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SIMPLICIAL COHOMOLOGY IS HOCHSCHILD COHOMOLOGY

Murray GERSTENHABER
University of Pennsylvania, Philadelphia, PA 19104, USA

Samuel D. SCHACK
State University of New York at Buffalo, Buffalo, NY 14214, USA

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0. Introduction

The central result of this paper is the following.

Theorem (SC = HC). *Let k be a commutative unital ring. There is a functor, $\Sigma \mapsto \mathbf{k}_\Sigma!$, from Locally Finite Simplicial Complexes to Associative Unital k-Algebras inducing an isomorphism*

$$H^*(\Sigma, k) \cong H^*(\mathbf{k}_\Sigma!, \mathbf{k}_\Sigma!)$$

between the simplicial cohomology of Σ and the Hochschild cohomology of $\mathbf{k}_\Sigma!$. The isomorphism preserves the cup product. □

The ring $\mathbf{k}_\Sigma!$ has a tidy description: for simplices $\sigma, \tau \subset \Sigma$ write $\sigma \geq \tau$ when σ is incident to τ, (i.e. $\sigma \subset \tau$). An element of $\mathbf{k}_\Sigma!$ is a formal (possibly infinite) sum $\sum_{\sigma \geq \tau} m_{\sigma,\tau}(\sigma, \tau)$ where $m_{\sigma,\tau} \in k$. The multiplication is determined by associativity, k-bilinearity, and the rule: $(\sigma,\tau)(\sigma',\tau') = 0$ if $\tau \neq \sigma'$ and $(\sigma,\tau)(\tau,\tau') = (\sigma,\tau')$. Note that $\mathbf{k}_\Sigma!$ is commutative if and only if Σ has dimension zero.

The proof of SC = HC requires the development of an intermediate theory of diagrams and their cohomology (Sections 2,4). The Cohomology Comparison Theorem (CCT), quoted in Section 5, then establishes an isomorphism between diagram and Hochschild cohomologies. Meanwhile there is an isomorphism between certain simplicial and diagram cohomologies (Section 4). These combine to yield the first statement of SC = HC (Section 6). For the second we realize the isomorphism by an explicit cochain map $\tau_\Sigma^* : C^*(\Sigma, k) \to C^*(\mathbf{k}_\Sigma!, \mathbf{k}_\Sigma!)$, (Sections 5,6), and show that it preserves the cup product (Section 6). Unfortunately τ_Σ^* is not a substitute for the CCT: we are unable to show by direct calculation that $H^*(\tau_\Sigma)$ is an isomorphism.

The Hochschild cohomology of any associative algebra carries a (possibly non-abelian) graded Lie product (Section 3); yet none has previously been noted on $H^*(\Sigma, k)$. The reason is simple (Section 6): when the algebra is $\mathbf{k}_\Sigma!$ the product is abelian. The proof requires τ_Σ^* and the introduction of Steenrod's $\cup_i$ (Section 4). The latter also propel us along a small detour in Sections 4,5 to take care of some unfinished business from [3]: the definition of cohomology operations on diagram cohomology.

The utility of diagrams in the simplicial setting is not limited to SC = HC. They naturally lend themselves to the description of some 'generalized simplicial cohomology theories' (Section 7). These theories satisfy the usual Eilenberg–Steenrod axioms with the exception of the dimension axiom.

A detailed account of the basic foundational work on diagrams and their cohomology – as well as the CCT – is presented in our earlier paper, "On the deformation of algebra morphisms and diagrams" [3]. However, faith in the cited theorems is an adequate substitute for that paper since we have repeated all relevant definitions here.

This paper is not the first attempt to recognize $H^*(\Sigma, k)$ as an algebraic cohomology theory. Clarke [1] developed a cohomology theory for semigroups with zero along the pattern of Hochschild cohomology. He then showed that, when Σ is finite, $C^*(\Sigma, k)$ is isomorphic to his $C^*(S, k)$, where $S = \{0\} \cup \{(\sigma, \tau)\} \subseteq \mathbf{k}_\Sigma!$. In fact, it is immediate from Clarke's definitions that $C^*(S, k) = \operatorname{im} \tau_\Sigma^*$, which we have elsewhere termed the *strict* cochains of $\mathbf{k}_\Sigma!$. However, the force of the CCT is then required to bridge the gap between the strict (= semigroup) and Hochschild cohomologies.

In a different vein, Watts [9] showed that $H^*(\Sigma, k)$ is given by derived functors. Specifically he showed it isomorphic to a Yoneda cohomology calculated in the category of *left* modules over a diagram. That Yoneda cohomology is, in turn, trivially seen to be just $\operatorname{Ext}^*(\mathbf{k}_\Sigma!, \mathbf{k}_\Sigma!)$, calculated in the category of *left*-$\mathbf{k}_\Sigma!$-modules. Unfortunately this does not give Hochschild cohomology; for that one must start with bimodules (over the same diagram) and then use the CCT, which is nontrivial. However, Watt's approach yields two bounties which ours has not (so far):

(1) It applies when Σ is a semi-simplicial complex.

(2) It can be used to show that $H_*(\Sigma, k)$ is given by derived functors.

We shall adhere to certain conventions throughout this paper: To begin, we fix a commutative associative unital ring k. A *k-algebra* is an associative unital ring A equipped with a unital ring morphism $f: k \to A$, (the *structure map*), whose image is contained in the center of A. We suppress f and write a for $f(a)$, even though f need not be a monomorphism. Of course, a k-algebra morphism is just a unital ring morphism $\phi: A \to \Gamma$ which commutes with the structure maps of A and Γ. The category of k-algebras will be denoted by ALG; also $\mathrm{ALG}^{\mathscr{C}}$ will denote the category of contravariant functors $\mathscr{C} \to \mathrm{ALG}$. (We have removed the traditional 'op'.) A two-sided A-module M is a *A-bimodule* if $am = ma$ for all $a \in k$, $m \in M$; (i.e., M is a *symmetric* k-module). [Better, but nonstandard, terminology would be: M is a

$(k \to \Lambda)$-bimodule.] Finally, note that if $\phi : \Lambda \to \Gamma$ is a k-algebra morphism and M is a Γ-bimodule, then M becomes a Λ-bimodule through ϕ: define λm and $m\lambda$ to be $\phi(\lambda)m$ and $m\phi(\lambda)$.

1. Posets and simplicial complexes

We shall employ two familiar functors: one, Σ, embeds the category of partially ordered sets (posets) in the category of simplicial complexes; the other, I, does just the reverse. These are not inverses; indeed $\Sigma I(\Sigma)$ is the barycentric subdivision of Σ. The definitions are:

The *geometric realization* of a poset I is the simplicial complex $\Sigma(I)$ in which $\Sigma(I)_p$ = the p-simplices = $\{\sigma = (i_p < \cdots < i_0)\}$. The rth face of σ is $\sigma_r = (i_p < \cdots < \hat{i}_r < \cdots < i_0)$ and the boundary is $\partial\sigma = \sum_{r=p}^{0} (-1)^{p-r}\sigma_r$. An order-preserving map $I \to J$ clearly induces a simplical map $\Sigma(I) \to \Sigma(J)$.

Next, the simplices of a simplicial complex Σ are partially ordered by the incidence relation and, so, form a poset $I(\Sigma)$. [The incidence relation is the transitive, reflexive closure of the face relation: $\sigma > \tau \Leftrightarrow \sigma$ is a face of τ.] A simplicial map $\Sigma \to \tilde{\Sigma}$ induces an order-preserving map $I(\Sigma) \to I(\tilde{\Sigma})$ whose image is a *filter*. [$\mathcal{F} \subset I$ is a filter if $i > j$, $j \in \mathcal{F} \Rightarrow i \in \mathcal{F}$.] This will be important in Section 7.

Note, for use in later sections, that a poset may be viewed as the object set of a small category. For every order relation $i \leq j$ there is a unique map, denoted ij, from i to j. Note that $\mathrm{id}_i = (i, i)$ and $(j, k)(i, j) = (i, k)$.

2. Diagrams and modules

A *diagram* (of k-algebras) over a small category $\mathscr{C}$ is a contravariant functor $\mathbb{A} : \mathscr{C} \to \mathrm{ALG}$, (a 'presheaf'); for $i \in \mathrm{ob}(\mathscr{C})$ and $w \in \mathrm{map}(\mathscr{C})$ we abbreviate $\mathbb{A}(i)$ and $\mathbb{A}(w)$ to $\mathbb{A}^i$ and ϕ^w $(=\phi^w_{\mathbb{A}})$. A diagram over the one point category $*$ is just a k-algebra; i.e. $\mathrm{ALG}^* \cong \mathrm{ALG}$. Now any functor $p : \mathscr{C} \to \mathscr{D}$ induces a functor $\tilde{p} : \mathrm{ALG}^{\mathscr{D}} \to \mathrm{ALG}^{\mathscr{C}}$. In particular, the unique $\mathscr{C} \to *$ induces a functor $\mathrm{ALG} \to \mathrm{ALG}^{\mathscr{C}}$ whose image consists of the *constant diagrams*. We represent the image of k by $\mathbf{k}_{\mathscr{C}}$ or simply $\mathbf{k}$; so $\mathbf{k}^i = k$ and $\phi^w_{\mathbf{k}} = \mathrm{id}$ for all i and w. When $\mathscr{C} = I(\Sigma)$ we shall write $\mathbf{k}_{\Sigma}$ for $\mathbf{k}_{\mathscr{C}}$.

Suppose that $\mathbb{A}$ is a diagram over $\mathscr{C}$ and $\mathbb{B}$ is a diagram over $\mathscr{D}$. A morphism $\mathbb{A} \to \mathbb{B}$ consists of a functor $p : \mathscr{C} \to \mathscr{D}$ together with a natural transformation $\mathbb{A} \to \tilde{p}\mathbb{B}$ in $\mathrm{ALG}^{\mathscr{C}}$. (*N.B.* We may have $\mathscr{C} = \mathscr{D}$, $p \neq \mathrm{id}$.) In this way diagrams form a category, DIAG.

A *bimodule* over a diagram $\mathbb{A} \in \mathrm{ALG}^{\mathscr{C}}$ is a contravariant functor $\mathbb{M} : \mathscr{C} \to$ Abelian Groups satisfying:

(1) Each M^i is an $\mathbb{A}^i$-bimodule.

(2) Each $T^w_{\mathbb{M}} : \mathbb{M}^j \to \mathbb{M}^i$ is an $\mathbb{A}^j$-bimodule morphism where $\mathbb{M}^i$ is an $\mathbb{A}^j$-bimodule through $\phi^w : \mathbb{A}^j \to \mathbb{A}^i$.

The $\mathbb{A}$-bimodules form an abelian category which is bicomplete and has enough projectives and enough injectives.

3. Review of Hochschild cohomology

Let Λ be a k-algebra and M, a Λ-bimodule. Set

$$C^q(\Lambda,M)=\{f:\prod^q \Lambda \to M \mid \text{(i) } f \text{ is } k\text{-multilinear and (ii) } f(x_q,\ldots,x_1)=0 \text{ whenever any } x_i=1\}.$$

(Condition (ii) is called *normality;* purists may wish to omit it, thereby obtaining 'the unnormalized Hochschild cochains'. Doing so will not affect the cohomology [3, §7].) Note that $C^0(k,M)=M$ and $C^{q>0}(k,M)=0$. Define $\delta: C^q(\Lambda,M)\to C^{q+1}(\Lambda,M)$ by

$$\delta f(\lambda_{q+1},\ldots,\lambda_1)=\lambda_{q+1}f(\lambda_q,\ldots,\lambda_1)+\sum_{i=q+1}^{2}(-1)^{q-i}f(\ldots,\lambda_i\lambda_{i-1},\ldots)$$
$$+(-1)^{q-1}f(\lambda_{q+1},\ldots,\lambda_2)\lambda_1.$$

The cohomology of the complex $C^*(\Lambda,M)$ is denoted $H^*(\Lambda,M)$ and is called the *Hochschild Cohomology* [4]. Of course, $H^*(\Lambda,-)$ is a functor, but it need not be a δ-functor (unless k is a field). That is, a short exact sequence (E) does not generally induce the long exact sequence of cohomology. However, it will if (E) is *allowable,* i.e. (E) splits when viewed as an exact sequence of k-modules. Such allowable exact sequences serve as the building blocks of a *relative* Yoneda extensions theory which is denoted $\mathrm{Ext}^*_\Lambda(-,-)$ [5, Chaps. IX,XII]. Then $H^*(\Lambda,-)\cong \mathrm{Ext}^*_\Lambda(\Lambda,-)$ [5, Chap. X].

In [2] the first author established that $H^*(\Lambda,\Lambda)$ carries an associative graded commutative cup product *and* a graded Lie product which acts as graded derivations on the cup product. [The Lie product can be non-trivial, but not when Λ arises from a simplicial complex as $\mathbf{k}_\Sigma$! (Section 6).]

These products are most naturally defined at the cochain level, as is at least one other product which, however, is *not* inherited by the cohomology. We sketch the details:

The *cup product* of $f^p\in C^p(\Lambda,\Lambda)$ and $g^q\in C^q(\Lambda,\Lambda)$ is the cochain $f^p\cup g^q\in C^{p+q}(\Lambda,\Lambda)$ given by

$$f^p\cup g^q(\lambda_{p+q},\ldots,\lambda_1)=f(\lambda_{p+q},\ldots,\lambda_{q+1})g(\lambda_q,\ldots,\lambda_1). \tag{3.1}$$

One readily shows:

$$\delta(f^p\cup g^q)=\delta f\cup g+(-1)^p f\cup \delta g. \tag{3.2}$$

So the cup product is well-defined on the Hochschild cohomology.

The cup product of cochains is not graded commutative, i.e. in general, $f^p\cup g^q\neq(-1)^{pq}g^q\cup f^p$. However, at the cohomology level this failure is rectified: define the *composition product* $f^p \bar{\circ}\, g^q\in C^{p+q-1}(\Lambda,\Lambda)$ by

$$f^p \bar{\circ} g^q(\lambda_{p+q-1},\dots,\lambda_1) = \sum_{i=1}^{p} (-1)^{(p-i)(q-1)} f(\dots,\lambda_{i+q}, q(\lambda_{i+q-1},\dots,\lambda_i),\lambda_{i-1},\dots). \tag{3.3}$$

Then

$$\delta(f^p \bar{\circ} g^q) = f \bar{\circ} \delta g + (-1)^{q-1}\delta f \bar{\circ} g + (-1)^q\{g \cup f - (-1)^{pq} f \cup g\}. \tag{3.4}$$

This equation demonstrates both the commutativity of $\cup$ at the cohomology level and the lack thereof at the cochain level.

To view $\bar{\circ}$ as a graded product define the *degree* of f^p to be $p-1$. As (3.4) shows, $\bar{\circ}$ is *not* well-defined on cohomology. However, its graded commutator is. Define $[f^p, g^q] \in C^{p+q-1}(\Lambda,\Lambda)$ by

$$[f^p, g^q] = f \bar{\circ} g - (-1)^{(p-1)(q-1)} g \bar{\circ} f. \tag{3.5}$$

Then (3.4) shows this is well-defined at the cohomology level. Moreover, calculation reveals that it is a graded Lie product, i.e.

$$[f^p, g^q] = -(-1)^{(p-1)(q-1)}[g,f] \tag{3.6}$$

and

$$(-1)^{(p-1)(r-1)}[[f^p, g^q], h^r] + (-1)^{(q-1)(p-1)}[[g^q, h^r], f^p] + (-1)^{(r-1)(q-1)}[[h^r, f^p], g^q] = 0. \tag{3.7}$$

Finally, let M be a Λ-bimodule. Observe that if $f^p \in C^p(\Lambda, M)$ while $g^q \in C^q(\Lambda,\Lambda)$, then (3.1) and (3.3) define cochains $f \cup g$, $g \cup f \in C^{p+q}(\Lambda, M)$ and $f \bar{\circ} g \in C^{p+q-1}(\Lambda, M)$. However, $g \bar{\circ} f$ and $[f,g]$ are meaningless.

4. Diagram, simplicial, and Yoneda cohomologies

Some of the results of this section can be formulated over an arbitrary small category. Nonetheless, for simplicity we restrict the setting of this section and consider only posets as base categories.

Fix a poset I. If $\sigma \in \Sigma(I)_p$ we shall write $(-1)^\sigma$ for $(-1)^p$. Now let $\mathbb{A}$ be a diagram over I and let $\mathbb{M}$ be an $\mathbb{A}$-bimodule. (So if $i \le j$ there is a ring map $\phi^{ij}: \mathbb{A}^j \to \mathbb{A}^i$ and an $\mathbb{A}^j$-bimodule map $T^{ij}: \mathbb{M}^j \to \mathbb{M}^i$.) The diagram cochains appear as the total complex of a double complex. Specifically, set $C^q = \prod_{i \le j} C^q(\mathbb{A}^j, \mathbb{M}^i)$ and

$$C^{q,p}(\mathbb{A},\mathbb{M}) = \{\Gamma: \Sigma(I)_p \to C^q \mid \Gamma^\sigma \in C^q(\mathbb{A}^{i_0}, \mathbb{M}^{i_p}) \text{ where } \sigma = (i_p < \cdots < i_0)\}.$$

(*Note.* We write Γ^σ for $\Gamma(\sigma)$.)

Define two anti-commuting coboundaries $\delta_h: C^{q,p} \to C^{q+1,p}$ and $\delta_s: C^{q,p} \to C^{q,p+1}$ as follows:

$$(\delta_h \Gamma)^\sigma = (-1)^\sigma \delta \Gamma^\sigma \qquad \text{(Hochschild coboundary)}$$

and, for $\sigma = (i_{p+1} < \cdots < i_0) \in \Sigma(I)_{p+1}$,

$$(\delta_s \Gamma)^\sigma = \Gamma^{\delta\sigma} = T^{i_{p+1}i_p}\Gamma^{\sigma_{p+1}} - \Gamma^{\sigma_p} + \Gamma^{\sigma_{p-1}} - \cdots + (-1)^{p+1}\Gamma^{\sigma_0}\phi^{i_1 i_0}.$$

Then $C^*(\mathbb{A}, \mathbb{M})$ is given by

$$C^n(\mathbb{A}, \mathbb{M}) = \coprod_{q+p=n} C^{q,p}(\mathbb{A}, \mathbb{M})$$

and $\delta = \delta_h + \delta_s$. (A comment for the purists: if diagrams cochains are constructed starting with unnormalized Hochschild cochains the cohomology is not affected [3, §7].)

Theorem. $H^*(\Sigma(I), k) \cong H^*(\mathbf{k}_I, \mathbf{k}_I)$.

Proof. Observe that, for any $\mathbb{M}$, $C^{q>0,p}(\mathbf{k}_I, \mathbb{M}) = 0$ since $C^{q>0}(k, M) = 0$. Hence $C^n(\mathbf{k}_I, \mathbb{M}) = C^{0,n}(\mathbf{k}_I, \mathbb{M})$ and $\delta_h = 0$. When $\mathbb{M} = \mathbf{k}_I$ define $i^* : C^*(\Sigma(I), k) \to C^*(\mathbf{k}_I, \mathbf{k}_I)$ by

$$\begin{aligned}(if^n)^\sigma &= f(\sigma) \quad \text{if } \sigma \in \Sigma(I)_n, \\ &= 0 \qquad \text{otherwise.}\end{aligned} \tag{4.1}$$

It is clear that i^* is a k-module isomorphism. That it is a cochain map follows immediately from the definition of δ_s and the fact that $T^{ij}_{\mathbf{k}} = \phi^{ij}_{\mathbf{k}} = \mathrm{id}_{\mathbf{k}}$ for all $i \le j \in I$. □

When $I = I(\Sigma)$ for a simplicial complex Σ this isomorphism asserts that $H^*(\mathbf{k}_\Sigma, \mathbf{k}_\Sigma)$ is the simplicial cohomology of Σ', the barycentric subdivision of Σ. Of course, in the most important cases the latter is the same as the simplicial cohomology of Σ, *e.g.* whenever Σ is locally finite. Observe further that a local coefficient system on Σ' is just a $\mathbf{k}_\Sigma$-bimodule $\mathbb{M}$ and, following the proof above, classical local cohomologies appear as $H^*(\mathbf{k}_\Sigma, \mathbb{M})$.

As in Section 3, $H^*(\mathbb{A}, -)$ is a δ-functor only relative to a special class of allowable short exact sequences. These, in turn, are the foundation of a relative Yoneda extension theory, $\mathrm{Ext}^*_{\mathbb{A}}(-,-)$ for $\mathbb{A}$-bimodules. As before (but less trivially) there is an isomorphism $H^*(\mathbb{A}, -) \cong \mathrm{Ext}^*_{\mathbb{A}}(\mathbb{A}, -)$. [$0 \to \mathbb{M}_1 \to \mathbb{M}_2 \to \mathbb{M}_3 \to 0$ is allowable if and only if every $0 \to \mathbb{M}^i_1 \to \mathbb{M}^i_2 \to \mathbb{M}^i_3 \to 0$ splits when viewed as a sequence of k-bimodules.]

Of course, we expect there to be cup and Lie products defined on $H^*(\mathbb{A}, \mathbb{A})$. Their description is intricate, involving – as one might anticipate – not only $\cup$ and $\bar{\circ}$ for $\{C^*(\mathbb{A}^j, \mathbb{A}^i)\}$ but also Steenrod's $\cup_i$ for $\Sigma = \Sigma(I)$ [8]. We begin with the latter, giving explicit definitions only for $\cup_0$ ($=\cup$) and $\cup_1$.

Suppose $\sigma' \in \Sigma_p$ and $\sigma'' \in \Sigma_q$, say $\sigma' = (i_p < \cdots < i_0)$ and $\sigma'' = (j_q < \cdots < j_0)$. Then if $i_0 = j_q$ define $\sigma' \cup \sigma'' \in \Sigma_{p+q}$ by

$$\sigma' \cup \sigma'' = (i_p < \cdots < i_0 = j_q < \cdots < j_0). \tag{4.2}$$

If, for some r we have $i_r = j_q$ and $i_{r-1} = j_0$ define a $(p+q-1)$-chain, $\sigma' \cup_1 \sigma''$, by

$$\sigma' \cup_1 \sigma'' = (-1)^{r(q-1)}(i_p < \cdots < i_r = j_q < \cdots < j_0 = i_{r-1} < \cdots < i_0). \tag{4.3}$$

Now for $\Gamma \in C^m(\mathbb{A}, \mathbb{M})$ and $\Delta \in C^n(\mathbb{A}, \mathbb{A})$ we define $\Gamma \cup \Delta \in C^{m+n}(\mathbb{A}, \mathbb{M})$ and $\Gamma \bar{\circ} \Delta \in C^{m+n-1}(\mathbb{A}, \mathbb{M})$ by

$$(\Gamma \cup \Delta)^\sigma = \sum_{\sigma = \sigma' \cup \sigma''} (-1)^{\sigma''(m-\sigma')} \Gamma^{\sigma'} \cup \Delta^{\sigma''} \tag{4.4}$$

and

$$(\Gamma \bar{\circ} \Delta)^\sigma = \sum_{\sigma = \sigma' \cup \sigma''} (-1)^{\sigma'} \Gamma^{\sigma'} \bar{\circ} \Delta^{\sigma''} - (-1)^{\sigma} \sum_{\sigma = \pm \sigma' \cup_i \sigma''} \pm \Delta^{\sigma''} \cup \Gamma^{\sigma'}, \tag{4.5}$$

(the $\pm$ is dictated by (4.3).) Naturally, when $\mathbb{M} = \mathbb{A}$ we define $[\Gamma, \Delta] \in C^{m+n-1}(\mathbb{A}, \mathbb{A})$ by

$$[\Gamma, \Delta] = \Gamma \bar{\circ} \Delta - (-1)^{(m-1)(n-1)} \Delta \bar{\circ} \Gamma. \tag{4.6}$$

Formulae (4.4–4.6) appear somewhat *ad hoc* here. In the next section we shall, among other things, show how they arise and prove that they are 'correct' when $\Sigma(I)$ is locally finite. For the movement we direct two further comments to the reader familiar with [3]:

(1) If $\Gamma \in Z^2(\mathbb{A}, \mathbb{A})$ is an infinitesimal, then the primary obstruction to integrating Γ to a deformation of $\mathbb{A}$ is just $\Gamma \bar{\circ} \Gamma$.

(2) The formulae in [3] for the case $I = \{0 < 1\}$ agree with (4.4–4.6).

5. The diagram ring and the CCT

We construct an embedding $!: \mathrm{DIAG} \to \mathrm{ALG}$, $\mathbb{A} \mapsto \mathbb{A}!$, and 'compatible' exact embeddings $!: \mathbb{A}$-Bimodules $\to \mathbb{A}!$-Bimodules, $\mathbb{M} \mapsto \mathbb{M}!$. In many cases the latter functors induce cohomology isomorphisms, even though they preserve neither projectives not injectives.

Let $\mathbb{A}$ be a diagram over $\mathscr{C}$. Its *diagram ring*, $\mathbb{A}!$, is defined as follows:

For every $w: i \to j$ in $\mathscr{C}$ let $\mathbb{A}^i \phi^w$ denote a copy of $\mathbb{A}^i$ indexed by w. We write a general element of $\mathbb{A}^i \phi^w$ as $a^i \phi^w$, but abbreviate $1\phi^w$ to ϕ^w. Set $\mathbb{A}! = \prod_i \coprod_w \mathbb{A}^i \phi^w$ with multiplication given by (infinite) linearity and

$$(a^i \phi^w)(a^j \phi^v) = \begin{cases} a^i \phi^w (a^j) \phi^{vw} & \text{if } vw \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{A}!$ is an associative unital k-algebra. (The unit is $\sum \phi^{\mathrm{id}_i}$.) Two special cases merit individual mention: When $\mathscr{C}$ is a group – that is, a one object category in which every map is an isomorphism – $\mathbf{k}_{\mathscr{C}}!$ is the group ring of $\mathscr{C}$ over k. If $\mathscr{C}$ is a poset, then $\mathbf{k}_{\mathscr{C}}!$ is a subring of the $\mathscr{C} \times \mathscr{C}$ matrices over k, namely those which have a 0 in cell (i,j) whenever $i \not\leq j$. (These have been called *tic tac toe algebras* by Mitchell [6].) Note that $\mathbf{k}_{I(\Sigma)}! = \mathbf{k}_\Sigma!$ is precisely the ring described in the introduction.

For each $\mathbb{A}$-bimodule $\mathbb{M}$ we define $\mathbb{M}!$ in a parallel fashion. So $\mathbb{M}! = \prod_i \coprod_w \mathbb{M}^i \phi^w$ and whenever vw exists we have both

$$(a^i \phi^w)(m^j \phi^v) = a^i T^w_{\mathbb{M}}(m^j) \phi^{vw} \quad \text{and} \quad (m^i \phi^w)(a^j \phi^v) = m^i \phi^w_{\mathbb{A}}(a^j) \phi^{vw}.$$

Now $\mathbb{M} \mapsto \mathbb{M}!$ is exact and, so, there is a natural transformation

$$\omega^*: \mathrm{Ext}^*_{\mathcal{C}}(-,-) \to \mathrm{Ext}^*_{\mathbb{A}!}(-!,-!).$$

In particular, there is a map $H^*(\mathbb{A},-) \to H^*(\mathbb{A}!,-!)$ from diagram cohomology to Hochschild cohomology. This is induced by a cochain map $\tau^*: C^*(\mathbb{A},-) \to C^*(\mathbb{A}!,-!)$ which we shall describe shortly. But first note that in dimension zero ω^* is just the canonical map $\mathrm{Hom}_{\mathbb{A}}(-,-) \to \mathrm{Hom}_{\mathbb{A}!}(-,-!)$. Hence if ! is not full – as may well happen – ω^* is certainly not an epimorphism. The *Cohomology Comparison Theorem* addresses the question: when is it an isomorphism? We quote the portion relevant to the simplicial theory.

Theorem (CCT). *ω^* is an isomorphism in the following cases:*

(i) *$\mathcal{C}$ is a finite poset; $\mathbb{A}$ is any diagram over $\mathcal{C}$.*

(ii) *$\mathcal{C}$ is a poset in which $\mathcal{C}_j = \{i \geq j\}$ is finite for every j; $\mathbb{A}$ is a diagram in which each $\mathbb{A}^i$ is k-flat and each $\mathrm{Hom}_{k-\mathcal{C}}(-,-) \to \mathrm{Hom}_{k-k}(-,-)$ is a natural epimorphism.* □

In case (i) we shall say that $\mathbb{A}$ is a *finite diagram*. Note that the conditions on $\mathbb{A}$ in case (ii) are automatic if each $\mathbb{A}^i$ is a localization of k.

To describe τ^* – which we do only when $\mathcal{C}$ is a poset – we need some notation and a definition. A *strict cochain* $F \in C^n(\mathbb{A}!, \mathbb{M}!)$ is one which satisfies

$$\begin{array}{lll} \text{(i)} & F(x_n, \dots, x_1) = 0 & \text{if any } x_r = \phi^{ii} \text{ for some } i. \\ \text{(ii)} & F(a_n\phi^{i_nj_n}, \dots, a_1\phi^{i_1j_1}) = 0 & \text{unless } j_r = i_{r-1} \text{ for all } r. \\ \text{(iii)} & F(a_n\phi^{i_ni_{n-1}}, \dots, a_1\phi^{i_1i_0}) \in \mathbb{M}^{i_n}\phi^{i_ni_0}. & \end{array} \tag{5.1}$$

[Here $a_r \in \mathbb{A}^{i_r}$.]

The strict cochains form a subcomplex $C^*_s(\mathbb{A}!, \mathbb{M}!)$ which is obviously closed under $\cup$ and, when $\mathbb{M} = \mathbb{A}$, $\circ$. Consequently the 'strict cohomology', $H^*_s(\mathbb{A}!, \mathbb{A}!)$, has cup and Lie products as in Section 3 and $H^*_s(\mathbb{A}!, \mathbb{A}!) \to H^*(\mathbb{A}!, \mathbb{A}!)$ is a morphism for both.

For $\sigma = (i_p < \cdots < i_0) \in \Sigma(\mathcal{C})_p$ define $\pi_\sigma \in C^p(\mathbb{A}^{i_p}, \mathbb{A}^{i_p})$ by $\pi_\sigma(a_p, \dots, a_1) = a_p \cdot \cdots \cdot a_1$, (the multiplication cochain). If $h \in C^m(\mathbb{A}^j, \mathbb{M}^k)$ and $i_m, \dots, i_1 \in \mathcal{C}_j$ there is only one reasonable interpretation for $h(a_m, \dots, a_1)$, where $a_r \in \mathbb{A}^{i_r}$, namely:

$$h(a_m, \dots, a_1) = h(\phi^{ji_m}(a_m), \dots, \phi^{ji_1}(a_1)).$$

At last, for $\Gamma \in C^n(\mathbb{A}, \mathbb{M})$ we define $\tau^n_{\mathcal{C}}\Gamma \in C^n(\mathbb{A}!, \mathbb{M}!)$ by

$$\tau^n_{\mathcal{C}}\Gamma \text{ is strict} \tag{5.2}$$

and

$$\tau^n_{\mathcal{C}}\Gamma(a_n\phi^{i_ni_{n-1}}, \dots, a_1\phi^{i_1i_0}) = \sum_{\sigma = \sigma'\cup\sigma''} \pi_{\sigma'} \cup \Gamma^{\sigma''}(a_n, \dots, a_1)\phi^{i_ni_0}$$

where $\sigma = (i_n \leq \cdots \leq i_0)$. [*N.B.* $i_r = i_{r-1}$ is permitted; i.e., σ may be a 'degenerate' n-simplex. If σ' is degenerate, interpret $\pi_{\sigma'}$ and $\Gamma^{\sigma'}$ as 0.]

Note that $\tau^*: C^*(\mathbb{A}, -) \to C^*(A!, -!)$ factors through $C_s^*(\mathbb{A}!, -!)$. In fact there is a cochain map $\hat{\tau}^*: C_s^*(\mathbb{A}!, -!) \to C^*(\mathbb{A}, -)$ which is a retract for τ^* (i.e. $\hat{\tau}^*\tau^* = \mathrm{id}$). Hence $H^*(\mathbb{A}, -) \to H_s^*(\mathbb{A}!, -!)$ is *always* a monomorphism. When $\mathbb{A}$ is finite this is an isomorphism and the CCT implies $H_s^*(\mathbb{A}!, -!) \cong H^*(\mathbb{A}!, -!)$. We do not know if the same is true in case (ii).

We shall not need the explicit description of $\hat{\tau}^*$ and, so, refer the interested reader to [3, §17]. Here we remark only that a straightforward calculation reveals

$$\Gamma \cup \Delta = \hat{\tau}(\tau\Gamma \cup \tau\Delta), \qquad \Gamma \bar{\circ} \hat{\tau}(\tau\Gamma \bar{\circ} \tau\Delta). \tag{5.3}$$

Of course, (5.3) justifies definitions (4.4–4.6). It also immediately implies that $\cup$ and $[\,,]$ are well-defined on $H^*(\mathbb{A}, \mathbb{A})$, since they are on $H_s^*(\mathbb{A}!, \mathbb{A}!)$.

Theorem. *If $\Sigma(\mathscr{C})$ is locally finite, then* (4.4) *and* (4.6) *define graded commutative and graded Lie products on $H^*(\mathbb{A}, \mathbb{A})$. Moreover, $H^*(\tau) = \omega^*$ is a morphism for both products.*

Proof. Observe that (5.3) and $\hat{\tau}^*\tau^* = \mathrm{id}$ imply

$$\tau(\Gamma \cup \Delta) - \tau\Gamma \cup \tau\Delta \in \ker \hat{\tau}^*;$$

it is clearly a cocycle if Γ and Δ are. Now if $\mathbb{A}$ is finite $H^*(\hat{\tau})$ is an isomorphism and, so,

$$\tau(\Gamma \cup \Delta) - \tau\Gamma \cup \tau\Delta \in B_s^*(\mathbb{A}!, \mathbb{A}!).$$

That is, $H^*(\tau)$ is a $\cup$-morphism; since it is also an isomorphism we find $\cup$ is graded commutative on $H^*(\mathbb{A}, \mathbb{A})$.

For the general case, set

$$\Gamma^m * \Gamma^n = \Gamma \cup \Delta - (-1)^{mn} \Delta \cup \Gamma.$$

We wish to find an $\Omega \in C^{m+n-1}(\mathbb{A}, \mathbb{A})$ with $\Gamma * \Delta = \delta\Omega$. To do so we construct a particular sequence of cochains $\{\Omega_0, \Omega_1, \ldots\}$ satisfying: for each $\sigma \in \Sigma(\mathscr{C})$, $\{r \mid \Omega_r^\sigma \neq 0\}$ is finite. From the construction it will be clear that $\sum \Omega_r$ may serve as Ω. If $\sigma \in \Sigma(\mathscr{C})$ write $\mathbb{A}_\sigma$ for $\bar{p}\mathbb{A}$ where p is the inclusion $\sigma \hookrightarrow \mathscr{C}$. The obvious restriction map $(-)|_\sigma : C^*(\mathbb{A}, \mathbb{A}) \to C^*(\mathbb{A}_\sigma, \mathbb{A}_\sigma)$ is a morphism for $\cup$ and $\bar{\circ}$. Moreover it has a retract: extend $\Pi \in C^*(\mathbb{A}_\sigma, \mathbb{A}_\sigma)$ to $\mathbb{A}$ by setting $\Pi^{\sigma'} = 0$ if $\sigma' \not\subset \sigma$. The collection $\{\Omega_r\}$ is obtained by an iterative procedure: since σ is finite the last paragraph shows $(\Gamma * \Delta)|_\sigma \in B^{m+n}(\mathbb{A}_\sigma, \mathbb{A}_\sigma)$. Suppose, for some s, that we have constructed $\{\Omega_r, r < s\}$ satisfying:

(i) $\Omega_r^\sigma = 0$ unless $\sigma \subset \sigma'$ for some $\sigma' \in \Sigma(\mathscr{C})_r$, and

(ii) $(\Gamma * \Delta|_\sigma = \delta \sum_{r<s} \Omega_r$ when $\dim \sigma < s$.

[These conditions clearly hold when $s = 0$.] Then for each $\sigma \in \Sigma(\mathscr{C})_s$ there is an $\Omega_\sigma \in C^{m+n-1}(\mathbb{A}_\sigma, \mathbb{A}_\sigma)$ for which $(\Gamma * \Delta - \delta \sum_{r<s} \Omega_r)|_\sigma = \delta\Omega_\sigma$. Extend Ω_σ to $\mathbb{A}$. It follows from local finiteness that $\Omega_s = \sum \Omega_\sigma$ is a well-defined cochain in $C^{m+n-1}(\mathbb{A}, \mathbb{A})$. Since conditions (i) and (ii) are immediate for $\{\Omega_r, r < s+1\}$, the induction proceeds. Note that local finiteness and (i) together imply $\{r \mid \Omega_r^\sigma \neq 0\}$ is finite, and so we may form $\Omega = \sum \Omega_r$. Then $\Gamma * \Delta = \delta\Omega$.

A similar argument shows $H^*(\tau)$ is a $\cup$-morphism: $\tau(\Gamma \cup \Delta) - \tau\Gamma \cup \tau\Delta$ is strict and, so, is determined by its evaluations at tuples of the form $(a_{m+n}\phi^{i_{m+n}i_{m+n-1}}, \ldots, a_1\phi^{i_1i_0})$. Such tuples correspond to finite pieces of the diagram. The first paragraph then asserts: $\tau(\Gamma \cup \Delta) - \tau\Gamma \cup \tau\Delta$ is a 'local' coboundary when Γ and Δ are cocycles. The local finiteness allows us to patch the cobounding cochains together as in the last paragraph. We omit the details.

With obvious modifications, the above proofs will yield the claims for [,]. □

6. Simplicial cohomology is Hochschild cohomology

Let I be a poset in which each $I_j = \{i \geq j\}$ is finite. We showed in Section 4 that $H^*(\Sigma(I), k) \cong H^*(\mathbf{k}_I, \mathbf{k}_I)$. On the other hand it is trivial that case (ii) of the CCT applies to $\mathscr{C} = I$, $\mathbb{A} = \mathbf{k}_I$. Hence, $H^*(\tau)$ is an isomorphism. Let

$$\bar{\tau}^* = \tau^* i^* : C^*(\Sigma(I), k) \to C^*(\mathbf{k}_I!, \mathbf{k}_I!).$$

Then we have

Theorem (SC = HC). *If each I_j is finite, then $H^*(\bar{\tau}) : H^*(\Sigma(I), k) \to H^*(\mathbf{k}_I!, \mathbf{k}_I!)$ is an isomorphism.* □

The finiteness condition is automatic when $I = I(\Sigma)$ for a simplicial complex Σ. Then $\Sigma(I) = \Sigma'$, the barycentric subdivision of Σ, and we find $H^*(\Sigma', k) \cong H^*(\mathbf{k}_\Sigma!, \mathbf{k}_\Sigma!)$. If, moreover, Σ is locally finite then we may replace Σ' with it in the last isomorphism.

Note that $H^*(\mathbf{k}_I!, \mathbf{k}_I!)$ has a graded Lie product [,] while none is known for $H^*(\Sigma(I), k)$. In fact, [,] is abelian. Also $H^*(\bar{\tau})$ is a morphism for the cup product. The proof of these claims requires Steenrod's extension of his $\cup_i$-products to $C^*(\Sigma(I), k)$ [8].

Recall that $C_p = C_p(\Sigma(I), k)$ is the free k-module generated by $\Sigma(I)_p$ and $C^p = C^p(\Sigma(I), k) = \mathrm{Hom}_k(C_p, k)$. Steenrod begins by generalizing $\cup$ ($= \cup_0$) and $\cup_1$ to obtain bilinear maps $\cup_i : C_p \times C_q \to C_{p+q-i}$. Then he defines $\cup_i : C^p \times C^q \to C^{p+q-i}$ by

$$f^p \cup_i g^q(\sigma) = \sum_{\sigma = \pm\sigma' \cup_i \sigma''} \pm f(\sigma')g(\sigma'') \tag{6.1}$$

where $\sigma \in \Sigma(I)_{p+q+i}$, $\sigma' \in \Sigma(I)_p$, and $\sigma'' \in \Sigma(I)_q$.

As with $\cup_1$ and δ, these are not well-defined at the cohomology level, although they lead to operations – the Steenrod squares – which are. We shall not need the explicit description of $\cup_i$ for $i \geq 2$; however, for $i \leq 2$ we shall require the coboundary formula (which applies for every i):

$$\delta(f^p \cup_i g^q) = \delta f \cup_i g + (-1)^p f \cup_i \delta g + (-1)^{p+q-i}\{f \cup_{i-1} g + (-1)^{pq-i} g \cup_{i-1} f\}. \tag{6.2}$$

Note the similarity of (6.2) to (3.2) and (3.4). As there, (6.2) implies both the existence and graded commutativity of $\cup$ ($=\cup_0$) at the cohomology level. [$\cup_{-1}=0$.]

The description of $\bar{\tau}^*$ is seen from (4.1) and (5.2) to be: $\bar{\tau}f^p$ is *the* strict (infinite) k-linear p-cochain determined by

$$\bar{\tau}f(\phi^{i_p i_{p-1}},\dots,\phi^{i_1 i_0})=f(i_p<\cdots<i_0)\phi^{i_p i_0}. \tag{6.3}$$

Proposition. (i) $\bar{\tau}(f^p\cup g^q)=\bar{\tau}f\cup\bar{\tau}g$.

(ii) $\bar{\tau}(f^p\cup_1 g^q)=(-1)^{p(q+1)}\bar{\tau}f\,\bar{\circ}\,\bar{\tau}g$.

Proof. (i) $\bar{\tau}f\cup\bar{\tau}g$ is strict and (infinite) k-linear since both $\bar{\tau}f$ and $\bar{\tau}g$ are. We compute:

$$\begin{aligned}&\bar{\tau}f\cup\bar{\tau}g(\phi^{i_{p+q}i_{p+q-1}},\dots,\phi^{i_1 i_0})\\&\quad=\bar{\tau}f(\phi^{i_{p+q}i_{p+q-1}},\dots,\phi^{i_{q+1}i_q})\cdot\bar{\tau}g(\phi^{i_q i_{q-1}},\dots,\phi^{i_1 i_0})\\&\quad=f(i_{p+q}<\cdots<i_q)\phi^{i_{p+q}i_q}\cdot g(i_q<\cdots<i_0)\phi^{i_q i_0}.\end{aligned}$$

Since $\phi^{ii}=\mathrm{id}_k$ in $\mathbf{k}_I$ and $\phi^{ij}\phi^{jk}=\phi^{ik}$ in $\mathbf{k}_I!$, the last expression above becomes

$$f(i_{p+q}<\cdots<i_q)g(i_q<\cdots<i_0)\phi^{i_{p+q}i_0}.$$

Of course, this is just $f\cup g(i_{p+q}<\cdots<i_0)\phi^{i_{p+q}i_0}$ and we have $\bar{\tau}f\cup\bar{\tau}g=\bar{\tau}(f\cup g)$.

(ii) First note that (4.3) and (6.1) assert

$$\begin{aligned}&f^p\cup_1 g^q(i_{p+q-1}<\cdots<i_0)\\&\quad=\sum_{j=1}^{p}(-1)^{j(q-1)}f(i_{p+q-1}<\cdots<i_{j+q-1}<i_{j-1}<\cdots<i_0)g(i_{j+q-1}<\cdots<i_{j-1}).\end{aligned}$$

As in (i), $\bar{\tau}f\,\bar{\circ}\,\bar{\tau}g$ is strict and (infinite) k-linear. Again we compute: set $\varepsilon_j=(-1)^{(p-j)(q-1)}$, then

$$\begin{aligned}&\bar{\tau}f\,\bar{\circ}\,\bar{\tau}g(\phi^{i_{p+q-1}i_{p+q-2}},\dots,\phi^{i_1 i_0})\\&\quad=\sum_{j=1}^{p}\varepsilon_j\,\bar{\tau}f(\phi^{i_{p+q-1}i_{p+q-2}},\dots,\bar{\tau}g(\phi^{i_{j+q-1}i_{j+q-2}},\dots,\phi^{i_j i_{j-1}}),\phi^{i_{j-1}i_{j-2}},\dots,\phi^{i_1 i_0})\\&\quad=\sum_{j=1}^{p}\varepsilon_j\,\bar{\tau}f(\phi^{i_{p+q-1}i_{p+q-2}},\dots,g(i_{j+q-1}<\cdots<i_{j-1})\phi^{i_{j+q-1}i_{j-1}},\dots,\phi^{i_1 i_0})\\&\quad=\sum_{j=1}^{p}\varepsilon_j g(i_{j+q-1}<\cdots<i_{j-1})\\&\qquad\qquad\cdot\bar{\tau}f(\phi^{i_{p+q-1}i_{p+q-2}},\dots,\phi^{i_{j+q}i_{j+q-1}},\phi^{i_{j+q-1}i_{j-1}},\dots,\phi^{i_1 i_0})\\&\quad=\sum_{j=1}^{p}\varepsilon_j g(i_{j+q-1}<\cdots<i_{j-1})\\&\qquad\qquad\cdot f(i_{p+q-1}<\cdots<i_{j+q-1}<i_{j-1}<\cdots<i_0)\phi^{i_{p+q-1}i_0}\\&\quad=(-1)^{p(q-1)}f\cup_1 g(i_{p+q-1}<\cdots<i_0)\phi^{i_{p+q-1}i_0}.\end{aligned}$$

and, so,

$$\bar{\tau}f\,\bar{\circ}\,\bar{\tau}g=(-1)^{p(q-1)}\bar{\tau}(f\cup_1 g). \qquad \square$$

Note that the proposition asserts that $\cup_1$ is essentially a special case of $\bar{\circ}$. This suggests that there may generally be other operations on Hochschild cochains which specialize to $\cup_i$ in this setting. The barrier to constructing such operations seems to be the possible non-triviality of [,].

Corollary. (i) *$H^*(\bar{\tau})$ is a $\cup$-morphism.*
(ii) *[,] is abelian on $H^*(\mathbf{k}_I!, \mathbf{k}_I!)$.*

Proof. (i) is trivial.
(ii) For cocycles f^p and g^q, (6.2) asserts

$$\delta(f \cup_2 g) = (-1)^{p+q}\{f \cup_1 g + (-1)^{pq} g \cup_1 f\}.$$

Hence

$$\begin{aligned} \bar{\tau}\delta(f \cup_2 g) &= (-1)^{p+q}\{(-1)^{p(q-1)}\bar{\tau}f \,\bar{\circ}\, \bar{\tau}g + (-1)^{pq+q(p-1)}\bar{\tau}g \,\bar{\circ}\, \bar{\tau}f\} \\ &= (-1)^{pq+q}\{\bar{\tau}f \,\bar{\circ}\, \bar{\tau}g + (-1)^{pq+p+q}\bar{\tau}g \,\bar{\circ}\, \bar{\tau}f\} \\ &= (-1)^{q(p+1)}[\bar{\tau}f, \bar{\tau}g]. \end{aligned}$$

Thus, $[\bar{\tau}f, \bar{\tau}g] = \delta\{(-1)^{q(p+1)}\bar{\tau}(f \cup_2 g)\}$ and [,] is trivial on the image of $H^*(\bar{\tau})$. But $H^*(\bar{\tau})$ is surjective. □

We close this section with a remark on Clarke's cohomology. Let Σ be a finite simplicial complex and set $S = \{0\} \cup \{\phi^{ij}\} \subset \mathbf{k}_\Sigma!$. Then S is a semigroup with zero and it is not hard to see that Clarke's cochain complex is just $C_s^*(\mathbf{k}_\Sigma!, \mathbf{k}_\Sigma!)$. This is not true when Σ is infinite. Indeed, we do not know if $H^*(\Sigma(I), k)$ is always given by the strict cohomology – precisely because we do not know if the latter always agrees with the Hochschild cohomology. In any event, to pass from Clarke's semigroup cohomology to Hochschild cohomology requires $H_s^*(\mathbf{k}_\Sigma!, \mathbf{k}_\Sigma!) = H^*(\mathbf{k}_\Sigma!, \mathbf{k}_\Sigma!)$ and this, in turn, requires at least the CCT.

7. Generalized simplicial cohomology theories

Let $\mathscr{A}$ be the category of simplicial complexes. A generalized simplicial theory arises when ever we fix a base complex $K \in \mathscr{A}$, a diagram $\mathbb{A}$ over $I(K)$ and an $\mathbb{A}$-bimodule $\mathbb{M}$. Specifically, the cohomology theory is then the functor defined on $\mathscr{A}/K$ by $(p: \Sigma \to K) \mapsto H^*(\tilde{p}\mathbb{A}, \tilde{p}\mathbb{M})$. [Recall: the objects of $\mathscr{A}/K$ are maps $p: \Sigma \to K$; a map $(p_1: \Sigma_1 \to K) \to (P_2: \Sigma_2 \to K)$ is an $\mathscr{A}$-map $q: \Sigma_1 \to \Sigma_2$ for which $p_2 q = p_1$.] Since $\mathscr{A}/* = \alpha$, a generalized simplicial theory on $\mathscr{A}$ is determined by a single k-algebra A and a A-bimodule M. In the very special case $A = M = k$ the cohomology theory is just $\Sigma \mapsto H^*(\mathbf{k}_\Sigma, \mathbf{k}_\Sigma) \cong H^*(\Sigma', k)$; i.e., it is the usual theory composed with the barycentric subdivision functor, $\Sigma \mapsto \Sigma'$. These generalized simplicial theories satisfy all the usual axioms except (possibly) dimension [7].

Functoriality. Any map $q: p_1 \to p_2$ in $\mathscr{A}/K$ induces a cochain map

$$q^*: C^*(\tilde{p}_2\mathbb{A}, \tilde{p}_2\mathbb{M}) \to C^*(\tilde{p}_1\mathbb{A}, \tilde{p}_1\mathbb{M}),$$

namely $(q^*\Gamma)^\sigma = \Gamma^{q\sigma}$ for $\sigma \in \Sigma_2'$. [If $q\sigma$ is degenerate set $(q^*\Gamma)^\sigma = 0$.] Clearly, $(q_1q_2)^* = q_2^*q_1^*$.

Exactness. To state this axiom we need an additional concept: relative cohomology modules. Let $q: p_1 \to p_2$ be an inclusion in $\mathscr{A}/K$. Define a submodule $\mathbb{M}_q$ of $\tilde{p}_2\mathbb{M}$ as follows: $\mathbb{M}_q^i = 0$ if $i \in I(\Sigma_1)$; $T_{\mathbb{M}_q}^{ij} = 0$ if either i or j is in $I(\Sigma_1)$; otherwise, $\mathbb{M}_q = \tilde{p}_2\mathbb{M}$. This defines a $\tilde{p}_2\mathbb{A}$-bimodule precisely because $I(\Sigma_1)$ is a filter in $I(\Sigma_2)$. Note that $E_q: 0 \to \mathbb{M}_q \to \tilde{p}_2\mathbb{M} \to \tilde{p}_2\mathbb{M}/\mathbb{M}_q \to 0$ is allowable. The generalized q-relative cohomology is $H^*(\tilde{p}_2\mathbb{A}, \tilde{p}_2\mathbb{M}/\mathbb{M}_q)$. [In the case $K = *$, $\mathbb{A} = \mathbb{M} = k$ this agrees with the usual cohomology of the pair $\Sigma_1 \subset \Sigma_2$.] There is an obvious cochain isomorphism $C^*(\tilde{p}_2\mathbb{A}, \tilde{p}_2\mathbb{M}/\mathbb{M}_q) \to C^*(\tilde{p}_1\mathbb{A}, \tilde{p}_1\mathbb{M})$ and, so, E_q induces the required long exact cohomology sequence.

Homotopy. First note that two maps $q, q': \Sigma_1 \to \Sigma_2$ induce chain maps on the chain complexes of their barycentric subdivisions. A homotopy $q \sim q'$ induces a chain homotopy, say $h: C_*(\Sigma_1', k) \to C_{*+1}(\Sigma_2', k)$. So $(q - q')\sigma = h\delta\sigma - \delta h\sigma$. We use h to construct a cochain homotopy $q^* \sim q'^*$: for $\Gamma \in Z^n(\tilde{p}_2\mathbb{A}, \tilde{p}_2\mathbb{M})$ define $\Delta \in C^{n-1}(\tilde{p}_1\mathbb{A}, \tilde{p}_1\mathbb{M})$ by $\Delta^\sigma = \Gamma^{h\sigma}$. Then

$$(\delta\Delta)^\sigma = \Gamma^{h\delta\sigma} + (-1)^\sigma \delta\Gamma^{h\sigma}.$$

But $(\delta\Gamma)^{h\sigma} = 0$ implies $-\delta\Gamma^{h\sigma} = (-1)^\sigma \Gamma^{\delta h\sigma}$. Hence

$$(\delta\Delta)^\sigma = \Gamma^{h\delta\sigma - \delta h\sigma} = \Gamma^{(q-q')\sigma} = \{q^*\Gamma - q'^*\Gamma\}^\sigma.$$

That is, $q^*\Gamma - q'^*\Gamma \in B^n(\tilde{p}_1\mathbb{A}, \tilde{p}_1\mathbb{M})$ and, consequently, $H^n(q) = H^n(q')$.

Excision. It is well known that this axiom is equivalent to exactness of the Mayer-Vietoris sequence [7]. For the latter, let $q_\alpha: (p_\alpha: \Sigma_\alpha \to K) \to (p: \Sigma \to K)$, $(\alpha = 1, 2)$, be comaximal inclusions, (i.e. $\Sigma = \Sigma_1 \cup \Sigma_2$). Their intersection is $p_{12}: \Sigma_1 \cap \Sigma_2 \to K$, where p_{12} is the restriction of p to $\Sigma_1 \cap \Sigma_2$. Denote the inclusion $p_{12} \to p_\alpha$ by j_α. Then

$$0 \to C^*(\tilde{p}\mathbb{A}, \tilde{p}\mathbb{M}) \xrightarrow{(q_1^* - q_2^*)} C^*(\tilde{p}_1\mathbb{A}, \tilde{p}_1\mathbb{M}) \oplus C^*(\tilde{p}_2\mathbb{A}, \tilde{p}_2\mathbb{M})$$
$$\xrightarrow{\binom{j_1^*}{j_2^*}} C^*(\tilde{p}_{12}\mathbb{A}, \tilde{p}_{12}\mathbb{M}) \to 0$$

is easily seen to be exact; it yields the required long exact sequence.

Dimension. Consider the case $K = *$. As we remarked earlier, we have $\mathbb{A} = \Lambda$, $\mathbb{M} = M$. Hence, $(\mathrm{id}: * \to *) \mapsto H^*(\Lambda, M)$ and the dimension axiom fails in general. Note, however, that if Λ is a k-central semi-simple algebra and $M = \Lambda$, then

$H^*(\Lambda, M) = H^*(*, k)$. Consequently, if Λ_Σ is the constant diagram over $I(\Sigma)$ determined by a k-central semi-simple algebra Λ, then

$$H^*(\Sigma', k) \cong H^*(\Lambda_\Sigma, \Lambda_\Sigma) \cong H^*(\Lambda_\Sigma!, \Lambda_\Sigma!).$$

[The last isomorphism follows from case (ii) of the CCT].

The subcategory of locally finite simplicial complexes merits special attention. For there, when $\mathbb{M} = \mathbb{A}$, the cohomologies have graded cup products and (generally nontrivial) Lie products. If we restrict further to finite simplicial complexes, then a generalized simplicial cohomology (in our sense) becomes

$$(p : \Sigma \to K) \mapsto H^*((\tilde{p}\mathbb{A})!, (\tilde{p}\mathbb{M})!).$$

References

[1] E. Clark, Cohomology of semigroups via topology – with an application to semigroup algebras, Comm. Algebra 4 (1976) 979–997.
[2] M. Gerstenhaber, On the cohomology structure of an associative ring, Ann. of Math. 78 (1963) 267–288.
[3] M. Gerstenhaber and S.D. Schack, On the deformation of algebra morphisms and diagrams, Trans. Amer. Math. Soc., to appear.
[4] G. Hochschild, On the cohomology groups of an associative algebra, Ann. of Math. 46 (1945) 58–67.
[5] S. MacLane, Homology (Springer, New York, 1963).
[6] B. Mitchell, Theory of Categories (Academic Press, New York, 1965).
[7] E.H. Spanier, Algebraic Topology (McGraw-Hill, New York, 1966).
[8] N.E. Steenrod, Products of cocycles and extensions of mappings, Ann. of Math. 48 (1947) 290–320.
[9] C.E. Watts, A homology theory for small categories, in: S. Eilenberg, ed., Proc. Conf. Categorical Algebra (Springer-Verlag, Berlin, 1966).