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Vyacheslav Futorny* · Steffen König · Volodymyr Mazorchuk**

$\mathcal{S}$-subcategories in $\mathcal{O}$

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Abstract. We prove that certain subcategories of $\mathcal{O}$, consisting of complete modules having a quasi-Verma flag with respect to a Levi subalgebra, admit a combinatorial description similar to Soergel's results on category $\mathcal{O}$. Using the Enright completion functor we also reprove Soergel's character formula for tilting modules in $\mathcal{O}$ and Ringel self-duality for the principal block of $\mathcal{O}$.

1. Introduction

Let $\mathfrak{G}$ be a simple complex finite-dimensional Lie algebra with a fixed Cartan subalgebra $\mathfrak{H}$ and a fixed triangular decomposition $\mathfrak{G} = \mathfrak{N}_- \oplus \mathfrak{H} \oplus \mathfrak{N}_+$. For such a situation, Bernstein, Gelfand and Gelfand [BGG] defined their celebrated category $\mathcal{O}$. Verma modules are produced by starting with a finite-dimensional $\mathfrak{H}$-module, inflating it to the Borel subalgebra $\mathfrak{B}_+ = \mathfrak{H} \oplus \mathfrak{N}_+$ and then inducing up to $\mathfrak{G}$-modules.

Basic properties of $\mathcal{O}$ are the following: There is a block decomposition such that each block has finitely many simple objects (up to isomorphism); there exist enough projective objects, and these are filtered by Verma modules – in modern terms: a block is equivalent to the module category of a quasi-hereditary algebra – and there is the BGG-reciprocity principle relating composition multiplicities in Verma modules to filtration multiplicities of Verma modules in projective objects.

* Regular Associate of the ICTP

** Alexander von Humboldt fellow at Bielefeld University

V. Futorny: Universidade de Sao Paulo, Instituto de Matematica e Estatistica, Caixa Postal 66281 – CEP 05315-970, Sao Paulo, Brazil. e-mail: futorny@ime.usp.br

S. König: Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany. e-mail: koenig@mathematik.uni-bielefeld.de.
Present address: Department of Mathematics and Computer Science, University of Leicester, University Road, Leicester LE1 7RH, UK

V. Mazorchuk: Algebra, Mechanics and Mathematics Department, Kyiv Taras Shevchenko University, 64 Volodymyrska st., 01033 Kyiv, Ukraine. e-mail: mazor@mechmat.univ.kiev.ua

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Moreover, Soergel ([S2]) has given a combinatorial description of the blocks of $\mathcal{O}$ which includes the following features: The endomorphism ring of the big projective module (in the principal block) is the coinvariant algebra (which is isomorphic to the cohomology algebra of the flag manifold). There is a double centralizer property relating the coinvariant algebra with the algebra of the principal block (via the mutual actions on the big projective module). Furthermore ([S3]), category $\mathcal{O}$ is its own Ringel dual.

The setup for defining $\mathcal{O}$ can be generalized as follows: Let $\mathcal{P} \supset \mathfrak{N}_+$ be a parabolic subalgebra of $\mathfrak{G}$, $\mathcal{P} = \mathfrak{A}' \oplus \mathfrak{N}$, where $\mathfrak{N}$ is nilpotent and $\mathfrak{A}'$ is reductive. Let also $\mathfrak{A}' = \mathfrak{A} \oplus \mathfrak{H}^{\mathfrak{A}}$, where $\mathfrak{A}$ is semi-simple and $\mathfrak{H}^{\mathfrak{A}}$ is abelian and central. Then we can start with $\mathfrak{A}'$-modules as input for inflation to $\mathcal{P}$ and then induce up to produce generalized Verma modules. At this point, there is no need to restrict attention to finite-dimensional $\mathfrak{A}'$-modules.

Therefore, in [FKM1] we defined and studied a certain parabolic generalization $\mathcal{O}(\mathcal{P}, \Lambda)$ of $\mathcal{O}$. It has been shown that, under some natural conditions, these categories correspond to projectively stratified (= standardly stratified) finite-dimensional algebras (but usually not to quasi-hereditary algebras) and there are analogues of the BGG-reciprocity principle.

The main problem now is to find a combinatorial description in the spirit of Soergel's approach. We have studied a basic example of $\mathcal{O}(\mathcal{P}, \Lambda)$ in [FKM2]. There we proved that in the case, when $\mathfrak{A}$ is isomorphic to $sl(2, \mathbb{C})$, certain categories do appear, whose blocks can be given a combinatorial description, analogous to Soergel's description of $\mathcal{O}$. In fact, the coinvariant algebra is the endomorphism algebra of the big projective module in the principal block and there is a double centralizer property. The main tool in proving these results was a special functor E, which produces an equivalence of $\mathcal{O}(\mathcal{P}, \Lambda)$ with a full subcategory of $\mathcal{O}$. We took the idea to define and study this functor from [M]. As soon as we constructed and investigated E, all the properties of $\mathcal{O}(\mathcal{P}, \Lambda)$ can be deduced from the analogous properties of $\mathcal{O}$.

The image of E carries an abelian structure coming from $\mathcal{O}(\mathcal{P}, \Lambda)$, and this abelian structure is not inherited from that in $\mathcal{O}$, hence it looks slightly mysterious from the Lie theoretic point of view.

In the present paper we want to consider a general situation, i.e. we assume that $\mathfrak{A}$ is an arbitrary semi-simple finite-dimensional Lie algebra. There are several examples of $\mathcal{O}(\mathcal{P}, \Lambda)$ for such situations (see, for example [FKM1, Sect. 11]), but we do not know how to construct an analogue of the functor E for them. In the hope that the machinery, worked out in [FKM2], should work in the general case, we tried to determine the candidate image of E (if such an E would exist and have all necessary properties). This approach led us to study certain subcategories in $\mathcal{O}$, which possess a combinatorial description similar to Soergel's results on classical category $\mathcal{O}$. We call such categories $\mathcal{S}$-subcategories. The main result of this paper is a construction of a series of $\mathcal{S}$-subcategories in $\mathcal{O}$, which consist of complete (in the sense of Enright, [E]) $\mathfrak{A}$-modules having a quasi-Verma flag. We also prove that our category has enough tilting modules and the algebra of the principal block is isomorphic to its Ringel dual.

The paper is organized as follows. In Sect. 2 we define the (new) notion of a module with a quasi-Verma flag in $\mathcal{O}$. The usual notion of a module having a Verma flag seems to be insufficient, at least we did not manage to work out the corresponding categories. In Sect. 3 we recall Mathieu's version of the Enright functor (which seems to be the most convenient one for us) and establish its basic properties. We also recall the notion of a complete module ([E]). In Sect. 4 we study the subcategory of $\mathcal{O}$, which consists of all complete modules having a quasi-Verma flag and prove that it is admissible in the sense of [FKM1]. The most difficult place here is to define an abelian structure (which, as we already know, cannot be inherited from $\mathcal{O}$). In Sect. 5 we study the corresponding $\mathcal{O}(\mathcal{P}, \Lambda)$ and prove that it is an $\mathcal{S}$-subcategory in $\mathcal{O}$ (the last $\mathcal{O}$ is with respect to $\mathfrak{G}$). In Sect. 6 we study a duality on $\mathcal{O}(\mathcal{P}, \Lambda)$, the tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$ and prove the Ringel self-duality for the principal block. Finally, in Sect. 7 we introduce a family of subcategories of the category of modules with quasi-Verma flag. These subcategories are parametrized by elements in the Weyl group, and we get the previously studied subcategory as a special case. The main result in this section provides us with equivalences between these categories. As a corollary, we obtain a new proof of Soergel's character formula for tilting modules.

2. Modules with quasi-Verma flag in $\mathcal{O}$

Let $\mathfrak{A}$ be a semi-simple complex finite-dimensional Lie algebra with a fixed Cartan subalgebra $\mathfrak{H}_{\mathfrak{A}}$ and the corresponding root system Δ. Fix a basis π in Δ and consider the corresponding decomposition $\Delta_+ \cup \Delta_-$ of Δ and the corresponding triangular decomposition $\mathfrak{N}_- \oplus \mathfrak{H}_{\mathfrak{A}} \oplus \mathfrak{N}_+$ of $\mathfrak{A}$. Consider the BGG category $\mathcal{O}$ of $\mathfrak{A}$ ([BGG]) with respect to the triangular decomposition above. We recall that $\mathcal{O}$ is a full subcategory in the category of all $\mathfrak{A}$-modules and consists of all finitely generated, $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable and locally $U(\mathfrak{N}_+)$-finite modules. For $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$ let $M(\lambda)$ (resp. $L(\lambda)$) denote the Verma module (resp. the unique simple quotient of the Verma module) with the highest weight $\lambda - \rho$, where ρ is half of the sum of all positive roots ([D, Chapter 7]). We also choose a Weyl-Chevalley basis $X_\alpha, \alpha \in \Delta$, $H_\alpha, \alpha \in \pi$ in $\mathfrak{A}$.

We will say that a module $M \in \mathcal{O}$ has a *quasi-Verma flag* if there is a filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = M, \tag{1}$$

such that M_i/M_{i-1} is isomorphic to a submodule of some $M(\lambda_i)$, $i = 1, 2, \ldots, k$. Denote by $\mathcal{F}$ the full subcategory of $\mathcal{O}$, which consists of all modules with quasi-Verma flag.

Further we will need some easy properties of $\mathcal{F}$. The filtration (1) will be called *non-degenerated* if M_i/M_{i-1} is not zero for all $i = 1, 2, \ldots, k$. We will also call k the *length* of the filtration (1). We recall ([BGG]) that any module in $\mathcal{O}$, and hence in $\mathcal{F}$ has finite length.

Lemma 1. *Let $M \in \mathcal{F}$. Then any two non-degenerate quasi-Verma flags of M have the same length. Moreover, this common length is equal to the number of simple* *subquotients of M (taken with their multiplicities), which are isomorphic to Verma modules.*

Proof. This follows from the fact that any Verma module has a simple socle, which is again a Verma module (the unique simple Verma module in this block) [D, Proposition 7.6.3]. □

If $M \in \mathcal{F}$ and (1) is a non-degenerate quasi-Verma flag of M, we will call k the *quasi-Verma length* of M and will denote it by $\mathrm{qVl}(M)$. According to Lemma 1 this notion is well-defined.

Lemma 2. *Let $M \in \mathcal{F}$. Then for any $\alpha \in \pi$ the operator $X_{-\alpha}$ acts injectively on M.*

Proof. A Verma module is free over $U(\mathfrak{N}_-)$, hence torsion-free. Its submodules are torsion-free as well. □

3. Enright functor (Mathieu's version)

Fix for some time a root $\alpha \in \pi$ and denote by $\mathfrak{A}^\alpha$ the corresponding $sl(2)$-subalgebra of $\mathfrak{A}$. Let U_α denote the Ore localization of $U(\mathfrak{A})$ with respect to the multiplicative set $\{X_{-\alpha}^i \mid i \in \mathbb{Z}_+\}$. It is well-defined according to [M, Lemma 4.2]. Denote by r_α the endofunctor of $\mathcal{O}$, obtained as a composition of the following functors:

- $U_\alpha \otimes_{U(\mathfrak{A})} -$,
- restriction to $U(\mathfrak{A})$,
- taking the locally X_α-finite part.

By [De, Sect. 2], r_α coincides on X_α-torsion free modules with the Enright completion functor C_α (see [E, Sect. 3]). It is straightforward to check that r_α is well-defined on $\mathcal{O}$ (see also [M, Appendix]) and $r_\alpha \circ r_\alpha = r_\alpha$. Order the elements of π in an arbitrary way: $\pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and set $r = r_{\alpha_1} \circ r_{\alpha_2} \circ \cdots \circ r_{\alpha_n}$.

Lemma 3. *Let $M, N \in \mathcal{O}$, $M \subset N$ and $\alpha \in \pi$. Then*

1. $r_\alpha(M) \subset r_\alpha(N)$,
2. $r_\alpha(N/M) \supset r_\alpha(N)/r_\alpha(M)$.

Proof. Follows from the left exactness of r_α ([E, Proposition 3.17]). □

Lemma 4. 1. *Let $P_\alpha = U(\mathfrak{A}) \otimes_{U(\mathfrak{A}^\alpha)} -$. Then $P_\alpha \circ r_\alpha = r_\alpha \circ P_\alpha$.*
2. *Let $\mathcal{P} = \mathfrak{A}^\alpha + \mathfrak{H}_\mathfrak{A} + \mathfrak{N}_+$ be a parabolic subalgebra in $\mathfrak{A}$ and let $P = U(\mathfrak{A}) \otimes_{U(\mathcal{P})} -$ be the functor of the corresponding induction. Then $P \circ r_\alpha = r_\alpha \circ P$.*

Proof. The first part follows from the natural identity $U_\alpha \otimes_{U(\mathfrak{A})} U(\mathfrak{A}) \otimes_{U(\mathfrak{A}^\alpha)} M \simeq U_\alpha \otimes_{U(\mathfrak{A}^\alpha)_\alpha} U(\mathfrak{A}^\alpha)_\alpha \otimes_{U(\mathfrak{A}^\alpha)} M$, where $U(\mathfrak{A}^\alpha)_\alpha$ is the localization of $U(\mathfrak{A}^\alpha)$ with respect to $\{X_{-\alpha}^i \mid i \in \mathbb{Z}_+\}$. The second part follows from the first one. □

Lemma 5. *Let $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$, $\alpha \in \pi$ and s_α be the reflection with respect to α. Then $r_\alpha(M(\lambda)) \simeq M(\lambda)$ if $M(\lambda) \not\subset M(s_\alpha(\lambda))$ and $r_\alpha(M(\lambda)) \simeq M(s_\alpha(\lambda))$ if $M(\lambda) \subset M(s_\alpha(\lambda))$.*

Proof. This is a standard property of the Enright functor ([E, De]). We show, how it can be easily deduced from Lemma 4. We recall that any Verma module over $\mathfrak{A}$ can be obtained from the Verma module over $\mathfrak{A}^\alpha$ using the functor P from Lemma 4. Now from Lemma 4 it follows that it is enough to check the statement in the case $\mathfrak{A} = \mathfrak{A}^\alpha$ for which it is trivial. □

Lemma 6. *Let $M \in \mathcal{O}$ and $\alpha \in \pi$. Then*

1. *$M \subset r_\alpha(M)$ for any $M \in \mathcal{F}$ and this inclusion is functorial;*
2. $\mathrm{qVl}(r_\alpha(M)) = \mathrm{qVl}(M)$ *for any $M \in \mathcal{F}$;*
3. *$M = r_\alpha(M)$ if and only if M, viewed as an $\mathfrak{A}^\alpha$-module, is an (infinite) direct sum of projective modules from the corresponding category $\mathcal{O}$. In particular, if $M \in \mathcal{F}$, then as an $\mathfrak{A}^\alpha$-module, $r_\alpha(M)$ coincides with the minimal direct sum of projective modules containing M.*

Proof. The first statement follows directly from Lemma 2. Let $M = M_k \supset M_{k-1} \supset \dots$ be a quasi-Verma flag of M. Then $r_\alpha(M) = r_\alpha(M_k) \supset r_\alpha(M_{k-1}) \supset \dots$ is a filtration of $r_\alpha(M)$. Moreover, $r_\alpha(M_i)/r_\alpha(M_{i-1})$ is contained in $r_\alpha(M_i/M_{i-1})$ by Lemma 3. We know that $M_i/M_{i-1} \subset M(\lambda_i)$ for some $\lambda_i \in \mathfrak{H}_{\mathfrak{A}}^*$. Hence, by Lemma 3 we have $r_\alpha(M_i/M_{i-1}) \subset r_\alpha(M(\lambda_i))$. By Lemma 5, $r_\alpha(M(\lambda_i))$ is a Verma module and hence $r_\alpha(M)$ has a quasi-Verma flag. Moreover, $\mathrm{qVl}(r_\alpha(M)) \leq \mathrm{qVl}(M)$. According to the first statement we have $M \subset r_\alpha(M)$ and hence $\mathrm{qVl}(r_\alpha(M)) = \mathrm{qVl}(M)$. The second statement is proved.

For the last statement we recall that $r_\alpha \circ r_\alpha = r_\alpha$, so it is enough to prove only the first part. By Lemma 4, it is enough to prove the last statement for $\mathfrak{A} = \mathfrak{A}^\alpha$. But in this case it is trivial. □

Recall that a module M is called *α-complete* if $r_\alpha(M) \simeq M$ ([E]) and *complete* if $r_\beta(M) \simeq M$ for all $\beta \in \pi$. It is clear that M is complete if and only if $r(M) = M$. We will denote by $\mathrm{st}\,\mathcal{F}$ the full subcategory of $\mathcal{F}$, consisting of all complete modules.

Lemma 7. *For any $M \in \mathcal{F}$ there exists $k \in \mathbb{N}$ such that $r^k(M) = r \circ r \circ \dots \circ r(M) \in \mathrm{st}\,\mathcal{F}$ (r occurs k times in the composition).*

Proof. Follows from Lemma 6 and the fact that any Verma module has finite length. □

Denote by st the composition $r \cdots \circ r \circ r$ in which r occurs $|\Delta_+|$ times. From [De] and [H, Section 1.8] it follows that $r_\alpha \circ \mathrm{st}(M) = \mathrm{st}(M)$ for any $M \in \mathcal{F}$ (even $M \in \mathcal{O}$) and $\alpha \in \pi$. Hence, st is a well-defined functor from $\mathcal{F}$ to $\mathrm{st}\,\mathcal{F}$. Moreover, the objects in $\mathrm{st}\,\mathcal{F}$ are precisely those objects in $\mathcal{F}$ which are invariant (i.e. stable) under st. This explains our notation.

Corollary 1. $\mathrm{st}(M(\lambda))$ *is a Verma module for any $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. In particular, for integral λ the module $\mathrm{st}(M(\lambda)) \simeq M(\mu)$, where μ is the element from the orbit of λ with respect to the Weyl group action, lying in the closure of the dominant Weyl chamber.*

Proof. Follows from Lemma 5. □

4. An abelian structure and admissibility of st$\mathcal{F}_{\text{int}}$

Lemma 8. *Let $M, N \in \operatorname{st}\mathcal{F}$ and $f : M \to N$ be an $\mathfrak{A}$-homomorphism. Then $r(\ker(f)) = \ker(f)$, i.e. $\ker(f)$ is complete.*

Proof. Clearly, it is enough to prove this for $\mathfrak{A} = \mathfrak{A}^\alpha$. But in this case the statement is trivial. □

Denote by $\operatorname{st}\mathcal{F}_{\text{int}}$ (resp. $\mathcal{O}_{\text{int}}$) the full direct summand of $\operatorname{st}\mathcal{F}$ (resp. $\mathcal{O}$), which consists of all modules, whose weights belong to the integral weight lattice. For $N \in \mathcal{O}$ by $a(N)$ we will denote the number of simple subquotients in N, which are Verma modules. Thus, for $N \in \mathcal{F}$ holds $a(N) = \operatorname{qVl}(N)$. The key statement of this section is the following.

Proposition 1. *Let $M \in \operatorname{st}\mathcal{F}_{\text{int}}$ and N be a complete submodule in M. Then $N \in \operatorname{st}\mathcal{F}_{\text{int}}$, moreover, there exists a quasi-Verma flag of M of the form* (1) *such that $M_i = N$ for some i.*

To prove this we will need the following lemmas.

Lemma 9. *Let $M \in \mathcal{O}$ and $\alpha \in \pi$. Assume that $X_{-\alpha}$ acts injectively on M. Then $r_\alpha(M)/M$ is a direct sum of finite-dimensional $\mathfrak{A}$-modules.*

Proof. Since $X_{-\alpha}$ acts injectively on M we have $M \subset r_\alpha(M)$. Now the statement follows from an $sl(2)$-computation. □

Lemma 10. *Let M and N be two complete modules in $\mathcal{O}_{\text{int}}$, $N \subset M$. Then any simple submodule in M/N is a simple Verma module. In particular, from $a(M/N) = 0$ follows $M = N$.*

Proof. Clearly it is enough to prove the first statement. Suppose that there is a simple submodule of M/N that has the form $L(\lambda)$, for integral λ which does not belong to the closure of the antidominant Weyl chamber. Hence there is $\alpha \in \pi$, such that $L(\lambda)$ contains a (non-zero) direct sum of finite-dimensional modules with respect to $\mathfrak{A}^\alpha$. Therefore, M/N has elements on which $X_{-\alpha}$ acts locally nilpotent. But by Lemma 3 $r_\alpha(M/N) \supset r_\alpha(M)/r_\alpha(N) \simeq M/N \supset L(\lambda)$, which is impossible, because $X_{-\alpha}$ acts injectively on $r_\alpha(M/N)$. □

Lemma 11. *Let $N \in \operatorname{st}\mathcal{F}_{\text{int}}$, M be a complete module in $\mathcal{O}_{\text{int}}$ and $N \subset M$. Assume that $a(M/N) = 1$. Then M/N is a submodule of some Verma module, in particular, $M \in \operatorname{st}\mathcal{F}_{\text{int}}$.*

Proof. We only have to prove that M/N is a submodule in some Verma module. Let $M(\mu)$ be the simple Verma subquotient of M/N. From Lemma 10 it follows that $M(\mu)$ is the simple socle of M/N. Indeed, consider the submodule $M' = M(\mu)+N$ in M. Let $M'' = \operatorname{st}(M')$. Since $a(M/M'') = 0$, we obtain $M = M''$ from Lemma 10. On the other hand, $M''/N \subset \operatorname{st}(M''/N) \subset \operatorname{st}(M(\mu))$. By Corollary 1 we have $\operatorname{st}(M(\mu)) \simeq M(\mu^+)$ for some μ^+. This completes the proof. □

Lemma 12. *Let $N, M \in \mathrm{st}\,\mathcal{F}_{\mathrm{int}}$, $N \subset M$. Then any quasi-Verma flag of N can be extended to a quasi-Verma flag of M.*

Proof. We use induction in $n = a(M) - a(N)$. If $n = 0$, then $M = N$ by Lemma 10. Now let $M' \supset N$ be a complete submodule of M such that $a(M') = a(M) - 1$. To find such M', extend $0 \subset N \subset M$ to a composition series of M. Then there is a submodule $\hat{M} \subset M$, such that $a(\hat{M}) = a(M) - 1$ and $\hat{M} \supset N$. Set $M' = \mathrm{st}(\hat{M})$. We only have to show that $a(M') = a(\hat{M})$. The last follows from Lemma 9 and the fact that simple Verma modules are direct sums of indecomposable strictly infinite-dimensional $\mathfrak{A}^{\alpha}$-modules for any $\alpha \in \pi$. Now assume that we have already constructed the extension of a quasi-Verma flag from N to M' (inductive assumption). By Lemma 11, M/M' is a submodule of a Verma module and we obtain the desired quasi-Verma flag for M. $\square$

Proof of Proposition 1. We will prove the statement by induction in $a(N)$. First suppose that $a(N) = 0$. Then, by Lemma 10, $N = 0$ and our statement for such N is obvious.

Now we prove the induction step. Suppose that the statement is true for any N' such that $a(N') < a(N)$ and consider a submodule $N' \subset N$ such that $a(N') = a(N) - 1$. Such submodule exists since N has a composition series. We recall that N is complete and set $N'' = \mathrm{st}(N') \subset N$. We have $a(N'') = a(N') < a(N)$ by the same arguments as in Lemma 12, and now N'' is complete. From the inductive assumption to N'', we get, in particular, $N'' \in \mathrm{st}\,\mathcal{F}_{\mathrm{int}}$ and furthermore $a(N/N'') = 1$. Hence, $N \in \mathrm{st}\,\mathcal{F}_{\mathrm{int}}$ by Lemma 11. We complete the proof applying Lemma 12. $\square$

Now the definition of an abelian structure on $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$ is quite transparent. Let $M, N \in \mathrm{st}\,\mathcal{F}_{\mathrm{int}}$, and $f : N \to M$ an $\mathfrak{A}$-homomorphism. By Lemma 8 and Proposition 1, $\ker(f) \in \mathrm{st}\,\mathcal{F}_{\mathrm{int}}$. We define the "image" of f inside this category as $\mathrm{st}(\mathrm{Im}(f))$, which belongs to $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$ by Proposition 1, and the "cokernel" of f as $\mathrm{st}(M/\,\mathrm{st}(\mathrm{Im}(f)))$. From Proposition 1 it follows that $M/\,\mathrm{st}(\mathrm{Im}(f)) \in \mathcal{F}_{\mathrm{int}}$ and hence $\mathrm{st}(M/\,\mathrm{st}(\mathrm{Im}(f))) \in \mathrm{st}\,\mathcal{F}_{\mathrm{int}}$. Moreover, one can see that $\mathrm{qVl}(N) = \mathrm{qVl}(\ker(f)) + \mathrm{qVl}(\mathrm{st}(\mathrm{Im}(f)))$ and $\mathrm{qVl}(M) = \mathrm{qVl}(\mathrm{st}(\mathrm{Im}(f))) + \mathrm{qVl}(\mathrm{st}(M/\,\mathrm{st}(\mathrm{Im}(f))))$.

Lemma 13. *The category $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$ with kernels, images and cokernels defined as above is an abelian category.*

Proof. We have to check the universal properties of the kernel and cokernel only. The universal property of the kernel is trivial since the kernel in $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$ coincides with the kernel in the category of $\mathfrak{A}$-modules. The universal property of cokernel follows easily from Lemma 8. $\square$

Proposition 2. $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$ *is admissible in the sense of [FKM1], i.e. it is an abelian category, a full subcategory in the category of $\mathfrak{A}$-modules, it consists of finitely generated modules and it is stable under tensoring with finite-dimensional $\mathfrak{A}$-modules.*

Proof. $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$ is abelian by Lemma 13 and is a full subcategory in $\mathcal{O}$. Hence we only have to check that $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$ is stable under tensoring with finite-dimensional

$\mathfrak{A}$-modules. Let $M \in \mathrm{st}\,\mathcal{F}_{\mathrm{int}}$ and F be a finite-dimensional $\mathfrak{A}$-module. Clearly $F \otimes M \in \mathcal{O}_{\mathrm{int}}$. We recall that $F \otimes \_$ is an exact functor (in the category of all $\mathfrak{A}$-modules) and any $F \otimes M(\lambda)$, $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$ has a Verma flag ([BGG, D]). Hence $F \otimes M \in \mathcal{F}$. From the exactness of $F \otimes \_$ and the third statement of Lemma 6 it follows also that $\mathrm{st}\,\mathcal{F}$ is stable under tensoring with finite-dimensional modules. Hence $F \otimes M \in \mathrm{st}\,\mathcal{F}$, which completes the proof. □

Lemma 14. *For any simple finite-dimensional module F the functor $F \otimes \_$ is exact on* $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$.

Proof. Follows from Lemma 8 and [De, Theorem 3.1]. □

5. $\mathcal{S}$-subcategories in $\mathcal{O}$ and the main result

The admissible category $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$ of $\mathfrak{A}$-modules constructed in the previous Section extends in a natural way to an admissible category $\Lambda = \Lambda(\mathrm{st}\,\mathcal{F}_{\mathrm{int}})$ of $\mathfrak{H}^{\mathfrak{A}}$-diagonalizable $\mathfrak{A}'$-modules. Now we turn back to the situation described in the Introduction, where we thought about $\mathfrak{A}$ as the semisimple part of the Levi factor of a parabolic subalgebra $\mathcal{P}$ in a simple finite-dimensional Lie algebra $\mathfrak{G}$. Consider the category $\mathcal{O}(\mathcal{P}, \Lambda)$ of $\mathfrak{G}$-modules consisting of finitely generated, $\mathfrak{H}^{\mathfrak{A}}$-diagonalizable, $\mathfrak{N}$-finite modules, which decompose into a direct sum of objects from Λ as $\mathfrak{A}'$-modules (this category was introduced in [FKM1]). It is clear that $\mathcal{O}(\mathcal{P}, \Lambda)$ is a full subcategory of the category $\mathcal{O}$ for $\mathfrak{G}$. At the same time $\mathcal{O}(\mathcal{P}, \Lambda)$ is an abelian category, whose abelian structure is derived from one on Λ. From Proposition 2 and Lemma 14 it follows that $\mathcal{O}(\mathcal{P}, \Lambda)$ is closed under tensoring with finite-dimensional modules, and $F \otimes \_$, F finite-dimensional, is an exact functor on $\mathcal{O}(\mathcal{P}, \Lambda)$. Denote by E the natural inclusion functor from $\mathcal{O}(\mathcal{P}, \Lambda)$ to $\mathcal{O}$. We will use E in order to emphasize that the abelian structures in $\mathcal{O}(\mathcal{P}, \Lambda)$ and $\mathcal{O}$ are different.

Denote by $\mathcal{O}_{\mathrm{triv}}$ the principal block of $\mathcal{O}$. We will say that a full subcategory $\mathcal{M} \subset \mathcal{O}$ is an *$\mathcal{S}$-subcategory* if the following conditions are satisfied.

S1. $\mathcal{M}$ is an abelian category.
S2. $\mathcal{M}$ is stable under tensoring with finite-dimensional $\mathfrak{G}$-modules.
S3. $\mathcal{M}_{\mathrm{triv}} = \mathcal{M} \cap \mathcal{O}_{\mathrm{triv}}$ is a direct summand of $\mathcal{M}$.
S4. $\mathcal{M}$ has enough projective objects, which are also projective in $\mathcal{O}$.
S5. The big projective module P in $\mathcal{O}_{\mathrm{triv}}$ (i.e. the projective cover of the unique simple Verma module in $\mathcal{O}_{\mathrm{triv}}$) belongs to $\mathcal{M}$.
S6. (*Soergel's double centralizer property*) The finite-dimensional algebra corresponding to $\mathcal{M}_{\mathrm{triv}}$ is isomorphic to the endomorphism algebra of P, viewed as a module over its endomorphism ring.

It is clear that $\mathcal{O}$ itself is an $\mathcal{S}$-subcategory in $\mathcal{O}$. Another example of an $\mathcal{S}$ subcategory in $\mathcal{O}$ was constructed in [FKM2] and coincides with the image of the functor E considered in that paper.

The main result of this paper is the following statement.

Theorem 1. $\mathcal{O}(\mathcal{P}, \Lambda(\mathrm{st}\,\mathcal{F}_{\mathrm{int}}))$ *is an $\mathcal{S}$-subcategory in $\mathcal{O}$.*

The rest of this section will be dedicated to the proof of Theorem 1. In fact we will prove a bit more than what is claimed in this theorem. We start with the description of projective modules in Λ and in $\mathcal{O}(\mathcal{P}, \Lambda)$.

Lemma 15. Λ *has a block decomposition with a unique simple module in each block.*

Proof. It is enough to prove this for $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$, which we can decompose with respect to the central characters $\mathrm{st}\,\mathcal{F}_{\mathrm{int}} = \oplus_{\chi\in Z(\mathfrak{A})^*} \mathrm{st}\,\mathcal{F}_{\mathrm{int}}(\chi)$ in a natural way. Let M be a simple module in $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}(\chi)$. Clearly, $\mathrm{qVl}(M) = 1$ and hence $M = M(\lambda)$, where λ is an integral weight lying in the closure of the dominant Weyl chamber, since M is complete. Now the uniqueness of such $M(\lambda)$ in $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}(\chi)$ follows from the Harish-Chandra Theorem ([D, Theorem 7.4.5, Proposition 7.4.7]).

Lemma 16. Λ *has enough projective modules.*

Proof. Again it is enough to prove the statement for $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$. Since any module in $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$ has finite length it is enough to construct a projective cover of the simple module $M(\lambda)$, where λ is an integral weight lying in the closure of the dominant Weyl chamber. Let $M(\lambda) \in \mathrm{st}\,\mathcal{F}_{\mathrm{int}}(\chi)$ for some $\chi \in Z(\mathfrak{A})^*$. Let w_0 be the longest element in the Weyl group and consider the projective module $P(w_0(\lambda))$. Obviously, there is an epimorphism $P(w_0(\lambda)) \to M(\lambda)$ (in $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$, not in $\mathcal{O}$). However, we have to show that $P(w_0(\lambda))$ is a projective object in $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$. From Lemma 6 it follows easily that $P(w_0(\lambda)) \in \mathrm{st}\,\mathcal{F}_{\mathrm{int}}$. Moreover, since $P(w_0(\lambda))$ is projective in $\mathcal{O}$ and by virtue of Lemma 1, we have $\dim(\mathrm{Hom}_{\mathfrak{A}}(P(w_0(\lambda)), M)) = \mathrm{qVl}(M)$ for any $M \in \mathrm{st}\,\mathcal{F}_{\mathrm{int}}(\chi)$. From this it follows that the functor $\mathrm{Hom}\,(P(w_0(\lambda)), \_)$ is exact on $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}(\chi)$ and hence $P(w_0(\lambda))$ is projective in $\mathrm{st}\,\mathcal{F}_{\mathrm{int}}$. Clearly, $P(w_0(\lambda))$ is indecomposable since it is indecomposable in $\mathcal{O}$ and the top of $P(w_0(\lambda))$ coincides with $\mathrm{st}(L(w_0(\lambda))) = M(\lambda)$. This completes the proof. □

Corollary 2. 1. $\mathcal{O}(\mathcal{P}, \Lambda)$ *has a block decomposition with a finite number of simple modules in each block.*
2. $\mathcal{O}(\mathcal{P}, \Lambda)$ *has enough projective modules.*
3. *Any block of* $\mathcal{O}(\mathcal{P}, \Lambda)$ *is equivalent to the module category over a finite-dimensional projectively stratified algebra.*

Note that the definition of "projectively stratified algebra" in the sense of [FKM1, FKM2] coincides with the earlier notion of "standardly stratified algebra" used in [AHLU].

Proof. This follows from Lemma15, Lemma 16 and [FKM1, Sects. 4 and 5], but we present a proof here in order to keep the paper self-contained. First we note that, as $\mathcal{O}(\mathcal{P}, \Lambda)$ is a subcategory of $\mathcal{O}$, the first claim is trivial.

Now, using an analogue of Rocha–Wallach's construction ([RW]), we construct projective modules in $\mathcal{O}(\mathcal{P}, \Lambda)$ as projections on blocks of the modules

$$P(V, k) = U(\mathfrak{G}) \bigotimes_{U(\mathcal{P})} \Big((U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)) \otimes V\Big),$$

where k is big enough and V is projective in Λ. By the same arguments as in [BGG, Theorem 1] this forces the existence of enough projectives in $\mathcal{O}(\mathcal{P}, \Lambda)$ giving us the second claim.

At this stage abstract nonsense tells us that each block of $\mathcal{O}(\mathcal{P}, \Lambda)$ with finitely many simple objects is equivalent to the module category over a finite-dimensional algebra. For an indecomposable projective $V \in \Lambda$ choose the module $M(V) = U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$ to be standard. Under such choice the simple modules in $\mathcal{O}(\mathcal{P}, \Lambda)$ will be exactly the unique simple quotients $L(V)$ of the modules $M(V)$. Fix a block of $\mathcal{O}(\mathcal{P}, \Lambda)$ with a partial order on the isoclasses of simple modules, induced from $\mathfrak{H}^*$. As $\mathcal{P}$ is a parabolic subalgebra, the kernel of the natural surjection from $M(V)$ onto $L(V)$ is filtered by simples, which are less or equal V with respect to our order. Let $P(V)$ be the projective cover of $L(V)$. From the other hand, as tensoring with a finite-dimensional module is exact, $P(V, k)$, and hence any projective in $\mathcal{O}(\mathcal{P}, \Lambda)$ is filtered by standard modules (this follows using the arguments analogous to [BGG, Proposition 2]). From the construction of $P(V, k)$ it follows that $M(V)$ occurs exactly one time in such a filtration and all other standard modules correspond to strongly bigger simples (again with respect to our order). Clearly, this means that the same is true for $M(V)$ and by definition we obtain that the finite-dimensional algebra of our block is projectively stratified. This completes the proof. □

Proposition 3. *Any indecomposable projective module in $\mathcal{O}(\mathcal{P}, \Lambda)$ is also an indecomposable projective module in $\mathcal{O}$. Furthermore, $P(\lambda) \in \mathcal{O}(\mathcal{P}, \Lambda)$ (for $\lambda \in \mathfrak{H}^*$) if and only if λ is $\mathfrak{A}$-integral and belongs to the closure of the $\mathfrak{A}$-antidominant Weyl chamber.*

Proof. Let V be a projective module in Λ. From the proof of Lemma 16 it follows that as an $\mathfrak{A}$-module, P is a direct sum of some $P(\mu)$, where μ is $\mathfrak{A}$-integral and belongs to the closure of the antidominant Weyl chamber. Recall one more time that that any projective in $\mathcal{O}(\mathcal{P}, \Lambda)$ can be constructed as a projection on a block of the module $P(V, k)$, defined in the proof of Corollary 2. Directly from this construction it follows that any projective in $\mathcal{O}(\mathcal{P}, \Lambda)$ is projective in $\mathcal{O}$. Since the functor $F \otimes \_$ is exact for any finite-dimensional F, the module $P(V, k)$, as an $\mathfrak{A}$-module is also a direct sum of some $P(\mu)$, where μ is $\mathfrak{A}$-integral and belongs to the closure of the antidominant Weyl chamber. This completes the proof. □

Set $\mathcal{O}(\mathcal{P}, \Lambda)_{\mathrm{triv}} = \mathcal{O}(\mathcal{P}, \Lambda) \cap \mathcal{O}_{\mathrm{triv}}$. It is clear that $\mathcal{O}(\mathcal{P}, \Lambda)_{\mathrm{triv}}$ is a direct summand of $\mathcal{O}(\mathcal{P}, \Lambda)$.

Corollary 3. *The big projective module from $\mathcal{O}$ belongs to $\mathcal{O}(\mathcal{P}, \Lambda)$, moreover it is an indecomposable projective object in $\mathcal{O}(\mathcal{P}, \Lambda)$.*

Proof. Obvious. □

Corollary 4. *The projectively stratified finite-dimensional algebra B associated with the block $\mathcal{O}(\mathcal{P}, \Lambda)_{\mathrm{triv}}$ is a subalgebra of the quasi-hereditary finite-dimensional algebra A associated with $\mathcal{O}_{\mathrm{triv}}$. Futhermore, the canonical duality on A restricts to a duality on B.*

Proof. The first statement follows directly from Proposition 3. Let * denote the canonical duality on $\mathcal{O}$. To prove the second statement it is enough to show that $P^* \in \mathcal{O}(\mathcal{P}, \Lambda)$ for any projective $P \in \mathcal{O}(\mathcal{P}, \Lambda)$. We have already seen that as an $\mathfrak{A}$-module, P is a direct sum of some $P(\mu)$, where μ is $\mathfrak{A}$-integral and belongs to the closure of the antidominant Weyl chamber. Now the statement follows from the fact that each such $P(\mu)$ is self-dual (i.e. a tilting module for $\mathfrak{A}$). $\square$

We note that since $\mathcal{O}(\mathcal{P}, \Lambda)$ is a full subcategory (E is a full functor), the endomorphism ring of the big projective module is the coinvariant algebra ([S2]). Now we can prove our main result.

Proof of Theorem 1. We only have to prove the Soergel's double centralizer property. For that purpose we are going to use abstract notations. Let A (resp. B) denote the algebra associated with $\mathcal{O}_{\text{triv}}$ (resp. $\mathcal{O}(\mathcal{P}, \Lambda)_{\text{triv}}$). According to Corollary 4, B is a matrix subalgebra of A. Let e be the primitive idempotent of A such that Ae is the big projective module in $\mathcal{O}_{\text{triv}}$. Then Be is the big projective module in $\mathcal{O}(\mathcal{P}, \Lambda)_{\text{triv}}$ and $C = eAe = eBe$ is the coinvariant algebra, which is the endomorphism algebra of Ae and Be. Let $T = \text{Hom}_A(Ae, \_)$ denote the functor used in Soergel's proof ([S2]). Recall that by Soergel's Theorem ([S2, Struktursatz 2]) for any $M \in \mathcal{O}_{\text{triv}}$ and any projective $Q \in \mathcal{O}_{\text{triv}}$ holds

$$\text{Hom}_A(M, Q) \simeq \text{Hom}_{C=eAe}(T(M), T(Q)).$$

We start from $B = \text{Hom}_B(B, B)$. Since E is a full functor, we have $\text{Hom}_B(B, B) \simeq \text{Hom}_A(E(B), E(B))$. Now applying Soergel's result we obtain that $\text{Hom}_A(E(B), E(B)) \simeq \text{Hom}_{eAe}(T(E(B)), T(E(B)))$. We know that $eAe = eBe$. Recall that $E(Be) = Ae$, hence $T(E(B)) = \text{Hom}_A(Ae, E(B)) = \text{Hom}_A(E(Be), E(B)) \simeq \text{Hom}_B(Be, B) = eB$. Finally,

$$\text{Hom}_{eAe}(T(E(B)), T(E(B))) \simeq \text{Hom}_{eBe}(eB, eB).$$

By Corollary 4, the algebra B has a duality, from which it follows that $\text{Hom}_{eBe}(eB, eB) \simeq \text{Hom}_{eBe}(Be, Be)$. This completes the proof. $\square$

6. Tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$

In Corollary 4 we have shown that there exists a natural duality on B, or more generally on $\mathcal{O}(\mathcal{P}, \Lambda)$. We will denote this duality by *, as for $\mathcal{O}$. We also know that for any projective $P \in \mathcal{O}(\mathcal{P}, \Lambda)$ the corresponding dual module $P^* \in \mathcal{O}(\mathcal{P}, \Lambda)$ can be computed in $\mathcal{O}$ (i.e. the dual modules to P in $\mathcal{O}(\mathcal{P}, \Lambda)$ and in $\mathcal{O}$ are isomorphic). Having a duality it is natural to consider the tilting modules.

Let V be an indecomposable projective module in Λ. Setting $\mathfrak{N}V = 0$, we define an induced module $M_{\mathcal{P}}(V) = U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$, which we will call a *standard module*. Since V is projective in Λ, as an $\mathfrak{A}$-module $M_{\mathcal{P}}(V)$ is a direct sum of projective modules in Λ and hence is self-dual as an $\mathfrak{A}$-module.

Lemma 17. *Let $M_{\mathcal{P}}(V) \in \mathcal{O}(\mathcal{P}, \Lambda)$ be a standard module. Then the dual modules to $M_{\mathcal{P}}(V)$ in $\mathcal{O}(\mathcal{P}, \Lambda)$ and in $\mathcal{O}$ are isomorphic.*

Proof. We reduce our consideration to a block of $\mathcal{O}(\mathcal{P}, \Lambda)$, which corresponds to a projectively stratified finite-dimensional algebra. Let S be the partially ordered set of simple modules. Then S also parametrizes the standard modules. From the construction of the projective modules in $\mathcal{O}(\mathcal{P}, \Lambda)$ it follows that $M_{\mathcal{P}}(V)$ can be written as $P(V)/N$, where $P(V)$ is an indecomposable projective module, N has a standard filtration and all the standard subquotients of this filtration are bigger than $M_{\mathcal{P}}(V)$ with respect to S. We know that the dual modules for $P(V)$ in $\mathcal{O}(\mathcal{P}, \Lambda)$ and $\mathcal{O}$ coincides. Now the statement follows by induction in S. □

We will call $M_{\mathcal{P}}(V)^*$, V is an indecomposable projective in Λ, *costandard modules*. Consider the full subcategory $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\bigtriangledown)$) of $\mathcal{O}(\mathcal{P}, \Lambda)$ which consists of all modules having a *standard filtration* (resp. *costandard filtration*), i.e. a filtration, whose quotients are standard (resp. costandard) modules.

Corollary 5. *Let $M \in \mathcal{F}(\Delta) \cup \mathcal{F}(\bigtriangledown)$. Then the dual modules to M in $\mathcal{O}(\mathcal{P}, \Lambda)$ and in $\mathcal{O}$ are isomorphic.*

Proof. Follows from Lemma 17 and exactness of the dualities. □

A module $M \in \mathcal{F}(\Delta) \cap \mathcal{F}(\bigtriangledown)$ will be called a *tilting* module. Hence, by virtue of Corollary 5, it should be a tilting module in $\mathcal{O}$. It is known (see for example [KK]) that any tilting module in $\mathcal{O}$ is a direct sum of indecomposable tilting modules and there is a natural bijection between indecomposable tilting modules and simple modules in $\mathcal{O}$. Let $T(\lambda)$, $\lambda \in \mathfrak{H}^*$ denote the unique indecomposable tilting module in $\mathcal{O}$, whose Verma flag starts with $M(\lambda)$. First of all we determine the $T(\lambda)$ belonging to $\mathcal{O}(\mathcal{P}, \Lambda)$.

Lemma 18. *$T(\lambda) \in \mathcal{O}(\mathcal{P}, \Lambda)$ if and only if λ is $\mathfrak{A}$-integral and belongs to the closure of the $\mathfrak{A}$-dominant Weyl chamber.*

Proof. Let $M \in \mathcal{O}(\mathcal{P}, \Lambda)$ be a module having a standard filtration. This filtration can be refined to a Verma flag in $\mathcal{O}$. Let $M(\lambda)$ be a Verma submodule in M occuring in this Verma flag. Then $M(\lambda)$ is complete in Λ and hence λ is $\mathfrak{A}$-integral and belongs to the closure of the $\mathfrak{A}$-dominant Weyl chamber. Therefore, the only candidates for being in $\mathcal{O}(\mathcal{P}, \Lambda)$ are $T(\lambda)$, which satisfy the condition of our Lemma.

Let w_0 denote the longest element in the Weyl group of $\mathfrak{A}$. First consider $T(\mu)$, where μ is $\mathfrak{A}$-integral and belongs to the closure of the $\mathfrak{A}$-dominant Weyl chamber, such that $M(w_0(\mu))$ is simple. Then $T(\mu)$ is a self-dual standard module and hence $T(\mu) \in \mathcal{O}(\mathcal{P}, \Lambda)$. To complete the proof we recall that $\mathcal{O}(\mathcal{P}, \Lambda)$, $\mathcal{F}(\Delta)$ and $\mathcal{F}(\bigtriangledown)$ are closed under tensoring with finite-dimensional modules and any $T(\lambda)$ such that λ satisfies the condition of our Lemma can be obtained as a direct summand in $T(\mu) \otimes F$ for some finite-dimensional F and some $T(\mu)$ as above ([CI]). □

Theorem 2. *Any tilting module in $\mathcal{O}(\mathcal{P}, \Lambda)$ is a direct sum of indecomposable tilting modules of the form $T(\lambda)$, where λ is $\mathfrak{A}$-integral and belongs to the closure of the $\mathfrak{A}$-dominant Weyl chamber.*

Proof. We have already proved that all $T(\lambda)$, where λ is $\mathfrak{A}$-integral and belongs to the closure of the $\mathfrak{A}$-dominant Weyl chamber, are tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$. Recall that blocks of $\mathcal{O}(\mathcal{P}, \Lambda)$ correspond to projectively stratified finite-dimensional algebras. Now the uniqueness of an indecomposable tilting module corresponding to a given simple module follows from an abstract result [AHLU, 2.1 and 2.2] on tilting modules over stratified algebras. □

Consider again the algebra B, which corresponds to $\mathcal{O}(\mathcal{P}, \Lambda)_{\text{triv}}$ and let T be the direct sum of all indecomposable tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)_{\text{triv}}$. We recall that End (T) is usually called the *Ringel dual* of B.

Theorem 3. *B is isomorphic to its Ringel dual.*

Proof. Let S denote the semi-regular $U(\mathfrak{G})$-bimodule ([S1, S3]) and let λ_0 be the highest weight of the trivial $\mathfrak{G}$-module. Let w_0 be the longest element in the Weyl group W of $\mathfrak{G}$. Then the functor $S \otimes \_$ maps $P(w(\lambda_0))$ to $T(ww_0(\lambda_0))$ for any $w \in W$ ([S3]). Note that if $w(\lambda_0)$ belongs to the closure of the $\mathfrak{A}$-antidominant Weyl chamber, then $ww_0(\lambda_0)$ belongs to the closure of the $\mathfrak{A}$-dominant Weyl chamber. Hence $S \otimes \_$ transfers projective modules from $\mathcal{O}(\mathcal{P}, \Lambda)_{\text{triv}}$ to tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)_{\text{triv}}$. Thus $S \otimes \_$ produces an isomorphism between B and its Ringel dual. □

7. Some other subcategories of $\mathcal{F}_{\text{int}}$

In this Section we again restrict our attention to the algebra $\mathfrak{A}$. Above we have been working with the subcategory st $\mathcal{F}_{\text{int}}$ of $\mathcal{F}$. Here we introduce other subcategories of $\mathcal{F}$ and study a connection between them and st $\mathcal{F}_{\text{int}}$. As a corollary we construct a functor with properties analogous to $S\otimes\_$. We begin with the following observation.

Lemma 19. *Let $M, N_i \in \mathcal{F}$, $N_i \subset M$ and* $\text{qVl}(M) = \text{qVl}(N_i)$, *$i \in I$. Then $N = \cap_{i\in I} N_i \in \mathcal{F}$ and* $\text{qVl}(N) = \text{qVl}(M)$.

Proof. The assumption $\text{qVl}(M) = \text{qVl}(N_i)$ means that each module N_i contains all the composition factors of M which are simple Verma modules. The same then is true for the intersection N.

We use induction on $\text{qVl}(M)$. For $\text{qVl}(M) = 1$ the statement follows from the fact that any Verma module has a simple socle, which is a simple Verma submodule. Let us prove the induction step. Fix a quasi-Verma flag of M as in (1) and $\text{qVl}(M) = k$. Then for $i = 1, 2$

$$0 = M_0 \cap N_i \subset M_1 \cap N_i \subset M_2 \cap N_i \subset \cdots \subset M_{k-1} \cap N_i \subset M_k \cap N_i = N_i$$

is a quasi-Verma flag of N_i. Moreover, this quasi-Verma flag is non-degenerate since $\text{qVl}(N_i) = \text{qVl}(M)$. By the inductive assumption, $M_{k-1} \cap N \in \mathcal{F}$ and its quasi-Verma length equals $k - 1$.Then $N/(M_{k-1} \cap N)$ is a non-trivial submodule in M_k/M_{k-1} and the lemma follows. □

Let W be the Weyl group of Δ. We will denote by $\leq$ the Bruhat order on W (assuming that the identity is the maximal element of W). Fix $w \in W$ and let w_0 be the minimal element in W with respect to $\leq$ (i.e. w_0 is the longest element in W). For $M \in \mathcal{F}$ let $\varphi_w(M)$ denote the intersection of all submodules N in M which satisfy the following condition: $(M : L(w'(\lambda))) = (N : L(w'(\lambda)))$ for any $w' \leq w$ and any dominant λ. By Lemma 19, $\varphi_w(M) \in \mathcal{F}$. For any $M, N \in \mathcal{F}$ and any homomorphism $f : M \to N$ one has $f(\varphi_w(M)) \subset \varphi_w(N)$, hence φ_w can be considered as a well-defined endofunctor of $\mathcal{F}$, which acts on the homomorphisms by restriction. Let $\min_w \mathcal{F}_{\text{int}}$ (resp. $\min_w \mathcal{F}$) denote the full subcategory of $\mathcal{F}_{\text{int}}$ (resp. $\mathcal{F}$), which consists of all M such that $f_w(M) = M$. The key result of this Section is the following statement.

Theorem 4. 1. *The functors* $\text{st} : \min_w \mathcal{F}_{\text{int}} \to \text{st}\,\mathcal{F}_{\text{int}}$ *and* $\varphi_w : \text{st}\,\mathcal{F}_{\text{int}} \to \min_w \mathcal{F}_{\text{int}}$ *are mutually inverse equivalences of categories. In particular,* $\min_w \mathcal{F}_{\text{int}}$ *has a natural abelian structure.*

2. *For any* $M \in \text{st}\,\mathcal{F}_{\text{int}}$ *(resp.* $M \in \min_w \mathcal{F}_{\text{int}}$*) and any finite-dimensional module* F *holds* $\varphi_w(M \otimes F) \simeq \varphi_w(M) \otimes F$ *(resp.* $\text{st}(M \otimes F) \simeq \text{st}(M) \otimes F$*). In particular,* $\min_w \mathcal{F}_{\text{int}}$ *is closed under tensoring with finite-dimensional modules.*

Proof. The second statement follows from the first one and [De, Theorem 3.1], so we have to prove the first statement only. From Lemma 6 and Lemma 21 it follows that $\text{st}(\varphi_w(M)) \simeq M$ and $\varphi_w(\text{st}(N)) \simeq N$ for any $M \in \text{st}\,\mathcal{F}_{\text{int}}$ and $N \in \min_w \mathcal{F}_{\text{int}}$. From Lemma 8 and Lemma 10 it follows that φ_w is faithful. Hence, to complete the proof we only have to show that φ_w is full. Denote by $\min_w \mathcal{F}_{\text{int}}$ the image $\varphi_w(\text{st}\,\mathcal{F}_{\text{int}})$. Then $\min_w \mathcal{F}_{\text{int}}$ is an abelian category, whose abelian structure is inherited from $\text{st}\,\mathcal{F}_{\text{int}}$ via φ_w. Moreover, we know that φ_w is faithful. Hence, $\min_w \mathcal{F}_{\text{int}}$ and $\text{st}\,\mathcal{F}_{\text{int}}$ are equivalent and it remains to show that $\min_w \mathcal{F}_{\text{int}}$ is a full subcategory in $\min_w \mathcal{F}_{\text{int}}$. We have to prove a lemma before we can complete the proof.

Lemma 20. 1. *Category* $\min_w \mathcal{F}_{\text{int}}$ *has a block decomposition with a unique simple module in each block.*

2. *Category* $\min_w \mathcal{F}_{\text{int}}$ *has enough projective modules, in particular the big projective module in the principal block of* $\mathcal{O}$ *is projective in* $\min_w \mathcal{F}_{\text{int}}$.

Proof. The proof is analogous to that of Lemma 15 and Lemma 16. We only note that simple modules in $\min \mathcal{F}_{\text{int}}$ are Verma modules $M(w(\lambda))$, and indecomposable projectives are $P(w_0(\lambda))$, λ integral dominant. □

Now we return to the proof of Theorem 4 in order to prove that $\min_w \mathcal{F}_{int}$ is a full subcategory in $\min_w \mathcal{F}_{\text{int}}$. Let P be a projective in $\text{st}\,\mathcal{F}_{\text{int}}(\chi)$, $\chi \in Z(\mathfrak{A})^*$. Then $\varphi_w(P) = P$ by Lemma 16 and Lemma 20. Moreover, we have for any $M \in \text{st}\,\mathcal{F}_{\text{int}}(\chi)$ an equality $\dim \text{Hom}\,(P, M) = \dim \text{Hom}\,(P, \varphi_w(M)) = \text{qVl}(M)$, since P is projective in $\mathcal{O}$.

Let $M, N \in \text{st}\,\mathcal{F}_{\text{int}}(\chi)$ and $f : \varphi_w(M) \to \varphi_w(N)$ be a morphism. Let P be the projective cover of M, which is also the projective cover of $\varphi_w(M)$. Let $a : P \to \varphi_w(M)$ be a canonical epimorphism. Since P is projective, there exists $x : P \to M$ and $y : P \to N$ such that $\varphi_w(x) = a$ and $\varphi_w(y) = f \circ a$. We also have $\ker a \subset \ker f \circ a$. Hence $\ker x \subset \ker y$, since φ_w is a restriction. Therefore, for $m \in M$ we can define $\psi(m) = y \circ x^{-1}(m)$ and obtain that ψ is a well-defined morphism and $\varphi_w(\psi) = f$. This completes the proof. □

Corollary 6. *All categories* $\min_w \mathcal{F}_{\text{int}}$ *are blockwise equivalent.*

Proof. Follows immediately from Theorem 4. □

Remark 1. Having Theorem 4 available, one can produce more natural equivalences between the categories $\min_w \mathcal{F}_{\text{int}}$. Let $\alpha \in \pi$ and s_α be the simple reflection with respect to α. Assume that $w \le w'$ and $w = s_\alpha w'$. Then r_α and f_w are mutually inverse equivalences between $\min_w \mathcal{F}_{\text{int}}$ and $\min_{w'} \mathcal{F}_{\text{int}}$. Indeed, we already have the abelian structure on both $\min_w \mathcal{F}_{\text{int}}$ and $\min_{w'} \mathcal{F}_{\text{int}}$ inherited from $\text{st}\,\mathcal{F}_{\text{int}}$ and we know that r_α and f_w are full, faithful and exact with respect to the image. Moreover, r_α sends simple (resp. projectives) from $\min_w \mathcal{F}_{\text{int}}$ to simples (resp. projectives) in $\min_{w'} \mathcal{F}_{\text{int}}$. Since any object in $\min_w \mathcal{F}_{\text{int}}$ has finite length, everything follows by standard induction in the length of a module.

Set $\min \mathcal{F}_{\text{int}} = \min_{w_0} \mathcal{F}_{\text{int}}$ and $\min = f_{w_0}$.

Lemma 21. $M \in \min \mathcal{F}_{\text{int}}$ *if and only if for any* $\alpha \in \pi$*, the module* M*, when viewed as an* $\mathfrak{A}^\alpha$*-module, is a direct sum of tilting modules in the corresponding category* $\mathcal{O}$*.*

Proof. Consider M as an $\mathfrak{A}^\alpha$-module. Let N be a maximal direct sum of tilting modules contained in M. From the definition of r_α we have $r_\alpha(N) \supset M$. Hence it is enough to show that N is an $\mathfrak{A}$-submodule of M. The last follows by standard arguments from the fact that tilting modules are stable under tensoring with finite-dimensional modules and $U(\mathfrak{A})$ is a direct sum of finite-dimensional $U(\mathfrak{A}^\alpha)$-modules under the adjoint action. □

Corollary 7. *Category* $\min \mathcal{F}_{\text{int}}$ *contains all tilting modules from* $\mathcal{O}_{\text{int}}$.

Proof. Follows directly from Lemma 21, Theorem 4, and the fact that tilting modules are stable under tensoring with finite-dimensional modules and [CI]. □

It is easy to see that the intersection of $\text{st}\,\mathcal{F}_{\text{int}}$ and $\min \mathcal{F}_{\text{int}}$ is an additive closure of the sum of all $P(\lambda)$, λ integral antidominant. Moreover, the functors min and st have some properties, which are analogous to that of Soergel's functor $S \otimes \_$ (see Theorem 3). In fact, one has the following.

Proposition 4. *Let* λ *be an integral dominant weight. Then for any element* $w \in W$ *holds* $\min(P(w(\lambda))) \simeq T(ww_0(\lambda))$ *and* $\text{st}(T(ww_0(\lambda))) \simeq P(w(\lambda))$.

Proof. It is enough to prove the first equality. In the simplest case we have $\min(P(\lambda)) = \min(M(\lambda)) \simeq M(w_0(\lambda)) = T(w_0(\lambda))$. Moreover, for any finite-dimensional module F we have $\min(P(\lambda) \otimes F) \simeq T(w_0(\lambda)) \otimes F$ by Theorem 4. Now the statement follows by induction applying the projection on the corresponding block of $\mathcal{O}$. □

Corollary 8. *For integral dominant* λ *and* $w_1, w_2 \in W$ *there is an equality* $[T(w_1(\lambda)) : M(w_2(\lambda))] = [P(w_1 w_0(\lambda)) : M(w_2 w_0(\lambda))]$.

Proof. Follows from Proposition 4 and the fact that min and st are exact with respect to the abelian structures on st $\mathcal{F}_{\rm int}$ and min $\mathcal{F}_{int}$ (the last coming from st $\mathcal{F}_{\rm int}$ via min) by induction with respect to the Bruhat order on W. □

In particular, this also gives an independent proof of Soergel's character formulae for tilting modules in the case of finite-dimensional Lie algebras ([S3]) and of Theorem 3.

In the same way as in Section 5, we can associate with $\min_w \mathcal{F}_{\rm int}$ an admissible category $\hat{\Lambda} = \hat{\Lambda}(\min_w \mathcal{F}_{\rm int})$ of $\mathfrak{H}^{\mathfrak{A}}$-diagonalizable $\mathfrak{A}'$-modules. It is not a big surprise that $\mathcal{O}(\mathcal{P}, \Lambda)$ and $\mathcal{O}(\mathcal{P}, \hat{\Lambda})$ are closely connected. In fact, the following statement is true.

Theorem 5. $\mathcal{O}(\mathcal{P}, \Lambda)$ *and* $\mathcal{O}(\mathcal{P}, \hat{\Lambda})$ *are blockwise equivalent.*

Proof. It is easy to verify that st and f_w can be extended to mutually inverse equivalences between $\mathcal{O}(\mathcal{P}, \Lambda)$ and $\mathcal{O}(\mathcal{P}, \hat{\Lambda})$. The main point to be checked here is that $f_w(M)$, $M \in \mathcal{O}(\mathcal{P}, \Lambda)$ is a $\mathfrak{G}$-module. This follows from the second statement of Theorem 4 and the fact that $U(\mathfrak{G})$ is a direct sum of finite-dimensional $\mathfrak{A}$-modules under the adjoint action. The rest is standard. □

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