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SMALL CAUCHY COMPLETIONS*

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In this work it is shown that if the underlying category $\mathscr{V}_0$ of a symmetric closed monoidal category $\mathscr{V}$ is locally presentable, then the Cauchy completion of any small $\mathscr{V}$-category is small.

Introduction

It has been observed (e.g. by Kelly in [6]) that for many common monoidal categories $\mathscr{V}$ such as $\mathscr{V}=$ **Set**, **Cat**, $\mathbb{R}^+$, or **AbGp**, the Cauchy completion of a small $\mathscr{V}$-category is always small. Although Kelly gives a counterexample in [6] to show that this is not true for every closed, complete and cocomplete $\mathscr{V}$, it has been conjectured to be true for those $\mathscr{V}$ such that $\mathscr{V}_0$ is locally presentable. In some informal notes Kelly [5] proves this conjecture under the additional assumption that the unit I of $\mathscr{V}$ is projective for strong epis. Here we drop this assumption and prove that the Cauchy completion of a small $\mathscr{V}$-category is always small when the underlying category of $\mathscr{V}$ is locally presentable.

0. Notation

We use $\mathscr{V}$ (or $\mathscr{S}$) to denote a complete, cocomplete, symmetric monoidal closed category. If $\mathscr{A}$ is a small $\mathscr{V}$-category, then $\mathscr{P}\mathscr{A}$ will denote the $\mathscr{V}$-functor category $[\mathscr{A}^{\mathrm{op}},\mathscr{V}]$ which, by [6, Theorem 4.51], is the free cocompletion of $\mathscr{A}$ under small colimits. We let $Y:\mathscr{A}\to\mathscr{P}\mathscr{A}$ denote the Yoneda embedding. If F and G are elements of $\mathscr{P}\mathscr{A}$, then G^F will abbreviate $\mathscr{P}\mathscr{A}(F,G)\in\mathscr{V}$. The identity of F is denoted by $j_F:I\to F^F$. We let K_G denote the canonical morphism: $\operatorname{colim}(G,Y^F)\to\operatorname{colim}(G,Y)^F\cong G^F$. If the underlying category of our base monoidal category is a

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† On 8 December 1988, shortly after submitting this article, and on the verge of the successful completion of his doctoral thesis, Scott Johnson met an untimely death at the age of 28. The parts of his thesis not contained in the present paper or reference [4] are being prepared for publication by Ross Street.

category of presheaves, then we shall denote the base monoidal category by $\mathcal{S}$. Throughout, $\mathcal{A}$ and $\mathcal{B}$ will denote *small* enriched categories.

1. Preliminaries

The equivalence of two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ in the bicategory $\mathcal{V}$-**Mod** (of modules between $\mathcal{V}$-categories as in [9]) is weaker than the equivalence of $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{V}$-**Cat**. This observation has led to the definition of the *Cauchy completion* $\mathcal{Q}\mathcal{A}$ of $\mathcal{A}$ such that $\mathcal{A} \simeq \mathcal{B}$ in $\mathcal{V}$-**Mod** if and only if $\mathcal{Q}\mathcal{A} \simeq \mathcal{Q}\mathcal{B}$ in $\mathcal{V}$-**Cat**. Lawvere [7] indicated a definition (made explicit in a more general context in [9]) of $\mathcal{Q}\mathcal{A}$ as the $\mathcal{V}$-category of modules $\mathcal{I} \to \mathcal{A}$ which possess a right adjoint in $\mathcal{V}$-**Mod**. Alternatively, $\mathcal{Q}\mathcal{A}$ is equivalent to the full subcategory of $\mathcal{P}\mathcal{A} = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ consisting of the *small projectives*: those F such that $\mathcal{P}\mathcal{A}(F,-) = (-)^F : \mathcal{P}\mathcal{A} \to \mathcal{V}$ preserves small colimits (see [6, Section 5.5] or [8]).

The following example from Kelly [6, Section 5.5] shows that $\mathcal{Q}\mathcal{A}$ need not be small when $\mathcal{A}$ is. Let $\mathbf{CL}_0$ be the category of complete lattices with sup-preserving functions and let $\otimes : \mathbf{CL}_0 \times \mathbf{CL}_0 \to \mathbf{CL}_0$ be such that the sup-preserving functions $A \otimes B \to C$ are the functions $A \times B \to C$ which are sup-preserving in each variable separately. This gives a monoidal category $\mathbf{CL}$ with the ordered set $\{0, 1\}$ as unit.

Claim. *The Cauchy completion of a small* $\mathbf{CL}$*-category* $\mathcal{A}$ *is the full subcategory of* $[\mathcal{A}^{\mathrm{op}}, \mathbf{CL}]$ *consisting of those functors which are retracts of arbitrary (small) products (= coproducts) of representables. In particular,* $\mathcal{Q}\mathcal{A}$ *is not small unless* $\mathcal{A}$ *is equivalent to the one-object* $\mathbf{CL}$*-category with* $\mathcal{A}(*,*) = 0$.

Proof. Clearly, the coproduct of $[A_i : i \in I\}$ in $\mathcal{V}_0$ is the same as the product $\prod_{i\in I} A_i$ with coprojection defined by

$$A_i \to \prod_{i\in I} A_i \to A_j,$$
$$a \mapsto \begin{cases} a & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we will denote this coproduct by $\bigoplus_{i\in I} A_i$. For any family $\{a_i : i \in I\}$ of objects of $\mathcal{A}$, any $F : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$, and any $G : \mathcal{K} \to \mathcal{P}\mathcal{A}$ with I and $\mathcal{K}$ small:

$$\mathcal{P}\mathcal{A}\Big(\bigoplus_{i\in I} \mathcal{A}(-, a_i), \mathrm{colim}(F, G)\Big) \cong \bigoplus_{i\in I} \mathrm{colim}(F, Ga_i) \cong \mathrm{colim}\Big(F, \bigoplus_{i\in I} Ga_i\Big)$$
$$\cong \mathrm{colim}\Big(F, \mathcal{P}\mathcal{A}\Big(\bigoplus_{i\in I} \mathcal{A}(-, a_i), G\Big)\Big).$$

Thus arbitrary products of representables, and hence their retracts (by [8, Corollary 3.6]) are small projective and so are in the Cauchy completion of $\mathcal{A}$.

Conversely, if F is small projective, the canonical morphism

$$K_F : \mathrm{colim}(F, Y^F) \to F^F$$

must be an isomorphism. In particular, K_F takes some element of its domain to 1_F. Since $\mathrm{colim}(F, Y^F)$ is a quotient of

$$\bigoplus_{a \in \mathscr{A}} Fa \otimes \mathscr{A}(-,a)^F \cong \bigoplus_{a \in \mathscr{A}} F^{\mathscr{A}(-,a)} \otimes \mathscr{A}(-,a)^F,$$

and since each $A \otimes B$ is itself a quotient of the complete lattice of all subsets of $A \times B$, there is a set I, and an I-indexed collection of pairs of morphisms $\{\langle x_i, y_i \rangle : i \in I\}$ with $x_i : F \to \mathscr{A}(-, a_i)$, and $y_i : \mathscr{A}(-, a_i) \to F$ such that $1_F = \sup_{i \in I}(y_i \circ x_i) : F \to F$. Thus F is a retract of $\bigoplus_{i \in I} \mathscr{A}(-, a_i)$. □

In the above proof, all that was needed for F to be small projective was that K_F map something onto the identity of F. A generalization of this idea to arbitrary $\mathscr{V}$ is given by Gouzou and Grunig [2, Theorem 1.1].

Proposition 1 (Gouzou and Grunig). *For any $\mathscr{V}$, if $F : \mathscr{A}^{\mathrm{op}} \to \mathscr{V}$ then F is small projective if and only if there is a morphism $\varphi : I \to \mathrm{colim}(F, Y^F)$ such that*

$$(*) \qquad \begin{array}{ccc} I & \xrightarrow{\ \varphi\ } & \mathrm{colim}(F, Y^F) \\ & {\scriptstyle j_F}\searrow \quad \swarrow{\scriptstyle K_F} & \\ & F^F. & \end{array}$$

Proof. If F is small projective, we may take φ to be $K_F^{-1} \circ j_F$. So suppose φ satisfies $(*)$. To show that F is small projective, we need only show that $(-)^F$ preserves colimits of the form $\mathrm{colim}(G, Y)$ for $G : \mathscr{A}^{\mathrm{op}} \to \mathscr{V}$ since, for $G : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ and $H : \mathscr{K} \to \mathscr{P}\mathscr{A}$, $\mathrm{colim}(G, H) \cong \mathrm{colim}(\mathrm{colim}(G, H), Y) \cong \mathrm{colim}(G, \mathrm{colim}(H, Y))$. If $G : \mathscr{A}^{\mathrm{op}} \to \mathscr{V}$, then the composite

$$G^F \xrightarrow{\ \cong\ } G^F \otimes I \xrightarrow{\ 1 \otimes \varphi\ } G^F \otimes \mathrm{colim}(F, Y^F) \xrightarrow{\ \mathrm{can.}\ } \mathrm{colim}(G, Y^F),$$

is readily seen, using $(*)$, to be the inverse of the canonical $K_G : \mathrm{colim}(G, Y^F) \to G^F$. □

2. The presheaf case

Throughout this section, we assume that the underlying category of our base monoidal category is the category of presheaves $S^{\mathbb{C}^{\mathrm{op}}}$ for some small category $\mathbb{C}$ (where S is the category of sets). We denote our base category by $\mathscr{S} = (S^{\mathbb{C}^{\mathrm{op}}}, \otimes, I)$.

If X is a set, let $\|X\|$ denote its cardinality. If h is an $\mathrm{Obj}(\mathbb{C})$-graded set, let $\|h\| = \sum_{c \in \mathbb{C}} \|h(c)\|$ and if h is the underlying object function of a functor $H : \mathbb{C}^{\mathrm{op}} \to S$, let

$\|H\| = \|h\|$. If $\mathscr{A}$ is a small $\mathscr{S}$-category and if f is an $\mathrm{Obj}(\mathbb{C}) \times \mathrm{Obj}(\mathscr{A})$-graded set, let $\|f\| = \sum_{a \in \mathscr{A}} \|fa\|$ and if f is the underlying object function of a functor $F : \mathscr{A}^{\mathrm{op}} \to \mathscr{S}$, let $\|F\| = \|f\|$. Finally, let $\|\mathbb{C}\|$ denote the cardinality of the set of arrows of $\mathbb{C}$.

We now fix a small $\mathscr{S}$-category $\mathscr{A}$ and choose a cardinal κ such that

(1) $\|\mathbb{C}\| \leq \kappa$.

(2) $\|I\| \leq \kappa$.

(3) $\|\mathbb{C}(-,c) \otimes \mathbb{C}(-,d)\| \leq \kappa$ for all $c, d \in \mathbb{C}$.

(4) $\|\mathscr{A}(a,b)\| \leq \kappa$ for all $a, b \in \mathscr{A}$ and $\|\mathrm{Obj}(\mathscr{A})\| \leq \kappa$.

Since $\otimes : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ is separately cocontinuous, we have, for $F, G \in \mathscr{S}$;

$$F \otimes G = \int^{c, d \in \mathbb{C}} Fc \times Gd \times (\mathbb{C}(-,c) \otimes \mathbb{C}(-,d))$$

which together with (1) and (3) (and the construction of coends in $S^{\mathbb{C}^{\mathrm{op}}}$) gives:

(5) If $F, G \in \mathscr{S}$ with $\|F\| \leq \kappa$ and $\|G\| \leq \kappa$, then $\|F \otimes G\| \leq \kappa$.

Lemma 2. *Suppose $F : \mathscr{A}^{\mathrm{op}} \to \mathscr{S}$ is a functor and f is a sub* $\mathrm{Obj}(\mathbb{C}) \times \mathrm{Obj}(\mathscr{A})$*-graded set of F with $\|f\| \leq \kappa$. Then there is a subfunctor $[f]$ of F, containing f, such that $\|[f]\| \leq \kappa$.*

Proof. Let $U : (\mathscr{S}^{\mathscr{A}^{\mathrm{op}}})_0 \rightarrowtail S^{\mathrm{Obj}(\mathbb{C}) \times \mathrm{Obj}(\mathscr{A})}$ be the ordinary functor taking an $\mathscr{S}$-functor $F : \mathscr{A}^{\mathrm{op}} \to \mathscr{S}$ to its underlying $\mathrm{Obj}(\mathbb{C}) \times \mathrm{Obj}(\mathscr{A})$-graded set. Then U is a (not necessarily fully faithful) inclusion with left adjoint $L : S^{\mathrm{Obj}(\mathbb{C}) \times \mathrm{Obj}(\mathscr{A})} \to (\mathscr{S}^{\mathscr{A}^{\mathrm{op}}})_0$ given by

$$Lf = \coprod_{\substack{c \in \mathbb{C} \\ a \in \mathscr{A}}} \mathscr{A}(-,a) \otimes \mathbb{C}(-,c) \times fac.$$

If f and F are as in the statement of the lemma, let $J : Lf \to F$ correspond under the adjunction $L \dashv U$ to $f \rightarrowtail UF$ and let $Lf \twoheadrightarrow [f] \rightarrowtail F$ be the epi-mono factorization (calculated pointwise) of J. The natural transformation $Lf \twoheadrightarrow [f]$ corresponds by adjunction to the inclusion $f \rightarrowtail U[f]$. Since $\|f\| \leq \kappa$, (4) and (5) give $\|[f]\| \leq \|(Lf)\| \leq \kappa$. $\square$

Lemma 3. *Suppose $F : \mathscr{A}^{\mathrm{op}} \to \mathscr{S}$ is a functor, $\xi \in \mathscr{S}$ with $\|\xi\| \leq \kappa$, and suppose $T : \xi \to \mathrm{colim}(F, Y^F)$. Then*

$$\begin{array}{ccc} \xi & \xrightarrow{\;T\;} & \mathrm{colim}(F, Y^F) \\ {\scriptstyle \upsilon}\downarrow & & \downarrow{\scriptstyle K_F} \\ G^F & \overset{i^F}{\rightarrowtail} & F^F \end{array}$$

for some natural transformation υ and some inclusion $i : G \rightarrowtail F$ with $\|G\| \leq \kappa$.

Proof. By [6, (3.70)],

$$\begin{aligned}\operatorname{colim}(F, Y^F) &\cong \int^{a\in\mathscr{A}} Fa \otimes \mathscr{A}(-,a)^F \\ &\cong \int^{a\in\mathscr{A}}\int^{d\in\mathbb{C}} Fad \times (\mathbb{C}(-,d)\otimes\mathscr{A}(-,a)^F).\end{aligned}$$

Thus there exist functions $\{t_c\colon c\in\mathbb{C}\}$ such that for all $c\in\mathbb{C}$,

$$\begin{array}{ccc} & \xi c & \\ {}^{t_c}\swarrow & & \searrow^{T_c} \\ \coprod_{\substack{a\in\mathscr{A}\\ d\in\mathbb{C}}} Fad\times(\mathbb{C}(-,d)\otimes\mathscr{A}(-,a)^F)c & \xrightarrow{(e_F)c} & \operatorname{colim}(F, Y^F)c \end{array}$$

where e_F is the canonical natural transformation. Let $f=\{\pi_1(t_c(x))\in F\colon x\in\xi c$ for some $c\in\mathbb{C}\}$ and let $G=[f]:\mathscr{A}^{\mathrm{op}}\to\mathscr{S}$. By Lemma 2 and the assumption $\|\xi\|\le\kappa$, we have $i:G\rightarrowtail F$ and $\|G\|\le\kappa$. For all $c\in\mathbb{C}$,

$$\begin{array}{ccc} \xi c & \xrightarrow{\quad T_c \quad} & \operatorname{colim}(F, Y^F)c \\ {\scriptstyle t_c}\downarrow & & \uparrow{\scriptstyle \operatorname{colim}(i,1)_c} \\ \coprod_{\substack{a\in\mathscr{A}\\ d\in\mathbb{C}}} Gad\times(\mathbb{C}(-,d)\otimes\mathscr{A}(-,a)^F)c & \xrightarrow{(e_G)_c} & \operatorname{colim}(G, Y^F)c. \end{array}$$

Now let υ_c be the composite $\xi c \xrightarrow{(e_G)_c\circ t_c} \operatorname{colim}(G,Y^F)c \xrightarrow{(K_G)_c} G^Fc$. Then for all $c\in\mathbb{C}$,

$$\begin{array}{ccc} \xi c & \xrightarrow{T_c} & \operatorname{colim}(F, Y^F)c \\ {\scriptstyle \upsilon_c}\downarrow & & \downarrow{\scriptstyle (K_F)_c} \\ G^Fc & \overset{i_c^F}{\rightarrowtail} & F^Fc. \end{array}$$

Since each i_c^F is a monomorphism, the naturality of υ follows from the naturality of K_FT. □

In particular, suppose $F:\mathscr{A}^{\mathrm{op}}\to\mathscr{S}$ is small projective. Then by Proposition 1 and Lemma 3 (with $\xi=I$) there is a υ and an $i:G\rightarrowtail F$ with $\|G\|\le\kappa$ such that

$$\begin{array}{ccc} I & & \\ {\scriptstyle \upsilon}\downarrow & \searrow^{j_F} & \\ G^F & \overset{i^F}{\rightarrowtail} & F^F. \end{array}$$

This immediately gives a factorization

$$\begin{array}{ccc} F & & \\ {\scriptstyle \tilde{v}}\downarrow & \searrow^{1_F} & \\ G & \overset{i}{\rightarrowtail} & F \end{array}$$

whence $F \cong G$. That is, for any small projective F, $\|F\| \leq \kappa$. Since $\mathscr{A}$ is small, there is only a small number of non-isomorphic such F and we have

Theorem 4. *If $\mathscr{A}$ is a small $\mathscr{S}$-category, then the Cauchy completion $\mathscr{Q}\mathscr{A}$ of $\mathscr{A}$ is small.* □

3. The locally-presentable case

We will now generalize Theorem 4 from $\mathscr{S}$ to those $\mathscr{V} = (\mathscr{V}_0, \otimes, I, [-,-])$ such that $\mathscr{V}_0$ is locally presentable. For ease of exposition we consider only the case where $\mathscr{V}_0$ is locally *finitely* presentable, the generalization to locally presentable being entirely straightforward. From Gabriel and Ulmer [1] there is, for such a $\mathscr{V}$, a small finitely-cocomplete category $\mathbb{C}$ such that $\mathscr{V}_0 \simeq \mathrm{Lex}(S^{\mathbb{C}^{op}}) =$ the full subcategory of $S^{\mathbb{C}^{op}}$ consisting of the left-exact (or finitely continuous) functors. We will therefore identify $\mathscr{V}_0$ with $\mathrm{Lex}(S^{\mathbb{C}^{op}})$ for the rest of this section.

We let $y : \mathbb{C} \to \mathscr{V}_0$ be the Yoneda embedding seen as landing in $\mathscr{V}_0$ and we let $Y : \mathbb{C} \to S^{\mathbb{C}^{op}}$ denote the usual Yoneda embedding. From [6, Section 5.10], $F : \mathbb{C}^{op} \to S$ is left exact if and only if it is a filtered colimit of representables. Thus, *$\mathscr{V}_0$ is the free filtered-colimit completion of $\mathbb{C}$*. From [1], the inclusion $i : \mathscr{V}_0 \to S^{\mathbb{C}^{op}}$ has a reflection $\sigma : S^{\mathbb{C}^{op}} \to \mathscr{V}_0$.

Theorem 5. *Let $\mathscr{V}$, i and σ be as above. Then*

(i) *There is a unique (up to isomorphism) symmetric closed monoidal structure $\mathscr{S}$ $(=(S^{\mathbb{C}^{op}}, \otimes, I, [-,-]))$ on $S^{\mathbb{C}^{op}}$ such that $i : \mathscr{V}_0 \to S^{\mathbb{C}^{op}}$ has a strong monoidal enrichment $i : \mathscr{V} \to \mathscr{S}$.*

(ii) *The inclusion i preserves the internal homs of $\mathscr{V}$ so that we may view any $\mathscr{V}$-category (respectively $\mathscr{V}$-functor, respectively $\mathscr{V}$-natural transformation) as an $\mathscr{S}$-category (respectively $\mathscr{S}$-functor, respectively $\mathscr{S}$-natural transformation). Since i preserves limits, limits and colimits in a $\mathscr{V}$-category are the same as for the corresponding $\mathscr{S}$-category.*

(iii) *There is a strong monoidal enrichment $(\sigma, \sigma^0, \tilde{\sigma}) : \mathscr{S} \to \mathscr{V}$ of σ. This makes $\mathscr{V}$ a strong monoidal reflective subcategory of $\mathscr{S}$.*

(iv) *There is an isomorphism $[\sigma X, V] \cong [X, V]$ natural in $X \in S^{\mathbb{C}^{op}}$ and $V \in \mathscr{V}_0$.*

(v) *The ordinary functor $\sigma : S^{\mathbb{C}^{op}} \to \mathscr{V}_0$ is the underlying functor of an $\mathscr{S}$-functor $\sigma : \mathscr{S} \to \mathscr{V}$.*

Proof. (i) Since $\otimes : \mathscr{V}_0 \times \mathscr{V}_0 \to \mathscr{V}_0$ is separately cocontinuous and since $i : \mathscr{V}_0 \to S^{\mathbb{C}^{op}}$ preserves filtered colimits, the composite $i\otimes$ preserves filtered colimits separately in both variables. We let S-**Coc**$[S^{\mathbb{C}^{op}} \times S^{\mathbb{C}^{op}}, S^{\mathbb{C}^{op}}]$ denote the full subcategory of $[S^{\mathbb{C}^{op}} \times S^{\mathbb{C}^{op}}, S^{\mathbb{C}^{op}}]$ consisting of the separately cocontinuous functors and we let S-**FilCoc**$[\mathscr{V}_0 \times \mathscr{V}_0, S^{\mathbb{C}^{op}}]$ denote the full subcategory of $[\mathscr{V}_0 \times \mathscr{V}_0, S^{\mathbb{C}^{op}}]$ consisting of the functors which preserve filtered colimits separately in both variables. By a result of Im and Kelly [3], and its generalization in [4] to arbitrary classes of weights for colimits we get:

(a) $\quad [\mathbb{C} \times \mathbb{C}, S^{\mathbb{C}^{op}}] \simeq S\text{-}\mathbf{Coc}[S^{\mathbb{C}^{op}} \times S^{\mathbb{C}^{op}}, S^{\mathbb{C}^{op}}]$,

(b) $\quad [\mathbb{C} \times \mathbb{C}, S^{\mathbb{C}^{op}}] \simeq S\text{-}\mathbf{FilCoc}[\mathscr{V}_0 \times \mathscr{V}_0, S^{\mathbb{C}^{op}}]$.

These equivalences are given, from left to right by left Kan extension along $Y \times Y$ (in (a)) and $y \times y$ (in (b)) and from right to left by restriction along $Y \times Y$ (in (a)) and $y \times y$ (in (b)). Thus, if we first restrict $i\otimes \in S$-**FilCoc**$[\mathscr{V}_0 \times \mathscr{V}_0, S^{\mathbb{C}^{op}}]$ along $y \times y : \mathbb{C} \times \mathbb{C} \to \mathscr{V}_0 \times \mathscr{V}_0$ and then take its left Kan extension along $Y \times Y : \mathbb{C} \times \mathbb{C} \to S^{\mathbb{C}^{op}} \times S^{\mathbb{C}^{op}}$ we get a separately cocontinuous tensor product on $S^{\mathbb{C}^{op}}$ (which we will also denote by $\otimes$). This tensor product is, by [6, Theorem 4.47], the left Kan extension of $i\otimes$ along $i \times i$ and restricts (to within isomorphism) to the tensor product of $\mathscr{V}$.

$$\begin{array}{ccc}
\mathscr{V}_0 \times \mathscr{V}_0 & \xrightarrow{\ i \times i\ } & S^{\mathbb{C}^{op}} \times S^{\mathbb{C}^{op}} \\
{\scriptstyle \otimes}\downarrow & \cong & \downarrow{\scriptstyle \otimes = \mathrm{Lan}_{i \times i}\, i\otimes} \\
\mathscr{V}_0 & \xrightarrow{\quad i \quad} & S^{\mathbb{C}^{op}}.
\end{array}$$

The equivalences (a) and (b), together with their one- and three-dimensional analogues allow us to induce the symmetry, unity and associativity isomorphisms of $\mathscr{V}$ to $S^{\mathbb{C}^{op}}$. Verification that these isomorphisms satisfy the coherence axioms for a monoidal category is an easy exercise which gives a monoidal structure $\mathscr{S} = (S^{\mathbb{C}^{op}}, \otimes, I)$ on $S^{\mathbb{C}^{op}}$. This structure is unique such that $\otimes$ is separately cocontinuous and such that i preserves $\otimes$ and I. Since the tensor product of $\mathscr{S}$ is separately cocontinuous, $\mathscr{S}$ is closed.

(ii) Let $\{-,-\}$ denote the internal-hom functor of $\mathscr{S}$. For $U, V, W \in \mathscr{V}$, $S^{\mathbb{C}^{op}}(W, \{U, V\}) \cong S^{\mathbb{C}^{op}}(W \otimes U, V) \cong \mathscr{V}_0(W \otimes U, V) \cong \mathscr{V}_0(W, [U, V]) \cong S^{\mathbb{C}^{op}}(W, [U, V])$. Since $\mathscr{V}_0$ is dense in $S^{\mathbb{C}^{op}}$, $[U, V] \cong \{U, V\}$, i.e. the strong monoidal inclusion $i : \mathscr{V}_0 \to S^{\mathbb{C}^{op}}$ preserves internal homs. Henceforth, we will let $[-,-]$ denote the internal-hom functor in $\mathscr{S}$ as well as in $\mathscr{V}$.

(iv) For $U, V \in \mathscr{V}$ and $X \in \mathscr{S}$, $S^{\mathbb{C}^{op}}(U, [X, V]) \cong S^{\mathbb{C}^{op}}(X, [U, V]) \cong S^{\mathbb{C}^{op}}(\sigma X, [U, V]) \cong S^{\mathbb{C}^{op}}(U, [\sigma X, V])$. Again, since $\mathscr{V}_0$ is dense in $S^{\mathbb{C}^{op}}$, $[X, V] \cong [\sigma X, V]$.

(iii) For $X, Y \in \mathscr{S}$ and $V \in \mathscr{V}$, $[\sigma(X \otimes Y), V] \cong [X \otimes Y, V] \cong [X, [Y, V]] \cong [\sigma X, [\sigma Y, V]] \cong [\sigma X \otimes \sigma Y, V]$, which gives a natural isomorphism $\tilde{\sigma}_{X,Y} : \sigma X \otimes \sigma Y \cong \sigma(X \otimes Y)$. The counit of the adjunction $\sigma \dashv i$ gives an isomorphism $\sigma^0 : \sigma I \cong I$ and $(\sigma, \sigma^0, \tilde{\sigma}) : \mathscr{S} \to \mathscr{V}$ is a strong monoidal enrichment of σ.

(v) It is easy to check that σ is the underlying functor of an $\mathcal{S}$-functor $\sigma : \mathcal{S} \to \mathcal{V}$ with

$$\begin{array}{ccc} [X,Y] & \xrightarrow{\sigma_{X,Y}} & [\sigma X, \sigma Y] \\ & {\scriptstyle [1,\eta_Y]}\searrow \quad \nearrow{\scriptstyle \cong} & \\ & [X, \sigma Y] & \end{array}$$

where $\eta : 1_{S^{\mathbb{C}^{\mathrm{op}}}} \to i\sigma$ is the unit of the adjunction $\sigma \dashv i$. $\square$

Of course, limits in $\mathcal{V}_0$ are calculated as in $S^{\mathbb{C}^{\mathrm{op}}}$, and any colimit in $\mathcal{V}_0$ is given by taking the reflection of the corresponding colimit in $S^{\mathbb{C}^{\mathrm{op}}}$. We reserve the usual notation for colimits, (including coproducts and coends) for the colimits as calculated in $S^{\mathbb{C}^{\mathrm{op}}}$. We will write $\sigma(\mathrm{colim}(F,G))$ to denote the F-weighted colimit of G as calculated in $\mathcal{V}_0$. From now on we will identify σV with V for $V \in \mathcal{V}_0$ since these are naturally isomorphic.

Letting $\mathrm{Fin}\,\mathbb{C}$ denote the finite-colimit closure of $\mathbb{C}$ in $S^{\mathbb{C}^{\mathrm{op}}}$, we have, by [6, Proposition 5.41] that $S^{\mathbb{C}^{\mathrm{op}}}$ is the free filtered-colimit completion of $\mathrm{Fin}\,\mathbb{C}$. Since $i : \mathcal{V}_0 \to S^{\mathbb{C}^{\mathrm{op}}}$ preserves filtered colimits, $i\sigma : S^{\mathbb{C}^{\mathrm{op}}} \to S^{\mathbb{C}^{\mathrm{op}}}$ is the left Kan extension of its restriction to $\mathrm{Fin}\,\mathbb{C}$.

$$\begin{array}{ccc} \mathrm{Fin}\,\mathbb{C} & \hookrightarrow & S^{\mathbb{C}^{\mathrm{op}}} \\ \searrow & \cong \quad \swarrow{\scriptstyle i\sigma} & \downarrow{\scriptstyle \sigma} \\ S^{\mathbb{C}^{\mathrm{op}}} & \underset{i}{\hookleftarrow} & \mathcal{V}_0 . \end{array}$$

Thus $\sigma(G) = \int^{\xi \in \mathrm{Fin}\,\mathbb{C}} \sigma(\xi) \times S^{\mathbb{C}^{\mathrm{op}}}(\xi, G)$. Since $y : \mathbb{C} \to \mathcal{V}_0$ preserves finite colimits, $\sigma(\xi) = \mathbb{C}(-, \mathrm{colim}(\xi, 1_{\mathbb{C}}))$ for $\xi \in \mathrm{Fin}\,\mathbb{C}$.

Theorem 6. *If $\mathcal{V}_0$ is locally (finitely) presentable and if $\mathcal{A}$ is a small $\mathcal{V}$-category, then the Cauchy completion $\mathcal{Q}\mathcal{A}$ of $\mathcal{A}$ is also small.*

Proof. Let $\mathcal{A}$ be a small $\mathcal{V}$-category and let κ be as in Section 2. If $F, G : \mathcal{A}^{\mathrm{op}} \to \mathcal{S}$, G^F will denote $[\mathcal{A}^{\mathrm{op}}, \mathcal{S}](F,G) = \int_{a \in \mathcal{A}} [Fa, Ga] \in \mathcal{S}$ which is isomorphic to $[\mathcal{A}^{\mathrm{op}}; \mathcal{V}](F,G)$ if F and G land in $\mathcal{V}$ since limits and internal homs in $\mathcal{V}$ are preserved by the inclusion $i : \mathcal{V} \to \mathcal{S}$. Note that any $\mathcal{S}$-functor $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is a $\mathcal{V}$-functor.

By Proposition 1, if $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is small projective, then there is a morphism φ in $\mathcal{V}$ such that

$$\begin{array}{ccc} I & \xrightarrow{\varphi} & \sigma(\mathrm{colim}(F, Y^F)) \cong \displaystyle\int^{\xi \in \mathrm{Fin}\,\mathbb{C}} \sigma(\xi) \times S^{\mathbb{C}^{\mathrm{op}}}(\xi, \mathrm{colim}(F, Y^F)). \\ & {\scriptstyle j_F}\searrow \quad \swarrow{\scriptstyle \sigma(K_F)} & \\ & F^F & \end{array}$$

For each $c \in \mathbb{C}$ we can assign to each $h \in Ic$ a triple $\xi_h \in \operatorname{Fin}\mathbb{C}$, $g_c(h) \in \sigma(\xi_h)c$ and $T_h : \xi_h \to \operatorname{colim}(F, Y^F)$ (representing the value $\varphi_c(h)$) such that there is a commutative diagram of *functions* (note that g may not be a natural transformation)

$$\begin{array}{ccc} & Ic & \\ {}^{g_c}\swarrow & & \searrow^{\varphi_c} \\ \coprod_{h \in Ic} \sigma(\xi_h)c & \xrightarrow{(\sigma(T_h)_c)_h} & \sigma(\operatorname{colim}(F, Y^F))c. \end{array}$$

Clearly $\|\xi\| \leq \kappa$ for all $\xi \in \operatorname{Fin}\mathbb{C}$. By Lemma 3 there is, for each $c \in \mathbb{C}$ and $h \in Ic$, a functor $G_h : \mathscr{A}^{\mathrm{op}} \to \mathscr{S}$ with $\|G_h\| \leq \kappa$, an inclusion $i_h : G_h \rightarrowtail F$ and a morphism υ_h such that

$$\begin{array}{ccc} \xi_h & \xrightarrow{T_h} & \operatorname{colim}(F, Y^F) \\ {\scriptstyle \upsilon_h}\downarrow & & \downarrow{\scriptstyle K_F} \\ G_h^F & \overset{i_h^F}{\rightarrowtail} & F^F. \end{array}$$

Hence, for each $c \in \mathbb{C}$

$$\begin{array}{ccc} & Ic & \\ {}^{g_c}\swarrow & & \searrow^{\varphi_c} \\ \coprod_{h \in Ic} \sigma(\xi_h)c & \xrightarrow{(\sigma(T_h)_c)_h} & \sigma(\operatorname{colim}(F, Y^F))c \\ {\scriptstyle \coprod_{h \in Ic} \sigma(\upsilon_h)_c}\downarrow & & \downarrow{\scriptstyle \sigma(K_F)_c} \\ \coprod_{h \in Ic} \sigma(G_h^F)c & \xrightarrow{(\sigma(i_h^F)_c)_h} & F^F c. \end{array}$$

Since I and $\mathbb{C}$ are bounded by κ there is a subfunctor $G_0 : \mathscr{A}^{\mathrm{op}} \to \mathscr{S}$ of F with $\|G_0\| \leq \kappa$ which contains each G_h for $h \in Ic$, $c \in \mathbb{C}$. We have inclusions

$$\begin{array}{ccc} G_h & \overset{i_h}{\rightarrowtail} & F \\ {}_{i_h'}\searrow & & \nearrow_{i_0} \\ & G_0 & \end{array}$$

for $h \in Ic$, $c \in \mathbb{C}$. Since σ is an $\mathscr{S}$-functor, the composite $\sigma \circ G_0 : \mathscr{A}^{\mathrm{op}} \to \mathscr{V}$ is an $\mathscr{S}$-functor between two $\mathscr{V}$-categories and is therefore a $\mathscr{V}$-functor. In the ordinary category $[\mathscr{A}^{\mathrm{op}}, \mathscr{V}]_0$ let $\sigma \circ G_0 \overset{e}{\twoheadrightarrow} G \overset{m}{\rightarrowtail} F$ be a strong epi-mono factorization of $\sigma \circ i_0 : \sigma \circ G_0 \to F$. This exists since $\mathscr{A}(A, B) \otimes - : \mathscr{V}_0 \to \mathscr{V}_0$ preserves strong epimorphisms. Then

$$\begin{array}{ccc}
\sigma(G_h^F) & \xrightarrow{\quad\sigma(i_h^F)\quad} & F^F \\
{\scriptstyle \sigma(i_h'^F)}\downarrow & \nearrow{\scriptstyle \sigma(i_0^F)} & \\
\sigma(G_0^F) & \nearrow{\scriptstyle (\sigma i_0)^F} & \uparrow{\scriptstyle m^F} \\
{\scriptstyle \sigma((\eta G_0)^F)}\searrow & & \\
 & (\sigma\circ G_0)^F \xrightarrow{\quad e^F\quad} & G^F.
\end{array}$$

Now let μ_c be the composite

$$Ic \xrightarrow{\quad g_c \quad} \coprod_{h\in Ic} \sigma(\xi_h)c \xrightarrow{\quad \coprod_{h\in Ic}\sigma(\upsilon_h)_c \quad} \coprod_{h\in Ic} \sigma(G_h^F)c \longrightarrow G^Fc$$

where the last arrow is derived from the composite of the three lower arrows in the previous diagram. Then we have (since $i:\mathscr{V}_0\to S^{\mathbb{C}^{\mathrm{op}}}$ preserves monomorphisms)

$$\begin{array}{ccc}
Ic & \xrightarrow{\quad \varphi_c \quad} & \sigma(\mathrm{colim}(F, Y^F))c \\
{\scriptstyle \mu_c}\downarrow & & \downarrow{\scriptstyle \sigma(K_F)_c} \\
G^Fc & \rightarrowtail{\scriptstyle (m^F)_c} & F^Fc
\end{array}$$

and the naturality of μ follows from that of $\sigma(K_F)\varphi$. Hence, as in the presheaf case, $F\cong G$. Since $\|G_0\|\leq\kappa$ and since, by [1], any object of $\mathscr{V}$ has only a small number of quotients, there can only be a small number of such F. □

In [9], Street defines the Cauchy completion $\mathscr{Q}\mathscr{A}$ of $\mathscr{A}$ where $\mathscr{A}$ is a small category enriched over a bicategory $\mathscr{W}$ such that $\mathscr{W}$ and $\mathscr{W}^{\mathrm{op}}$ admit right liftings. Suppose $\mathscr{W}(U, V)$ is locally representable for all objects U and V of $\mathscr{W}$. Then the proof here can be modified to show that for small $\mathscr{A}$, the set of objects in $\mathscr{Q}\mathscr{A}$ over any given object U of $\mathscr{W}$ is small. In particular, if Obj($\mathscr{W}$) is small, then so is $\mathscr{Q}\mathscr{A}$.

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