just described. We will now sketch the general context in which this comparison

takes place. (We will not formulate things in all generality, however. Adjoint retractions will be more extensively discussed elsewhere.)

Let $\mathbb{C}$ and $\mathbb{D}$ be categories with finite left limits, both equipped with a subcanonical Grothendieck topology. If $P : \mathbb{C} \to \mathbb{D}$ is a left-exact functor which preserves covers, P induces a geometric morphism $Sh(\mathbb{D}) \xrightarrow{P} Sh(\mathbb{C})$. Explicitly, the inverse image functor $p^* : Sh(\mathbb{C}) \to Sh(\mathbb{D})$ is the left Kan extension of

$$\mathbb{C} \xrightarrow{P} \mathbb{D} \xrightarrow{Y} Sh(\mathbb{D})$$

(Y is the Yoneda embedding), while the direct image functor is defined by "compose with P" : $Sh(\mathbb{D}) \to Sh(\mathbb{C})$.

A (left-exact) functor $P : \mathbb{C} \to \mathbb{D}$ sometimes also induces a geometric morphism $Sh(\mathbb{C}) \to Sh(\mathbb{D})$. The functor of presheaves

$$P^* = \text{compose with } P : Sets^{\mathbb{D}^{op}} \to Sets^{\mathbb{C}^{op}}$$

has both adjoints, the Kan extensions,

$$\varinjlim_P \dashv P^* \dashv \varprojlim_P,$$

and thus P induces a geometric morphism

$$q : Sets^{\mathbb{C}^{op}} \to Sets^{\mathbb{D}^{op}}$$

with $P^* = q^*$ as inverse image.

2.1. Lemma. *Let* $P : \mathbb{C} \to \mathbb{D}$ *be any functor. Then the geometric morphism*

$$q : Sets^{\mathbb{C}^{op}} \to Sets^{\mathbb{D}^{op}}$$

described above restricts to a geometric morphism $Sh(\mathbb{C}) \to Sh(\mathbb{D})$ *iff* P *has the following*

Covering Lifting Property (CLP). *For every cover* $\{D_\alpha \xrightarrow{f_\alpha} PC\}_\alpha$ *in* $\mathbb{D}$ *there exists a cover* $\{C_\beta \xrightarrow{g_\beta} C\}_\beta$ *in* $\mathbb{C}$ *such that every* $P(g_\beta) : PC_\beta \to PC$ *factors through some* f_α.

Proof. ($\Leftarrow$) We need to show (cf. [6, Theorem 3.47]) that the restriction of q^* to representables,

$$\mathbb{D} \to Sets^{\mathbb{C}^{op}}, \quad D \mapsto \mathbb{D}(P(-), D)$$

maps covers to dense families. To see that this follows from the covering lifting property, take a cover $\{D_\alpha \xrightarrow{f_\alpha} D\}_\alpha$ in $\mathbb{D}$ and an element $h \in \mathbb{D}(P(C), D)$ for some $C \in \mathbb{C}$. Make the pullbacks

$$
\begin{array}{ccc}
D_\alpha & \xrightarrow{f_\alpha} & D \\
\uparrow{\scriptstyle h'} & & \uparrow{\scriptstyle h} \\
D'_\alpha & \xrightarrow{f'_\alpha} & PC
\end{array}
$$

and apply the covering lifting property to the family $\{D_\alpha \xrightarrow{f'_\alpha} PC\}_\alpha$ to get a covering $\{C_\beta \xrightarrow{g_\beta} C\}$ in $\mathbb{C}$. Each Pg_β is of the form $f'_\alpha \circ k_\alpha$, so for each restriction $h 1 g_\beta$ of h we have

$$h 1 g_\beta =_{\mathrm{def}} h \circ Pg_\beta = h \circ f'_\alpha \circ k_\alpha = f_\alpha \circ h' \circ k_\alpha.$$

Thus, given $h \in \mathbb{D}(PC, D)$ we have for each $C_\beta \xrightarrow{g_\beta} C$ an element $h' \circ k_\alpha \in \mathbb{D}(PC_\beta, D_\alpha)$ which is mapped to $h 1 g_\beta$ by

$$\mathbb{D}(P(-), D_\alpha) \xrightarrow{f_\alpha \circ -} \mathbb{D}(P(-), D).$$

In other words, the family $\{\mathbb{D}(P(-), D_\alpha) \to \mathbb{D}(P(-), D)\}_\alpha$ is dense.

($\Rightarrow$) For the converse, we can just reverse this argument for the particular case that $PC \xrightarrow{h} D$ is the identity on $D = PC$. $\square$

Given a geometric morphism $q : \mathrm{Sets}^{\mathbb{C}^{\mathrm{op}}} \to \mathrm{Sets}^{\mathbb{D}^{\mathrm{op}}}$ induced by a functor $P : \mathbb{C} \to \mathbb{D}$ having the CLP, its restriction $\mathrm{Sh}(\mathbb{C}) \xrightarrow{q} \mathrm{Sh}(\mathbb{D})$ can thus be described as

$$q^* = \text{``compose with } P \text{, then sheafify''},$$

$$q_*(X)(D) = \mathrm{Sh}(\mathbb{C})[\mathbb{D}(P(-), D), X]$$

($q_*(X)$ is a sheaf if X is, when P has the CLP).

If P is left-exact, preserves covers, and has the CLP, we have two geometric morphisms

$$\mathrm{Sh}(\mathbb{C}) \underset{p}{\overset{q}{\rightleftarrows}} \mathrm{Sh}(\mathbb{D})$$

and q^* is just "compose with p", i.e. $q^* = p_*$. In the case that we consider in this paper, the functor P has a left adjoint $L : \mathbb{D} \to \mathbb{C}$ such that $P \circ L = 1$. It then follows that this adjunction lifts to an adjunction "compose with L" $\dashv$ "compose with P" between the categories of presheaves $\mathrm{Sets}^{\mathbb{C}^{\mathrm{op}}} \rightleftarrows \mathrm{Sets}^{\mathbb{D}^{\mathrm{op}}}$. Hence in this case the left adjoint $p^* : \mathrm{Sh}(\mathbb{C}) \to \mathrm{Sh}(\mathbb{D})$ to p_* is "compose with L" followed by sheafification. Therefore $p^* p_* \cong 1$, i.e. p is an inclusion, and consequently q is a surjection and $qp \cong 1$.

We put all this together in the following theorem.

2.2. Theorem. *Let* $P : \mathbb{C} \to \mathbb{D}$ *be a left-exact functor which preserves covers and has the covering lifting property. Then P induces two geometric morphisms*

$$\mathrm{Sh}(\mathbb{C}) \underset{p}{\overset{q}{\rightleftarrows}} \mathrm{Sh}(\mathbb{D})$$

with $p^* \dashv p_* = q^* \dashv q_*$. *If P has a left adjoint-right inverse, then p is an inclusion, q a surjection, and $qp \cong 1$.* $\square$

We will call a pair of geometric morphisms as described in this theorem an *adjoint*

retraction. Let us note some properties of adjoint retractions that we will need later on.

2.3. Proposition. *Let* $\text{Sh}(\mathbb{C}) \xrightarrow[p]{\overset{q}{\longrightarrow}} \text{Sh}(\mathbb{D})$ *be an adjoint retraction induced by a functor* $P: \mathbb{C} \to \mathbb{D}$ *as in Theorem* 2.2, *and let* A *be a* T_1-*locale. Then* $q^* = p_*$ *preserves the sheaf of A-models.*

Proof. First note that if $P: \mathbb{C} \to \mathbb{D}$ satisfies the hypothesis of Theorem 2.2, so does every localization $P/C: \mathbb{C}/C \to \mathbb{D}/PC$. Hence we have adjoint retractions

(*) $\text{Sh}(\mathbb{C})/C \rightleftarrows \text{Sh}(\mathbb{D})/PC$

for every $C \in \mathbb{C}$. The object of A-models in $\text{Sh}(\mathbb{C})$ is given by $A_{\text{Sh}(\mathbb{C})}(C) = \text{Cts}(\Omega(C), A)$, and similarly for $A_{\text{Sh}(\mathbb{D})}$. But the pair of geometric morphisms (*) induces an isomorphism

$$\text{Cts}(\Omega(C), A) \rightleftarrows \text{Cts}(\Omega(PC), A)$$

(natural in C), by Lemma 1.4. Thus $q^*(A_{\text{Sh}(\mathbb{C})}) \cong A_{\text{Sh}(\mathbb{D})}$. $\square$

Proposition 2.3 tells us for example what the Dedekind reals are in $\text{Sh}(\mathbb{D})$, if we know them already in $\text{Sh}(\mathbb{C})$: if $\mathbb{R}_{\text{Sh}(\mathbb{C})}$ denotes the sheaf of Dedekind reals in $\text{Sh}(\mathbb{C})$, $\mathbb{R}_{\text{Sh}(\mathbb{D})}$ is (isomorphic to) the sheaf $\mathbb{R}_{\text{Sh}(\mathbb{C})} \circ P$.

2.4. Proposition. *Let* $\text{Sh}(\mathbb{C}) \xrightarrow[p]{\overset{q}{\longrightarrow}} \text{Sh}(\mathbb{D})$ *be as in Theorem* 2.2. *Then* q *is locally connected, in particular* q^* *preserves exponentials.*

Proof. The preservation of exponentials by q^* is equivalent to the existence of a left adjoint $q_!$ to q^*, satisfying Frobenius reciprocity; i.e. for $X \in \text{Sh}(\mathbb{C})$, $Y \in \text{Sh}(\mathbb{D})$, the canonical map

$$q_!(X \times q^*(Y)) \to q_!(X) \times Y$$

is an isomorphism (cf. Barr & Paré [2]).

Clearly in our case, p^* witnesses the existence of such a $q_!$. To show that q is locally connected, we have to check that for every $X \in \text{Sh}(\mathbb{D})$, the inverse image of $q/X: \text{Sh}(\mathbb{C})/q^*X \to \text{Sh}(\mathbb{D})/X$ preserves exponentials. But this is clear from the fact that if q is (part of) an adjoint retraction, then so is q/X. $\square$

2.5. Corollary. *Let* $\text{Sh}(\mathbb{C}) \xrightarrow[p]{\overset{q}{\longrightarrow}} \text{Sh}(\mathbb{D})$ *be as in Theorem* 2.2. *Then* q^* *is an open geometric morphism, i.e.* q^* *preserves the universal quantifiers* $\forall f$ (*any map* $f: X \to X'$ *in* $\text{Sh}(\mathbb{D})$). *Hence* q^* *preserves and reflects all first order logic.*

Proof. Every locally connected morphism is open (cf. Johnstone [7]). But the preservation of universal quantification also follows easily directly from the surjectivity of P in this case. $\square$

2.6. Example. Let $\text{Sh}(\mathbb{C}) \underset{p}{\overset{q}{\rightleftarrows}} \text{Sh}(\mathbb{D})$ be as above. 2.3–2.5 together give very strong preservation properties of q^*. Any property of models of T_1-locales like the Dedekind reals, elements of Bairespace $\mathbb{N}^{\mathbb{N}}$, etc. that does not involve quantification over arbitrary subsets holds in $\text{Sh}(\mathbb{C})$ iff it holds in $\text{Sh}(\mathbb{D})$. The logically minded reader would perhaps like to rephrase this by saying that the theory of $\text{Sh}(\mathbb{C})$ (in the appropriate language without arbitrary powersets) is a *conservative extension* of that of $\text{Sh}(\mathbb{D})$. As an illustration of this phenomenon, let us mention the following instance:

Claim. $\text{Sh}(\mathbb{C})$ *satisfies* "*all functions* $\mathbb{R} \to \mathbb{R}$ *are continuous*" *iff* $\text{Sh}(\mathbb{D})$ *does.*

Proof. Let us write the relevant statement as

$$\forall f \in \mathbb{R}^{\mathbb{R}} \quad \forall x \in \mathbb{R} \quad \forall p, q \in \mathbb{Q} \quad (p < f(x) < q \to \exists p', q' \in \mathbb{Q}(p' < x < q' \&$$

$$\& \forall y(p' < y < q' \to p < f(y) < q))).$$

Of course, q^* preserves the interpretation of the rationals $\mathbb{Q}$, since this is a constant sheaf. By 2.3, q^* preserves the sheaf of Dedekind reals, and hence by 2.4 it also preserves the interpretation of the exponential $\mathbb{R}^{\mathbb{R}}$. Thus by 2.5, this statement holds in $\text{Sh}(\mathbb{C})$ iff it holds in $\text{Sh}(\mathbb{D})$. $\square$

3. C^∞-rings

Here we will collect some of the basic properties of C^∞-rings that we will need later on. Almost nothing in this section is new, and most of the proofs will be omitted. For more details, the reader is referred to Kock's book, [10, §§III.5 and 6]; see also Dubuc [3], Reyes [16, fasc. 1], and Lawvere [11].

Let C^∞ be the category whose objects are the spaces $\mathbb{R}^n$, $n \in \mathbb{N}$, and whose morphisms $\mathbb{R}^n \to \mathbb{R}^m$ are the C^∞-maps. C^∞ is an algebraic theory à la Lawvere, and is called *the theory of C^∞-rings*. By definition, a C^∞-ring (in Sets) is a finite product preserving functor $A : C^\infty \to \text{Sets}$, and homomorphisms of C^∞-rings are just natural transformations.

If A is a C^∞-ring, $A(\mathbb{R})$ is its *underlying set*, and every smooth map $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ has an *interpretation* $A(f) : A(\mathbb{R})^n \to A(\mathbb{R})^m$. Note that since all constant functions and the ring operations of $\mathbb{R}$ are smooth, a C^∞-ring has an $\mathbb{R}$-algebra structure, and morphisms of C^∞-rings are particular $\mathbb{R}$-algebra homomorphisms. (As usual, we will often use the same symbol A for the functor $A : C^\infty \to \text{Sets}$ and its underlying set.)

Here are some examples: The Yoneda lemma implies that for each n, the free C^∞-ring on n generators is the functor $C^\infty(\mathbb{R}^n)$ defined by $C^\infty(\mathbb{R}^n)(\mathbb{R}^m) = C^\infty(\mathbb{R}^n, \mathbb{R}^m)$, the set of smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, and $C^\infty(\mathbb{R}^n)(f) = $ compose with f, f a morphism of C^∞. As was just pointed out, $C^\infty(\mathbb{R}^n)$ will most of the time not stand for this functor, but for its underlying set $C^\infty(\mathbb{R}^n, \mathbb{R})$. Its n generators are

the projections. More generally, if M is a (smooth) manifold, we have a C^∞-ring $C^\infty(M)$ which is defined just as $C^\infty(\mathbb{R}^n)$ was: the underlying set of $C^\infty(M)$ is the set of C^∞-maps $M \to \mathbb{R}$, and the morphisms of C^∞ are interpreted via composition. (Of course, $C^\infty(M)$ looks like a *dual* of M, and indeed it is, as we will see below.)

A host of other examples can be derived from the following proposition:

3.1. Proposition. *Any (algebraic) ideal I in a C^∞-ring A is a C^∞-congruence. Thus, the canonical projection $p : A \to A/I$ induces a C^∞-ring structure on A/I making p into a homomorphism of C^∞-rings.* $\square$

In particular, for any ideal I in $C^\infty(\mathbb{R}^n)$ we have a C^∞-ring $C^\infty(\mathbb{R}^n)/I$. All finitely generated C^∞-rings are of this form, and we write

$$(C^\infty\text{-rings})_{\mathrm{ft}} = C^\infty\text{-rings of } finite\ type \subseteq C^\infty \text{ rings}$$

for the full subcategory of C^∞-rings whose objects are of the form $C^\infty(\mathbb{R}^n)/I$. Note that homomorphisms of $(C^\infty\text{-rings})_{\mathrm{ft}}$ can be described quite explicitly: a homomorphism

$$C^\infty(\mathbb{R}^n)/I \to C^\infty(\mathbb{R}^m)/J$$

is an equivalence class of smooth maps $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ with the property that $I \subseteq \varphi^*(J) = \{f \mid f \circ \varphi \in J\}$, two such maps φ and φ' being equivalent if for each projection $\pi_k : \mathbb{R}^n \to \mathbb{R}$ ($k = 1, \ldots, n$), $\pi_k \circ \varphi = \pi_k \circ \varphi' \mod J$.

A C^∞-ring is *finitely presented* iff it is of the form $C^\infty(\mathbb{R}^n)/I$, where $I = (g_1, \ldots, g_p)$ is a finitely generated ideal. $(C^\infty\text{-rings})_{\mathrm{fp}}$ is the full subcategory of C^∞-rings whose objects are finitely presented. An important observation, due to Lawvere, gives some finitely presented C^∞-rings:

3.2. Proposition. *For any smooth manifold M, $C^\infty(M)$ is a finitely presented C^∞-ring.* $\square$

In fact, it suffices to show this for open subspaces of $\mathbb{R}^n$, since every manifold is a retract of one of its open neighbourhoods. If U is an open subspace of some $\mathbb{R}^n$, then there exists a (smooth) characteristic function $\chi_U : \mathbb{R}^n \to [0, 1]$ for U (that is, $\chi_U(x) > 0$ iff $x \in U$), and χ_U can be used to embed U as a *closed* set $\bar{U}$ in $\mathbb{R}^{n+1}$:

$$\xi : U \hookrightarrow \mathbb{R}^{n+1}, \quad \xi(x) = (x, \chi_U(x)^{-1}).$$

We have a 'restriction map'

$$\Phi : C^\infty(\mathbb{R}^{n+1}) \twoheadrightarrow C^\infty(U), \quad \Phi(f) = f \circ \xi$$

which is a homomorphism of C^∞-rings, and $\ker(\Phi) = \mathcal{U}_{\bar{U}}^0$, the ideal of functions that vanish on $\bar{U}$. What Theorem 3.2 says in this case is that

$$\mathcal{U}_{\bar{U}}^0 = (y \cdot \chi_U(x) - 1).$$

Recall that if $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are arbitrary subsets, a C^∞-map $X \to Y$ is a function $X \xrightarrow{f} Y$ which is the restriction of a C^∞-map $g: U \to \mathbb{R}^m$ defined on some open nbd of X. Thus, we may define a C^∞-ring $C^\infty(X)$ in exactly the same way as we defined $C^\infty(M)$, M a manifold. If X is closed, $C^\infty(X)$ is of finite type: $C^\infty(X) \cong C^\infty(\mathbb{R}^n)/\mathscr{M}_X^0$, $\mathscr{M}_X^0$ being the ideal of functions g with $g|X = 0$. Therefore we have a contravariant functor from the category $\mathbb{E}$ of (locally) closed subspaces of some $\mathbb{R}^n$ and smooth maps (see Section 1) to $(C^\infty\text{-rings})_{ft}$, and it follows easily from the explicit description of homomorphisms of C^∞-rings of finite type that this contravariant functor is *full* and *faithful*. We will come back to this functor shortly.

But first we discuss *coproducts* of C^∞-rings (of finite type). If A and B are arbitrary C^∞-rings, we write $A \otimes_\infty B$ for the coproduct, and $A \xrightarrow{i_A} A \otimes_\infty B \xleftarrow{i_B} B$ for the canonical inclusions. Note first that it follows from the universal property defining the coproduct that if $I \subseteq A$ and $J \subseteq B$ are ideals,

$$(A/I) \otimes_\infty (B/J) \cong (A \otimes_\infty B)/(I, J),$$

where (I, J) is the ideal generated by $i_A(I) \cup i_B(J)$. Also, since $C^\infty(\mathbb{R}^n)$ is free on n generators,

$$C^\infty(\mathbb{R}^n) \otimes_\infty C^\infty(\mathbb{R}^m) \cong C^\infty(\mathbb{R}^n \times \mathbb{R}^m),$$

and the coproduct inclusions come from the projections

$$\mathbb{R}^n \xleftarrow{\pi_1} \mathbb{R}^n \times \mathbb{R}^m \xrightarrow{\pi_2} \mathbb{R}^m.$$

Thus,

$$C^\infty(\mathbb{R}^n)/I \otimes_\infty C^\infty(\mathbb{R}^m)/J \cong C^\infty(\mathbb{R}^n \times \mathbb{R}^m)/(I, J),$$

and (I, J) is generated by $\{f \circ \pi_1 \mid f \in I\} \cup \{g \circ \pi_2 \mid g \in J\}$. For manifolds, we can do better: If M and N are manifolds,

$$C^\infty(M) \otimes_\infty C^\infty(N) \cong C^\infty(M \times N).$$

In order to be able to express our geometric (as opposed to algebraic) intuitions about C^∞-rings, we define the category of *loci* (or formal C^∞-varieties, cf. Reyes [16]) as

$$\mathbb{L} = (C^\infty\text{-rings})_{ft}^{op}.$$

Thus, a locus is the dual of a C^∞-ring $A = C^\infty(\mathbb{R}^n)/I$, and we will write $\bar{A}$ for this dual. One advantage of passing to duals is that the category $\mathbb{E}$ now becomes a subcategory of $\mathbb{L}$: we have a full embedding

$$\mathbb{E} \hookrightarrow \mathbb{L}, \quad X \mapsto \overline{C^\infty(X)}.$$

From the preceding remarks, we conclude that the image of $\mathbb{E}$ under this embedding consists (up to isomorphism) of the duals of C^∞-rings of the form $C^\infty(\mathbb{R}^n)/\mathscr{M}_F^0$, where F is a closed subset of $\mathbb{R}^n$.

Another important subcategory of $\mathbb{L}$ will be the full subcategory of duals of *germ-determined* C^∞-rings, defined as follows. A C^∞-ring is called *local* if it has exactly

one maximal ideal (i.e. if its underlying ring is a local ring). A C^∞-ring L of finite type is local with residue field $\mathbb{R}$ iff it is isomorphic to a quotient of a ring of germs, i.e. iff L is of the form $C_j^\infty(\mathbb{R}^n)/I$, where $C_0^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)/\mathscr{H}_{\{0\}}^g$ is the C^∞-ring of germs at 0 of smooth functions $\mathbb{R}^n \to \mathbb{R}$ ($\mathscr{H}_{\{0\}}^g$ is the ideal of functions f whose germ $f|_0$ at 0 is the zero-germ). A C^∞-ring A is *germ-determined* iff it is embedded in a product of local C^∞-rings with residue field $\mathbb{R}$, i.e. iff for any $a \in A$,

$$a = 0 \text{ in } A \iff \text{ for all } A \xrightarrow{h} L, L \text{ a local } C^\infty\text{-ring with}$$

$$\text{residue field } \mathbb{R}, h(a) = 0.$$

Most of the time we will work with the following equivalent description of germ-determined C^∞-rings of finite type. An ideal $I \subseteq C^\infty(\mathbb{R}^n)$ is called a *germ-determined ideal* if for all $f \in C^\infty(\mathbb{R}^n)$, $\forall x \in \mathbb{R}^n \; \exists g_x \in I \; f|_x = g_x|_x$ implies $f \in I$. Note that it suffices to check this for points x in the set $Z(I) = \{x \in \mathbb{R}^n \mid f(x) = 0, \text{ all } f \in I\}$ of zeros of I, since $Z(I)$ is closed.

3.3. Lemma. *A C^∞-ring of finite type $C^\infty(\mathbb{R}^n)/I$ is germ-determined iff I is a germ-determined ideal.* $\square$

This implies a *Nullstellensatz* for germ-determined ideals: a germ-determined C^∞-ring $C^\infty(\mathbb{R}^n)/I$ is the zero ring iff $Z(I) = \emptyset$.

If $I \subseteq C^\infty(\mathbb{R}^n)$ is germ-determined, and $f \in C^\infty(\mathbb{R}^n)$, (I, f) is also germ-determined; in particular, every finitely generated ideal is germ-determined.

Clearly, any ideal of the form $\mathscr{H}_F^0$ is germ-determined, so the inclusion $\mathbb{E} \hookrightarrow \mathbb{L}$ factors as

$$\mathbb{E} \xrightarrow{\; i \;} \mathbb{G} \xrightarrow{\; j \;} \mathbb{L}.$$

3.4. Proposition. *The full embeddings $\mathbb{E} \xrightarrow{\; i \;} \mathbb{G}$ and $\mathbb{G} \xrightarrow{\; j \;} \mathbb{L}$ have right adjoints $\gamma : \mathbb{G} \to \mathbb{E}$ and $\lambda : \mathbb{L} \to \mathbb{G}$ respectively.*

Proof. If $I \subseteq C^\infty(\mathbb{R}^n)$, let $\bar{I}$ be its germ-determined reflection,

$$\bar{I} = \{ f \mid \forall x \, f|_x \in I|_x \}$$

where $I|_x = \{g|_x \mid g \in I\}$ is the set of germs at x of elements of I. Then λ is defined by

$$\lambda(\overline{C^\infty(\mathbb{R}^n)/I}) = \overline{C^\infty(\mathbb{R}^n)/\bar{I}}.$$

λ is functorial (hence independent of the presentation of a C^∞-ring), since for any C^∞-map $f : \mathbb{R}^n \to \mathbb{R}^m$, $(f^*(I))^- \subseteq f^*(\bar{I})$. It is straightforwardly checked that $j \dashv \lambda$. The functor γ is defined by

$$\gamma(\overline{C^\infty(\mathbb{R}^n)/I}) = Z(I).$$

Functoriality of γ is again easily deduced from the explicit description of morphisms in $(C^\infty\text{-rings})_{\mathrm{ft}}$, and $i \dashv \gamma$ is again straightforward. $\quad\square$

Observe that Proposition 3.4 gives us an explicit description of inverse limits in $\mathbb{G}$: first take the limit in $\mathbb{L}$, then apply λ. Also note that since i and j are full, we have isomorphisms

$$\gamma i X \cong X, \qquad \lambda j \bar{A} \cong \bar{A},$$

natural in $X \in \mathbb{E}$, $\bar{A} \in \mathbb{G}$. Also, there are canonical embeddings

$$i\gamma \bar{A} \hookrightarrow \bar{A} \quad \text{in } \mathbb{G}, \qquad j\lambda \bar{B} \hookrightarrow \bar{B} \quad \text{in } \mathbb{L}.$$

In the sequel we will mainly work with germ-determined C^∞-rings of the form $C^\infty(U)/I$ (U open in $\mathbb{R}^n$). By the isomorphism $\Phi: C^\infty(\mathbb{R}^{n+1})/.\#_U^0 \to C^\infty(U)$ discussed above, these are of finite type. To see when they are germ-determined, we need the following extension of Lemma 3.3.

3.5. Lemma. *Let $I \subseteq C^\infty(U)$ be any ideal, and consider the isomorphism*

$$\Phi/I: C^\infty(\mathbb{R}^{n+1})/\Phi^{-1}(I) \to C^\infty(U)/I.$$

Then I is a germ-determined ideal iff $\Phi^{-1}(I)$ is. Or equivalently, (since $\Phi^{-1}(I) = \xi^*(I) = \{f \mid f \circ \xi \in I\}$),

$$\xi^*(\tilde{I}) = (\xi^*(I))^{\sim}.$$

Thus, the functor λ of 3.4 sends $C^\infty(U)/I$ to $C^\infty(U)/\tilde{I}$.

Proof. $(\xi^*(I))^{\sim} \subseteq \xi^*(\tilde{I})$ always holds (functoriality of λ). For the converse, take an $F: \mathbb{R}^{n+1} \to \mathbb{R}$ with $F \circ \xi \in \tilde{I}$. So for each $x \in U \subseteq \mathbb{R}^n$ we have a function $g_x \in I$ such that $F \circ \xi|_x \equiv g_x|_x$. But then if $(x_0, y_0) \in Z(\xi^*(I)) = \xi(Z(I))$, $x_0 \in Z(I)$, and we let $G: U \times \mathbb{R} \to \mathbb{R}$ be the function

$$G(x, y) = g_{x_0}(x) - F \circ \xi(x) + F(x, y),$$

and let $H: \mathbb{R}^{n+1} \to \mathbb{R}$ be a function which agrees with G on some open neighbourhood of $\hat{U}$. Then $H|_{(x_0, y_0)} \equiv F|_{(x_0, y_0)}$, and $H \circ \xi(x) = G \circ \xi(x) = g_{x_0}(x)$ for all x in some neighbourhood of x_0, so $H \in (\xi^*(I))^{\sim}$, and therefore $F \in (\xi^*(I))^{\sim}$. $\quad\square$

4. The smooth topos $\mathscr{S}$ and its relation to the euclidean topos $\mathscr{E}$

In the previous section, we introduced the category $\mathbb{G}$ of duals of germ-determined C^∞-rings of finite type, and we noted that these are precisely the C^∞-rings which have a representation of the form

$$C^\infty(U)/I$$

where U is an open subset of some $\mathbb{R}^n$, and I is a germ-determined ideai. From now on, we will work with representations of this form. $\mathbb{G}$ has finite limits, and products in $\mathbb{G}$ are given by the formula

$$C^\infty(U)/I \otimes_\infty C^\infty(V)/J \cong C^\infty(U \times V)/(I, J)^-$$

for coproducts of germ-determined C^∞-rings.

We equip $\mathbb{G}$ with a Grothendieck topology whose basic covers are families of canonical inclusions of the form

$$\{\overline{C^\infty(U_\alpha)/(I\,|\,U_\alpha)^-} \hookrightarrow \overline{C^\infty(U)/I}\}_\alpha$$

where $\{U_\alpha\}_\alpha$ is an open cover of U. The resulting site will also be denoted by $\mathbb{G}$, and we write

$$\mathscr{G} = \mathrm{Sh}(\mathbb{G})$$

for the topos of sheaves on $\mathbb{G}$.

4.1. Lemma. *The topology of the site $\mathbb{G}$ is subcanonical.*

Proof. Although in general a homomorphism $C^\infty(U)/I \to C^\infty(V)/J$ need not be induced by a smooth map $V \to U$, this is true if $U = \mathbb{R}^n$. From this, the result follows easily: if $\{U_\alpha\}_\alpha$ is an open cover of U, and

$$\{f_\alpha : \overline{C^\infty(U_\alpha)/(I\,|\,U_\alpha)^-} \to \bar{A}\}_\alpha$$

is a compatible family of maps to the dual $\bar{A}$ of a C^∞-ring $A = C^\infty(\mathbb{R}^m)/J$, each f_α comes from a smooth function $f_\alpha = (f_\alpha^1, \dots, f_\alpha^m) : U_\alpha \to \mathbb{R}^m$, and $f_\alpha^i\,|\,U_\beta - f_\beta^i\,|\,U_\alpha \in (I\,|\,U_\alpha \cap U_\beta)^-$ by compatibility. The unique $f : \overline{C^\infty(U)/I} \to \bar{A}$ which extends all the f_α is now obtained as $f = \sum f_\alpha \theta_\alpha$, where $\{\theta_\alpha\}_\alpha$ is a partition of unity subordinate to the cover $\{U_\alpha\}_\alpha$ of U. $\square$

This lemma was already observed by E. Dubuc, who introduced the topos $\mathscr{G}$ as a model for synthetic differential geometry (cf. [3]). The ring of linetype R in $\mathscr{G}$ is the representable object $\overline{C^\infty(\mathbb{R})}$, i.e.

$$R : \mathbb{G}^{\mathrm{op}} \to \mathrm{Sets}, \quad \bar{A} \mapsto \text{the underlying set of } A.$$

The object D of first order infinitesimals is the representable object $\overline{C^\infty(\mathbb{R})/(x^2)}$. $\mathscr{G}$ satisfies the Kock-Lawvere axiom ('axiom 1' of Kock [10]), that is

$$R \times R \cong R^D.$$

The integration axiom which ensures the existence of primitives holds in $\mathscr{G}$ (Van Quê & Reyes [14]), and $(-)^D$ has a right adjoint $(-)_D$ (see Kock [10]). Generalizations of these axioms for infinitesimal spaces other than D also hold in the model.

$\mathcal{G}$ classifies a geometric theory of loci or formal smooth varieties, which can be described as follows. Let us consider the language L with one sort R, having an n-ary function symbol f for each $f \in C^\infty(\mathbb{R}^n)$, and an n-ary relation symbol $\mathrm{Loc}(I)$ ('the locus of I') for each germ-determined ideal $I \subseteq C^\infty(\mathbb{R}^n)$ ($n = 0, 1, \ldots$). Our basic theory T_0 consists of the following groups of axioms:

(I) $(R, f)_f$ is a C^∞-ring.

(II) $\forall x \in R^n$ ($f(x) = 0 \leftrightarrow \mathrm{Loc}(f)(x)$), all $n \in \mathbb{N}$, where (f) is the principle ideal generated by f in $C^\infty(\mathbb{R}^n)$.

(III) $\forall x \in R^n$ $\forall y \in R^m$ ($\mathrm{Loc}(I)(x) \wedge \mathrm{Loc}(J)(y) \leftrightarrow \mathrm{Loc}(I, J)(x, y)$), where $I \subseteq C^\infty(\mathbb{R}^n)$, $J \subseteq C^\infty(\mathbb{R}^m)$, and (I, J) is the ideal generated by $I \circ \pi_1 \cup J \circ \pi_2$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, all $n, m \in \mathbb{N}$.

(IV) $\forall x \in R^n$ ($\mathrm{Loc}(I)(x) \rightarrow \mathrm{Loc}(J)(\varphi(x))$), where $I \subseteq C^\infty(\mathbb{R}^n)$, $J \subseteq C^\infty(\mathbb{R}^m)$, and $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ is such that $\varphi^*(J) \subseteq I$.

4.2. Proposition. *For any* (*Grothendieck*) *topos* $\mathcal{F}$, *there is a natural bijection between* T_0-*models in* $\mathcal{F}$ *and leftexact functors from* $\mathbb{G}^{\mathrm{op}}$ *to* $\mathcal{F}$.

Proof. Routine. □

In the sequel, we shall identify T_0-models with the corresponding functors.

The category $\mathbb{G}$ of germ-determined loci introduced in Section 3 can be given the structure of a site, $\mathbb{G}_{\mathrm{fin}}$, by defining the Grothendieck (pre-)topology to be generated by the duals of cocoverings of the form

(i) $A \nearrow A[1/a]$
$\qquad A $ whenever $a + b = 1$,
$\qquad \searrow A[1/b]$

and the empty family for the zero ring 0. (Here $A[1/a]$ is the universal solution to the problem of inverting $a \in A$ in the category of germ-determined C^∞-rings.) As in the case of the Zariski topos, the coverings of this pretopology are precisely the families of the form

$$\{\overline{A[1/a_\alpha]} \to \bar{A}\}_\alpha$$

such that the ideal generated by the $\{a_\alpha\}_\alpha$ is all of A.

Since by the Implicit Function Theorem,

$$C^\infty(\mathbb{R}^{n+1})/(I, x_{n+1}\varphi(x) - 1) \cong C^\infty(U)/(I \mid U) \quad \text{where } U = \mathbb{R}^n \setminus \varphi^{-1}(0),$$

this algebraic description of the site is equivalent to the following topological one: the covers of $\overline{C^{\infty}(\mathbb{R}^n)}/I$ are the finite families

(ii) $\{\overline{C^{\infty}(U_\alpha)/(I \mid U_\alpha)}^{\sim} \rightarrow \overline{C^{\infty}(U)/I}\}_\alpha$

where $\{U_\alpha\}_\alpha$ is an open cover of $\mathbb{R}^n$.

In the presheaf topos $\mathrm{Sets}^{\mathbb{G}^{op}}$ we have the canonical C^{∞}-ring $R = \overline{C^{\infty}(\mathbb{R})}$, and the covers in (i) clearly just force R to be local, so we have

4.3. Proposition. *Let M be a T_0-model in a topos $\mathscr{F}$. Then $M \models ``R$ is local'' iff the left-exact functor $M : \mathbb{G}^{op} \rightarrow \mathscr{F}$ preserves the covers of $\mathbb{G}_{\mathrm{fin}}$ (i.e. M is continuous).* $\square$

In addition to forcing R to be local by the coverings in (i) (or equivalently (ii)), we can force R to be Archimedean by declaring the family

(iii) $\{(-m, m) \hookrightarrow R\}_{m \in \mathbb{N}}$

to be a cover, where $(-m, m)$ here stands for

$$\overline{C^{\infty}(\{x \in \mathbb{R} \mid -m < x < m\})} \in \mathbb{G}.$$

Topologically, this comes down to adding $\{\overline{C^{\infty}(-m, m)} \hookrightarrow \overline{C^{\infty}(\mathbb{R})}\}_m$ as a cover to (ii). Since for any open cover $\{U_\alpha\}_\alpha$ of $\mathbb{R}$, finitely many of the U_α already cover $(-m, m)$, a straightforward induction argument shows that this comes down to having all families of the form

(iv) $\{\overline{C^{\infty}(U_\alpha)/(I \mid U_\alpha)}^{\sim} \hookrightarrow \overline{C^{\infty}(\mathbb{R}^n)/I}\}_\alpha,$

where $\{U_\alpha\}$ is an open cover of $\mathbb{R}^n$, as coverings in the site. (This was first noted by E. Dubuc & A. Joyal.) But the site that we have now obtained is precisely the site $\mathbb{G}$ that we started this section with. So for the topos $\mathscr{S} = \mathrm{Sh}(\mathbb{G})$ we have

4.4. Proposition. *$\mathscr{S}$ classifies the theory T_0 of germ-determined loci with the additional axioms saying that R is local and Archimedean.* $\square$

Let us therefore return to the site $\mathbb{G}$, and have a look at the functors γ and i that we introduced in Section 3, in order to see whether we can apply the results of Section 2.

4.5. Lemma. (a) *The functor $\gamma : \mathbb{G} \rightarrow \mathbb{E}$ is left-exact, preserves covers and has the covering lifting property.*

(b) *The functor i preserves and reflects covers.*

Proof. (a) γ is left-exact because it has a left adjoint. γ sends a basic cover $\overline{C^{\infty}(U_\alpha)/(I \mid U_\alpha)}^{\sim} \hookrightarrow \overline{C^{\infty}(U)/I}\}_\alpha$ to the family $\{Z(I) \cap U_\alpha \hookrightarrow Z(I)\}_\alpha$, so it is clear that γ preserves covers. To check the CLP, take $\bar{A} = \overline{C^{\infty}(U)/I}$ in $\mathbb{G}$, and suppose that

$\{V'_\alpha\}_\alpha$ is a cover of $Z(I)$ by open subsets of U. Then let $W_\alpha = V_\alpha \cup U \setminus Z(I)$, so that $\{(\cdot^\infty(W_\alpha)/(I \mid W_\alpha)^- \hookrightarrow C^\infty(U)(I)\}_\alpha$ is a basic cover in $\mathbb{G}$ which is mapped by γ onto $\{V_\alpha \cap Z(I) \hookrightarrow Z(I)\}_\alpha$.

(b) Suppose X is closed in $\mathbb{R}^n$, and $\{V_\alpha\}_\alpha$ is a cover of X with each V_α open in X. We need to prove that $\{C^\infty(V_\alpha) \hookrightarrow C^\infty(X)\}_\alpha$ is a covering family in $\mathbb{G}$. Write $V_\alpha = X \cap U_\alpha$, U_α open in $\mathbb{R}^n$. Then $C^\infty(X) \cong C^\infty(\mathbb{R}^n)/\mathcal{U}_X^0$, and $C^\infty(V_\alpha) \cong C^\infty(U_\alpha)/\mathcal{U}_{X \cap U_\alpha}^0$, so it suffices to show that $(\mathcal{U}_X^0 \mid U_\alpha)^- = \mathcal{U}_{X \cap U_\alpha}^0$. $\subseteq$ is clear, and for $\supseteq$ we note that a smooth function $f: U_\alpha \to \mathbb{R}$ with $f \mid U_\alpha \cap X \equiv 0$ extends locally to a function $g: \mathbb{R}^n \to \mathbb{R}$ with $g \mid X \equiv 0$: for pick $x \in U_\alpha$, and choose nbds V_x and W_x of x with $x \in V_x \subseteq \bar{V}_x \subseteq W_x \subseteq \bar{W}_x \subseteq U$. Let h be a smooth function $\mathbb{R}^n \to \mathbb{R}$ with $h \mid \bar{V}_x \equiv 1$, $h \mid (\mathbb{R}^n \setminus W_x) \equiv 0$, and let $g: \mathbb{R}^n \to \mathbb{R}$ be defined by $g(x) = f(x) \cdot h(x)$ if $x \in U$, $g(x) = 0$ if $x \notin U$. Then g is smooth, and g coincides with f on V_x.

It is clear that i reflects covers, since $\gamma i = 1$ and γ preserves them. $\square$

We can now apply Theorem 2.2 with γ as the functor P. Since $\gamma i = 1$, this gives us

4.6. Theorem. *There is an adjoint retraction*

$$\mathcal{G} \underset{\gamma}{\overset{\varrho}{\underset{\longleftarrow}{\rightleftarrows}}} \mathcal{E}$$

of toposes, i.e. $\gamma = 1$, *and* $\gamma^* \dashv \gamma_* = \varrho^* \dashv \varrho_*$. *The functor* $\gamma_* = \varrho^*: \mathrm{Sh}(\mathbb{E}) \to \mathrm{Sh}(\mathbb{G})$ *is "compose with* $\gamma: \mathbb{G} \to \mathbb{E}$*", and the functor* $\gamma^*: \mathrm{Sh}(\mathbb{G}) \to \mathrm{Sh}(\mathbb{E})$ *is "compose with* $i: \mathbb{E} \to \mathbb{G}$*".* $\square$

Note that the explicit description of γ^* in this case comes from lifting the adjunction $i \dashv \gamma$ of Proposition 3.4 first to the level of presheaves ("compose with i" $\dashv$ "compose with γ": $\mathrm{Sets}^{\mathbb{G}^{op}} \rightleftarrows \mathrm{Sets}^{\mathbb{E}^{op}}$) and then restricting it to sheaves by Lemma 4.5.

4.7. Remark. In fact $\mathcal{G}$ and $\mathcal{E}$ both have a canonical point, and the retraction of 4.6 is a retraction of pointed toposes. This is so because the global sections functor $\Gamma: \mathcal{G} \to \mathrm{Sets}$ has a right adjoint B: If $S \in \mathrm{Sets}$, we define a sheaf $B(S)$ on $\mathbb{G}$ by

$$B(S)(\bar{A}) = S^{\gamma(\bar{A})}.$$

Since $\gamma(\bar{A}) = \Gamma(\bar{A})$ as sets, it is clear that we have a bijective correspondence

$$\frac{\Gamma(\bar{A}) \to S}{\bar{A} \to B(S)}.$$

This extends to arbitrary objects of $\mathcal{G}$ (not just representables), since $\Gamma: \mathcal{G} \to \mathrm{Sets}$ preserves all colimits. Thus, we have a canonical point $p_\mathcal{G}: \mathrm{Sets} \to \mathcal{G}$, and in precisely the same way we can define a point $p_\mathcal{E}: \mathrm{Sets} \to \mathcal{E}$ of $\mathcal{E}$ whose inverse image is the global sections functor. Since ϱ^* and γ^* from 4.6 preserve global sections, it is clear that $\varrho \circ p_\mathcal{G} = p_\mathcal{E}$, $\gamma \circ p_\mathcal{E} = p_\mathcal{G}$, i.e. ϱ and γ are maps of pointed toposes.

As a consequence of this observation, the global sections functor $\mathscr{G} \xrightarrow{\Gamma}$ Sets (or $\mathscr{E} \xrightarrow{\Gamma}$ Sets) preserves all the constructions performed in $\mathscr{G}$ (or $\mathscr{E}$) by taking limits and colimits. This can be used for example to compare the De Rham cohomology of $\mathscr{G}$ with the one of Sets, as in Moerdijk & Reyes [12].

5. Dedekind reals and smooth reals in $\mathscr{G}$, the standard map

On the basis of the results obtained so far we will now investigate some of the properties that the Dedekind reals $\mathbb{R}$, the geometric line R which models SDG, and related spaces have in the topos $\mathscr{G}$. We will begin by investigating $\mathbb{R}$ and some other spaces of models of T_1-locales, and then we will turn to some of the logical properties of R. The emphasis however, will be on the interaction between $\mathbb{R}$ and R induced by the standard map.

As stated in the introduction, the properties that are valid in $\mathscr{G}$ for the Dedekind reals $\mathbb{R}$ tell us in which sense $\mathscr{G}$ is a model for *intuitionistic* analysis, and those valid for R tell us in which sense $\mathscr{G}$ is a model for *synthetic* calculus. And it is one of the interesting aspects of $\mathscr{G}$ that it gives us a means to compare the two.

The first consequence of Theorem 4.6 that we should note is the following one.

5.1. Theorem (Representation Theorem for models of T_1-locales). *Let X be a T_1-locale in* Sets, *and let $X_\mathscr{A} : \mathbb{G}^{op} \to$ Sets be the sheaf of models of (the propositional theory corresponding to) X in $\mathscr{G}$. Then*

$$X_\mathscr{A}(\bar{A}) \cong \mathrm{Cts}(\gamma\bar{A}, X).$$

(*This describes for example the Dedekind reals* $\mathbb{R}$, *Cantor space, Baire space, the function space* $\mathbb{R}^\mathbb{R}$, *etc. in* $\mathscr{G}$.)

Proof. By 1.5, the sheaf $X_\mathscr{E}$ of models of X in the euclidean topos $\mathscr{E}$ is given by $X_\mathscr{E}(-) \cong \mathrm{Cts}(-, X) : \mathbb{E}^{op} \to$ Sets. By 2.3 and 4.6, $X_\mathscr{A} \cong X_\mathscr{E} \circ \gamma$, which yields the isomorphism asserted in the theorem. $\square$

Note that there is no ambiguity in the notation $\mathbb{R}^\mathbb{R}$ that we used in 5.1, since by Example 2.6, and Proposition 1.10,

5.2. Theorem. *In* $\mathscr{G}$, *all functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *are continuous* (*and more generally, cf. Remark* 1.11).

Theorem 5.1 describes the *points* of the formal spaces corresponding to the Dedekind reals, the Cantor space, Baire space, etc. In fact these spaces of points all coincide with the corresponding formal spaces, by the following theorem. The

proof is almost literally the same as the one for $\mathcal{E}$ that we gave in 1.8.

5.3. Theorem. *Bar Induction holds in* $\mathcal{G}$, *i.e. formal Baire space has enough points. Consequently, in* $\mathcal{G}$ *every separable locale has enough points. In particular, the Dedekind unit interval* $[0,1] \subseteq \mathbb{R}$ *and the Cantor set are compact in* $\mathcal{G}$. $\quad\square$

(We do not have a general argument for deriving the validity of Bar Induction in $\mathcal{G}$ from its validity in $\mathcal{E}$, despite the similarity of the proofs; Bar Induction involves universal quantification over bars in the tree $\mathbb{N}^{<\mathbb{N}}$, and therefore it is not taken care of by the conservation properties 2.4 and 2.5.)

Let us turn to R, the object of $\mathcal{G}$ represented by $C^{\infty}(\mathbb{R}) \in \mathbb{G}$. R carries a strict order relation $<$ defined by $x < y$ iff there exists an invertible $z \in R$ such that $z^2 = y - x$. In the model, this gives the order relation you would expect to get: for example, if $\alpha : \bar{A} \to R = C^{\infty}(\mathbb{R})$ is an element of R at stage $\bar{A}$, $\bar{A} \Vdash \alpha > 0$ iff α factors through $C^{\infty}(\mathbb{R}_{>0}) \hookrightarrow C^{\infty}(\mathbb{R})$ in $\mathcal{G}$, iff $\gamma(\alpha) : \gamma(\bar{A}) \to \mathbb{R}$ factors through $\mathbb{R}_{>0} \subseteq \mathbb{R}$. It follows that this order is total, i.e.

$$x \neq 0 \leftrightarrow (x < 0 \text{ or } x > 0) \leftrightarrow x \text{ is invertible}$$

holds in $\mathcal{G}$. The rationals are dense in this order, so the order topology on R coincides with the topology generated by rational open intervals, and this is the topology that we consider when speaking about topological properties of R. Here is one such property.

5.4. Proposition. *The smooth unit interval* $[0,1] \subseteq R$ *is compact in* $\mathcal{G}$.

Proof. Let $\mathcal{U}$ be a subsheaf of $\mathcal{P}(R)$ at stage $\bar{A}$, $A = C^{\infty}(\mathbb{R}^n)/I$, such that $\bar{A} \Vdash$ "$\mathcal{U}$ covers $[0,1]$". For each external $\alpha \in [0,1]$, we have for the corresponding constant element $\alpha \in R(\bar{A})$ that $\bar{A} \Vdash \exists U \in \mathcal{U} \; \alpha \in U$. Without loss we may assume that $\mathcal{U}$ consists of rational intervals, so we find a cover $\{U_i^{\alpha}\}_i$ of $\mathbb{R}^n$ and rational intervals $(p_i^{\alpha}, q_i^{\alpha})$ such that

$$\bar{A}_i^{\alpha} \Vdash \alpha \in (p_i^{\alpha}, q_i^{\alpha}) \in \mathcal{U}$$

(where $A_i^{\alpha} = C^{\infty}(U_i^{\alpha})/(I \mid U_i^{\alpha})^-$). For $x \in Z(I) \subseteq \mathbb{R}^n$, the set $\{(p_i^{\alpha}, q_i^{\alpha}) \mid x \in U_i^{\alpha}\}$ covers $[0,1] \subseteq \mathbb{R}$ externally, so by taking a finite subcover and the intersection of the corresponding U_i^{α}'s, we find a neighbourhood V_x of x and intervals $(p_1, q_1), \ldots, (p_n, q_n)$ such that

$$C^{\infty}(V_x)/(I \mid V_x)^- \Vdash ([0,1] \subseteq (p_1, q_1) \cup \cdots \cup (p_n, q_n), \text{ and each } (p_k, q_k) \in \mathcal{U}).$$

This holds for each $x \in Z(I)$, so (by extending the cover $\{V_x \mid x \in Z(I)\}$ to $\{V_x \cup \mathbb{R}^n \setminus Z(I) \mid x \in Z(I)\}$) we find

$$\bar{A} \Vdash \text{"}\mathcal{U} \text{ has a finite subcover"}. \quad\square$$

The representability of R forces strong properties of continuous choice upon us.

The first of these is continuous choice for lawlike types, RL-choice. (Lawlike types are the types interpreted by constant sheaves, cf. van der Hoeven & Moerdijk [5].)

5.5. Theorem (RL-choice). *Let L be a lawlike type, such as* $\mathbb{N}$. *Then*

$$\mathscr{G} \vDash \forall P \subseteq R \times L \; (\forall x \; \exists l \; P(x,l) \rightarrow \exists \text{ open cover } \mathscr{U} \text{ of } R$$

$$\forall U \in \mathscr{U} \; \exists l \; \forall x \in U \; P(x,l)).$$

Proof. Suppose such a $P \subseteq R \times L$ is given at stage $\bar{A}$, $A = C^{\infty}(\mathbb{R}^n)/I$, and $\bar{A} \Vdash \forall x \in R \; \exists l \in L \; P(x,l)$. Now consider the projections

$$\bar{A} \times \overline{C^{\infty}(\mathbb{R})} \xrightarrow{\;\;p_1\;\;} \bar{A}$$

$$\downarrow p_2$$

$$\overline{C^{\infty}(\mathbb{R})}$$

p_2 acts as a generic smooth real at $\bar{A} \times \overline{C^{\infty}(\mathbb{R})}$: From

$$\bar{A} \times \overline{C^{\infty}(\mathbb{R})} \Vdash \exists l \in L \; P(p_2, l)$$

we obtain an open cover $\{U_\alpha \times V_\alpha\}_\alpha$ of $\mathbb{R}^n \times \mathbb{R}$, $U_\alpha \subseteq \mathbb{R}^n$, $V_\alpha \subseteq \mathbb{R}$, and *constant* elements l_α of the constant sheaf L, such that

$$\bar{A}_\alpha \times \overline{C^{\infty}(V_\alpha)} \Vdash P(p_2, l_\alpha)$$

where $A_\alpha = C^{\infty}(U_\alpha)/(I \,|\, U_\alpha)^-$.

Now define the subsheaf $\mathscr{U}$ of $\Gamma(\mathbb{R})$ to be the sheaf generated by the conditions

$$\bar{A}_\alpha \Vdash V_\alpha \in \mathscr{U}.$$

We have to show two things to complete the proof:

(1) $\bar{A} \Vdash \mathscr{U}$ is a cover,

(2) $\bar{A} \Vdash \forall U \in \mathscr{U} \; \exists l \in L \; \forall x \in \mathscr{U} \; P(x,l)$.

For (1), take $\bar{B} \xrightarrow{f} \bar{A}$ and a smooth real y at $\bar{B}$, i.e. $\bar{B} \xrightarrow{y} \overline{C^{\infty}(\mathbb{R})}$. Then (f, y) factors locally through the cover $\bar{A}_\alpha \times \overline{C^{\infty}(V_\alpha)}$ of $\bar{A} \times \overline{C^{\infty}(\mathbb{R})}$; more precisely, make a pullback

$$\begin{array}{ccc}
\bar{B}_\alpha & \xrightarrow{(f_\alpha, y_\alpha)} & \bar{A}_\alpha \times \overline{C^{\infty}(V_\alpha)} \\
\cap \; \downarrow & & \cap \; \downarrow \\
\bar{B} & \xrightarrow{(f, y)} & \bar{A} \times \overline{C^{\infty}(\mathbb{R})}
\end{array}$$

Then the $\bar{B}_\alpha$ cover $\bar{B}$, and $\bar{B}_\alpha \Vdash \exists U \in \mathscr{U} \, y \in U$. For (2), it suffices to consider the generating elements of $\mathscr{U}$, i.e. we need to show

$$\bar{A}_\alpha \Vdash \exists l \in L \, \forall x \in V_\alpha \, P(x, l)$$

But $\bar{A}_\alpha \times \overline{C^\infty(V_\alpha)} \Vdash P(p_2, l_\alpha)$ and from this we immediately obtain $A_\alpha \Vdash \forall x \in V_\alpha$ $P(x, l_\alpha)$. $\square$

The importance of the analog of Theorem 5.5 for $\mathbb{R}$ or $\mathbb{N}^\mathbb{N}$ (instead of R) is well-known in intuitionistic analysis. In our case, this is illustrated by the following two corollaries.

5.6. Corollary. *In $\mathscr{G}$, every countable (i.e. indexed by the natural numbers) cover of R has an open refinement.* $\square$

5.7. Corollary. *Let (X, δ) be a metric space in $\mathscr{G}$ having a dense subset D which is a constant sheaf. Then all functions $R \to X$ are continuous. In particular, all functions $R \to R$ are continuous in $\mathscr{G}$.*

Proof. Apply Theorem 5.5 to the predicates $P_n(x, d) = \delta(x, d) < 2^{-n}$. $\square$

Continuity of all functions $R \to R$ follows also from properties of the standard map. Yet another proof which applies in a more general context, will be given in [13], cf. Theorem 6.1 below.

The proof of Theorem 5.5 is based on the fact that from a stage $\bar{A}$, the projection $\bar{A} \times C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ acts as a generic element of R. This trick also easily yields

5.8. Theorem. (a) (Local RR-choice)

$$\mathscr{G} \models \forall P \subseteq R \times R \, (\forall x \, \exists y \, P(x, y) \to \exists \text{ open cover } \mathscr{U} \text{ of } R \, \forall U \in \mathscr{U}$$

$$\exists U \xrightarrow{f} R \, \forall x \in U \, P(x, fx)).$$

(b) (Local continuous choice) *Let Y be an external space, and $\mathbf{Y}$ the space in $\mathscr{G}$ given by $\mathbf{Y}(\bar{A}) = \mathrm{Cts}(\gamma\bar{A}, Y)$ (as the spaces in Theorem 5.1 above). Then*

$$\mathscr{G} \models \forall P \subseteq R \times \mathbf{Y} \, (\forall x \, \exists y \, P(x, y) \to \exists \text{ open cover } \mathscr{U} \text{ of } R \, \forall U \in R$$

$$\exists \text{cts function } U \xrightarrow{f} \mathbf{Y} \, \forall x \in U \, P(x, fx).$$

In particular, $\mathscr{G} \models$ "All functions $R \to \mathbf{Y}$ are continuous". $\square$

Another feature of the proof of 5.5 is the lifting of an external cover ($\{U_\alpha \times V_\alpha\}_\alpha$ of $\bar{A} \times R$) to an internal one ($\{\overline{C^\infty(V_\alpha)}\}_\alpha$ of R, at stage $\bar{A}$). Here is another application of the same idea.

5.9. Proposition. *In $\mathscr{G}$, R has partitions of unity subordinate to every open cover.*

Proof. Let $\mathscr{U}$ be an open cover of R at stage $\bar{A} \in \mathbb{G}$, $A = C^{\infty}(\mathbb{R}^n)/I$, which we may without loss take to consist of rational intervals, i.e.

$$\bar{A} \Vdash \text{``} \forall x \in R \, \exists U \in \mathscr{U} \, x \in U = \text{some } (p, q)\text{''}.$$

Again consider the generic real p_2 at stage $\bar{A} \times \overline{C^{\infty}(\mathbb{R})}$. Then

$$\bar{A} \times \overline{C^{\infty}(\mathbb{R})} \Vdash \text{``} \exists U \in \mathscr{U} \, p_2 \in U = \text{some } (p, q)\text{''},$$

so we find an open cover $\{U_\alpha \times V_\alpha\}_\alpha$ of $\mathbb{R}^n \times \mathbb{R}$ and rational intervals $\{(p_\alpha, q_\alpha)\}_\alpha$ such that

$$\bar{A}_\alpha \times \overline{C^{\infty}(V_\alpha)} \Vdash p_2 \in (p_\alpha, q_\alpha) \in \mathscr{U}$$

(where $A_\alpha = C^{\infty}(U_\alpha)/(I \mid U_\alpha)^-$). Thus $V_\alpha \subseteq (p_\alpha, q_\alpha)$, so by shrinking the (p_α, q_α) we may without loss assume that $V_\alpha = (p_\alpha, q_\alpha)$.

Now let $\{\theta_\alpha\}_\alpha$ be an external partition of unity on $\mathbb{R}^n \times \mathbb{R}$ subordinate to the cover $\{U_\alpha \times V_\alpha\}_\alpha$. Each $\theta_\alpha : \mathbb{R}^n \times \mathbb{R} \to [0, 1]$ is smooth, and can be regarded as an element of the sheaf $[0, 1]^R$ at stage $\bar{A}$. We now define $S \subseteq [0, 1]^R$ to be the subsheaf generated by the conditions

$$\bar{A}_\alpha \Vdash \theta_\alpha \in S.$$

It is then straightforward to check that

$$\bar{A} \Vdash \text{``}\{\theta \mid \theta \in S\} \text{ is a partition of unity subordinate to } \mathscr{U}\text{''}. \qquad \square$$

A property of R of a rather different nature is a version of Markov's principle, due to A. Kock. An object $\bar{B} \in \mathbb{G}$ is called *point-determined* if for any $\bar{B} \xrightarrow{\ f\ } R = \overline{C^{\infty}(\mathbb{R})}$, if $f\alpha = 0$ for all points $\mathbf{1} \xrightarrow{\alpha} \bar{B}$ of $\bar{B}$, then $f = 0$. Clearly, if $B = C^{\infty}(\mathbb{R}^n)/I$, $\bar{B}$ is point-determined iff $I = \mathscr{U}^0_{Z(I)}$.

5.10. Proposition (Kock). *An object $\bar{B} \in \mathbb{G}$ is point-determined iff*

$$\mathscr{U} \vDash \forall f \in R^B \, (\neg \forall x \, f(x) = 0 \to \exists x \, f(x) \neq 0).$$

Proof. ($\Rightarrow$) Suppose $\bar{B}$ is point-determined, say $B = C^{\infty}(\mathbb{R}^n)/\mathscr{U}^0_X$, and let $f \in R^B(\bar{A})$, i.e. $f : \bar{A} \times \bar{B} \to R$ where $A = C^{\infty}(\mathbb{R}^m)/J$, be such that $\bar{A} \Vdash \neg \forall x \in \bar{B} \, f(x) = 0$. Then in particular in all points $\mathbf{1} \xrightarrow{p} \bar{A}$ of $\bar{A}$,

$$\mathbf{1} \Vdash \neg \forall x \in \bar{B} \, (f \mid p)(x) = 0,$$

so $\mathbf{1} \Vdash \exists x \in \bar{B} \, (f \mid p) \neq 0$, so there is an $x \in X$ with $f(p, x) \neq 0$. Then also on a neighbourhood U_p of p, $f(y, x) \neq 0$ for all $y \in U_p$, so if we put $V_p = U_p \cup \mathbb{R}^m \setminus Z(J)$, we find that

$$\overline{A \cap V_p} = \overline{C^{\infty}(V_p)/(J \mid V_p)^-} \Vdash f(x) \neq 0$$

for this particular $x \in X$. But the V_p, $p \in Z(J)$, cover $\mathbb{R}^m$, so

$$\bar{A} \Vdash \exists x \in \bar{B} \, f(x) \neq 0.$$

($\Leftarrow$) Suppose $\mathscr{G} \Vdash \forall f \in R^B (\neg \forall x\, f(x) = 0 \rightarrow \exists x\, f(x) \neq 0)$, where $B = C^\infty(\mathbb{R}^n)/I$, and choose an $f : \bar{B} \rightarrow R$ with $f \notin I$. Then $1 \Vdash \neg \forall x\, f(x) = 0$, for if $\bar{A} \in \mathbb{G}$ and $A \neq 0$, $\bar{A} \Vdash \forall x\, f(x) = 0$ would imply that $\bar{B} \Vdash \forall x\, f(x) = 0$ (since $\bar{A}$ has a point p, so there is a map $\bar{B} \rightarrow 1 \overset{p}{\longrightarrow} \bar{A}$), i.e. $f \notin I$, contradiction. Thus by hypothesis, $1 \Vdash \exists x\, f(x) \neq 0$, $\exists x \in Z(I)\, f(x) \neq 0$. $\square$

Note that as a consequence of 5.10, in $\mathscr{G}$, R is a field in the following sense (recall that in $\mathscr{G}$, $x \neq 0$ iff x is invertible, for all $x \in R$):

$$\mathscr{G} \models \forall x_1, \ldots, x_n \in R\, (\neg(x_1 = 0 \wedge \cdots \wedge x_n = 0) \rightarrow (\text{one of the } x_i \text{ is invertible})).$$

5.11. Corollary (Markov's principle)

$$\mathscr{G} \models \forall P \subseteq \mathbb{N}\,(\forall n(Pn \vee \neg Pn) \wedge \neg \neg \exists n\, Pn \rightarrow \exists n\, Pn).$$

Proof. This follows from 5.10, since a decidable $P \subseteq \mathbb{N}$ defines a map $\mathbb{N} \rightarrow 2 \subseteq R$, and $\mathbb{N} \cong C^\infty(\mathbb{R})/\mathscr{M}_\mathbb{N}^0$ is point-determined. $\square$

We now turn to the standard map. Since R is Archimedean and the rationals are dense, we may define (synthetically, cf. Veit [18]) for each $x \in R$ a Dedekind cut $(\{p \in \mathbb{Q} \mid p < x\}, \{q \in \mathbb{Q} \mid q > x\})$, and this defines the so-called *standard map*

$$\text{st} : R \rightarrow \mathbb{R}$$

which sends each $x \in R$ to its *standard part* $\text{st}(x) \in \mathbb{R}$. In a similar way, for each $\varphi \in R^R$ we can define its *standardization* $\text{st}(\varphi) \in \mathbb{R}^{\mathbb{R}}$, and this defines a map

$$\text{st} : R^R \rightarrow \mathbb{R}^{\mathbb{R}}.$$

In the topos $\mathscr{G}$, these standard maps have very simple representations. If f is a smooth real at $\bar{A} \in \mathbb{G}$, $\bar{A} \overset{f}{\longrightarrow} C^\infty(\mathbb{R})$, we obtain a continuous (even smooth) functions $\gamma(\bar{A}) \rightarrow \mathbb{R}$ by applying γ, which is precisely a Dedekind real at stage $\bar{A}$, by the representation Theorem 5.1. This function $\gamma(\bar{A}) \rightarrow \mathbb{R}$ is the standard part of f. Thus, $R \overset{\text{st}}{\longrightarrow} \mathbb{R}$ is just "apply γ". A similar description can be given of $R^R \overset{\text{st}}{\longrightarrow} \mathbb{R}^{\mathbb{R}}$: if $\bar{A} \Vdash \varphi \in R^R$, that is $\varphi : \bar{A} \times C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ in $\mathbb{G}$, application of γ yields a continuous function $\text{st}(\varphi) : \gamma(\bar{A}) \times \mathbb{R} \rightarrow \mathbb{R}$, or $\text{st}(\varphi) : \gamma(\bar{A}) \rightarrow \mathbb{R}^{\mathbb{R}}$. Thus by Theorem 5.1, $\bar{A} \Vdash \text{st}(\varphi) \in \mathbb{R}^{\mathbb{R}}$. (Note that there is no ambiguity in notation when we write the exponentials R^R, $\mathbb{R}^{\mathbb{R}}$, since all functions are continuous.) It is clear that we have

$$\text{st}(\varphi)(\text{st}(x)) = \text{st}(\varphi(x)).$$

Observe also that

$$\forall p, q \in \mathbb{Q}\, \forall x \in R\, (p < x < q \leftrightarrow p < \text{st}(x) < q),$$

so $\text{st} : R \rightarrow \mathbb{R}$ is *continuous* (in fact, st^{-1} gives an isomorphism of complete Heyting algebras, i.e. R and $\mathbb{R}$ define the same locale). From this and the fact that all functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous, it follows that all fuctions from R to R are con-

tinuous (cf. the remark following 5.7). Similarly, if we equip R^R with the C^∞-topology (uniform convergence on compacts of all derivatives), $\mathrm{st}: R^R \to \mathbb{R}^r$ is continuous.

The standard maps $\mathrm{st}: R \to \mathbb{R}$ and $\mathrm{st}: R^R \to \mathbb{R}^{\mathbb{R}}$ are almost surjective, in the sense that in $\mathscr{G}$ it is valid that $\forall x \in \mathbb{R} \; \neg \neg \exists y \in R \; \mathrm{st}(y) = x$, and $\forall \varphi \in \mathbb{R}^{\mathbb{R}} \; \neg \neg \exists \psi \in R^R \; \mathrm{st}(\psi) = \varphi$, since in the points of $\mathbb{G}$, R and $\mathbb{R}$ are indistinguishable.

Another way of looking at the standard maps, which explain their universality, is by moving the relevant spaces along the geometric inclusion $\mathscr{E} \xrightarrow{\;\gamma\;} \mathscr{G}$ of 4.6. Since γ^* is "compose with $i: \mathbb{E} \hookrightarrow \mathbb{G}$", we find

$$\gamma^*(R) = \mathbb{E}(-, \mathbb{R}), \qquad \gamma^*(R^R) = \mathbb{E}(- \times \mathbb{R}, \mathbb{R}).$$

Also, from Theorem 5.1 it follows that

$$\mathbb{R}_r = \gamma_*(\mathbb{R}_r), \qquad (\mathbb{R}^{\mathbb{R}})_r = \gamma_*((\mathbb{R}^{\mathbb{R}})_r).$$

Now a function $R \xrightarrow{\;\varphi\;} \mathbb{R}$ in $\mathscr{G}$ defined at stage $\bar{A}$, $\varphi \in \mathbb{R}^R(\bar{A})$ corresponds to a map $\bar{A} \to \mathbb{R}^R$ in $\mathscr{G}$, or $\bar{A} \times R \to \mathbb{R}$ in $\mathscr{G}$. By applying the adjunction $\gamma^* \dashv \gamma_*$ (and the fact that $\gamma_*(\mathbb{R}) = \mathbb{R}$), this corresponds to a map $\gamma^*(\bar{A} \times R) \to \mathbb{R}$ in $\mathscr{E}$, i.e. a continuous function $\gamma(\bar{A}) \to \mathbb{R}^{\mathbb{R}}$ in Sets, by theorem 5.1. Thus we have bijections

$$\frac{\bar{A} \longrightarrow \mathbb{R}^R \text{ in } \mathscr{G}}{\gamma(\bar{A}) \xrightarrow{\text{cts}} \mathbb{R}^{\mathbb{R}} \text{ in Sets}}$$

$$\gamma(\bar{A}) \longrightarrow \mathbb{R}^{\mathbb{R}} \text{ in } \mathscr{E}$$

But again by Theorem 5.1, an element of $\mathbb{R}^{\mathbb{R}}$ at $\gamma(\bar{A})$ in $\mathscr{E}$ is the same as an element of $\mathbb{R}^{\mathbb{R}}$ at $\bar{A}$ in $\mathscr{G}$. Therefore we have

$$\mathbb{R}^R \cong \mathbb{R}^{\mathbb{R}}.$$

This isomorphism is easily seen to come from the standard map $R \xrightarrow{\;\mathrm{st}\;} \mathbb{R}$ by composition, i.e. in $\mathscr{G}$ the map

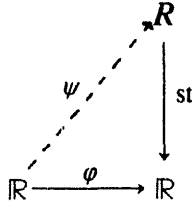$$\mathbb{R}^{\mathbb{R}} \to \mathbb{R}^R, \qquad \varphi \mapsto \varphi \circ \mathrm{st}$$

is a bijection. In other words, $R \xrightarrow{\;\mathrm{st}\;} \mathbb{R}$ is the universal function $R \to \mathbb{R}$ (internally in $\mathscr{G}$). Thus we have shown:

5.12. Theorem. $\mathscr{G} \models \mathbb{R}^R \cong \mathbb{R}^{\mathbb{R}}$, *via composition with the standard map* $R \xrightarrow{\;\mathrm{st}\;} \mathbb{R}$. *Thus, elements of* $\mathbb{R}^R(\bar{A})$ *correspond to continuous functions* $\gamma(A) \times \mathbb{R} \to \mathbb{R}$ *in Sets. (Consequently,* $\mathscr{G} \models$ "*all functions* $R \to \mathbb{R}$ *are continuous*", *but we knew this already, cf. 5.7.)* $\square$

Curiously enough, functions from $\mathbb{R}$ to R behave rather differently:

5.13. Proposition. *In* $\mathscr{G}$, *every function* $\mathbb{R} \xrightarrow{\;\psi\;} R$ *is constant 'up to an infinitesimal*

bit', in the sense that $\text{st} \circ \psi$ *is constant. In other words, in* $\mathscr{G}$ *a function* $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ *lifts to a function* $\mathbb{R} \xrightarrow{\psi} R$ *iff* φ *is constant.*

$$
\begin{array}{ccc}
 & & {}_*R \\
 & {}^{\psi}\nearrow & \Big\downarrow \text{st} \\
\mathbb{R} & \xrightarrow{\ \ \varphi\ \ } & \mathbb{R}
\end{array}
$$

Proof. (i) For a start, let us look at a global map $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ which lifts to $\mathbb{R} \xrightarrow{\psi} R$, i.e. ψ is a natural transformation

$$\psi : \mathrm{Cts}(\gamma(-), \mathbb{R}) \longrightarrow \mathbb{C}(-, R).$$

Now take the case $\bar{A} = \overline{C^\infty(\mathbb{R})}$, and apply ψ to the identity to find a *smooth* $\psi_{\bar{A}}(1) = \sigma : \mathbb{R} \to \mathbb{R}$.

A pointwise argument as in the proof of 1.10 now shows that $\text{st} \circ \psi$ comes from "compose with σ", so σ has to be such that for any continuous $X \xrightarrow{f} \mathbb{R}$, where X is locally closed $\subseteq$ some $\mathbb{R}^n$, the composite $\sigma \circ f$ is smooth. But then σ must be constant, for if $\sigma'(y) \neq 0$, we can find a neighbourhood U_y of y on which σ is invertible, so if we take for $X \xrightarrow{f} \mathbb{R}$ the function $x \mapsto \sigma^{-1}(\sigma y + |x|)$ defined on the neighbourhood $X = \{x \mid \varrho y + |x| \in \sigma(U_y)\}$ of 0, $(\text{st} \circ \psi)_X(f)(x) = \sigma y + |x|$, which is not smooth.

(ii) Now suppose $\bar{A} \times \mathbb{R} \xrightarrow{\psi} R$ is a map $\mathbb{R} \to R$ in $\mathscr{G}$ defined at stage $\bar{A}$, where $A = C^\infty(\mathbb{R}^n)/I$. By applying γ we find a C^∞-map $Z(I) \times \mathbb{R} \xrightarrow{\sigma} \mathbb{R}$ such that for each $\mathbb{G}$-map $\bar{B} \xrightarrow{f} \bar{A}$ and each continuous map $\alpha : \gamma(\bar{B}) \to \mathbb{R}$, $\sigma \circ (f, \alpha)$ is in the image of st, i.e. is a C^∞-map. In particular, as in the global case we may conclude that for each $t \in Z(I)$, $\sigma(t, -)$ is constant. In other words, σ does not depend on its second coordinate, and from this it easily follows that

$$\bar{A} \Vdash \forall x, y \in \mathbb{R} \ \text{st}(\psi(x)) = \text{st}(\psi(y)),$$

which completes the proof. $\square$

We can make an analysis similar to the one that led to 5.13 for the case of functionals $F : R^R \to \mathbb{R}^{\mathbb{R}}$, but things turn out to be somewhat more difficult here. Suppose we are given such a functional F in $\mathscr{G}$ at stage $\bar{A}$, i.e.

$$F : \bar{A} \times R^R \to \mathbb{R}^{\mathbb{R}} \quad \text{in } \mathscr{G}.$$

Via $\gamma^* \dashv \gamma_*$ (recall that γ_* preserves $\mathbb{R}^{\mathbb{R}}$) this corresponds to a map

$$\gamma^*(\bar{A}) \times \gamma^*(R^R) \to \mathbb{R}^{\mathbb{R}} \quad \text{in } \mathscr{E},$$

that is, a natural transformation (over $\mathbb{E}$)

$$\tau : \mathbb{E}(-, \gamma(\bar{A})) \times \mathbb{E}(- \times \mathbb{R}, \mathbb{R}) \longrightarrow \mathrm{Cts}(- \times \mathbb{R}, \mathbb{R}).$$

Our aim is to show that such a natural transformation τ corresponds to a continuous map

$$\gamma(\bar{A}) \times C^{\infty}(\mathbb{R}, \mathbb{R}) \xrightarrow{\varphi} C^0(\mathbb{R}, \mathbb{R}),$$

where $C^0(\mathbb{R}, \mathbb{R})$ is given the compact-open topology, and $C^{\infty}(\mathbb{R}, \mathbb{R})$ is equipped with the C^{∞}-topology (or Fréchet-topology), i.e. $f_n \to f$ in $C^{\infty}(\mathbb{R}, \mathbb{R})$ if for all m, the sequence $\{f_n^{(m)}\}_n$ converges to $f^{(m)}$ in $C^0(\mathbb{R}, \mathbb{R})$.

Now clearly, such a map φ determines a natural transformation τ by composition. (To be explicit, if $X \xrightarrow{f} \gamma(\bar{A})$ and $X \times \mathbb{R} \xrightarrow{\alpha} \mathbb{R}$ are maps in $\mathbb{E}$, the function $\tau_X(f, \alpha): X \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\tau_X(f, \alpha)(x, r) = \varphi(f(x), \alpha(x, -))(r).$$

Let us introduce an abbreviation, and write $\varphi[f, \alpha]$ for this function.)

Conversely, a transformation $\tau: \mathbb{E}(-, \gamma\bar{A}) \times \mathbb{E}(- \times \mathbb{R}, \mathbb{R}) \to \mathrm{Cts}(- \times \mathbb{R}, \mathbb{R})$ determines at least a *function* $\gamma\bar{A} \times C^{\infty}(\mathbb{R}, \mathbb{R}) \xrightarrow{\varphi} C^0(\mathbb{R}, \mathbb{R})$ by calculating the action of τ on points (just as we did in the proof of Theorem 1.10): If $1 = \{*\}$ is the one-point space in $\mathbb{E}$, we may define φ by $\varphi(x, y) = \tau_1(\hat{x}, \hat{y})$, where $\hat{x}: 1 \to \gamma\bar{A}$ and $\hat{y}: 1 \times \mathbb{R} \to \mathbb{R}$ are the maps in $\mathbb{E}$ corresponding to $x \in \gamma\bar{A}$, $y \in C^{\infty}(\mathbb{R}, \mathbb{R})$. Indeed, by naturality of τ we get that if $X \xrightarrow{f} \gamma\bar{A}$ and $X \times \mathbb{R} \xrightarrow{\alpha} \mathbb{R}$ are maps in $\mathbb{E}$, and x is a point of X (corresponding to $\hat{x}: 1 \to X$ in $\mathbb{E}$),

$$\tau_X(f, \alpha)(x, r) = \tau_1(f, \alpha)(\hat{x}(*), r)$$

$$= \tau_1(f \circ \hat{x}, \alpha \circ (\hat{x} \times 1))(*, r)$$

$$= \varphi(f(x), \alpha(x, -))(r),$$

in other words,

$$\tau_X(f, \alpha) = \varphi[f, \alpha].$$

This function $\varphi: \gamma\bar{A} \times C^{\infty}(\mathbb{R}, \mathbb{R}) \to C^0(\mathbb{R}, \mathbb{R})$ has the property that for any pair of C^{∞}-maps $X \xrightarrow{f} \gamma\bar{A}$ and $X \times \mathbb{R} \xrightarrow{\alpha} \mathbb{R}$, $\varphi[f, \alpha]$ is a *continuous* function $X \times \mathbb{R} \to \mathbb{R}$. We may now apply the following lemma to conclude that φ is continuous.

5.14. Lemma. *Let E be a Fréchet vector space (over $\mathbb{R}$), and X a locally closed subspace of some $\mathbb{R}^n$. If $X \times E \xrightarrow{\varphi} \mathbb{R}$ is 'pathcontinuous' in the sense that for each C^{∞}-map $f: Y \to X \times E$ (with Y a locally closed subspace of some $\mathbb{R}^m$), $\varphi \circ f: Y \to \mathbb{R}$ is continuous, then φ is continuous.*

Proof. Assume that φ is not continuous at (x, e). Then we can find an $\varepsilon > 0$ and a sequence $(x_n, e_n) \to (x, e)$ such that $|\varphi(x_n, e_n) - \varphi(x, e)| \geq \varepsilon$, for all n. Proceeding as in Van Quê & Reyes [14], we can define a C^{∞}-function $g: \mathbb{R} \to E$ with $g(0) = e$, $g(1/n) = e_n$. (Indeed, define $g(t) = e + \sum_{n \geq 3} \chi_n(t) e_n'$, with $e_n' = e_n - e$, in the notation of loc. cit.) Now let $Y = X \times \mathbb{R}$, and $f = X \times g$. Since $(x_n, 1/n) \to (x, 0)$ in Y, $\varphi(x_n, g(1/n)) \to \varphi(x, g(0))$, i.e. $\varphi(x_n, e_n) \to \varphi(x, e)$, a contradiction. $\square$

Thus we have shown a natural one-to-one correspondence between maps $\bar{A} \times R^R \xrightarrow{F} \mathbb{R}^\mathbb{R}$ in $\mathcal{G}$ and continuous functions $\gamma(\bar{A}) \times C^\infty(\mathbb{R}, \mathbb{R}) \xrightarrow{\varphi} C^\infty(\mathbb{R}, \mathbb{R})$ in Sets. From this we immediately derive that in $\mathcal{G}$ all functions $F: R^R \to \mathbb{R}^\mathbb{R}$ (at an arbitrary stage $\bar{A}$) are continuous.

The universality of $R^R \xrightarrow{\text{st}} \mathbb{R}^\mathbb{R}$ appears as follows. We have seen that (continuous) maps $\mathbb{R} \to \mathbb{R}$ in $\mathcal{G}$ at stage $\bar{A}$ correspond to continuous functions $\gamma(\bar{A}) \times \mathbb{R} \to \mathbb{R}$ (Theorems 5.1, 5.2). A subsheaf of these is formed by the functions $\gamma(\bar{A}) \times \mathbb{R} \to \mathbb{R}$ that have all continuous partial derivatives with respect to the second (the $\mathbb{R}$-) variable, and it is easily checked that this is precisely the interpretation of "C^∞-maps $\mathbb{R} \to \mathbb{R}$" in $\mathcal{G}$. Thus we may reformulate the above correspondence between maps $R^R \xrightarrow{F} \mathbb{R}^\mathbb{R}$ at stage $\bar{A}$ in $\mathcal{G}$ and continuous maps $\gamma(\bar{A}) \times C^\infty(\mathbb{R}, \mathbb{R}) \xrightarrow{\varphi} C^0(\mathbb{R}, \mathbb{R})$ as an isomorphism in $\mathcal{G}$,

$$(\mathbb{R}^\mathbb{R})^{R^R} \cong (\mathbb{R}^\mathbb{R})^{C^\infty(\mathbb{R}, \mathbb{R})}.$$

This correspondence may be unwinded by using the standard map $\text{st}: R^R \to \mathbb{R}^\mathbb{R}$. Clearly, st factors through $C^\infty(\mathbb{R}, \mathbb{R}) \subset \mathbb{R}^\mathbb{R}$, and if we regard it as a map $R^R \to C^\infty(\mathbb{R}, \mathbb{R})$, the above isomorphism comes about through the bijection "compose with st"

$$(\mathbb{R}^\mathbb{R})^{C^\infty(\mathbb{R}, \mathbb{R})} \to (\mathbb{R}^\mathbb{R})^{R^R}, \quad G \mapsto \text{st} \circ G.$$

(Let us stress again that this is a bijection in $\mathcal{G}$, i.e. it works not only for global sections, but at all stages.)

For the record,

5.15. Theorem $\mathcal{G} \models (\mathbb{R}^\mathbb{R})^{R^R} \cong (\mathbb{R}^\mathbb{R})^{C^\infty(\mathbb{R}, \mathbb{R})}$, *again via composition with the standard map* $R^R \xrightarrow{\text{st}} C^\infty(\mathbb{R}, \mathbb{R})$. *Thus, elements of* $(\mathbb{R}^\mathbb{R})^{R^R}$ *at stage* $\bar{A}$ *correspond to continuous functions* $\gamma \bar{A} \times C^\infty(\mathbb{R}, \mathbb{R}) \to C^0(\mathbb{R}, \mathbb{R})$ *in Sets. Consequently,* $\mathcal{G} \models$ "*All functions* $R^R \to \mathbb{R}^\mathbb{R}$ *are continuous*". $\square$

6. Some remarks on other models, the finite cover topology

In the previous sections we have seen that if we force the generic C^∞-ring R to be local and Archimedean, i.e. (see Section 4) if we work in the smooth topos $\mathcal{G}$, intuitionistic analysis in general, and the Dedekind reals in particular, inherit some of the good properties of R, such as continuity of all functions. Our main tool for comparing R and $\mathbb{R}$ in this context was the adjoint retraction between $\mathcal{G}$ and the euclidean topos $\mathcal{E}$. The purpose of this section is to point out that many of these good properties for intuitionistic analysis (analysis on $\mathbb{R}$) fail if we do not force R to be Archimedean, while much of the smooth analysis (analysis on R) remains the same.

Here it will be convenient to look at the analog of the topos $\mathcal{G}$ of sheaves on $\mathbb{G}$, but with arbitrary covers replaced by finite ones. More precisely, let $\mathbb{E}_{\text{fin}}$ be the site

with the same underlying category as $\mathbb{E}$ (cf. Definition 1.9), but with the Grothendieck topology generated by *finite* open covers (i.e., a family $K = \{ Y_\alpha \xrightarrow{f_\alpha} X \}_\alpha$ covers X iff there is a finite open cover $\{U_1, \ldots, U_n\}$ of X such that each embedding $U_i \hookrightarrow X$ is in K), and let ℓ_{fin} be the topos $\mathrm{Sh}(\mathbb{E}_{\mathrm{fin}})$. Similarly, define a site $\mathbb{G}_{\mathrm{fin}}$ with underlying category $\mathbb{G}$ (defined in Section 3) and with the Grothendieck topology generated by finite open covers, and let $\mathscr{G}_{\mathrm{fin}}$ be the topos $\mathrm{Sh}(\mathbb{G}_{\mathrm{fin}})$.

It is clear that if we regard the functor γ described in 3.5 as a functor $\mathbb{G}_{\mathrm{fin}} \xrightarrow{\gamma} \mathbb{E}_{\mathrm{fin}}$, Theorem 2.3 still applies, and we obtain the analog of Theorem 4.6 for the toposes $\mathscr{G}_{\mathrm{fin}}$ and ℓ_{fin}. We still write γ and ϱ for the geometric morphisms involved; so there is a commutative diagram of geometric morphisms

$$
\begin{array}{ccc}
\ell_{\mathrm{fin}} & \underset{\gamma}{\overset{\varrho}{\rightleftarrows}} & \mathscr{G}_{\mathrm{fin}} \\
\Big\uparrow\Big\downarrow & & \Big\uparrow\Big\downarrow \\
\ell & \underset{\gamma}{\overset{\varrho}{\rightleftarrows}} & \mathscr{G}
\end{array}
$$

and the adjoint retraction $\ell_{\mathrm{fin}} \leftrightarrows \mathscr{G}_{\mathrm{fin}}$ gives us that $\varrho^* = \gamma_* : \ell_{\mathrm{fin}} \to \mathscr{G}_{\mathrm{fin}}$ preserves the spaces of models of T_1-locales, such as $\mathbb{R}$, $\mathbb{N}^\mathbb{N}$, etc.

We just said that in many respects smooth analysis does not change too much if we move from $\mathscr{G}$ to $\mathscr{G}_{\mathrm{fin}}$. In fact, the presheaf topos $\mathrm{Sets}^{(C^\infty\text{-rings})_{\mathrm{fi}}} = \mathrm{Sets}^{\mathbb{L}^{\mathrm{op}}}$ is already an adequate model for synthetic integration theory, for example. Here, however, we can never have an adjoint retraction from $\mathrm{Sets}^{\mathbb{L}^{\mathrm{op}}}$ to a topological topos like the euclidean one, since there are C^∞-rings $C^\infty(\mathbb{R}^n)/I$ which are far from trivial, while the corresponding set of points $\Gamma(C^\infty(\mathbb{R}^n)/I) = Z(I) \subseteq \mathbb{R}^n$ can be empty (for example, let I be the functions with compact support). This does not occur if we restrict our attention to germ-determined ideals, thus explaining why germ-determined ideals are so convenient to work with.

We should point out here that there is some loss also: if one restricts ones attention to germ-determined ideals, there can never be any *invertible infinitesimals* in the model. Such an object of invertible infinitesimals does exist, for example, in the *smooth Zariski topos* of sheaves on $\mathbb{L}$ with the finite cover topology (this topos is the precise analog (for C^∞-rings) of the Zariski topos), and we expect that this feature will make the smooth Zariski topos an important object of future study (cf. Reyes [17], Moerdijk & Reyes [13]).

Let us now give some examples of properties that fail to hold if we pass from the topos $\mathscr{G}$ to $\mathscr{G}_{\mathrm{fin}}$. First something that does not fail, however. From the proof that we gave of $\mathscr{G} \models$ "All functions $R \to R$ are continuous" (cf. Corollary 5.7) it may seem that this fact depends on the topology of $\mathbb{G}$, but it does not. It even holds in the presheaf topos:

6.1. Theorem. *In* $\mathrm{Sets}^{\mathbb{L}^{\mathrm{op}}}$, *all functions from the smooth unit interval* $[0, 1] \subset R$ *to* R *are uniformly continuous.*

Proof. See Moerdijk & Reyes [13]. □

6.2. Remark. It is quite surprising that we obtain uniform continuity instead of just continuity in 6.1, since $[0,1] \subset R$ is not compact in $\mathrm{Sets}^{L^{op}}$ (or in the smooth Zariski topos): if δ is an invertible infinitesimal, $(x-\delta, x+\delta)$ is an open neighbourhood of x for every $x \in [0,1]$, and the cover $\{(x-\delta, x+\delta) \mid x \in [0,1]\}$ cannot have a finite subcover.

6.3. Corollary. *In the smooth Zariski topos, in $\mathscr{G}_{fin}$ and in $\mathscr{G}$, all functions from R to R are continuous.* □
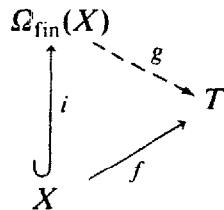
The analog of 6.3 for the Dedekind reals fails in $\mathscr{G}_{fin}$. To see this, let us first determine the Dedekind reals, and the space of continuous functions $C^0(\mathbb{R}, \mathbb{R})$ in $\mathscr{G}_{fin}$. These spaces coincide with the corresponding formal spaces or locales, since

6.4. Lemma. *Let $(\mathbb{C}, J)$ be an arbitrary coherent site (J is generated by finite covers). Then in $\mathrm{Sh}(\mathbb{C}, J)$, Bar Induction holds. In particular, $\mathscr{G}_{fin} \models$ "Bar Induction", and all separable locales have enough points in $\mathscr{G}_{fin}$.*

Proof. Take $C \in \mathbb{C}$, and let $P \subset \mathbb{N}^{<\mathbb{N}}$ be a monotone inductive bar at C. If $\alpha \in \mathbb{N}^{\mathbb{N}}$ externally, α acts as a constant element α of $\mathbb{N}^{\mathbb{N}}$ internally, so $C \Vdash \exists n\, \bar{\alpha}(n) \in P$. Hence since covers of C are finite and P is monotone, $C \Vdash \bar{\alpha}(n) \in P$ for some n. Thus $\hat{P} = \{u \in \mathbb{N}^{<\mathbb{N}} \mid C \Vdash u \in P\}$ bars $\mathbb{N}^{\mathbb{N}}$ externally, and is monotone and inductive. Therefore $\langle\,\rangle \in \hat{P}$. □

For the following lemma, we have to return to the notation of Section 1. Let $\mathbb{C}$ be a topological site, and $\mathbb{C}_{fin}$ the site with the same underlying category as $\mathbb{C}$, but with the topology generated by *finite* open covers. Assume that all spaces in $\mathbb{C}$ are locally compact, and for a space $X \in \mathbb{C}$ write $\Omega_{fin}(X)$ (resp. $\Omega(X)$) for the locale of subobjects of $\mathbb{C}(-, X)$ in $\mathrm{Sh}(\mathbb{C}_{fin})$ (resp. $\mathrm{Sh}(\mathbb{C})$). (M. Fourman pointed out to us that this lemma also follows from an unpublished result of A. Joyal.)

6.5. Lemma. *Let $X \in \mathbb{C}$ as above, and let T be a regular (Hausdorff) space. Then a continuous map $X \xrightarrow{f} T$ extends to a map $\Omega_{fin}(X) \xrightarrow{g} T$ iff $\overline{f(X)}$ is compact. Moreover, if this is the case, the extension g is unique.*

$$
\begin{array}{ccc}
\Omega_{fin}(X) & & \\
\uparrow{\scriptstyle i} & \searrow{\scriptstyle g} & \\
& & T \\
U & \nearrow{\scriptstyle f} & \\
X & &
\end{array}
$$

Proof. Recall that $X \xrightarrow{\ i\ } \Omega_{fin}(X)$ is the composition of the embeddings $X \xrightarrow{\ j_1\ }$
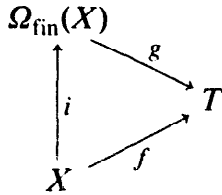
$\Omega(X)$ and $\Omega(X) \overset{j_2}{\hookrightarrow} \Omega_{\text{fin}}(X)$. As observed in Section 1, X is a retract of $\Omega(X)$, and the pair $X \underset{r}{\overset{j_2}{\rightleftarrows}} \Omega(X)$ induces an isomorphism $\mathrm{Cts}(X, T) \cong \mathrm{Cts}(\Omega(X), T)$, provided T is a T_1-locale. j_2^{-1} sends a crible $K \in \Omega_{\text{fin}}(X)$ to its closure as a crible in $\Omega(X)$, i.e.

$$j_2^{-1}(K) = \{ Y \overset{\varphi}{\longrightarrow} X \mid \exists \text{ open cover } \{U_\alpha\}_\alpha \text{ of } Y \text{ such that}$$

$$\text{each composite } U_\alpha \hookrightarrow Y \overset{\varphi}{\longrightarrow} X \text{ is in } K \}.$$

Thus,

$$i^{-1}(K) = \bigcup \{ U \text{ open} \subseteq X \mid U \subseteq X \in K \}.$$

Now suppose we have a commutative diagram

First note that from the fact that T is a T_1-space, it follows that

$$\Omega(X) \overset{j_2}{\hookrightarrow} \Omega_{\text{fin}}(X) \overset{g}{\longrightarrow} T = \Omega(X) \overset{r}{\longrightarrow} X \overset{f}{\longrightarrow} T.$$

Thus, if U is open in T and $Y \overset{\varphi}{\longrightarrow} X$ is a map in $\mathbb{C}$,

(*) $Y \overset{\varphi}{\longrightarrow} X \in g^{-1}(U)$ implies $Y \overset{\varphi}{\longrightarrow} X \in r^{-1}f^{-1}(U)$,   i.e. $f\varphi(Y) \subseteq U$.

As another preparatory remark, we note that for an arbitrary subset A of T and an open neighbourhood $U \supseteq A$, the following are equivalent:
   (i) $\bar{A}$ is compact, and $\bar{A} \subseteq U$.
   (ii) There is a compact $K \subseteq T$ with $A \subseteq K \subseteq U$.
   (iii) For every open cover $\{U_\alpha\}_\alpha$ of U, finitely many of the U_α's cover A.
   If these conditions hold, it is common to write

$$A \ll U.$$

Now we show that from $gi = f$ it follows that $f(X) \ll T$: Suppose $\{V_\alpha\}_\alpha$ is an open cover of T. Then $\bigvee_\alpha g^{-1}(V_\alpha) = \text{true}$ in $\Omega_{\text{fin}}(X)$, so there is a finite cover $\{U_1, \ldots, U_n\}$ of X such that each inclusion $U_i \hookrightarrow X$ is in some $g^{-1}(V_\alpha)$. Therefore by (*), $f(U_i) \subseteq V_{\alpha_i}$, so $f(X) \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. Thus $f(X) \ll T$.

Conversely, if $f(X)$ is compact, we may define an extension $\Omega_{\text{fin}}(X) \overset{\bar{f}}{\longrightarrow} T$ by

$$\bar{f}^{-1}(U) = \left\{ Y \overset{\varphi}{\longrightarrow} X \mid f\varphi(Y) \ll U \right\}.$$

Clearly, $\bar{f}^{-1}$ is orderpreserving, and $\mathrm{id}_X \in \bar{f}^{-1}(T)$ since $\overline{f(X)}$ is compact, i.e. $\bar{f}^{-1}$ preserves the top element. It is trivial to check that $\bar{f}^{-1}$ preserves binary meets. For sups, suppose $\{U_\alpha\}_\alpha$ is an open cover of U, and $Y \overset{\varphi}{\longrightarrow} X \in \bar{f}^{-1}(U)$. We need to

show that $Y \xrightarrow{\varphi} X \in V_\alpha f^{-1}(U_\alpha)$. By hypothesis, $f\varphi(Y) \subset$ some compact $K \subseteq U$, so since T is regular we may choose for each $t \in K$ a neighbourhood V_t of t such that $\bar{V}_t \subseteq$ some U_{α_t}. Then there exists a finite cover $\{V_{t_1}, \ldots, V_{t_n}\}$ of K, and if we let $W_i = \varphi^{-1} f^{-1}(V_{t_i})$, $i = 1, \ldots, n$, we find that $W_i \hookrightarrow Y \xrightarrow{\varphi} X \in f^{-1}(U_{\alpha_{t_i}})$, since $f\varphi(W_i) \subseteq \bar{V}_{t_i} \cap K \subseteq V_{t_i} \cap K$ compact $\subseteq U_{\alpha_t}$. Thus $\varphi \in V_\alpha f^{-1}(U_\alpha)$.

To show that $\hat{f}i = f$, we use local compactness of X:

$$i^{-1}\hat{f}^{-1}(U) = \{x \mid V \text{ open nbd } W_x \text{ of } x \text{ with } W_x \hookrightarrow X \in \hat{f}^{-1}(U)\}$$

$$= \{x \mid \exists \text{ open nbd } W_x \text{ of } x \text{ with } f(W_x) \ll U\}$$

$$= f^{-1}(U), \qquad \text{provided } X \text{ is locally compact.}$$

To show that $\hat{f}$ is the unique extension of f, it suffices to show that $\hat{f}$ is *minimal* among extensions of f, since T is a T_1-space. To this end, suppose g is another such extension, and suppose $Y \xrightarrow{\varphi} X \in g^{-1}(U)$. We show that $\varphi \in \hat{f}^{-1}(U)$, i.e. that $\varphi \in \hat{f}^{-1}(U)$, i.e. that $f\varphi(Y) \ll U$. So let $\{U_\alpha\}_\alpha$ be an open cover of U. Then $U \xrightarrow{\varphi} X \in g^{-1}(U) = V_\alpha g^{-1}(U_\alpha)$, so there is a finite cover $\{V_1, \ldots, V_n\}$ of Y such that each restriction $\varphi \mid V_i$ is in some $g^{-1}(U_{\alpha_i})$. Then by $(*)$ above, $f\varphi(V_i) \subseteq U_{\alpha_i}$, i.e. $f\varphi(Y) \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. Thus $g^{-1} \leq \hat{f}^{-1}$, or $\hat{f} \leq g$, and the proof is complete. $\square$

6.6. Theorem. *Let T be a regular Hausdorff space in* Sets, *and let $T_{\mathscr{S}_{\mathrm{fin}}}$ be the sheaf of models of (the propositional theory corresponding to) T in $\mathscr{S}_{\mathrm{fin}}$. Then*

$$T_{\mathscr{S}_{\mathrm{fin}}}(\bar{A}) \cong \{f : \gamma(\bar{A}) \to T \mid f \text{ is continuous, and } \overline{\mathrm{im}(f)} \text{ is compact}\}$$

Proof. Combine the preceding lemma with Corollary 1.5 and Proposition 2.3 (i.e. the version of 4.6 for $\mathscr{S}_{\mathrm{fin}}$). $\square$

6.7. Example. In $\mathscr{S}_{\mathrm{fin}}$, the Dedekind reals are interpreted by the sheaf of bounded continuous functions,

$$\mathbb{R}_{\mathscr{S}_{\mathrm{fin}}}(\bar{A}) = \text{set of bounded cts functions } \gamma\bar{A} \to \mathbb{R}.$$

6.8. Corollary. *In $\mathscr{S}_{\mathrm{fin}}$, "All functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous" fails to hold.*

Proof. Note that by 6.6 and 6.7, the interpretation of the Dedekind unit interval $[0, 1]$ is just the sheaf of continuous functions,

$$[0, 1]_{\mathscr{S}_{\mathrm{fin}}}(\bar{A}) \cong \mathrm{Cts}(\gamma\bar{A}, [0, 1]).$$

Therefore we will replace $\mathbb{R}$ by $[0, 1]$ for notational convenience, and show that here is a function $[0, 1] \to [0, 1]$ in $\mathscr{S}_{\mathrm{fin}}$ which is not continuous at, say, $\frac{1}{2}$.

Any continuous map $f : \mathbb{N} \times [0, 1] \to [0, 1]$ induces by composition a natural transformation $[0, 1]_{\mathscr{S}_{\mathrm{fin}}} \to [0, 1]_{\mathscr{S}_{\mathrm{fin}}}$ over $\mathbb{G}/\mathbb{N}$, i.e. f gives an element F of $[0, 1]^{[0,1]}(\mathbb{N})$. On the other hand, by Theorem 6.6 an internal continuous map $F : [0, 1] \to [0, 1]$ at stage $\mathbb{N}$ corresponds to a continuous $f : \mathbb{N} \to [0, 1]^{[0,1]}$ in Sets such that $\mathrm{im}(f)$ is

relatively compact in the space $[0,1]^{[0,1]}$ equipped with the compact-open toplogy. In particular, if we regard f as a family $\{f_n\}_n$ of functions $[0,1] \to [0,1]$, this family must be equicontinuous by the Arzela-Ascoli theorem. So any family $\{f_n\}_n$ which is not equicontinuous (at $\frac{1}{2}$, say) induces an internal map F at the level $\mathbb{N}$, with

$$\mathbb{N} \Vdash \text{``}F \text{ is a function } [0,1] \to [0,1]\text{''},$$

$$\mathbb{N} \nVdash \text{``}F \text{ is continuous (at } \tfrac{1}{2}\text{)''}. \qquad \square$$

We give one more example of what we lose when we pass from $\mathscr{S}$ to $\mathscr{S}_{\text{fin}}$.

6.9. Proposition. *Markov's principle*

$$\forall P \subseteq \mathbb{N} \, ((\forall n \, (Pn \vee \neg Pn) \wedge \neg\neg \exists n \, Pn) \to \exists n \, Pn)$$

fails to hold in $\mathscr{S}_{\text{fin}}$.

Proof. Again consider the object $\mathbb{N}$ of $\mathbb{G}_{\text{fin}}$, and consider the subsheaf P of $\mathbb{N}$ at $\mathbb{N}$ (two different $\mathbb{N}$'s!) generated by

$$[-n,n] \Vdash n \in P.$$

Then $\mathbb{N} \Vdash \forall n \, (Pn \vee \neg Pn)$ since $\mathbb{N}$ is discrete, and $\mathbb{N} \Vdash \neg\neg \exists n \, n \in P$, since at every point $p \in \mathbb{N}$, $p \Vdash \exists n \, n \in P$. But $\mathbb{N} \nVdash \exists n \, m \in P$, clearly. $\quad\square$

6.10. Remark. The following *finite* form of Markov's principle holds in $\mathscr{S}_{\text{fin}}$, expressing that in $\mathscr{S}_{\text{fin}}$, R is still a field in Kock's sense (cf. the remark following 5.10).

$$\mathscr{S}_{\text{fin}} \models \text{``}\neg(x_1 = 0 \wedge \cdots \wedge x_n = 0) \to \text{one of } x_1, \ldots, x_n \text{ is invertible''}.$$

(where the x_i's range over R).

Proof. Let $f_1, \ldots, f_n$ be elements of R at stage $\bar{A}$. Say $A = C^\infty(\mathbb{R}^n)/I$, so $f_1, \ldots, f_n \in A$ and if

$$\bar{A} \Vdash \neg(f_1 = 0 \wedge \cdots \wedge f_n = 0),$$

then $C^\infty(\mathbb{R}^n)/(I, f_1, \ldots, f_n)^\sim$ is the zero ring, i.e. $1 \in (I, f_1, \ldots, f_n)^\sim$, or equivalently, $Z(I, f_1, \ldots, f_n) = \emptyset$. But then $\mathbb{R}^m = \mathbb{R} \setminus Z(I) \cup U(f_1) \cup \cdots \cup U(f_n)$, where $U(f_i) = \{x \mid f(x) \neq 0\}$, and this gives a finite cover of $\bar{A}$ such that at each element of the cover it is forced that $f_1 \neq 0 \vee \cdots \vee f_n \neq 0$. Thus $\bar{A} \Vdash f_1 \neq 0 \vee \cdots \vee f_n \neq 0$. But in $\mathscr{S}_{\text{fin}}$, $x \neq 0$ iff x is invertible, just as in $\mathscr{S}$. $\quad\square$

Acknowledgements

References

[1] M. Artin, A. Grothendieck and J.L. Verdier, Théorie de Topos et Cohomologie Étale des Schémas (SGA4), Vol. 1, Lecture Notes in Math. 269 (Springer, Berlin, 1972).

[2] M. Barr and R. Paré, Molecular toposes, J. Pure Appl. Algebra 17 (1980) 127–152.

[3] E.J. Dubuc, C^∞-schemes, Amer. J. Math. 103 (4) (1981).

[4] M.P. Fourman and R.J. Grayson, Formal spaces, in: A.S. Troelstra and D. van Dalen, eds., The L.E.J. Brouwer Centenary Symposium, (North-Holland, Amsterdam, 1982) 107–122.

[5] G. van der Hoeven and I. Moerdijk, Sheaf models for choice sequences, Annals Pure Appl. Logic (1984), to appear.

[6] P.T. Johnstone, Topos Theory (Academic Press, New York, 1977).

[7] P.T. Johnstone, Open maps of toposes, Manuscripta Math. 31 (1980).

[8] A. Joyal and G.E. Reyes, Separable real closed local rings, in: Proc. Vth Symposium of Logic in Latin America (Bogotá, 1981), to appear.

[9] A. Joyal and M. Tierney, An extension of the Galois theory of Grothendieck (1982), preprint.

[10] A. Kock, Synthetic Differential Geometry, London Math. Soc. Lecture Notes Series 51 (Cambridge Univ. Press, Cambridge, 1981).

[11] F.W. Lawvere, Categorical dynamics, topos theoretic methods in geometry, in: A. Kock, ed., Various Publications Series 30 (Matematisk Institut, Aarhus Universitet, 1979).

[12] I. Moerdijk and G.E. Reyes, Cohomology theories in models of differential geometry, in: A. Kock, ed., Category Theoretic Methods in Geometry, Aarhus, 1983.

[13] I. Moerdijk and G.E. Reyes, A smooth version of the Zariski topos, Advances in Math. (1983), to appear.

[14] N. van Quê and G.E. Reyes, Smooth functors and synthetic calculus, in: A.S. Troelstra and D. van Dalen, eds., The L.E.J. Brouwer Centenary Symposium (North-Holland, Amsterdam, 1982) 377–395.

[15] G.E. Reyes, Théorie de modèles et faisceaux, Advances in Math. 30 (2) (1978).

[16] G.E. Reyes, Editor, Géometrie Différentielle Synthétique, Rapports de Recherche 11 (Dept. de Math. et Stat., Université de Montréal, 1980).

[17] G.E. Reyes, Synthetic reasoning and variable sets (Models for synthetic differential geometry), in: Proc. Conference on Categories and Foundations of Continuum Physics (Buffalo, 1982), to appear.

[18] B. Veit, L'application "standard", Unpublished manuscript (1980).