Demographic Research Monographs

David P. Smith
Nathan Keyfitz

Mathematical Demography

Selected Papers

Edited by
Kenneth W. Wachter
and Hervé Le Bras

Second, revised edition

Springer

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Foreword to New Edition of Smith and Keyfitz

The test of a classic is whether its echo can be heard today. We at the Max Planck Institute for Demographic Research are very pleased to be publishing this new edition of Smith and Keyfitz' collection of demographic classics. We hope that this new edition will amplify the echoes of the great demographic discoveries of the last centuries.

This new edition comes from both practical and intellectual needs of demographers. On a practical side, Smith and Keyfitz' 1977 edition is out of print and unavailable to anyone wishing to have it on their shelves. With this re-publication, we are not only making hardcopies once again available but also, as is our tradition with the Springer Demographic Research Monographs, ensuring that the entire publication is accessible on-line as open access at www.demogr.mpg.de/books/drm. It will be easier than ever for modern scholars and students to obtain the classics and their accompanying commentary.

On the intellectual side, the classics of the field represent essential nourishment for the modern demographer, giving a sense of the long history of demographic science as well as to see the insights of classic discoveries as their discoverers framed them. We are thankful to Kenneth W. Wachter and Hervé Le Bras, the editors of this new edition, for updating Smith and Keyfitz' collection, selecting the most enduring works and adding new commentaries where needed. As they explain in their editors' preface, the inspiration for this new edition had its source in Rostock at the Institute's 2009 celebration of Demography's birthday on February 27th, the day when John Graunt first read his ”Observations” at the Royal Society in London.

The publication of this new edition coincides with two important anniversaries – 2012 is the 350th anniversary of John Graunt's foundational work, and 2013 is the centenary of Nathan Keyfitz' birth – a propitious time for the ring of demography's classics to be heard once again.

Rostock
December, 2012

Joshua R. Goldstein
James W. Vaupel

Preface to the Second Edition

Mathematical demography has a distinguished early history, intertwined with discoveries by the great figures of mathematics. Our generation of demographers learned this history from the volume which we now in large part republish. David Smith and Nathan Keyfitz in their collection of 1977 made the founding works available to all. But this collection has long been out of print, making it far more difficult for the present generation of demographers to locate their own works within the tradition of their forerunners.

The idea of republishing "Smith and Keyfitz" arose during a symposium at the Max Planck Institute for Demographic Research in Rostock, Germany, on 27 February 2009. The symposium, held in the Süssmilch Auditorium, was convened by James Vaupel and Joshua Goldstein to inaugurate several years of celebration of the 350th birthday of demography. The birthday follows on the February 1662 appearance of the *Natural and Political Observations ... Made Upon the Bills of Mortality...* published under the name of "John Graunt, Citizen of London". The event in Rostock also celebrated the contributions of Peter Laslett. Peter inspired and guided the fusion of mathematical demography with historical sociology and studies of aging which marks the discipline of population studies today. He also organized and edited the series of full facsimile reprintings by Gregg International Publishers of the classics of demography which have served over the intervening decades as indispensable companions to the Smith and Keyfitz volume.

The past cannot just be archived. New steps forward bring into the light new meanings for previous formulations and give new salience to what may have been obscure. Likewise, the genealogy of an idea has power to reshape its evolving future form. We hope that readers of this volume will feel encouraged to submit to current demographic journals studies of other works important to the history of mathematical demography and write about the meaning of those works in their own time and ours.

The lasting influence of Smith and Keyfitz is found not only in the works they made accessible, but even more in the commentaries they wrote about these works. These commentaries explain the terminology, notation, and intention of the original authors and go on to situate their results in the framework of Twentieth-Century mathematical demography. The commentaries combine historical scholarship with scientific insight.

The 514 pages of the 1977 volume exceed the size desirable for a reprinting. A selection of papers to retain was made at the outset of this project. In their original collection, Smith and Keyfitz included works as late as the 1970s which are now readily available on the world-wide web and hardly need reprinting. Not all the choices of works that were recent in 1977 have stood the test of time and some then-recent works not chosen have come into the limelight now. A decision was made to omit from this reprinting the nearly contemporary works and concentrate on older already classic contributions.

A decision was also made not to reprint the section from the 1977 volume devoted to "Branching Processes and Other Stochastic Processes". This subject remains central to mathematical demography, but our understanding of its history was subsequently transformed by the rediscovery of Bienyamé's contributions. The story is told by Heyde and Seneta [18] and the original works are readily accessible.

We retain the full commentaries by Smith and Keyfitz including discussion of works omitted from this reprinting. New editorial notes are inserted within bold curly brackets, marked "Ed. Note:", to alert the reader to anything omitted and to set the commentaries into the context of our own Twentyfirst Century.

The date of original publication of this collection, at the end of the 1970s, marked a watershed in the discipline. Three central themes dominate the volume – lifetables, stable populations, and measures of fertility and its component processes. These themes all go back to the early days of demography. Advances continue to be made in these areas, but by 1977 the established core was coming to completion. Stable population theory was being rounded out and turned into a versatile part of the repertory, particularly by Ansley Coale and Nathan Keyfitz himself. Digital computers were ending the need for ingenuity in numerical computations that had gone hand-in-hand with the study of lifetables and projections. The spread of sample surveys into developing countries eased the challenges for indirect measures of fertility championed by William Brass.

Today, a third of a century later, mathematical demography is enlivened by new directions, which have taken their place beside the themes featured by Smith and Keyfitz. We think of non-stable theory, demographic feedback models, heterogeneity and frailty, spatial demography, and the mathematical innovations of biodemography. These endeavors trace some of their ancestry to the founders represented in this volume, but also to other forerunners whose roles we hope future historical studies will bring to center stage.

As this reprinting was in preparation, Nathan Keyfitz passed away at the age of 96. A few months before his death, speaking with him by telephone, one of us (KWW) told him about plans for this project. He was delighted. He still took an active interest in mathematical demography, and in the whole range of demographic thinking, and he looked forward to having this cherished work back in print. In preparing it we have benefited from encouragement and advice from Barbara

Keyfitz. We also salute David R. Smith, who took the lead in crafting the original 1977 volume. We dedicate this reprinting to Nathan. His own works are classics in our field.

Berkeley, California, June 2013 Kenneth W. Wachter
Paris, June 2013 Hervé Le Bras

Preface to the First Edition

This volume is an effort to bring together important contributions to the mathematical development of demography and to suggest briefly their historical context. We have tried to find who first thought of the several concepts and devices commonly used by demographers, what sort of problem he was facing to which the device or concept seemed the solution, and how his invention developed subsequently in the hands of others.

Historically, the book starts with a Roman table of life expectancies from the third century a.d. about which we know little, and with John Graunt's explorations in an area that was still popularly suspect when he wrote in 1662. These are followed by the astronomer Halley, who looked into the field long enough to invent the life table and to notice that Their Majesties would take a sizeable loss on the annuity scheme they had just launched; and by Euler, who was first to devise the formulas of stable population theory and to apply them to filling gaps in data. To these we add the handful of further contributions in the 19th century and many pieces from the explosion of contributions that began in this century with Lotka. We doubt that we have managed to trace everything back to its ultimate beginning, and suspect that our nominees in some cases have been anticipated by predecessors who will be turned up by other students.

The works we include form a living heritage in demography: Graunt; Halley; Euler; Lotka; Milne, who formalized life table construction; Lexis, who was preoccupied with the way members of a population are situated simultaneously in age and in time, and showed how a plane chart, now known as a Lexis diagram, can help analysis. Much less alive, and largely excluded here, are such notions as that of George King, that graduation of data for a life table was more accurate from pivotal death rates calculated at five-year intervals; John Graunt's belief that the right way to describe the dynamics of a population was as the *ratio* of births to deaths, without considering age; and devices that once reduced the labor of numerical calculation but are obsolete in a computer age. These and many other ideas that have proved to be dead ends and are now of merely antiquarian interest we tried to distinguish from those that were part of a chain of development that is still advancing. As far as we could discriminate our excerpts are confined to the latter.

To determine which works most deserve attention among the large number written has not been easy, and we have undoubtedly made mistakes both of inclusion and of omission. We were far from insisting on subtle mathematical ideas, but did look for the effective uses of mathematics that have come to be assimilated into population work. Articles that profess to deal with population but whose main interest was mathematics we tried to avoid, and we avoided them doubly if they were a mere import from some other subject that seemed unlikely ever to be naturalized in population analysis. Some ideas and techniques have a kind of *droit de la cité* in contemporary population study, and we hope these are the ones that predominate in our selections.

To find passages that were self-contained and suitable for contemporary reading was occasionally difficult. Writers often used symbols well known to their place and time, and their immediate readership had no need for definitions we would now miss. To this the earlier works add key formulas with no hint as to how they are derived. Where we expect readers to have trouble as a result, because we did, we include a brief explanation of what is being done.

The choice of excerpts from the classic articles and books rather than complete reprints in all cases was dictated partly by economy of publication, but this was not the only constraint. Benjamin Gompertz fairly compactly introduces his Law of mortality, but spends above fifty pages fitting it to life tables and working out its implications for annuity payments. Harro Bernardelli published the first article on the use of matrices in population projection in the *Journal of the Burma Research Society*, which is not a source that most of us would come across in our ordinary reading. He has top priority for inclusion, but he deals partly with problems of the Burmese economy under British colonial rule and with speculations on cyclic events that do not carry much interest for readers today. Leslie, whose reading in a sickbed had taken him deep into the mathematical properties of matrices, went into cogredient and contragredient transformations that are unlikely to have demographic application. We saw no need to burden the reader with these only to have him discover at the end that he would never need them.

In editing we did not strip down our authors to the point of losing the context of their contribution to our subject. We learned much of an incidental character in our reading and have tried to retain that richness. Where substantive omissions are made we note these for the reader's benefit.

Several topics that fall in the province of demography are not included, among them treatments of human spatial ecology, urbanization, and migration. Omission is partly due to space limitations, and partly to lack of confidence in our ability to decide what is basic in fields whose mathematical explorations are recent and expanding rapidly.

We expect from the reader at least some background in calculus and matrix algebra, and several papers will require an understanding of stochastic processes. The reader lacking a background in elementary mathematics will find the greater part of the book difficult.

Secondary accounts of much of what we present can be found in Keyfitz (1968), and stochastic processes are well handled in Feller (1968) and Chiang (1968). Our chief sources for the early histories given here are Hendriks (1852, 1853), Westergaard (1969), and Lorimer (1959).

Cambridge, MA 1977

David Smith
Nathan Keyfitz

Nathan Keyfitz: A Biographical Sketch

Nathan Keyfitz was the world's leading figure in mathematical demography in the second half of the Twentieth Century. His books *Introduction to the Mathematics of Population* in 1968 and *Applied Mathematical Demography* in 1977, with expanded editions in 1985 and 2005, defined the subject for generations of students and scientists. They were accompanied by other major volumes, the collection *World Population: An Analysis of Vital Data* from 1968 with Wilhelm Flieger, with successor volumes in 1972 and 1991, as well as *Causes of Death: Life Tables for National Populations* with Samuel Preston and Robert Schoen, and the collection from 1977 with David Smith, *Mathematical Demography: Selected Papers,* reprinted here. These works tied together practical demography with mathematical analysis and continue in use today.

Nathan Keyfitz was born on 29 June 1913 in Montreal, Canada. His family had emigrated from Mogilev, now in Belarus, around 1900. After taking a degree in Mathematics at McGill University in 1934, he looked for an academic position, but there were few if any opportunities at that time of great depression. For some months he worked in life insurance societies. In 1936 he was hired as a clerk at the Dominion Bureau of Statistics in Ottawa (later Statistics Canada) where he had to correct the inconsistencies in the individual census results, a monotonous and boring activity. Taking an interest in the data themselves, comparing results from the Canadian census with those from the American and English censuses, led to admonishment by his immediate chief. But it attracted the attention of R.H. Coats, head of the Bureau, and Coats encouraged Keyfitz' vocation as a researcher alongside his mundane duties.

John Robbins, head of education statistics at the Bureau, encouraged Keyfitz to consider studying for a Ph.D. and directed him toward an opportunity at the University of Chicago. William Ogburn, head of the Sociology Department at Chicago offered him a Carnegie Foundation Fellowship, and in 1942 Keyfitz began graduate study there. After a year, Keyfitz returned to the Dominion Bureau of Statistics but continued work on a dissertation and received a Ph.D. in Sociology from the University of Chicago in 1952. In Chicago, he attended lectures by Louise

Wirth and Ernest Burgess, leaders of what became known as the Chicago School of urban sociology.

In 1946, Keyfitz was sent to Raleigh, North Carolina, to a six week session of courses in statistics taught by Ronald Fischer, Abraham Wald, William Cochran, and Frank Yates. It was the beginning of what he called “chance meetings that establish firm friendships” which he enjoyed throughout his long life. A year later he was sent to Geneva to represent Canada at a United Nations subcommission on sampling, meeting such great statisticians as Fisher, Prasanta Chandra Mahalanobis and Georges Darmois.

Having climbed a number of steps in the hierarchy of the Dominion Bureau of Statistics, Keyfitz was taken as an expert to advise South Asian statistical and census offices: in 1951 in Burma where he became friends with travel writer Norman Lewis; in Indonesia in 1952–1953 where he he met his life-long friend, the French mathematician and academician Marc-Paul Schützenberger. At that time Schützenberger was serving as a medical expert for the World Health Organization. All his life Keyfitz remained a lover of Indonesia and he returned in 1964, 1979, 1984, and 1989 for long stays.

Keyfitz served for eighteen months as technical director for the Colombo Plan for Cooperative Development. Stationed in Ceylon, now Sri Lanka, his duties took him to Laos, Vietnam, Singapore and India. At the invitation of Mahalanobis he resided some months at the Statistical Institute in Calcutta together with the geneticist J.B.S. Haldane. During these years of statistical practice, he did not abandon his desire to do research.

In 1959 he took up his first academic position at the University of Toronto in the Department of Economics, Political Science, and Sociology. He described leaving the Dominion Bureau of Statistics with more hopes than regrets. He was 45 years old, but on the threshold of what would be a prodigiously long career in academic demography.

Keyfitz remained on the faculty at Toronto until 1962, teaching the works of Weber and Durkheim among others. The next decade brought a series of moves from university to university. In 1962, figuratively speaking, he climbed Montreal’s Mont Royal to teach for a year at the French Université de Montréal. It was only a kilometer from his undergraduate university, McGill, but for a person born and educated in the English-speaking part of the city, it was a venture into unknown territory. In 1963 he returned to the University of Chicago as Professor of Sociology. From that time on, he could concentrate on mathematical demography, a field that he practically invented as he drew together into a unified whole piecemeal work from many disciplines. In 1968 he headed west to the University of California, Berkeley.

On the urging of Fred Mosteller, founder of the Statistics Department at Harvard University, Keyfitz set about writing what would become his influential *Introduction to the Mathematics of Population.* The book galvanized the field and established his reputation. He served as president of the Population Association of America in 1970 and was elected to the American Academy of Arts and Sciences in 1971 and the National Academy of Sciences, USA, in 1977. It was rapid recognition, although not quite as swift as his forerunner in demography, John Graunt, who after signing the *Natural and Political Observations ...* in 1661 was elected a member of the Royal Society in England the same year.

The move to U.C. Berkeley came at the invitation of Kingsley Davis and Judith Blake who were leading a new Department of Demography there. The four years at Berkeley can be considered the acme of Keyfitz' scientific production. He wrote seminal papers on topics ranging from the evaluation of demographic projections and the structure of causes of mortality around the world to the number of people who have lived on the earth.

The Berkeley years were also, as he later described them, the most intense period of his broader academic life. The entire university system was under pressure from protests and demonstrations. Keyfitz felt close to students involved in condemnation of the war in Viet Nam. The department became politically polarized, especially after an arson fire struck a building in which Kingsley Davis had an office. There were tensions between Judith Blake and others, some surrounding a joint course of lectures which Keyfitz delivered on contraception and its history with John T. Noonan at the U.C. Berkeley School of Law, later a U.S. Appeals Court judge.

In the wake of turmoil at Berkeley, Keyfitz accepted an invitation by Roger Revelle to come to Harvard. Other demography faculty also departed, Samuel Preston to the University of Washington and Etienne van de Walle to the University of Pennsylvania. The Berkeley Department of Demography was abolished. It was refounded in 1978 as a Graduate Group and later again as a Department under the leadership of Eugene A. Hammel and has been known for carrying forward the tradition of mathematical demography which Keyfitz initiated.

At Harvard, Keyfitz was based partly at the Center for Population Studies led by Roger Revelle in a quaint building on Bow Street in Cambridge, Massachusetts, partly in the Department of Sociology, and partly across the Charles River at the Harvard School of Public Health. One of the editors of the present volume, Kenneth Wachter, shared an office with him at Bow Street. These were the years of the consolidation of Stable Population Theory at Keyfitz' hands and the years when his celebrated book *Applied Mathematical Demography* took shape.

Harvard was not Keyfitz' final harbor. He taught there from 1972 to 1983. He had a part-time position at the Ohio State University between 1981 and 1983, and a little later spent three months at Stanford working with Marc Feldman, Brian Arthur, and Paul Ehrlich. In 1984 he took on leadership of the population program at the International Institute for Applied Systems Analysis (IIASA) in Laxenburg outside Vienna, Austria. IIASA had been founded to promote East-West scientific cooperation under the shadow of the Cold War. Keyfitz made it a center for creative mathematical demography, fostering the work of a new generation of leaders including James W. Vaupel. While based at IIASA up to 1993, Keyfitz travelled throughout eastern Europe, although he never obtained the opportunity of visiting Mogilev, the cradle of his ancestors.

After 1995, Keyfitz returned to live by the Charles River in Cambridge, Massachusetts. He remained intellectually active. There was hardly a week without a visit to his former department at Harvard and phone-calls and exchanges with a wide scientific circle. Each year a reflective and thought-provoking Christmas letter from the Keyfitz family was eagerly awaited by their many friends.

Keyfitz married Beatrice Orkin in 1939. They were a devoted couple throughout their life, until her death in 2009. His achievements would have been unthinkable without her love, support, and intellectual stimulation, and the welcome she gave to the many people whose lives they both touched. Their daughter Barbara was born in 1944 and their son Robert in 1947. Barbara went on to be Professor of Mathematics at Ohio State University and Robert to be Senior Economist at the World Bank and at the International Monetary Fund.

By grasping opportunities offered partly by chance, Keyfitz found himself able to combine the very different aspects of being a demographer. As a sociologist, he wrote on the role of the middle class in the developed as well as in the developing world. As a statistician, he had 23 years of practice in a national statistical office. As a mathematician, he gave a new unity to the mathematical study of populations. Interested in biology, he was one of the founders of the journal *Theoretical Population Biology*. He was engaged with economics and wrote on family allowances and on retirement schemes.

In this variety, Keyfitz was helped by the diversity of his friends around the world, by his ability to collaborate, and by his sense of balance. He could converse on the one hand with Paul Ehrlich, who at one time advocated a maximum population of a billion humans, and on the other with Roger Revelle who did not exclude the possibility of feeding forty billion humans. He was on friendly terms with spokespersons for the theory of evolution and with critics of it, with Stephen Jay Gould and with Marc-Paul Schützenberger. It was too much, perhaps, to hope at Berkeley to reconcile Judith Blake with John T. Noonan. But for him what was important was not to reach the same opinion but to engage in free discussion.

Keyfitz had a strong sense of history. In collaboration with David Smith, he hearkened back to founding works of demographic theory and method, gave them careful scholarly readings, and thought hard about their relationship to the progress of his own time. The collection which is reprinted here represents the fruits of this collaboration and this endeavor. It stands as a tribute to Nathan Keyfitz' legacy.

(Principal sources for Nathan Keyfitz' life are the Memoir that he wrote, available on the Keyfitz family website at www.keyfitz.org, and the bibliography posted by Harvard University at oasis.lib.harvard.edu. The editors have also drawn on their own reminiscences for this brief sketch.)

David P. Smith: A Biographical Sketch

David P. Smith became well-known in the demographic world already as a Harvard graduate student in 1977 when he edited, together with Nathan Keyfitz, the collection reprinted here. He went on to a notable career in applied demography at the World Fertility Survey and the University of Texas at Houston. Retiring in 2005, he remains active in the field.

David Paul Smith was born on 26 September 1944 in Detroit, Michigan. He matriculated at Albion College in 1962, where he took classes and came to know John A. Ross, who was a professor at Albion at the time. In 1964, Smith transferred to the University of California, Berkeley and completed a B.A. in the study of developing nations in 1966. He joined the Peace Corps for two years, serving in Turkey and renewing his friendship with the Ross family, who were stationed in Ankara with the Population Council. After his Peace Corps service, Smith spent a semester at the School of Public Health at the University of Michigan. He then arranged to work for the Population Council in Seoul, working on family planning surveys and data analysis with John Ross in the Korean Family Planning Program. *Mathematical Demography: Selected papers* would later be dedicated to John Ross.

In 1971, after two years in Korea, Smith applied for graduate study at Berkeley, but the Demography Department was being closed down. Nathan Keyfitz had moved to Harvard. John Ross recommended Smith to Keyfitz, who invited Smith to visit. Smith immediately drove from Michigan to Massachusetts, and after an hour's conversation with Keyfitz was admitted to Harvard's graduate program in Sociology. His classes included Keyfitz' memorable introduction to mathematical demography. As Smith recently recalled,

> The book got written because Nathan mentioned in class one day that he had a collection of the pioneering papers in mathematical demography he was hoping to put into a book. I went to his office to find out who the pioneers were, thinking it was one of the things a graduate student in demography ought to know. After a short

> conversation l left with a stack of xeroxes and a vague commitment to help him on the book. By the next summer I was starting to edit papers to get them down to essential content, filling in some holes in the collection, tracking down the originals in Widener Library to be photographed, and starting on the chapter introductions.

Smith's dissertation, completed in 1978 under Keyfitz' supervision, tested empirical evidence for the theory put forward by Richard Easterlin to explain cycles in U.S. fertility. Papers based on the dissertation in *Population Studies* and *Theoretical Population Biology* played an important role in empirical critiques of Easterlin's theory. At the time, mathematical understanding of the population dynamics implied by Easterlin-style systems of demographic feedback was limited, but would flourish in the following decade.

While at Harvard, Smith married Joan M. Smith and moved to Hartford a couple of years before his Ph.D. was awarded in 1979. Stepsons Kenneth Clair Smith and Russell L. Smith had been born in 1965 and 1968. From Hartford, David Smith commuted periodically to Cambridge, Massachusetts to assist in classes Keyfitz was teaching. Then Smith was hired by the World Fertility Survey and moved to London, where he implemented methods for computing fertility measures, family planning indices, and life table estimates from computerized files of survey responses.

In 1981 Smith returned to the United States and joined the faculty of the University of Texas at the Health Science Center in Houston as Associate Professor of Demography. He served at Texas throughout the next 24 years of his career. Early research at Texas included studies of duration of breastfeeding in Sri Lanka and in the United States and patterns of prevalence of ischemic heart disease. From 1989 to 1994 he served as Director and Principal Investigator of the TexNet Project funded by the Hogg Foundation. The TexNet Project was devoted to improving the quality of evaluation research in the analysis of teen service programs in Texas relating to pregnancy. Guides for assisting programs in analyzing data and appreciating the limitations that small non-random samples place on statistical analysis were published in 1992 and 1994.

Over the next decade Smith continued his contributions to evaluation research, including analysis of the Episcopal Health Charities Healthy Communities Project with funding from the Center for Health Policy Studies at the University of Texas, Houston, analysis of substance abuse prevention programs in San Antonio directed at pregnant and parenting teens, and a major assessment of perinatal health needs at the turn of the millenium.

In 1992, Smith returned to his early interests in mathematical demography, publishing the volume *Formal Demography* with Plenum Press. Toward the end of the decade, in collaboration with Hélène Rossert, he completed a translation of Alfred Lotka's *Théorie Analytique des Associations Biologique,* a volume in which Lotka summed up his discoveries and a valuable supplement to the papers by Lotka reprinted in the present volume.

In the following years, in collaboration with Benjamin Bradshaw, Smith undertook an extensive project on the estimation of cause-specific death rates for 72 causes of death, by Hispanic and non-Hispanic ethnicity, race, and age. This work, commissioned by the National Center for Health Statistics, led Smith to focus on the role of inconsistencies in self-identification of ethnicity between Census returns and death certificates. It culminated in an important paper by Smith and Bradshaw in 2006, "Rethinking the Hispanic Paradox", in the *American Journal of Public Health.* The extent to which apparently favorable health and survival outcomes among U.S. Hispanics reflect artifacts of the estimation process, a question much in the public eye, continues to be a focus of Smith's research.

(The authors are grateful to David Smith for supplying information and bibliographic data on which this biographical sketch is based.)

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Part I
The Life Table

Mathematical demography has its modern beginnings in the gradual development of correct procedures for forming life tables, and in a single remarkable paper by Leonard Euler (1760), paper 11 in Part II, that introduced stable age distributions. One far older work of at least some quality survives: a third century A.D. table of life expectancies attributed to Ulpian (paper 1) that remained in use in northern Italy through the 18th century. The table and an accompanying discussion have been taken from C. F. Trenerry (1926).

The mathematician Girolamo Cardano took up the problem briefly in 1570, but without substantive results. Cardano made the assumption that a man who took great care in all things would have a certain life expectancy α, so that $\overset{\circ}{e}_x = \alpha - x$ for all ages x, and then asked what part of this might be forfeited by a relaxation of prudence. He proposed letting life expectancy fall by $\frac{1}{40}$ of its value during each year in which a man was reasonably careful but not fastidious: by the nature of the life expectancies, a man might be born with the prospect of living say 260 years and yet die at age 80, having every year thrown away by inattention a part of what remained to him (Cardano 1570, pp. 204—211). A modern interpretation would be that a cohort carries its life table with it. The result was not generalized to populations.

John Graunt's *Natural and Political Observations Upon the Bills Mortality* (1662) is the first substantive demographic work to have been written. The book is occasionally curious but most often impressive, even from a perspective of three hundred years: Graunt culled a remarkable amount of information from the christening and death lists begun in the later plague period and usually understood its implications. Parts of the treatise are included here as paper 2. { **Ed. Note:** This founding work of demography is surrounded by some mystery. A reconstruction of the algorithm for the 1662 lifetable and an investigation into questions of authorship may be found in [22] }

A second work of great importance followed upon Graunt's, Edmund Halley's (1693) presentation of the Breslau (Wroclaw Poland) life table (paper 3). Halley had made an effort to obtain the Breslau lists in order to see what might be done with them, after learning of their apparent quality.

The methods of calculation Halley used in his life table were partly informal, as in his remarks on stationarity and in his unorthodox subtraction (where ℓ_x is the number of survivors to exact age x in the life table from among ℓ_0 births and L_x represents the number between ages x and $x+1$)

$$890[=\ell_1] - 198 = 692[= L_6],$$

explained by the oblique statement: "198 do die in the *Five Years* between 1 and 6 compleat, taken at a *Medium*."

The terminology has created confusion down to the present century. Raymond Pearl (1922, p. 83), apparently reading the L_x terms that make up Halley's table as ℓ_x, calculated life expectancy at birth as 33.5 years by the table instead of the correct 27.5. The mistake is carried over in Dublin, Lotka and Spiegelman (1949, p. 34).

Johan DeWit (1671) preceded Halley in the correct calculation of annuities, using exact (ℓ_x) as against Halley's approximate (L_x) denominators, and most of Halley's other *Uses* can be answered differently, but the quality of Halley's table and discussion much surpasses the few earlier works and several of the subsequent ones. His table is graphed below alongside Ulpian's, Graunt's, DeWit's, and as references the middle level table for Crulai c. 1700 (Gautier and Henry 1958, pp. 163, 190) and one of the Coale and Demeny (1966) model life tables.

After Halley, the next impressive contribution was the series of life tables for annuitants and monastic orders by Antoine Deparcieux, printed in 1746. The accuracy of Deparcieux's data was sufficient for him to show that adult life expectancies had been increasing over the previous half century. Deparcieux calculated his $\overset{\circ}{e}_x$ values by the simple but adequate formula

$$\overset{\circ}{e}_x = \frac{\sum_{i=x}^{\omega} (\ell_i - \ell_{i+1})(i + 0.5 - x)}{\ell_x} = \frac{\sum_{i=x}^{\omega} \ell_i}{\ell_x} - 0.5 .$$

Of their utility he writes (1760, pp. 58—59):

"Les vies moyennes [i.e., $\overset{\circ}{e}_x$] sont ce qui m'a paru de plus commode pour faire promptement & sans aucun calcul, la comparaison des different ordres de mortalité qu'on a établis ... [Life expectancies are what have appeared to me most convenient for making promptly and without any calculation a comparison of different orders of mortality that one has established...]."

Two later efforts merit attention here: Daniel Bernoulli (1766) introduced continuous analysis and suggested the force of mortality [$\mu(x)$] in an application of differential calculus to the analysis of smallpox rates. Later Emmanuel Etienne Duvillard (1806), in an article that also introduced the T_x column (defined as $T_x = \sum_{x}^{\omega} L_i$), applied Bernoulli's method to estimate the increase in life expectancy that would follow if smallpox were eliminated by Edward Jenner's vaccine. The calculus, which these and most modern work employ, dates to a seventy year period

(1665—1736) between Isaac Newton's first investigations and the publication of his principal works. Westergaard (1969, pp. 92—93) comments however that it was not until the late nineteenth century that continuous analysis was widely enough understood for Bernoulli's work to be appreciated.

Joshua Milne in his excellent *Treatise on the Valuation of Annuities* (1815), which includes a careful analysis of life tables made prior to his, was first to suggest a formula by which ℓ_x values could be calculated for real populations. His is the well known expression

$$d_x = \ell_x \left(\frac{D_x}{P_x + \frac{1}{2} D_x} \right)$$

where d_x is the number dying between ages x and $x + 1$ in the life table population, D_x represents calendar year deaths between these ages in an observed population, and $P_x + \frac{1}{2} D_x$ constructs an initial population analogous to ℓ_x except for scale by adding half of the yearly deaths to the observed midyear population P_x.

Excerpts from Milne's discussions of the life table and of age-specific fertility rates (due to Henrie Nicander (1800, pp. 323—324)) are given in paper 4. Milne's method for graduating data from grouped to single ages, not the best, has been omitted. His footnoted criticism of Thomas Simpson's (1742) work opens an area of discussion that is easily missed: Milne's life table was misread by William Sutton (1884) – whose clarification in 1874 of the construction of Richard Price's 1771 Northampton Table is a more competent work – but was immediately reestablished by George King (1884). Like other fields, demography does not only move forward.

The fifth article in this section excerpts from George King (1902), whose notation is contemporary, his explanations of terms of the life table. From the middle of the 19th century William Farr standardized much of the life table, but he did not put his work in a form at all comparable to King's excellent textbook. A recent addition to the life table, from C. L. Chiang (1960a), is the term ${}_na_x$. This is King's unremembered "average amount of existence between ages x and $x + n$, belonging to those who die between these ages," i.e.: ${}_na_x = \frac{{}_nL_x - \ell_{x+n}}{{}_nd_x}$.

Out of sequence, the Lexis (1875) diagram is introduced in paper 6. For most of a century it has been a standby of all analysis attempting to relate age and time. Among contemporary works, those of Roland Pressat (1969, 1972) exploit it most fully. { **Ed. Note:** Wilhelm Lexis' own version of the diagram is not the one in use today. Several of Lexis' contemporaries played a role in the invention of diagrams of this kind and controversies around priority occurred in the later 1800s, as discussed in [37]. }

The important contributions to the life table in this [20-th] century have been competent abridgement techniques for generating tables by five or ten year age groupings in place of single years of age. The Lowell Reed and Margaret Merrell (1939) article, paper 7, did much to establish the validity of abridgement techniques by its introduction of an attractive expression:

$$ {}_nq_x = 1 - \exp[-n\,{}_nm_x - an^3\,{}_nm_x^2] $$

for estimating ${}_nq_x$ from ${}_nm_x$ values where wide age groupings are used. In the expression, ${}_nm_x$ is the age-specific death rate in the life table population ages x to $x + n$ (that is, ${}_nm_x = {}_nd_x/{}_nL_x$ and ${}_nq_x$ the probability of dying within the interval for a person of exact age x, (${}_nq_x = {}_nd_x/\ell_x$). By empirical examination the authors found that the constant a required by the expression could be the same for all ages above infancy. Reed and Merrell examine two other approximations to ${}_nq_x$, the first of which:

$$ {}_nq_x = \frac{n\,{}_nm_x}{1 + \frac{n}{2}\,{}_nm_x} $$

can be derived from Milne's formula for d_x, the other due apparently to Farr (1864, pp. xxiii—xxiv), and evident earlier to Gompertz (1825, paper 25 in Part III.

$$ {}_nq_x = 1 - \exp[-n\,{}_nm_x]. $$

Following their article we include derivations for both expressions.

T. N. E. Greville (1943) was able through ingenious expansions to derive each of these equations by working with the definitions of ${}_nm_x$ and ${}_nq_x$, and to show that the Reed-Merrell formula incorporates Gompertz' Law that the force of mortality is an exponential function of age. The assumption is appropriate at older ages and inappropriate for infancy, and thus defines the age range over which the Reed-Merrell formula is applicable. In the same article Greville discusses approximations to ${}_nL_x$ values where, as before, age groupings are wide (paper 8).

The methods used by Greville can be generalized to take advantage of the observed age structure of a population as well as its mortality schedule. In the simplest case this gives rise to the formula, due to Nathan Keyfitz and James Frauenthal (1975),

$$ {}_nq_x = 1 - \exp\left[-n\,{}_nM_x + \frac{n}{48\,{}_nP_x}({}_nP_{x+n} - {}_nP_{x-n})({}_nM_{x+n} - {}_nM_{x-n})\right] $$

with as before the caveat that infancy requires separate consideration.

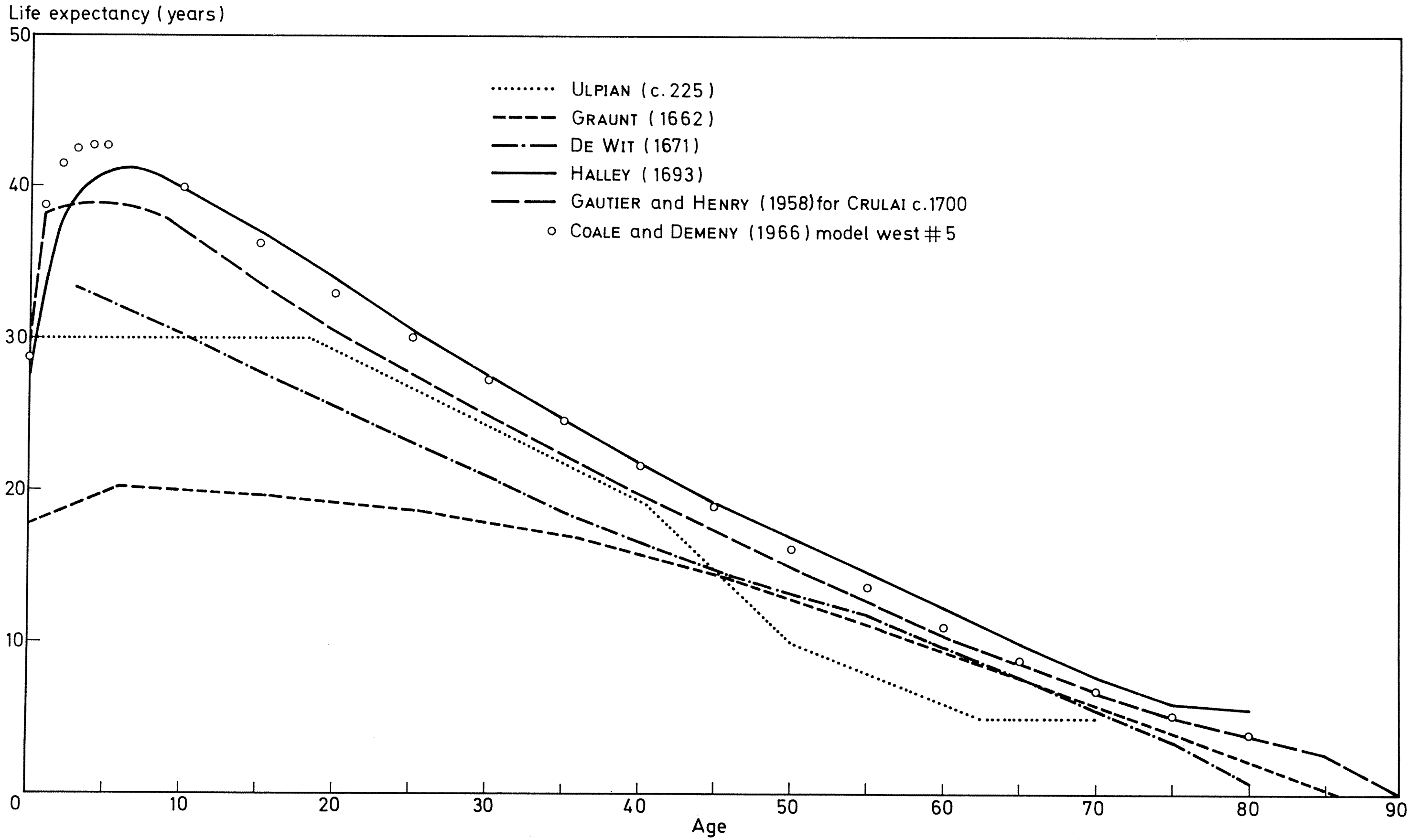
The chapter concludes with excerpts from the well known article by Edward Deevey (1947) in which he evaluates efforts that had been made up to that time to develop life tables for animal populations.

{ **Ed. Note:** The methods for abridged lifetables featured in papers 7 and 8 are now of mainly historical interest. It has become standard practice to calculate abridged lifetables directly from tables with single-year intervals, selecting the required ℓ_x values and cumulating the ${}_1L_x$ values for the table. Exposition is found in the documentation for the Human Mortality Database developed by John Wilmoth and Vladimir Shkolnikov on the World Wide Web, the leading resource for high-quality estimates of mortality rates and lifetables.

The subject of lifetables is taken up again in Part IV of this volume, which includes parametric mortality models and model lifetables. By far the most important has come to be the 1825 model of Benjamin Gompertz [16] discussed there.

In the first decade of the Twentyfirst Century, the subject of lifetables has been dominated by applications of proportional hazard models capitalizing on an extension of statistical regression techniques to survival analysis from 1972 by Sir David Cox [10]. A proportional hazards approach to unobserved heterogeneity, the frailty modeling introduced by James Vaupel, Kenneth Manton, and Eric Stallard [38] has come into widespread use.

Estimation of non-human lifetables has gone far beyond the works discussed by Deevey. Progress is described, for instance, in [29] for animals and [35] for plants. Our knowledge of life-history parameters for populations of individuals from other species continues to grow rapidly, stimulated by progress in biodemography. An introduction to the field of biodemography is found in the collection by Wachter and Finch [42]. }


Life expectancy (years)
50
40
30
20
10
0
10
20
30
40
50
60
70
80
90
Age
Ulpian (c. 225)
Graunt (1662)
De Wit (1671)
Halley (1693)
Gautier and Henry (1958) for Crulai c.1700
Coale and Demeny (1966) model west # 5

1. Tables of Annuity Values Which Were Sanctioned by the Roman Law for the Purposes of the Lex Falcidia

C. F. TRENERRY (1926)

From *The Origin and Early History of Insurance*, pp. 151—152. London: P.S. King & Son.

On the question of whether Ulpian's table contains an interest factor Hendriks (1852, p. 224) writes: "... as it was the special duty of the jurist to look to a sufficient fund being created for the discharge of the obligations under such annuities, he is not likely in those times to have taken much account of *interest*, which at best could only have been continuously obtained with great difficulty." The table values should be recognized as excessive if an interest rate much above 1% is incorporated, and we believe with Hendriks that none is. The table is reproduced below:

Age	Provision
0—19	30 years
20—24	28
25—29	25
30—34	22
35—39	20
40	19
41	18
42	17
43	16
44	15
45	14
46	13
47	12
48	11
49	10
50—54	9
55—59	7
60+	5

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_1,

No trace of tables of premiums, etc., used by Romans.

Lex Falcidia.

Two tables of annuity values authorized.

Although there is no trace of any attempt on the part of the Romans to construct a table of mortality or rates of subscriptions payable by members of burial clubs, yet there are records of two tables which were sanctioned by the civil law for the capitalization of annuities. The necessity for such tables arose from the provisions of the *Lex Falcidia*,[1] by which it was ruled that the heir or heirs to an estate should receive not less than one-quarter of the total property left by the testator. As it frequently happened that the will was made when the testator was in better circumstances than at the time of his death, it became necessary in such cases for the values of the various legacies to be proportionately reduced in order that the total value should not exceed three-quarters of the estate. In the case of the ordinary legacy, this reduction presented no difficulty, but when the testator had left a life annuity to one or more legatees, the question arose as to the basis of reduction of the values of such annuities. In order to meet this requirement, it was necessary to capitalize the annuities, and, in order to permit of this being done, a table of annuity values, known subsequently as "Macer's Table," was used. This table was, as will be seen from the quotation below, authorized in a *responsum* of the jurisconsult Macer, who, at the same time, sanctioned the use of another and more correct table, the authorship of which he attributed to Ulpian.[1'] In the case of the first table,

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[1] Passed 40 B.C. See *Digest*, XXXV, ii, 1.

[1'] *Digest*, XXXV, ii, 68. Æmilius Macer, Lib. 2, ad legem vicesimam hereditatium. " Computationi in alimentis faciendæ hanc formam esse Ulpianus scribit, ut a prima ætate usque ad annum vicesimum quantitas alimentorum triginta annorum computetur, ejusque quantitatis Falcidia præstetur : ab annis vero viginti usque ad annum vicesimum-quintum annorum viginti octo : ab annis vigintiquinque usque ad annos triginta, annorum vigintiquinque : ab annis triginta usque ad annos trigintaquinque, annorum vigintiduo : ab annis trigintaquinque usque ad annos quadraginta, annorum viginti : ab annis quadraginta usque ad annos quinquaginta, (tot) annorum computatio fit, quot ætati ejus ad annum sexagesimum deerit, remisso uno anno : ab anno vero quinquagesimo usque ad annum quinquagesimum quintum, annorum novem : ab annis quinquagentaquinque usque ad annum sexagesimum annorum septem : ab annis sexaginta cujuscunque ætatis sit, annorum quinque. . . . Solitum est tamen a prima ætate usque ad annum trigesimum computationem annorum triginta fieri, ab annis vero triginta, tot annorum computationem inire, quot ad annum sexagesimum deesse videntur, nunquam ergo amplius quam triginta annorum computatio initur."

Construction of Macer's Table quite arbitrary.

it is clear on inspection that the values are arbitrary, and in no way based on a mortality experience. Ulpian's Table, however, shows distinct signs of an attempt to represent more nearly the actual values of annuities.

Ulpian's Table based on life experience.

It has been held by various writers[2] that the values given by Ulpian are those of the expectation of life rather than those of annuities, and the reason put forward for this assumption is that the Romans were unlikely to have used the factor of interest when compiling the table. There seems, however, to be no particular justification for this assumption. It is generally admitted that the table of Ulpian shows signs of having been calculated with direct reference to either a mortality or a subscription experience, but, so far, no satisfactory suggestion has been made of the way in which the table was constructed.

Table of Ulpian probably table of values of a_x, not of e_x

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[2] Cf. Hendriks, *J.I.A.*, VI, p. 313.

2. Natural and Political Observations Mentioned in a Following Index, and Made Upon the Bills of Mortality

JOHN GRAUNT (1964 (1662))

From *Journal of the Institute of Actuaries* 90. Excerpts are from pages 15, 19—23, 35—38, 44—47.

In extracting from Graunt's observations we have tried to give the reader a feel both for Graunt's imaginativeness and tenacity and for the materials he had to work with. We include both of his estimates for London's population: the first (460,000) he found by attributing to the city one-fourteenth of a national population estimate of 6,440,000 not well derived. The second (384,000), from his more careful knowledge about London, is probably of the correct order of magnitude, and suggests a reasonable total population figure (5 to 5.5 million) (cf. Wrigley 1967, pp. 44—45). Graunt's life table entries and his comments on London's doubling time are not of this quality.

We have mostly avoided Graunt's discussion of specific diseases, and have omitted his remarks on the expansion of London beyond the old city walls, on the healthfulness of the city and countryside, and on the country bills.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_2,

CHAPTER I

OF THE BILLS OF MORTALITY, THEIR BEGINNING, AND PROGRESS

THE first of the continued weekly Bills of Mortality extant at the Parish-clerks Hall, begins the 29th of December, 1603, being the first year of King James his reign; since when, a weekly account hath been kept there of burials and christenings. It is true, there were Bills before, viz. for the years 1592,–93,–94, but so interrupted since, that I could not depend upon the sufficiency of them, rather relying upon those accounts which have been kept since, in order, as to all the uses I shall make of them.

2. I believe that the rise of keeping these accounts, was taken from the Plague: for the said Bills (for aught appears) first began in the said year 1592, being a time of great mortality; and after some disuse, were resumed again in the year 1603, after the great Plague then happening likewise. ...

10. We have hitherto described the several steps, whereby the Bills of Mortality are come up to their present state; we come next to shew how they are made and composed, which is in this manner, viz. when anyone dies, then, either by tolling or ringing of a bell, or by bespeaking of a grave of the Sexton, the same is known to the searchers, corresponding with the said Sexton.

11. The Searchers hereupon (who are ancient matrons, sworn to their Office) repair to the place where the dead corpse lies, and by view of the same, and by other enquiries, they examine by what disease or casualty the corpse died. Hereupon they make their Report to the Parish-clerk and he, every Tuesday night, carries in an account of all the burials and christenings, happening that week, to the clerk of the Hall. On Wednesday the general account is made up and printed, and on Thursdays published, and dispersed to the several families, who will pay four shillings per annum for them.

12. *Memorandum.* That although the general yearly Bills have been set out in the several varieties afore-mentioned, yet the original entries in the Hall-books were as exact in the very first year as to all particulars, as now; and the specifying of casualties and diseases, was probably more.

CHAPTER II

GENERAL OBSERVATIONS UPON THE CASUALTIES

IN my discourses upon these Bills I shall first speak of the casualties, then give my observations with reference to the places and parishes comprehended in the Bills; and next of the years, and seasons.

1. There seems to be good reason why the Magistrate should himself take notice of the numbers of burials and christenings, viz. to see whether the City increase or decrease in people; whether it increase proportionately with the rest of the Nation; whether it be grown big enough, or too big, etc. But why the same should be made known to the People, otherwise than to please them as with a curiosity, I see not.

2. Nor could I ever yet learn (from the many I have asked, and those not of the least sagacity) to what purpose the distinction between males and females is inserted, or at all taken notice of; or why that of marriages was not equally given in? Nor is it obvious to everybody, why the account of casualties (whereof we are now speaking) is made? The reason, which seems most obvious for this latter, is, that the state of health in the City may at all times appear.

3. Now it may be objected that the same depends most upon the accounts of epidemic diseases, and upon the chief of them all, the *Plague*; wherefore the mention of the rest seems only a matter of curiosity.

4. But to this we answer, that the knowledge even of the numbers which die of the *Plague*, is not sufficiently deduced from the mere report of the Searchers, which only the Bills afford; but from other ratiocinations, and comparings of the *Plague* with some other casualties.

5. For we shall make it probable that in years of Plague a quarter part more dies of that disease than are set down; the same we shall also prove by the other casualties. Wherefore, if it be necessary to impart to the world a good account of some few casualties, which since it cannot well be done without giving an account of them all, then is our common practice of so doing very apt, and rational.

6. Now, to make these corrections upon the perhaps, ignorant, and careless Searchers' Reports, I considered first of what authority they were in themselves, that is, whether any credit at all were to be given to their distinguishments: and finding that many of the casualties were but matter of sense, as whether a child were *Abortive*, or *Stillborn*; whether men were *Aged*, that is to say, above sixty years old, or thereabouts, when they died, without any curious determination whether such aged persons died purely of *Age*, as for that the innate heat was quite extinct, or the radical moisture quite dried up (for I have heard some candid physicians complain of the darkness which themselves were in hereupon) I say, that these distinguishments being but matter of sense, I concluded the Searchers' Report might be sufficient in the case.

7. As for *Consumptions*, if the Searchers do but truly report (as they may) whether the dead corpse were very lean and worn away, it matters not to many of our purposes whether the disease were exactly the same as physicians define it in their books. Moreover, in case a man of 75 years old died of a cough (of which had he been free, he might have possibly lived to ninety) I esteem it little error (as to many of our purposes) if this person be in the table of casualties, reckoned among the *Aged* and not placed under the title of *Coughs*.

8. In the matter of Infants I would desire but to know clearly, what the Searchers mean by Infants, as whether children that cannot speak, as the word Infant seems to signify, or children under two or three years old, although I should not be satisfied, whether the Infant died of *Wind*, or of *Teeth*, or of the *Convulsion*, etc. or were choked with *Phelgm*, or else of *Teeth*, *Convulsion*, and *Scowring*, apart, or together, which, they say, do often cause one another: for, I say, it is somewhat, to know how many die usually before they can speak, or how many live past any assigned number of years.

9. I say it is enough if we know from the Searchers but the most predominant symptoms; as that one died of the *Head-Ache*, who was sorely tormented with it, though the physicians were of opinion that the disease was in the stomach. Again, if one died suddenly, the matter is not great, whether it be reported in the Bills, *Suddenly*, *Apoplexy*, or *Planet-strucken*, etc.

10. To conclude, in many of these cases the Searchers are able to report the opinion of the physician who was with the patient, as they receive the same from the friends of the defunct, and in very many cases, such as *Drowning*, *Scalding*, *Bleeding*, *Vomiting*, *Making-away themselves*, *Lunaticks*, *Sores*, *Small-Pox*, etc. their own senses are sufficient, and the generality of the world, are able pretty well to distinguish the *Gowt*, *Stone*, *Dropsy*, *Falling-sickness*, *Palsy*, *Agues*, *Pleurisy*, *Rickets*, etc. one from another.

11. But now as for those casualties which are aptest to be confounded, and mistaken, I shall in the ensuing discourse presume to touch upon them so far as the learning of these Bills hath enabled me.

12. Having premised these general advertisements, our first observation upon the casualties shall be, that in twenty years there dying of all diseases and casualties, 229,250, that 71,124 died of the *Thrush*, *Convulsion*, *Rickets*, *Teeth*, and *Worms*; and as *Abortives*, *Chrysoms*, *Infants*, *Liver-grown*, and *Overlaid*; that is to say, that about one-third of the whole died of those diseases, which we guess did all light upon children under four or five years old.

13. There died also of the *Small-Pox*, *Swine-Pox*, and *Measles*, and of *Worms* without *Convulsions*, 12,210, of which number we suppose likewise, that about one-half might be children under six years old. Now, if we consider that 16 of the said 229 thousand died of that extraordinary and grand casualty the *Plague*, we shall find that about 36 per centum of all quick conceptions died before six years old.

14. The second observation is, that of the said 229,250 dying of all diseases, there died of acute diseases (the *Plague* excepted) but about 50,000 or 2/9 parts. The which proportion doth give a measure of the state and disposition of this climate and air, as to health, these acute and epidemic diseases happening suddenly and vehemently, upon the like corruptions and alterations in the air.

15. The third observation is, that of the said 229 thousand about 70 died of chronic diseases, which shews (as I conceive) the state and disposition of the country (including as well its food, as air) in reference to health, or rather to longevity: for as the proportion of acute and epidemic diseases shews the aptness of the air to sudden and vehement impressions, so the chronic diseases shew the ordinary temper of the place, so that upon the proportion of chronic diseases seems to hang the judgment of the fitness of the country for long life. For, I conceive, that in countries subject to great epidemic sweeps men may live very long, but where the proportion of the chronic distempers is great, it is not likely to be so; because men being long sick, and always sickly, cannot live to any great age, as we see in several sorts of Metal-men, who although they are less subject to acute diseases than others, yet seldom live to be old, that is, not to reach unto those years which David says is the age of man. ...

17. In the next place, whereas many persons live in great fear and apprehension of some of the more formidable and notorious diseases following; I shall only set down how many died of each: that the respective numbers, being compared with the total 229,250, those persons may the better understand the hazard they are in.

Table of notorious diseases	
Apoplexy	1,306
Cut of the Stone	38
Falling Sickness	74
Dead in the streets	243
Gowt	134
Head-Ache	51
Jaundice	998
Lethargy	67
Leprosy	6
Lunatick	158
Overlaid, and *Starved*	529
Palsy	423
Rupture	201
Stone and *Strangury*,	863
Sciatica	5
Sodainly	454

Table of casualties	
Bleeding	69
Burnt, and *Scalded*	125
Drowned	829
Excessive drinking	2
Frighted	22
Grief	279
Hanged themselves	222
Killed by several accidents	1,021
Murdered	86
Poisoned	14
Smothered	26
Shot	7
Starved	51
Vomiting	136

18. In the foregoing observations we ventured to make a standard of the healthfulness of the air from the proportion of acute and epidemic diseases, and of the wholesomeness of the food from that of the chronic. Yet forasmuch as neither of them alone do shew the longevity of the inhabitants, we shall in the next place come to the more absolute standard, and correction of both, which is the proportion of the aged, viz. 15,757 to the total 229,250. That is of about 1 to 15 or 7 per cent. Only the question is, what

number of years the Searchers call *Aged*, which I conceive must be the same, that David calls so, viz. 70. For no man can be said to die properly of *Age*, who is much less: it follows from hence, that if in any other country more than seven of the 100 live beyond 70 such country is to be esteemed more healthful than this of our City. ...

CHAPTER VII

OF THE DIFFERENCE BETWEEN BURIALS AND CHRISTENINGS

THE next observation is, that in the said Bills there are far more burials than christenings. This is plain, depending only upon arithmetical computation; for, in 40 years, from the year 1603 to the year 1644, exclusive of both years, there have been set down (as happening within the same ground, space, or parishes) although differently numbered, and divided, 363,935 burials, and but 330,747 christenings within the 97, 16, and 10 out-parishes, those of Westminster, Lambeth, Newington, Redriff, Stepney, Hackney, and Islington, not being included.

2. From this single observation it will follow that London should have decreased in its people, the contrary whereof we see by its daily increase of buildings upon new foundations, and by the turning of great palacious houses into small tenements. It is therefore certain that London is supplied with people from out of the country, whereby not only to repair the overplus difference of burials above-mentioned, but likewise to increase its inhabitants, according to the said increase of housing. ...

4. But, if we consider what I have upon exact enquiry found true, viz. that in the country, within ninety years, there have been 6,339 christenings and but 5,280 burials, the increase of London will be salved without inferring the decrease of the people in the country; and withal, in case all England have but fourteen times more people than London, it will appear, how the said increase of the country may increase the people, both of London and itself; for if there be in the 97, 16, 10, and 7 parishes, usually comprehended within our Bills, but 460,000 souls as hereafter we shall shew, then there are in all England and Wales, 6,440,000 persons, out of which subtract 460,000, for those in and about London, there remains 5,980,000 in the country, the which increasing about 1/7 part in 40 years, as we shall hereafter prove doth happen in the country, the whole increase of the country will be about 854,000 in the said time, out of which number, if but about 250,000 be sent up to London in the said 40 years, viz. about 6,000 per annum, the said missions will make good the alterations which we find to have been in and about London, between the years 1603 and 1644 above-mentioned. But that 250,000 will do the same, I prove thus, viz. in the 8 years, from 1603 to 1612, the burials in all the parishes, and of

all diseases, the *Plague* included, were at a medium 9,750 per annum. And between 1635 and 1644 were 18,000, the difference whereof is 8,250, which is the total of the increase of the burials in 40 years, that is about 206 per annum. Now, to make the burials increase 206 per annum, there must be added to the City thirty times as many (according to the proportion of 3 dying out of 11 families) viz. 6,180 *advenae*, the which number multiplied again by the 40 years, makes the product 247,200, which is less than the 250,000 above propounded; so as there remains above 600,000 of increase in the country within the said 40 years, either to render it more populous, or send forth into other colonies, or wars. But that England hath fourteen times more people is not improbable, for the reasons following.

1. London is observed to bear about the fifteenth proportion of the whole tax.
2. There is in England and Wales about 39,000 square miles of land, and we have computed that in one of the greatest parishes in Hampshire, being also a market town and containing twelve square miles, there are 220 souls in every square mile, out of which I abate 1/4 for the overplus of people more in that parish, than in other wild counties. So as the 3/4 parts of the said 220, multiplied by the total of square miles, produces 6,400,000 souls in all, London included.
3. There are about 10,000 parishes in England and Wales, the which, although they should not contain the 1/3 part of the land nor the 1/4 of the people of that country parish which we have examined, yet may be supposed to contain about 600 people, one with another, according to which account there will be six millions of people in the Nation. I might add, that there are in England and Wales about five and twenty millions of acres at 16½ foot to the perch; and if there be six millions of people, then there is about four acres for every head, which how well it agrees to the rules of plantation, I leave unto others, not only as a means to examine my assertion, but as an hint to their enquiry concerning the fundamental trade, which is husbandry and plantation. ...

CHAPTER VIII

OF THE DIFFERENCE BETWEEN THE NUMBERS OF MALES AND FEMALES

THE next observation is, that there be more males than females.

There have been buried from the year 1628 to the year 1662, exclusive, 209,436 males and but 190,474 females: but it will be objected, that in London it may indeed be so, though otherwise elsewhere; because London is the great stage and shop of business, wherein the masculine sex bears the greatest part. But we answer, that there have been also christened within the same time, 139,782 males and but 130,866 females, and that the country accounts are consonant enough to those of London upon this matter.

2. What the causes hereof are, we shall not trouble ourselves to conjecture, as in other cases, only we shall desire that travellers would enquire whether it be the same in other countries. ...

CHAPTER XI

OF THE NUMBER OF INHABITANTS

I HAVE been several times in company with men of great experience in this City, and have heard them talk seldom under millions of people to be in London, all which I was apt enough to believe until, on a certain day, one of eminent reputation was upon occasion asserting that there was in the year 1661 two millions of people more than in the year 1625, before the great *Plague*; I must confess that, until this provocation, I had been frighted with that misunderstood example of David, from attempting any computation of the people of this populous place; but hereupon I both examined the lawfulness of making such enquiries and, being satisfied thereof, went about the work itself in this manner: viz.

2. First, I imagined that, if the conjecture of the worthy person aforementioned had any truth in it, there must needs be about six or seven millions of people in London now; but repairing to my Bills I found that not above 15,000 per annum were buried, and consequently, that not above one in four hundred must die per annum if the total were but six millions.

3. Next considering, that it is esteemed an even lay whether any man lives ten years longer, I supposed it was the same, that one of any 10 might die within one year. But when I considered, that of the 15,000 aforementioned about 5,000 were *Abortive*, and *Stillborn*, or died of *Teeth*, *Convulsion*, *Rickets*, or as *Infants*, and *Chrysoms*, and *Aged*. I concluded that of men and women, between ten and sixty, there scarce died 10,000 per annum in London, which number being multiplied by 10, there must be 100,000 in all, that is not the one-sixtieth part of what the Alderman imagined. These were but sudden thoughts on both sides, and both far from truth, I thereupon endeavoured to get a little nearer, thus: viz.

4. I considered, that the number of child-bearing women might be double to the births: forasmuch as such women, one with another, have scarce more than one child in two years. The number of births I found, by those years wherein the registries were well kept, to have been somewhat less than the burials. The burials in these late years at a medium are about

13,000 and consequently the christenings not above 12,000. I therefore esteemed the number of teeming women to be 24,000: then I imagined, that there might be twice as many families as of such women; for that there might be twice as many women aged between 16 and 76, as between 16 and 40, or between 20 and 44; and that there were about eight persons in a family, one with another, viz. the man and his wife, three children and three servants, or lodgers: now 8 times 48,000 makes 384,000.

5. Secondly, I find by telling the number of families in some parishes within the Walls, that 3 out of 11 families per annum have died: wherefore, 13,000 having died in the whole, it should follow there were 48,000 families according to the last mentioned account.

6. Thirdly, the account which I made of the trained bands and auxiliary soldiers, doth enough justify this account.

7. And lastly I took the map of London set out in the year 1658 by Richard Newcourt, drawn by a scale of yards. Now I guessed that in 100 yards square there might be about 54 families, supposing every house to be 20 foot in the front: for on two sides of the said square there will be 100 yards of housing in each, and in the two other sides 80 each; in all 360 yards: that is 54 families in each square, of which there are 220 within the Walls, making in all 11,880 families within the Walls. But forasmuch as there die within the walls about 3,200 per annum, and in the whole about 13,000; it follows that the housing within the Walls is 1/4 part of the whole, and consequently, that there are 47,520 families in and about London, which agrees well enough with all my former computations: the worst whereof doth sufficiently demonstrate that there are no millions of people in London, which nevertheless most men do believe, as they do, that there be three women for one man, whereas there are fourteen men for thirteen women, as elsewhere hath been said.

8. We have (though perhaps too much at random) determined the number of the inhabitants of London to be about 384,000: the which being granted, we assert that 199,112 are males and 184,886 females.

9. Whereas we have found that of 100 quick conceptions about 36 of them die before they be six years old, and that perhaps but one surviveth 76, we, having seven decades between six and 76, we sought six mean proportional numbers between 64, the remainder living at six years, and the one which survives 76, and find that the numbers following are practically near enough to the truth; for men do not die in exact proportions, nor in fractions: from whence arises this Table following:

Viz. of 100 there dies within the first six years	36
The next ten years, or decade	24
The second decade	15
The third decade	9
The fourth	6
The next	4
The next	3
The next	2
The next	1

10. From whence it follows, that of the said 100 conceived there remains alive at six years end 64.

At sixteen years end	40	At fifty-six	6
At twenty-six	25	At sixty-six	3
At thirty-six	16	At seventy-six	1
At forty-six	10	At eighty	0

11. It follows also, that of all which have been conceived, there are now alive 40 per cent above sixteen years old, 25 above twenty-six years old, *& sic deniceps*, as in the above Table: there are therefore of aged between 16 and 56, the number of 40, less by six, viz. 34; of between 26 and 66, the number of 25 less by three, viz. 22: *& sic deniceps.*

Wherefore, supposing there be 199,112 males, and the number between 16 and 56 being 34. It follows, there are 34 per cent of all those males fighting men in London, that is 67,694, viz. near 70,000: the truth whereof I leave to examination, only the 1/5 of 67,694, viz. 13,539, is to be added for Westminster, Stepney, Lambeth, and the other distant parishes, making in all 81,233 fighting men.

12. The next enquiry shall be, in how long time the City of London shall, by the ordinary proportion of breeding and dying, double its breeding people. I answer in about seven years, and (Plagues considered) eight. Wherefore since there be 24,000 pair of breeders, that is one-eighth of the whole, it follows that in eight times eight years the whole people of the City shall double without the access of foreigners: the which contradicts not our account of its growing from two to five in 56 years with such accesses.

13 According to this proportion, one couple viz. Adam and Eve, doubling themselves every 64 years of the 5,610 years, which is the age of the world according to the Scriptures, shall produce far more people than are now in it. Wherefore the world is not above 100 thousand years old as some vainly imagine, nor above what the Scripture makes it.

3. An Estimate of the Degrees of the Mortality of Mankind

EDMUND HALLEY (1693)

From *Philosophical Transactions* XVII. Excerpts are from pages 596—604, 610, 655—656.

Halley's text may be clearer if the reader notes that he intends: $l_0 = 1238$; $L_0 = 1000$; $l_1 = 890$; $L_1 = 855$. According to Westergaard (1969, p. 34) the Breslau lists showed 1218 births in 1691, among whom 992 were alive on January 1, 1692. Rounding the figure, Halley set $L_0 = 1000$ for 1238 births.

In his *Use* 5 Halley refers to "years purchase", the price charged for an annuity paying one dollar per year. Parliament granted William and Mary permission in 1691 to borrow money through an annuity scheme; the rate Halley refers to was set the following year.

We have omitted Halley's discussion of joint and group annuities.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_3,

The Contemplation of the *Mortality* of *Mankind*, has besides the *Moral*, its *Physical* and *Political* Uses, both which have been some years since most judiciously considered by the curious Sir *William Petty*, in his *Natural* and *Political* Observations on the Bills of *Mortality* of *London*, owned by Captain *John Graunt*. And since in a like Treatise on the Bills of *Mortality* of *Dublin*. But the Deduction from those Bills of *Mortality* seemed even to their Authors to be defective: First, In that the *Number* of the People was wanting. Secondly, That the *Ages* of the People dying was not to be had. And Lastly, That both *London* and *Dublin* by reason of the great and casual Accession of *Strangers* who die therein, (as appeared in both, by the great Excess of the *Funerals* above the *Births*) rendred them incapable of being Standards for this purpose; which requires, if it were possible, that the People we treat of should not at all be changed, but die where they were born, without any Adventitious Increase from Abroad, or Decay by Migration elsewhere.

This *Defect* seems in a great measure to be satisfied by the late curious Tables of the Bills of *Mortality* at the City of *Breslaw*, lately communicated to this Honourable Society by Mr. *Justell*, wherein both the *Ages* and *Sexes* of all that die are monthly delivered, and compared with the number of the *Births*, for Five Years last past, *viz.* 1687, 88, 89, 90, 91, seeming to be done with all the Exactness and Sincerity possible.

This City of *Breslaw* is the Capital City of the Province of *Silesia*; or, as the *Germans* call it, *Schlesia*, and is scituated on the Western Bank of the River *Oder*, anciently called *Viadrus*; near the Consines of *Germany* and *Poland*, and very nigh the Latitude of *London*. It is very far from the Sea, and as much a *Mediterranean* Place as can be desired, whence the Confluence of Strangers is but small, and the Manufacture of Linnen employs chiefly the poor People of the place, as well as of the Country round about; whence comes that sort of Linnen we usually call your *Sclesie Linnen*; which is the chief, if not the only Merchandize of the place. For these Reasons the People of this City seem most proper for a *Standard*; and the rather, for that the *Births* do, a small matter, exceed the *Funerals*. The only thing wanting is the Number of the whole People, which in some measure I have endeavoured to supply by comparison of the *Mortality* of the People of all Ages, which I shall from the said Bills trace out with all the Acuracy possible.

It appears that in the Five Years mentioned, *viz.* from 87 to 91 inclusive, there were *born* 6193 Persons, and *buried* 5869; that is, born *per Annum* 1238, and *buried* 1174; whence an *Encrease* of the People may be argued of 64 *per Annum*, or of about a 20th part, which may perhaps be ballanced by the Levies for the *Emperor*'s Service in his Wars. But this being contingent, and the Births certain, I will suppose the People of *Breslaw* to be encreased by 1238 *Births* annually. Of these it appears by the same Tables, that 348 do die *yearly* in the *first Year* of their *Age*, and that but 890 do arrive at a full *Years Age*; and likewise, that 198 do die in the *Five Years* between 1 and 6 compleat, taken at a *Medium*; so that but 692 of the Persons *born* do survive *Six* whole *Years*. From this *Age* the Infants being arrived at some degree of Firmness, grow less and less *Mortal*; and it appears that of the whole People of *Breslaw* there die *yearly*, as in the following Table, wherein the upper Line shews the *Age*, and the next under it the *Number* of Persons of that Age *dying yearly*.

7 . 8 9 . . 14 . 18 . 21 . 27 . 28 . . 35 .
11 . 11 . 6 . 5½ .2 . 3½ 5 6 4½ 6½ 9 . 8 . 7 .7 .

36 . 42 . 45 49 54 . 55 . 56 . 63
8 . 9½ 8 . 9 . 7 . 7 . 10 11 . 9 . 9 . 10 . 12

70 71 . 72 77 81 84 . 90 91 .
9½ 14 9 . 11 9½ 6 .7 .3 .4 .2 . 1 . 1 . 1 .

98 . 99 . 100 .
0 . ⅕ . ⅗

And where no *Figure* is placed over, it is to be understood of those that die between the Ages of the preceding and consequent *Column*.

From this Table it is evident, that from the Age of 9 to about 25 there does not die above 6 *per Annum* of each *Age*, which is much about one *per Cent.* of those that are of those *Ages*: And whereas in the 14, 15, 16, 17 *Years* there appear to die much fewer, as 2 and 3½, yet that seems rather to be attributed to Chance, as are the other Irregularities in the Series of Ages, which would rectifie themselves, were the number of Years much more considerable, as 20 instead of 5. And by our own Experience in *Christ-Church Hospital*, I am informed there die of the *Young Lads*, much about one *per Cent. per Annum*, they being of the foresaid *Ages*. From 25 to 50 there seem to die from 7 to 8 and 9 *per Annum* of each Age; and after that to 70, they growing more *crasie*, though the number be much diminished, yet the *Mortality encreases*, and there are found to die 10 or 11 of each Age *per Annum*: From thence the number of the *Living* being grown very small, they gradually decline till there be none left to *die*; as may be seen at one View in the Table.

From these Considerations I have formed the *adjoyned Table*, whose Uses are manifold, and give a more just *Idea* of the *State* and *Condition* of *Mankind*, than any thing yet extant that I know of. It exhibits the *Number* of *People* in the City of *Breslaw* of all Ages, from the *Birth* to extream *Old Age*, and thereby shews the Chances of *Mortality* at all *Ages*, and likewise how to make a certain Estimate of the value of *Annuities* for *Lives*, which hitherto has been only done by an imaginary *Valuation*: Also the *Chances* that there are that a *Person* of any *Age* proposed does live to any other *Age* given; with many more, as I shall hereafter shew. This *Table* does shew the *number* of *Persons* that are living in the *Age* current annexed thereto.

Thus it appears, that the whole People of *Breslaw* does consist of 34000 *Souls*, being the Sum *Total* of the Persons of all Ages in the *Table*: The first use hereof is to shew the Proportion of *Men* able to bear *Arms* in any *Multitude*, which are those between 18 and 56, rather than 16 and 60; the one being generally too weak to bear the *Fatigues* of *War* and the Weight of *Arms*, and the other too crasie and infirm from *Age*, notwithstanding particular Instances to the contrary. Under 18 from the *Table*, are found in this City 11997 Persons, and 3950 above 56, which together make 15947. So that the Residue to 34000 being 18053 are Persons between those *Ages*. At least one half thereof are Males, or 9027: So that the whole Force this City can raise of *Fencible Men*, as the *Scotch* call them, is about

Age Curt.	Per-sons	Age Curt.	Per-sons	Age Curt.	Per-sons	Age Curt.	Per-sons	Age Curt.	Per-sons	Age Curt	Per-sons
1	1000	8	680	15	628	22	586	29	539	36	481
2	855	9	670	16	622	23	579	30	531	37	472
3	798	10	661	17	616	24	573	31	523	38	463
4	760	11	653	18	610	25	567	32	515	39	454
5	732	12	646	19	604	26	560	33	507	40	445
6	710	13	640	20	598	27	553	34	499	41	436
7	692	14	634	21	592	28	546	35	490	42	427

Age Curt.	Per-sons	Age Curt.	Per-sons	Age Curt.	Per-sons	Age Curt.	Per-sons	Age Curt.	Per-sons	Age Curt.	Per-sons
43	417	50	346	57	272	64	202	71	131	78	58
44	407	51	335	58	262	65	192	72	120	79	49
45	397	52	324	59	252	66	182	73	109	80	41
46	387	53	313	60	242	67	172	74	98	81	34
47	377	54	302	61	232	68	162	75	88	82	28
48	367	55	292	62	222	69	152	76	78	83	23
49	357	56	282	63	212	70	142	77	68	84	20

Age	Per-sons
7	5547
14	4584
21	4270
28	3964
35	3604
42	3178
49	2709
56	2194
63	1694
70	1204
77	692
84	253
100	107
	34000
Sum Total	

9000, or $\frac{9}{34}$, or somewhat more than a quarter of the *Number* of *Souls*, which may perhaps pass for a Rule for all other places.

The *Second Use* of this Table is to shew the differing degrees of *Mortality*, or rather *Vitality* in all *Ages*; for if the number of Persons of any *Age* remaining after one year, be divided by the difference between that and the number of the Age proposed, it shews the *odds* that there is, that a Person of that Age does not die in a *Year*. As for Instance, a Person of 25 *Years* of *Age* has the odds of 560 to 7 or 80 to 1, that he does not *die* in a *Year*: Because that of 567, living of 25 years of Age, there do die no more than 7 in a *Year*, leaving 560 of 26 Years old.

So likewise for the *odds*, that any Person does not die before he attain any proposed *Age*: Take the *number* of the remaining Persons of the Age proposed, and divide it by the difference between it and the number of those of the *Age* of the Party proposed; and that shews the *odds* there is between the Chances of the Party's living or dying. As for Instance; What is the *odds* that a Man of 40 lives 7 Years: Take the number of Persons of 47 years, which in the Table is 377, and substract it from the number of Persons of 40 years, which is 445, and the *difference* is 68: Which shews that the *Persons dying* in that 7 years are 68, and that it is 377 to 68 or $5\frac{1}{2}$ to 1, that a Man of 40 does live 7 Years. And the like for any other *number* of *Years*.

Use III. But if it be enquired at what number of *Years*, it is an even Lay that a Person of any *Age* shall die, this Table readily performs it: For if the *number* of Persons *living* of the *Age* proposed be *halfed*, it will be found by the *Table* at what Year the said *number* is reduced to half by *Mortality*; and that is the Age, to which it is an even Wager, that a Person of the *Age* proposed shall arrive before he *die*. As for Instance; A Person of 30 Years of *Age* is proposed, the number of that Age is 531, the half thereof is 265, which number I find to be between 57 and 58 Years; so that a Man of 30 may reasonably expect to live between 27 and 28 Years.

Use IV. By what has been said, the *Price* of *Insurance* upon *Lives* ought to be regulated, and the difference is discovered between the *price* of ensuring the *Life* of a *Man* of 20 and 50, for Example: it being 100 to 1 that a Man of 20 dies not in a year, and but 38 to 1 for a Man of 50 Years of Age.

Use V. On this depends the Valuation of *Annuities* upon *Lives*; for it is plain that the *Purchaser* ought to pay for only such a part of the value of the *Annuity*, as he has Chances that he is living; and this ought to be computed yearly, and the Sum of all those yearly Values being added together, will amount to the value of the *Annuity* for the *Life* of the Person proposed. Now the present value of Money payable after a term of years, at any given rate of Interest, either may be had from Tables already computed; or almost as compendiously, by the Table of Logarithms: For the Arithmetical Complement of the Logarithm of Unity and its yearly Interest (that is, of 1.06 for Six *per Cent.* being 9.974694.) being multiplied by the number of years proposed, gives the present value of One Pound payable after the end of so many years. Then by the foregoing Proposition, it will be as the number of Persons living after that term of years, to the number dead; so are the Odds that any one Person is Alive or Dead. And by consequence, as the Sum of both or the number of Persons living of the Age first proposed, to the number remaining after so many years, (both given by the Table) so the present value of the yearly Sum payable after the term proposed, to the Sum which ought to be paid for the Chance the person has to enjoy such an Annuity after so many Years. And this being repeated for every year of the persons Life, the Sum of all the present Values of those Chances is the true Value of the Annuity. This will without doubt appear to be a most laborious Calculation, but it being one of the principal Uses of this Speculation, and having found some *Compendia* for the Work, I took the pains to compute the following Table, being the short Result of a not ordinary number of Arithmetical Operations; It shews the Value of Annuities for every Fifth Year of Age, to the Seventieth, as follows.

Age	Years Purchase	Age	Years Purchase	Age	Years Purchase
1	10.28	25	12.27	50	9.21
5	13.40	30	11.72	55	8.51
10	13.44	35	11.12	60	7.60
15	13.33	40	10.57	65	6.54
20	12.78	45	9.91	70	5.32

This shews the great Advantage of putting Money into the present *Fund* lately granted to their Majesties, giving 14 *per Cent. per Annum*, or at the rate of 7 years purchase for a Life; when young Lives, at the usual rate of Interest, are worth above 13 years Purchase. It shews likewise the Advantage of young Lives over those in Years; a Life of Ten Years being almost worth 13½ years purchase, whereas one of 36 is worth but 11. ...

It may be objected, that the different *Salubrity* of places does hinder this Proposal from being *universal*; nor can it be denied. But by the number that die, being 1174 *per Annum* in 34000, it does appear that about a 30th part die yearly, as Sir *William Petty* has computed for *London*; and the number that die in Infancy, is a good Argument that the Air is but indifferently salubrious. So that by what I can learn, there cannot perhaps be one better place proposed for a Standard. At least 'tis desired that in imitation hereof the Curious in other Cities would attempt something of the same nature, than which nothing perhaps can be more useful. ...

A second Observation I make upon the said Table, is that the Growth and Encrease of Mankind is not so much stinted by any thing in the Nature of the *Species*, as it is from the cautious difficulty most People make to adventure on the state of *Marriage*, from the prospect of the Trouble and Charge of providing for a Family. Nor are the poorer sort of People herein to be blamed, since their difficulty of subsisting is occasion'd by the unequal Distribution of Possessions, all being necessarily fed from the Earth, of which yet so few are Masters. So that besides themselves and Families, they are yet to work for those who own the Ground that feeds them: And of such does by very much the greater part of Mankind consist; otherwise it is plain, that there might well be four times as many Births as we now find. For by computation from the Table, I find that there are nearly 15000 Persons above 16 and under 45, of which at least 7000 are Women capable to bear Children. Of these notwithstanding there are but 1238 born yearly, which is but little more than a sixth part: So that about one in six of these Women do breed yearly; whereas were they all married, it would not appear strange or unlikely, that four of six should bring a Child every year. The Political Consequences hereof I shall not insist on, only the Strength and Glory of a King being in the multitude of his Subjects, I shall only hint, that above all things, Celibacy ought to be discouraged, as, by extraordinary Taxing and Military Service: And those who have numerous Families of Children to be countenanced and encouraged by such Laws as the *Jus trium Liberorum* among the *Romans*. But especially, by an effectual Care to provide for the Subsistence of the Poor, by finding them Employments, whereby they may earn their Bread, without being chargeable to the Publick.

4. A Treatise on the Valuation of Annuities and Assurances on Lives and Survivors

JOSHUA MILNE (1815)

London. Excerpts are from pages vi—xii, 89—91, 97—100, 487—489, 582.

Milne's notation and principal equations translate into modern form as:

${}^{n}\epsilon = l_n$ = survivors to exact age n from among l_0 births at observed age-specific death rates

$$S^{n}\epsilon = \sum_{x=n}^{\omega} l_x$$

$\overset{n}{L} = P_n$ = observed population at ages n to $n+1$

$\overset{n}{D} = D_n$ = annual deaths to persons ages n to $n+1$

$\delta = d_n$ = deaths to persons ages n to $n+1$ in the life table

$$\frac{S^{n}\epsilon - \frac{1}{2}{}^{n}\epsilon}{{}^{n}\epsilon} = \frac{\sum_{x=n}^{\omega} l_x - \frac{1}{2} l_n}{l_n} \doteq \mathring{e}_n = \text{life expectancy at age } n$$

$$\frac{{}^{n}\epsilon \overset{n}{D}}{\overset{n}{L} + \frac{1}{2}\overset{n}{D}} = l_n \left(\frac{D_n}{P_n + \frac{1}{2} D_n} \right) \doteq d_n \, .$$

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_4, © Springer-Verlag Berlin Heidelberg 2013

The general theorems throughout the work are adapted to any law of mortality whatever, and even to different laws, for all the different lives that may be involved; but they cannot be applied to practical purposes, unless one or more tables of mortality, adapted to the lives proposed, be given. The construction of such tables, although one of the most important parts of the subject, has remained the longest imperfect; principally from the want of *data*, and partly from its having occupied but little of the attention of mathematicians.

This is the object of the third chapter.

It is first assumed that the population remains stationary without being affected by migration, the method of constructing a table of mortality from the registers of burials in these circumstances is then shown, and the principal properties of such tables are demonstrated (143—165).

These might with more propriety be called tables of vitality than of mortality, as their principal use is, to show the mean duration of life, and the probability of its continuance; were it not, that the measure of vitality is often materially different from that of the continuance of life. For one individual may, and certainly often does, to every purpose of utility, enjoyment, or suffering, live much more than another during the same time. So that the absolute quantity of vitality depends upon its intensity, as well as its duration.

They are also, when properly constructed according to the general form of article 162, tables of population as well as mortality, since they show the proportion of the people of each sex in every interval of age, as well as the proportions of them that annually enter upon and die in each of these intervals, when the population is stationary, (art. 163 and 840).

Considered in this point of view, they are of importance in the science of Political Economy, and as the subject is treated here, in a manner which, at the same time that it is general and comprehensive, is perfectly elementary, this part of the work may, perhaps, be useful to those who cultivate that science, but have made little or no progress in mathematics.

When the state of the population is variable, and affected by migration, it is very difficult to ascertain the law of mortality, or even to approach near to it by means of the registers of burials alone. Mr. Simpson[1] endeavoured to make such allowances for the influence of the continual influx of new settlers upon the London bills, as to deduce the law of mortality from them, and from his great judgment and accuracy there is reason to believe that his success was considerable, but he did not explain distinctly how he proceeded.

Dr. Price afterwards, in his "Essay on the Method of forming Tables of Observations[2]," showed how the number of the inhabitants of a town, at every age, might be determined from the bills of mortality, when the number at each age was maintained stationary by the influx of new settlers, although the deaths of every year exceeded the births; provided that the annual number of settlers of each age could be ascertained, and remained always the same: But besides that the practical applications of that method are extremely limited by the hypotheses, the requisite *data* can hardly in any instance be obtained.

The principle of the method is here demonstrated in article 168, is extended in article 169 to places wherein the annual births exceed the deaths, while the population is kept stationary by emigration; and in article 170, a general theorem is given, that includes both these. But it is shown in article 172, that such a table could not exhibit the true law of mortality, nor enable us to determine the probabilities of life.

For that purpose, wherever the population is variable, that is, strictly speaking, in all cases, it is necessary that the number of the living, as well as that of the annual deaths at each age, should be known; and the method of constructing an accurate table of mortality from these *data*, is next given (174—183).

Almost all who have hitherto treated the subject of this chapter, have neglected the use of symbols, and have been content to illustrate their reasoning by examples in numbers. But in this, and all other inquiries that admit of the application of mathematical reasoning, the employment of symbols is attended with many advantages, since they enable us to treat of generals with almost the same facility as particulars; they relieve the mind from the fatigue and obscurity that attend circumlocution and ambiguity of language, and increase its powers much more than mechanical engines do those of the body.

It has generally been assumed that the number of the people when not affected by migration, increases in geometrical progression; a few theorems are given which determine the relations between the time elapsed, the rate of augmentation, and its amount, upon that hypothesis (184—187). Then it is shown how, upon the same hypothesis, a table of mortality may be constructed from the registers of deaths alone, when the annual rate of increase is given (188—191).

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[1] See his excellent little tract, entitled *The Doctrine of Annuities and Reversions*, etc. (8vo. London, 1742), and his supplement to it, printed first in his *Select Exercises for young proficients in the mathematics*, (8vo. London, 1752), but since separately.

In that supplement, the author complained with some bitterness of M. Deparcieux having criticised, with more severity than judgment, the alterations which he had made in Mr. Smart's table of mortality for London.

But Mr. Simpson's information of what M. Deparcieux had advanced in his *Essai sur les Probabilités de la durée de la vie humaine*, was derived from the account of the work given in the History of the Royal Academy of Sciences at Paris (An. 1746, p. 45.). What Mr. Simpson objected to, and was really injurious, appears to have been given by the writer of that account, rather as his own sentiments than those of M. Deparcieux, which he does not appear to have understood, any more than Mr. Simpson's. I consider the remarks which M. Deparcieux did make on Mr. Simpson's table, to be both candid and judicious, and, in justice to his memory, will here insert what is most material in them.

After observing upon Dr. Halley's table for Breslaw, and that of Mr. Smart for London, which Mr. Simpson corrected, he proceeds thus, "Il est bien difficile, pour ne pas dire impossible, qu'on puisse établir un ordre de mortalité approchant du vrai, par le moyen des regîtres d'une ville comme celle de Londres, à cause de la quantité prodigieuse d'étrangers qui vont s'y établir et mourir. Aussi M. Simpson a-t-il jugé à propos d'y faire quelques corrections, sans trop dire comment. On verra dans la suite par la comparaison qu'on fera de cette table, avec quelques autres, si on peut beaucoup y compter, malgré la correction." (p. 38.).

He then shows several of the difficulties that attend the construction of tables of mortality for large towns, from their mortuary registers, and adds, "Il suit de toutes ces raisons, que la table du Docteur Hallei doit être préférée à celle de M. Simpson. Il est vrai que ce dernier semble ne vouloir donner la sienne que pour les habitans de Londres, ce qui pourroit être approchant du vrai s'il n'entendoit parler seulement que de ceux qui naissent dans cette ville; ce qui ne peut servir de regle pour aucun autre endroit qu'on ne l'ait examiné." (p. 41.).

[2] See his valuable work, entitled Observations on Reversionary Payments, vol. ii. p. 73, of the 7th and last edition (2 vols. 8vo. London, 1812), which is that always quoted in this work.

In the Memoirs of the Academy of Sciences at Berlin for the year 1760, there is a paper by the celebrated EULER, entitled *Recherches génerales sur la mortalité et la multiplication du genre humain*, wherein the subject is treated algebraically. He assumes that the population is not affected by migration, and that both the annual births and deaths are always as the contemporaneous population; consequently, that the number of the people increases or decreases in geometrical progression. Then he gives several theorems, exhibiting the relations that would obtain between the annual births and deaths, and the population; and determines the law of mortality upon these hypotheses, but does not show how it may be deduced from actual observations independent of hypotheses; neither does he undertake the construction of any table of mortality, but, by way of example, gives that of M. Kerseboom, with the changes in the numbers which became necessary in consequence of his altering the radix from 1400 annual births to 1000[3].

M. Euler was perfectly aware how much the application of his theory was limited by the hypotheses it was derived from, he allows that his conclusions will not apply where migration has place, nor in case of any extraordinary increase of the people, such, he says, as takes place in new colonies; but he appears to have considered the increase in geometrical progression to prevail generally, though I believe it seldom does, either in countries that have been long or very recently settled (192—197). ...

162. In a society, none of the members of which enter it but by birth, nor go out of it but by death, where the number of the members remains always the same, and one uniform law of mortality obtains; if the number of deaths at every age, in any one year be given, a table may be constructed which will exhibit the numbers of the living in that society at all ages, and the law of mortality, according to which the members continually pass out of it by death, while their places are supplied by others that are continually rising from birth towards the greater ages, and which shall also show the expectation of life at every age—Thus:

Let there be five columns, in the first of which insert the ages 0, 1, 2, 3, ... $(\omega-1)$; then, against every age, insert in the fifth column the given number that died in the year between that and the next greater age; which being done, the numbers to be inserted in the third and fourth columns may be easily determined, by beginning at the greatest age in the table, and proceeding regularly, year by year, to the least—Thus:

To the number against any age in the fourth column, add that against the next less age in the fifth, and the sum will be the number to be inserted against that next less age in the fourth column (150).

To the sum of the numbers in the third and fourth columns, against any age, add half the number in the fifth column, against the next less age; and the sum last obtained will be the number to be inserted against that next less age in the third column (161).

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[3] He assumes unity for the number of the born, consequently, the number completing each year of age is expressed by the fraction which measures the probability that a child just born will attain to that age. Perhaps it is from this property that these have, by some writers, been called tables of *the probabilities of life;* but although they show those probabilities directly at birth, when the radix is any power of 10; they only furnish the means of determining them at all ages after that.

Lastly, divide the number against any age in the third column, by the number against the same age in the fourth; the quotient will be the expectation of life at that age (154 and 157) to be inserted in the second column:

And the general form of the table will be this:

No. of Col.	1	2	3	4	5
	Age	Expectation of life at that age	No. of the living at that age and upwards	No. that annually complete that year of their age	No. that die annually in their next succeeding year
	n	$\dfrac{S\,{}^{n}\epsilon - \frac{1}{2}\,{}^{n}\epsilon}{{}^{n}\epsilon}$	$S\,{}^{n}\epsilon - \frac{1}{2}\,{}^{n}\epsilon$	${}^{n}\epsilon$	${}^{n}\epsilon - {}^{n+1}\epsilon$
Greatest age	$\omega - 1$	$\dfrac{\frac{1}{2}\,{}^{\omega-1}\epsilon}{{}^{\omega-1}\epsilon} = \dfrac{1}{2}$	$\frac{1}{2}\,{}^{\omega-1}\epsilon$	${}^{\omega-1}\epsilon$	${}^{\omega-1}\epsilon$
Limiting age	ω	0	0	0	0

163. From the third and fourth columns of this table, it will be easy to determine the numbers both of the living, and of the annual deaths, between any two ages that may be assigned; for we have only to subtract the number in each column against the greater age, from that against the less; the remainder will, in each case, be the number sought. ...

174. But whether the population be stationary, or increasing, or decreasing; and whether such changes be produced by procreation, mortality, or migration, or by the joint operation of any two or more of those causes; provided that the mode of their operation be uniform, or nearly so, and not by sudden starts, the law of mortality may be approached near enough for any useful purpose, by actual enumeration and the bills of mortality—Thus:

175. Let the number of persons, in each year of their age, that are resident in a place at any one time be taken; and let an accurate register be kept of the number that die annually in each year of their age, during a term of eight or ten years at the least, whereof the first half may precede, and the second follow, the time of the enumeration.

Then, if the number of inhabitants of every age either increase or decrease uniformly during that term, the mean number of annual deaths in every year of age, thus registered, will be the same as if the population of the place had continued throughout that term the same as when the enumeration was made.

176. In consequence of the several causes of change mentioned in art. 174, by which the population we are considering may be affected, the proportion between the numbers that annually complete any two ages therein may differ materially from that which would obtain in a society not affected by migration, but subject to the same law of mortality at every age, and maintaining its population stationary, by a constant equality between the annual numbers of the births and deaths.

And therefore the probabilities of life in that place cannot be determined immediately from the proportions furnished by the enumeration.

But from the enumeration, and the register together, the law of mortality, and consequently those probabilities, may be determined in the following manner.

177. Let $\overset{n}{L}$ denote the number of the living in the place, at the time of the enumeration, of the age of n years, that is, in the $(n+1)$th year of their age; and $\overset{n}{D}$, the mean number that died annually in the same year of their age, during the term for which the register was kept (175); while ${}^{n}\mathscr{E}$ represents the number, in the society mentioned in the last article, that annually complete the nth year of their age; and $\delta = {}^{n}\mathscr{E} - {}^{n+1}\mathscr{E}$, the number of the same society that die annually in their next succeeding year, as in art. 149:

Then will $\overset{n}{L} : \overset{n}{D} :: \frac{1}{2}({}^{n}\mathscr{E} + {}^{n+1}\mathscr{E}) : {}^{n}\mathscr{E} - {}^{n+1}\mathscr{E} :: {}^{n}\mathscr{E} - \frac{1}{2}\delta : \delta$, and ${}^{n}\mathscr{E}\overset{n}{D} - \frac{1}{2}\delta\overset{n}{D} = \delta\overset{n}{L}$;

whence $\dfrac{{}^{n}\mathscr{E}\overset{n}{D}}{\overset{n}{L} + \frac{1}{2}\overset{n}{D}} = \delta$.

But $\overset{n}{L}$ and $\overset{n}{D}$ are always given by the enumeration and the register; therefore when ${}^{n}\mathscr{E}$ is given, δ, and thence ${}^{n+1}\mathscr{E} = {}^{n}\mathscr{E} - \delta$, may be easily obtained.

In this manner, assuming at pleasure ${}^{0}\mathscr{E}$ or $\mathscr{E}$, the number of annual births, and proceeding year by year, to determine successively the number that die in each year of their age, and, consequently, the number that complete that year, the numbers against every age in the fourth and fifth columns of the table in article 162 may be inserted; from whence the numbers for the second and third columns thereof may also be supplied, as directed in that article.

178. From about the age of seven or eight years, till forty or fifty, $\frac{1}{2}\overset{n}{D}$ is generally so small in comparison with $\overset{n}{L}$, that it might be neglected in the denominator of the fraction in the formula just given, without producing any considerable error in the value of δ, whereby that formula would become $\dfrac{{}^{n}\mathscr{E}\overset{n}{D}}{\overset{n}{L}} = \delta$, and this would facilitate the calculation a little.

179. This approximate value of δ would also result from the supposition that all who enter upon any year of their age continue alive throughout that year; and, that such of them as do not enter upon the next expire at its commencement; which hypothesis has generally been assumed in constructing tables of this kind.

But as the accuracy of such tables is of great importance, and the abridgment of the labour of constructing them in this way is but small, it will be better always to use the correct formula given in article 177. ...

788. In K. V. Ac. Handl. 1800, s. 323. Mr. Nicander has given statements of the mean number of women in all Sweden and Finland, with the annual average number of deliveries, and the proportion delivered annually in each of the under-

mentioned intervals of age; during sixteen years ending with 1795, which are presented at one view in the following table[4]:

Between the ages of	Mean Number of Females living	Annual average Number of Deliveries	One Woman of	Proportion of 1000 Deliveries
15 & 20	134,548	3,298	40.8	33
20 & 25	129,748	16,507	7.8	165
25 & 30	121,707	26,329	4.6	263
30 & 35	111,373	25,618	4.3	256
35 & 40	97,543	18,093	5.4	181
40 & 45	90,852	8,518	10.6	85
45 & 50	78,897	1,694	46.5	17
Above 50	69,268	39	1776.0	0.4
Above 15	833,936	100,096	8.3	1000

According to the proportion of legitimate and illegitimate children, Mr. Nicander found, that of 100,096[5] women delivered annually, 96,124 were married, and 3972 unmarried.

Upon comparing the legitimate births with the married women, and the illegitimate with the unmarried above fifteen years of age, he also found that 10 married women out of 54, and 10 unmarried out of 918, were annually brought to bed.

Some other results given by Mr. Nicander are presented under table XI., which table differs but little from that given above. What difference there is, arises from hence, that I have deduced the mean number of women in each interval of age from that gentleman's table marked Q, in K. V. Ac. Handl. 1801,1 Qu., and have calculated the two last columns from that and the column of deliveries.

It should be observed, that as the annual numbers here given are the average of sixteen years' observations, the number of deliveries they were derived from was 1,601,536.

789. The proportion of males born to females, appears to be the result of a uniform law of nature, and is probably always the same under the same circumstances, though it seems to vary with the situation of the parents, and I should rather suspect that variation to depend principally upon their age.

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[4] It will be observed that this table only shows the actual fecundity, which is always kept below the natural aptitude or physical power, by the obstacles to marriage; as appears by the table itself, for the proportion of women delivered annually between the ages of 30 and 35, is almost twice as great as between 20 and 25; which can only arise from the proportion of the married in the first of these intervals of age being at least so much greater than in the second.

[5] This number is printed 100,098 by Mr. Nicander, and I believe it should be so, as it occurs several times, but if that be right, some other number in the column of deliveries must be wrong, and as I have not the means of determining which it is, I have thought it better to state the amount of them as they stand, because the error bears a less proportion to that than any other, it is indeed of no kind of consequence.

The following table exhibits the proportion with considerable precision, under a few varieties in the circumstances.

Place	Term.		Number of Births of		No. of Male Births for every 10,000 Females
	Years	ending with	Males	Females	
England and Wales	29	1800	3,285,188	3,150,922	10,426
	10	1810	1,468,677	1,410,229	10,415
Sweden and Finland	9	1763	400,086	387,702	10,436
	20	1795	1,006,420	965,000	10,429
France	3	1802	110,312	105,287	10,477
England	29	1800	2,997,842	2,879,011	10,413
	10	1810	1,391,977	1,338,998	10,396
Wales	29	1800	177,401	166,593	10,649
	10	1810	76,700	71,231	10,768
Scotland	29	1800	67,353	62,636	10,753
Carlisle	18	1796	2,400	2,271	10,568
Montpellier	21	1792	12,919	12,145	10,637
Diocess of Borgo in Finland	18	1791	93,701	91,404	10,251
	22	1795	117,928	115,191	10,238
	4	1795	24,227	23,787	10,185
Stockholm	9	1763	12,015	11,706	10,264
Illegitimate					
In Sweden and Finland	20	1795	37,700	37,060	10,173
In Montpellier	21	1792	1,373	1,362	10,081

Table XI. Showing the Fecundity of Women at the different Periods of Life, in *Sweden and Finland*, from 1780 to 1795, both Years inclusive. (Art. 788).

	Absolute Fecundity		Intensity of Fecundity	
Between the Ages of	Mean Number of Females living	Annual average Number of Deliveries	That is, of 10,000 living	Or one of
15 & 20	132,765	3298	248	40.256
20 & 25	131,377	16,507	1257	7.959
25 & 30	121,650	26,329	2164	4.620
30 & 35	112,250	25,618	2282	4.382
35 & 40	98,710	18,093	1833	5.456
40 & 45	89,259	8518	954	10.479
45 & 50	74,002	1694	229	43.686
50 & 55	69,035	39	5.65	1770.1
20 & 40	463,987	86,547	1865	5.361
15 & 55	829,048	100,096	1207	8.283

Double Births, 1 of 58
Triple, 1 of 3,365.
Quadruple, 1 of 143,000.

One Birth in 104 produced the death of the mother.

And the number of deaths from Childbirth, was to the whole number of deaths of females between the ages of 15 and 55, as 2 to 19, or as 1 to 10 nearly.

5. Statistical Applications of the Mortality Table

GEORGE KING (1902)

From *Institute of Actuaries' Textbook*, Part II, Second Edition, pages 56—58, 63. London: Charles and Edward Layton.

We omit several numerical examples given by King of the uses of the life table, which concern populations at various ages, annuity payments, and the effects of immigration on observed death rates. The article introduces the life table in modern notation.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_5,

4. The fundamental column of the Life Table is the column of l_x. The first value in that column, l_0, called the radix, is the number of annual births in the imaginary population; and the succeeding numbers show how many persons, out of l_0 born alive, complete each year of age. In the table the number of annual births is 127,283; and we observe that 100,000 live to complete the tenth year of their age; 89,685 live to complete the thirtieth; and so on. We also see that only 4 live to complete the century; and that, although 1 survives 101 years, all die before reaching the age 102. Age 102 is therefore the *limiting age* of the table, being the year of age on which some lives enter, but which none complete. To the limiting age, the Greek letter ω is assigned for symbol; and therefore $l_\omega=0$. Also, the difference between the limiting age, and the present age, is called the *complement of life;* so that, at age x, the complement of life is $\omega-x$; in the case of our table, $102-x$.

5. The column of d_x contains the differences between the numbers in the column of l_x; and shows how many, out of l_0 persons born alive, die in each year of their age. Thus, by the table, out of 127,283 persons born alive, 14,358 die before completing their first year; 691 survive to age 30, but die before reaching 31; and so on. The number, then, in column d opposite any age, x, is the number who complete that year of age, but die before completing the next; that is, the number in column d opposite age x, is the number who die in the $(x+1)$th year of age. As all born must die, it follows that the sum of all the numbers in column d is equal to l_0: also, the sum of the numbers in column d, from age x to the oldest age, is equal to l_x.

6. To find how many die aged between x and $x+n$, we may take the sum of the numbers in column d, from d_x to d_{x+n-1}, inclusive; but unless n be very small, it will be easier to obtain the result by means of the l column; because $l_x-l_{x+n}=d_x+d_{x+1}+\&c.+d_{x+n-1}$. Thus, by the table, the number of persons, out of 127,283 born alive, who die between ages 20 and 30, is $l_{20}-l_{30}=6376$.

7. Passing now to the column L, we have the population living in a stationary community. Such a community, sustained by l_0 annual births, will, on the supposition of uniform distribution of births and deaths, always contain $\frac{1}{2}(l_0+l_1)=\mathrm{L}_0$ children in the first year of their

age; $\frac{1}{2}(l_1+l_2)=\mathrm{L}_1$ in the second year of their age; and so on. Thus, by the table, a population sustained by 127,283 annual births, will always contain 95,787 young persons aged between 20 and 21. The total population at all ages will be the sum of all the numbers in column L; and that is given in column T. By the table, the total population that would be supported by 127,283 annual births, is $6{,}082{,}031=\mathrm{T}_0$. The column T bears exactly the same relation to the column L, that the column l bears to the column d: that is, T_x is the sum of the numbers in column L, from age x to the oldest age: therefore T_x is the total population, aged x and upwards, in the community. In the community of T_0 inhabitants, there must be l_0 deaths annually; because there are l_0 births; and, the population being stationary, the deaths must be equal in number to the births. Similarly, there must be l_x deaths annually of persons aged x and upwards; and l_x-l_{x+n} deaths annually, of persons aged between x and $x+n$. Also, the number of inhabitants aged between x and $x+n$, is $\mathrm{T}_x-\mathrm{T}_{x+n}$; and, therefore, the proportion of deaths to population, between ages x and $x+n$, is $\frac{l_x-l_{x+n}}{\mathrm{T}_x-\mathrm{T}_{x+n}}$; and, for the whole community, the proportion of deaths to population is $\frac{l_0}{\mathrm{T}_0}$. When n is unity, $\frac{l_x-l_{x+n}}{\mathrm{T}_x-\mathrm{T}_{x+n}}=\frac{d_x}{\mathrm{L}_x}=m_x$, the central death rate at age x. By the table, the proportion of deaths to the population for the whole community is $\frac{127{,}283}{6{,}082{,}031}=\cdot 020928$ or about 21 per thousand. The proportion for the population aged less than 50, is $\frac{127{,}283-72{,}795}{6{,}082{,}031-1{,}475{,}603}$ $=\frac{54{,}488}{4{,}606{,}428}=\cdot 011829$, or not quite 12 per thousand, while the proportion for the population aged 50 and upwards, is $\frac{72{,}795}{1{,}475{,}603}=\cdot 049332$, or over 49 per thousand. These figures illustrate the remarks made in Chapter iii, Art. 13. They show that if from any cause there is an unusual proportion of young persons in a community, the ratio of deaths to population will be diminished; but, as previously remarked, it does not follow that therefore the members of that community enjoy unusual longevity. ...

We have seen that, of l_x persons who attain the precise age x, l_{x+1} will complete a year of life in the first year; and d_x will live on the average half a year each; therefore the quantity of existence in the first year due to the l_x persons will be $l_{x+1}+\frac{1}{2}d_x=\mathrm{L}_x$. Similarly for future years; therefore $\Sigma\mathrm{L}_x=\mathrm{T}_x$ will be the total future existence due to the l_x persons; giving $\frac{\mathrm{T}_x}{l_x}=\mathring{e}_x$ years to each; and the average age at death of the l_x persons will be $x+\mathring{e}_x$ years. The existence within the next n years due to the l_x persons, is $\mathrm{T}_x-\mathrm{T}_{x+n}$; giving $\frac{\mathrm{T}_x-\mathrm{T}_{x+n}}{l_x}={}_{|n}\mathring{e}_x$ years for each. Of these years, $n\times l_{x+n}$ are due to those who complete age $x+n$; leaving $\mathrm{T}_x-\mathrm{T}_{x+n}-nl_{x+n}$ for those who die between age x and age $x+n$. But l_x-l_{x+n} persons die between these ages; therefore the average amount of existence between ages x and $x+n$, belonging to those who die between these ages, is $\frac{\mathrm{T}_x-\mathrm{T}_{x+n}-nl_{x+n}}{l_x-l_{x+n}}$, and their average age at death is $x+\frac{\mathrm{T}_x-\mathrm{T}_{x+n}-nl_{x+n}}{l_x-l_{x+n}}$. For example, if $x=20$ and $n=10$, we find ${}_{|10}\mathring{e}_{20}=\frac{\mathrm{T}_{20}-\mathrm{T}_{30}}{l_{20}}=9{\cdot}680$. Also, the existence within the 10 years, due to all those who reach age 20, is $\mathrm{T}_{20}-\mathrm{T}_{30}=929{,}902$; the existence in the period due to those who survive it is $10l_{30}=896{,}850$; leaving 33,052 years to the $l_{20}-l_{30}=6{,}376$ persons who die in the 10 years; or 5·184 to each. The average age at death of those who die between 20 and 30, is therefore 25·184. Similarly the average age at death of those who die below 20 is $\frac{\mathrm{T}_0-\mathrm{T}_{20}-20l_{20}}{l_0-l_{20}}=3{\cdot}734$.

13. We have seen, Art. 7, that the ratio of deaths to population in a stationary community is $\frac{l_0}{\mathrm{T}_0}$: also (Chap. iii, Art. 16), that the complete expectation of life is $\frac{\mathrm{T}_0}{l_0}$. It therefore follows that the ratio of deaths to population is equal to the reciprocal of the complete expectation of life.

6. Formal Treatment of Aggregate Mortality Data

WILHELM LEXIS (1875)

From *Einleitung in die Theorie der Bevölkerungs-Statistik*, pp. 5—7. Strasbourg: Trubner. Translated by Nathan Keyfitz.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_6,

As an example of a population with a single source of variation, let us consider that of a fixed territory subject only to death. (One could consider birth as the alteration of condition, but it seems more natural to take the births as the absolute starting point of the elements). The proposed treatment is valid also for the presentation of any other kind of alteration of condition of those born.

The development of our example yields the formal theory of mortality statistics, which covers not only mortality, but also its correlate, survivorship.

Provisionally disregarding out- and in-migration, we consider a number of individuals who are distinguished for us only by being born at different times, and dying at different times and ages. Thus the population can only be decomposed according to these variables. The decomposition to reveal the inner structure of the population can be seen most simply in a graphical scheme.

Let (Fig. 1) N be an axis of coordinates, on which time is measured from an arbitrary fixed origin O, and let the birth times of individuals be marked on ON. These birth points occur discontinuously and more or less densely. The number of points that lie on equal intervals of the line ON would serve to compare the density of births at different times.

Two kinds of birth points are to be distinguished, those corresponding to live births, and those corresponding to stillbirths. We consider only the first.

Starting from the birth points, we now measure the ages of individuals vertically to the birth axis, and parallel to the axis of ages $O\Omega$. That age at which the event under consideration, namely death, occurs to an individual is represented by a point.

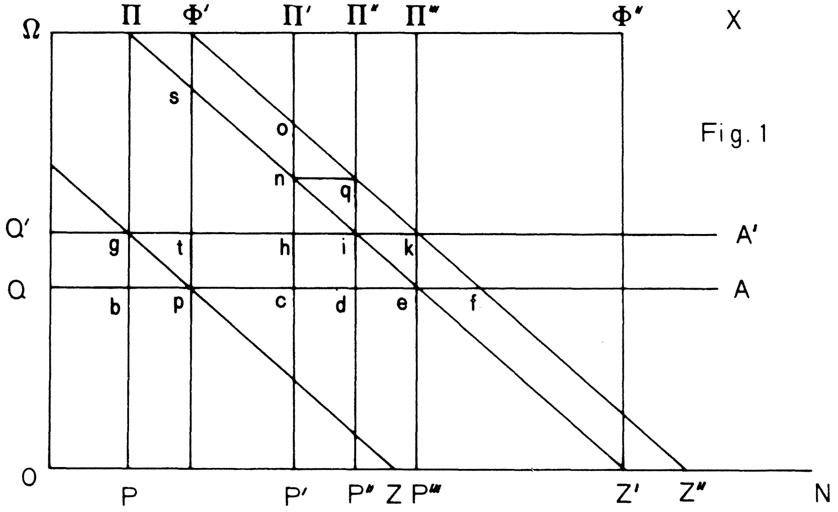

Fig. 1

In that way the plane ΩON is strewn with scattered death points, not shown in the diagram. The diagram is closed by the line ΩX, which runs parallel to ON at a distance equal to $O\Omega = \omega$, which corresponds to the highest age reached. The figure can be thought of as extending indefinitely to the right.

On every life-line there now lies only one death point, but with a high density of births there would tend to be in each small interval of births dying points of all the ages from O to ω.

The mass of points so presented can be divided by parallels to the two axes.

If the birth times of the abcissa are in general expressed by n, the equation $n=OP$ represents the vertical line $P\Pi$. This divides the dying times of the individuals who are born before the time OP from those born after.

If the ages, represented by the ordinates, are designated by a, then $a=OQ$ is the equation of the line QA, which distinguishes the dying points of those who die before reaching the age $O\Omega$ from those who die after.

Finally, there is a third division of the plane, such that the cases of death occurring before a certain absolute time are divided from those that lie beyond this time. If z is the general designation of the time of observation, then the equation

$$n+a=z$$

applies for the points that fall exactly at time z.

Let us now suppose that z takes a fixed value OZ' as measured on the birth axis, corresponding to the age O, and let z go through all values from O to ω, then n goes backwards through all values from z (or OZ') to $z-\omega$. The above equation thus represents the straight line $Z'\Pi$, which on further extension would thus cut the axis $O\Omega$ at a distance z from point O and makes an angle of 45° with either axis.

Consider now an age at death that belongs to an arbitrary birth point in the interval $z-\omega$ to z (or PZ'). If it is below that determined by the above equation, (that is, lower than $z-n$) then the death point of the individual concerned lies below line $Z'\Pi$. But if the age at death is higher than would be determined by the n from the above equation, then the death point lies above that line.

The line $Z'\Pi$ divides those who die before the time z from those whose death occurs after this time. For values of n that exceed z it is obvious that the corresponding cases of death can only occur after the time z. It is also plain that persons born before the time $z-\omega$ (or OP) necessarily die before the time z, for they otherwise would exceed the maximum age ω.

7. A Short Method for Constructing an Abridged Life Table

LOWELL J. REED and MARGARET MERRELL (1939)

From *American Journal of Hygiene* 30. Excerpts are from pages 33—38, 51.

For infancy Reed and Merrell revise their q_x formula to correct underenumeration, an approach that is not general. This has been omitted, as have their equations for L_x and T_x values by approximate integration. The article is followed by a note on other ${}_nq_x$ formulas.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_7,

The life table is so valuable a description of the age variation in danger of dying that it is desirable to have rapid methods of obtaining it from the recorded vital statistics. The principal value of the life table lies in the form in which age-specific risks of dying are stated. They are expressed as the probability of dying within a designated interval subsequent to an exact age, or as the probability of surviving from one exact age to another. Such probabilities are not only very descriptive, but they lend themselves readily to the treatment of mortality in a wide variety of problems.

The fundamental element in the construction of a life table for a particular time is thus to convert the observed age-specific death rates for that time into the probabilities of dying within stated age intervals. There is nothing in the definition of the probability of dying which fixes the length of these intervals. They may be chosen to suit the needs of the problem. The term "complete life table" has by custom come to designate a table in which the interval is 1 year, and the probability is stated for every year of age. It should be remembered, however, that this is pure convention, since a table computed for monthly intervals would be more complete and one for weekly intervals still more so.

The term "abridged life table" is less stabilized in its usage, since there are various ways in which a life table may depart from the form called "complete." One form of abridged table is illustrated by Foudray's 1920 Tables for the United States (1923) and by Dublin and Lotka's Appendix Tables in *Length of Life* (1936), in which the probabilities of dying are given for single years of age, but these values are tabulated at intervals of 5 or 10 years rather than for every year. They constitute, therefore, selected probabilities from a complete table, the intermediate values being omitted.

Another form of abridgement, which is the one considered in this paper, increases the interval over which the probabilities are stated, usually to 5 or 10 years. These probabilities cover the interval between the stated ages and therefore form a table which is not incomplete except in the defined technical sense. They differ from the values in a complete life table only in the fact that the interval chosen for study is greater than 1 year. The abridgement in this case, being through a condensation of the complete table rather than through omission, has certain advantages, since the survivorship values can be computed directly from the probabilities of dying, and the probabilities may be used as they stand, without interpolation, in any analysis where an interval greater than 1 year gives sufficient detail.

The method proposed is a short procedure for obtaining these probabilities of dying directly from the observed rates without going through the elaboration of constructing a complete table. The principal function to be derived is therefore ${}_nq_x$, the probability of dying within n years after age x, and the other life table functions flow directly from these values by procedures which are described.

Direct observation shows that there is a very high degree of association between the observed age-specific death rate over an age interval and the probability of dying within that interval. ...The basis of this relationship may be seen from the two following equations expressing, respectively, the age-specific death rate, ${}_nm_x$, in a stationary population, and the probability of dying, ${}_nq_x$, as functions of l_x, the number surviving to age x out of a given number born alive.

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[1] Paper no. 212 from the Department of Biostatistics, School of Hygiene and Public Health, The Johns Hopkins University, Baltimore, Md.

$$ {}_nm_x = \frac{l_x - l_{x+n}}{\int_x^{x+n} l_x dx}, \tag{1} $$

$$ {}_nq_x = \frac{l_x - l_{x+n}}{l_x}. \tag{2} $$

These equations show that ${}_nq_x$ is expressible in terms of ${}_nm_x$ and that if the equation for l_x were known, the function relating the two types of rates could be stated explicitly. The assumption that l_x is expressible as a straight line over the interval n leads to the well-known relationship[3]

$$ {}_nq_x = \frac{2n\ {}_nm_x}{2 + n\ {}_nm_x}, \tag{3} $$

which is the equation of a hyperbola, passing through the origin with a slope n, and having an upper asymptote, ${}_nq_x = 2. \ldots$

As an alternative, we might assume that l_x is an exponential curve over an interval n, since it is as reasonable to break up the arithlog graph of l_x into linear segments as it is to treat the arithmetic graph in this way. This assumption gives for the relationship between the age-specific rate of the life table and the probability of dying over the same age interval, the catalytic equation[4]

$$ {}_nq_x = 1 - e^{-n\ {}_nm_x}. \tag{4} $$

This equation, like the hyperbola (3), passes through the origin with a slope of n, but has an upper limit ${}_nq_x = 1$, where the hyperbola has a limiting value of 2. Both of these curves have, therefore, a rational position at the origin, the probability of dying being 0 when the death rate is 0. Furthermore, they have a rational slope at the origin since, as the average annual death rate becomes smaller, the probability of dying within n years approaches n times the value per year. With regard to the upper limit imposed on ${}_nq_x$, the catalytic with its limit of 1 is more rational than the hyperbola with its limit of 2, since the probability of dying cannot exceed unity. For intervals as small as 1 year, it is hard to say whether the hyperbola (3) or the catalytic (4) is to be preferred as an expression of the relationship between m_x and q_x, but neither proves to be satisfactory for longer intervals.

Other attempts to state an equation describing the l_x curve over longer intervals have led to rather involved equations of association. Furthermore, for practical applications it would be necessary to adjust such equations for the difference between observed and life table values of ${}_nm_x$. Therefore, the relationship between the observed ${}_nm_x$ and ${}_nq_x$ was studied directly without an explicit assumption with regard to the l_x equation. For this purpose Glover's 1910 life tables (1921) were examined as to the relation of the observed 5-year and 10-year death rates to the derived probabilities of dying over the same intervals.

Figure 1 presents for a 5-year interval the association between the observed ${}_5m_x$ and the derived ${}_5q_x$ for the 33 tables in Glover's 1910 series, the x values taking all the multiples of 5 through age 80. It is apparent from this graph that the observations fall exceedingly close to a smooth curve. This is especially impressive since the life tables are not at all similar in their mortality schedule. They represent tables for the different sexes and colors, for urban and rural areas, and for various combinations of these factors, with the variation in patterns that comes from such diverse groups. It would thus appear that the relationship of ${}_5m_x$ and ${}_5q_x$ is one which need not be stated for a particular age or

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[3] For the derivation of equation (3), see appendix 1.

[4] For the derivation of equation (4), see appendix 1.

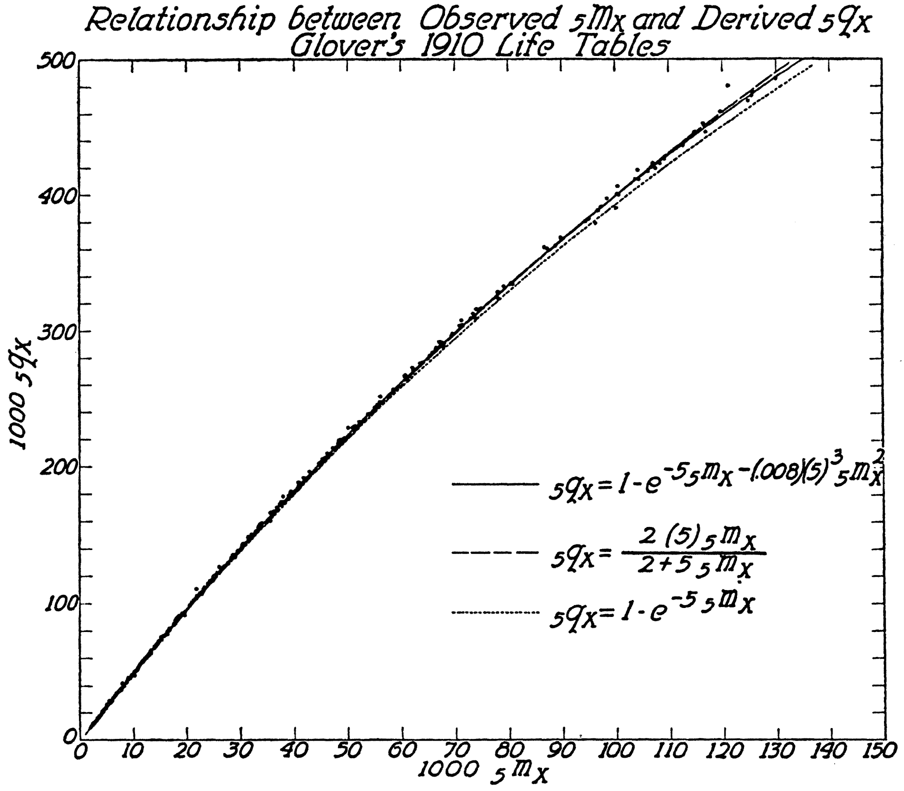

FIGURE 1. Correlation between the observed age specific death rate, $_5m_x$, and the probability of dying, $_5q_x$, stated per 1000, from Glover's 1910 life tables.

size of death rate, but is independent of these factors. The statement of this relationship in the form of an equation would clearly permit passing directly from the age-specific death rate to the probability of dying.

The observed relationship is of somewhat the same general form as the hyperbola (3) or the catalytic (4), as may be seen in figure 1. Neither of these theoretically derived equations, however, could be considered an adequate description of the relationship. The catalytic begins to fall below the points when $_5m_x$ is approximately 20 per 1000 (about age 50) and remains below the points throughout the rest of the range. The hyperbola differs from the points less consistently than the catalytic, but does not have the proper curvature. For the ages beyond 80 which are not shown on this graph, it is in general farther above the points than the catalytic is below them, since the l_x curve in extreme old age is better described by an exponential than a linear function. Thus, the catalytic and the hyperbola deviate from the points in different form but neither stays with them. This is due to the fact that the basic assumptions from which they were derived do not hold for intervals much longer than 1 year.

An equation was therefore sought which would describe the entire range of observations and one which would also be suitable for expressing the relationship of the rates for intervals other than 5 years. This requirement proved to be satisfactorily met by the equation

$$ {}_nq_x = 1 - e^{-n\,{}_nm_x - an^3\,{}_nm_x^2}, \qquad (5) $$

in which a is an arbitrary constant. Its value determined empirically for the method presented in figure 1 is $a = .008$. This equation is seen to be similar in form to (4), passing through the origin with a slope of n and having an upper asymptote ${}_nq_x = 1$, but it has a corrective term in the exponent which has the effect of lifting the latter part of the curve. Thus it is a rational curve in its position and slope at the origin and in its limiting value, and from figure 1, it is apparent that it fits the points well, rather remarkably so, in view of the fact that it contains only one arbitrary constant. It should be stated that the curve fits the points beyond age 80, which are not shown on this graph, as well as those presented, and the equation may therefore be used from age 5 to the end of life, to derive the life table probabilities from the observed rates.

The complexity of the l_x curve prevents the use of such a simple equation over very broad intervals. The grouping to which it may, with fair accuracy, be extended is indicated by the relationship for 10-year values. Figure 2 shows the scatter of points for all the 10-year rates from Glover's 1910 tables, at 5-year intervals from age 5 to 80. Here again, the points from the various life tables fall into the same pattern, none of the tables showing any marked or consistent divergence from the others. This scatter of points is compared graphically with the three equations just considered. The assumption that l_x could be expressed as a series of linear or exponential equations would naturally be more violent for an interval of 10 than of 5 years. It is not surprising, therefore, that both the hyperbola and the catalytic depart from the trend of the points in the same way as they did for the 5-year rates, but to a greater degree.

The empirical curve, derived above, with the same value of the arbitrary constant, is seen to give a very satisfactory description of the relationship, and the points hang closely to the central curve. The values beyond age 80 are also well described by the equation and thus for a grouping as broad as 10 years, the equation effects a satisfactory transformation of the observed age-specific death rates into probabilities of dying from age 5 to the end of life.

Although the observations have been presented only for an interval of 5 and of 10 years, it should not be inferred that the equation is limited to these values of n. The equation derived depends implicitly on the area and ordinate relationships in the l_x curve and is sufficiently elaborate to summarize these relationships with a high degree of approximation for an interval as large as 10 years. Consequently it holds equally well for smaller intervals, provided the age-specific rates over these intervals can be obtained, either by direct observation or adjustment of the observations, with the same accuracy as those observed for an n of 5 or 10 years. For an interval as small as 1 year, this curve, the catalytic (3), and the hyperbola (4) give virtually the same results over the entire age range.

The procedure for summarizing the relationship between the observed ${}_nm_x$ and ${}_nq_x$ statistically in the form of a regression equation does not take account of the individual variation about the central curve. However, the individual variation is, in general, well within the range of error of observation, and for areas such as states or smaller units, it is even within the variation of simple sampling. Thus the refinement of ${}_nq_x$ values beyond the value predicted by

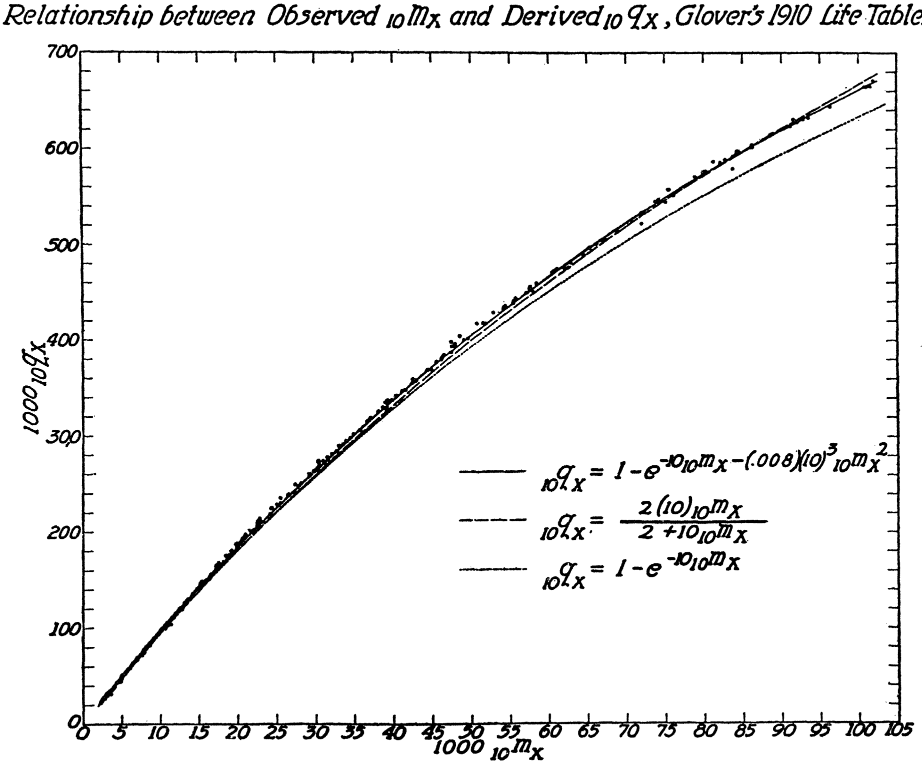

FIGURE 2. Correlation between the observed age specific death rate, $_{10}m_x$, and the probability of dying, $_{10}q_x$, stated per 1000, from Glover's 1910 life tables.

the regression curve from the observed specific rate is, for the usual case, carrying the statistical analysis beyond the point justified by the accuracy of the material. ...

Appendix 1

(*b*) *Derivation of equation (3) in text:*

Assume that over the interval n,

$$l_x = a + bx.$$

By definition

$$_nm_x = \frac{l_x - l_{x+n}}{\int_x^{x+n} l_x dx} \qquad (1)$$

and

$$_nq_x = \frac{l_x - l_{x+n}}{l_x}. \qquad (2)$$

Then

$$_nm_x = \frac{a + bx - a - b(x+n)}{\int_x^{x+n} (a+bx)dx}$$

$$= \frac{-b}{a + bx + \frac{bn}{2}},$$

$$_nq_x = \frac{-bn}{a + bx}.$$

Eliminating $a + bx$ between these two equations gives

$$_nq_x = \frac{2n\,_nm_x}{2 + n\,_nm_x}. \qquad (3)$$

(*c*) *Derivation of equation (4) in text:*

Assume that over the interval n

$$l_x = e^{a+bx}.$$

By substitution in equations (1) and (2)

$$_nm_x = \frac{e^{a+bx} - e^{a+b(x+n)}}{\int_x^{x+n} e^{a+bx}dx} = -b$$

and

$$_nq_x = \frac{e^{a+bx}(1 - e^{bn})}{e^{a+bx}} = 1 - e^{bn}.$$

Eliminating b between these two equations gives

$$_nq_x = 1 - e^{-n\,{}_nm_x}. \qquad (4)$$

References

Foudray, E.

1923 United States Life Tables, 1919–20. Washington (Bureau of the Census).

Dublin, L. I. and A. J. Lotka.

1936 Length of Life. New York. Ronald Press Company.

Glover, J. W.

1921 United States Life Tables, 1890, 1901, 1910, and 1901–1910. Washington (Bureau of the Census).

Editors' Note: Approximations to ${}_nq_x$

The equations for ${}_nq_x$ Reed and Merrell examine apart from their own derive from Milne and Farr. Their *equation (3)—equation (1)* in the article by Greville that follows—is found by setting $M_x = m_x$ and generalizing from 1 year to n year intervals Milne's equation (1815, p. 99)

$$\frac{P_x}{D_x} = \frac{\frac{1}{2}(l_x + l_{x+1})}{l_x - l_{x+1}} = \frac{l_x - \frac{1}{2}d_x}{d_x},$$

to yield

$${}_nm_x = \frac{{}_nD_x}{{}_nP_x} = \frac{l_x - l_{x+n}}{\frac{n}{2}(l_x + l_{x+n})} = \frac{{}_nd_x}{n(l_x - \frac{1}{2}{}_nd_x)}.$$

On dividing by l_x this becomes

$${}_nm_x = \frac{{}_nq_x}{n(1 - \frac{1}{2}{}_nq_x)} \quad \therefore \quad {}_nq_x = \frac{n\,{}_nm_x}{1 + \frac{n}{2}\,{}_nm_x}.$$

Reed and Merrell arrive at the same result by setting $l_x = a + bx$, which of course Milne assumed. The equation can also be identified as DeMoivre's (1725) hypothesis, discussed in chapter 4.

Reed and Merrell's *equation (4)*—Greville's *equation (2)*—derives from Farr's (1864, pp. xxiii—xxiv) English Life Table No. 3. Farr introduced a better form, which would generalize for abridged tables as

$${}_nq_x = 1 - \exp[-{}_nm_x r_x^{-n/2}(r_x^n - 1)/\ln r_x],$$

where

$$r_x = ({}_nm_{x+n}/{}_nm_x)^{1/n}.$$

Here, r is "the rate at which the rate of mortality increases or decreases." The derivation of the formula is explained by Wolfenden (1954, pp. 130—131). Assuming the force of mortality μ_x to be approximately ${}_nm_{x-n/2}$ and applying Gompertz' Law that the force of mortality increases exponentially with age, we have

$$\mu_{x+t} = {}_nm_{x+t-n/2} = r_x^{\,t-n/2}\,{}_nm_x.$$

This permits the substitution

$$
\begin{aligned}
{}_nq_x &= 1-\exp\left[-\int_0^n \mu(x+t)\,dt\right] \\
&= 1-\exp\left[-{}_nm_x r_x^{-n/2}\int_0^n r_x{}^t\,dt\right] \\
&= 1-\exp\left[-{}_nm_x r_x^{-n/2}(r_x^n-1)/\ln r_x\right].
\end{aligned}
$$

In application Farr used single year rates and dropped the minor term $r_x^{-n/2}$. If the force of mortality is assumed constant (i.e., $r=1$) the equation reduces to

$$
{}_nq_x = 1-\exp[-n\,{}_nm_x].
$$

This yields the l_x formula given in the appendix to Reed and Merrell:

$$
l_x = \alpha e^{-x\,{}_nm_x} = e^{a+bx}.
$$

Graphed with the simple exponential and the Reed-Merrell formula (see their Figure 2) the Farr equation would be the least accurate for the low ${}_nm_x$ rates at younger ages and intermediate for the higher rates that come later in life, reflecting its close dependence on Gompertz' Law. Greville's article completes this analysis by uncovering a derivation for the Reed-Merrell formula, which employs Gompertz' Law more effectively as a correction term.

8. Short Methods of Constructing Abridged Life Tables

T. N. E. GREVILLE (1943)

From *Record of the American Institute of Actuaries, Part I* 32. Excerpts are from pages 29, 34—40.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_8,

I. INTRODUCTION

In recent years sociologists, public health officers, and students of population problems have been taking an increasing interest in the life table as a description of the age-variation in the chances of death and survival. As a result of this interest in the subject, attention has been focused on rapid methods of constructing abridged life tables. Some of the most interesting and useful methods have been devised by nonactuaries and, probably for that reason, have not received the attention they deserve from the actuarial profession. It is the purpose of this paper to bring some of these methods to the notice of actuaries and also to show that certain formulas which have been put forward by their originators on purely empirical grounds actually have a valid mathematical basis. ...

III. RELATION BETWEEN ${}_nm_x$ AND ${}_nq_x$

The basic feature of the type of methods of life-table construction under consideration here is the mathematical relation assumed to hold between ${}_nm_x$ and ${}_nq_x$. There are a number of such relations which may be used. Perhaps the simplest is obtained by assuming that l_x can be regarded as a linear function in the age-interval. This leads to the equation

$$ {}_nq_x = \frac{2n \cdot {}_nm_x}{2 + n \cdot {}_nm_x}\,, \tag{1} $$

a formula which, for the particular case $n = 1$, is well known to actuaries [5, p. 5]. Another possible assumption is that l_x can be represented by an exponential function. This leads to the equation

$$ \operatorname{colog}_e {}_np_x = n \cdot {}_nm_x\,, \tag{2} $$

a relation which is also familiar to actuaries in the special case of $n = 1$ [5, p. 16].

More precise equations of the same general form as equations (1) and (2) will now be derived. In the life table

$$ {}_nm_x = \frac{l_x - l_{x+n}}{T_x - T_{x+n}} = -\frac{d}{dx}\log_e (T_x - T_{x+n}) = -\frac{d}{dx}\log_e {}_nL_x\,. \tag{3} $$

Integrating,

$$ {}_nL_x = Ce^{-\int {}_nm_x\,dx}\,. \tag{4} $$

Now, applying the Euler-Maclaurin summation formula, we have

$$T_x = \sum_{h=0}^{\infty} {}_nL_{x+nh} = C\left\{ \frac{1}{n}\int_x^{\infty} e^{-\int {}_nm_t\,dt}\,dt + \tfrac{1}{2}e^{-\int {}_nm_x\,dx} \right.$$
$$\left. + \frac{n}{12}\,{}_nm_x e^{-\int {}_nm_x\,dx} + \cdots\cdot \right\}.$$

Differentiating and substituting from equation (4) gives

$$l_x = {}_nL_x\left\{ \frac{1}{n} + \tfrac{1}{2}\,{}_nm_x + \frac{n}{12}\left({}_nm_x^2 - \frac{d}{dx}\,{}_nm_x\right) + \cdots\cdot \right\}. \tag{5}$$

Making use of equations (3) and (5), we get

$$\left.\begin{aligned} {}_nq_x &= \frac{l_x - l_{x+n}}{l_x} = \frac{{}_nm_x \cdot {}_nL_x}{l_x} \\ &= \frac{{}_nm_x}{\frac{1}{n} + \frac{1}{2}\,{}_nm_x + \frac{n}{12}\left({}_nm_x^2 - \frac{d}{dx}\,{}_nm_x\right) + \cdots\cdot}, \end{aligned}\right\} \tag{6}$$

or,

$${}_nq_x = \frac{2n \cdot {}_nm_x}{2 + n \cdot {}_nm_x + \frac{n^2}{6}\left({}_nm_x^2 - \frac{d}{dx}\,{}_nm_x\right) + \cdots\cdot}, \tag{7}$$

an equation of the same general form as equation (1).

In order to derive an equation of the form of equation (2), we write

$${}_np_x = 1 - {}_nq_x = \frac{2 - n \cdot {}_nm_x + \frac{n^2}{6}\left({}_nm_x^2 - \frac{d}{dx}\,{}_nm_x\right) + \cdots\cdot}{2 + n \cdot {}_nm_x + \frac{n^2}{6}\left({}_nm_x^2 - \frac{d}{dx}\,{}_nm_x\right) + \cdots\cdot},$$

whence

$$\text{colog}_e\,{}_np_x = \log_e\left[1 + \frac{n}{2}\,{}_nm_x + \frac{n^2}{12}\left({}_nm_x^2 - \frac{d}{dx}\,{}_nm_x\right) + \cdots\cdot\right]$$
$$- \log_e\left[1 - \frac{n}{2}\,{}_nm_x + \frac{n^2}{12}\left({}_nm_x^2 - \frac{d}{dx}\,{}_nm_x\right) + \cdots\cdot\right].$$

This gives, upon expansion and simplification,

$$\operatorname{colog}_e {}_n p_x = n \cdot {}_n m_x + \frac{n^3}{12} {}_n m_x \frac{d}{dx} {}_n m_x + \cdots , \tag{8}$$

a formula of the same general form as equation (2).

The first question that arises in the actual application of formulas (7) and (8) concerns the evaluation of the derivative of ${}_n m_x$. It turns out that great exactness is not necessary, as the terms into which this derivative enters are in the nature of minor adjustments which do not have a great effect on the resulting ${}_n q_x$ values. If it is assumed that the function ${}_n m_x$ can be represented by a polynomial, it is found that

$$\frac{d}{dx} {}_n m_x = \frac{{}_n m_{x+n} - {}_n m_{x-n}}{2n} \tag{9}$$

or

$$\frac{d}{dx} {}_n m_x = \frac{-{}_n m_{x+2n} + 8 {}_n m_{x+n} - 8 {}_n m_{x-n} + {}_n m_{x-2n}}{12n} , \tag{10}$$

according to whether a polynomial of the second or the fourth degree is assumed. These formulas, however, are not applicable when neighboring age-groups do not contain the same number of years.

As an alternative, it may be noted that the well-known formula for the derivative of ${}_n p_x$ [5, p. 17] gives

$$\frac{d}{dx} \operatorname{colog}_e {}_n p_x = \mu_{x+n} - \mu_x .$$

Therefore, differentiating equation (8) gives

$$\frac{d}{dx} {}_n m_x = \frac{1}{n} (\mu_{x+n} - \mu_x)$$

approximately. Substituting this approximation in equation (8) gives, finally,

$$\operatorname{colog}_e {}_n p_x = n \cdot {}_n m_x \left[1 + \frac{n}{12} (\mu_{x+n} - \mu_x) \right] .$$

This is the formula given (for the case of $n = 5$) by the editors of *JIA* [1, p. 301; 2, pp. 121–22]. However, in the practical application of this formula it was found necessary to approximate the expression in parentheses in terms of ${}_nm_x$ values. This makes the formula equivalent to equation (8), used in conjunction with the approximation (9).

Since ${}_nm_x$ is approximately equal to the force of mortality at the middle of the period, a more logical assumption is that ${}_nm_x$ is an exponential function, in accordance with Gompertz' Law. If

$${}_nm_x = Bc^x \, ;$$

then

$$\frac{d}{dx}\, {}_nm_x = k \cdot {}_nm_x \, ,$$

where

$$k = \log_c e \, .$$

Substituting this value in equations (6) and (8) gives us

$${}_nq_x = \frac{{}_nm_x}{\dfrac{1}{n} + {}_nm_x\left[\dfrac{1}{2} + \dfrac{n}{12}\left({}_nm_x - k\right)\right]} \tag{6a}$$

and

$$\mathrm{colog}_e\, {}_np_x = n \cdot {}_nm_x + \frac{k}{12}\, n^3 \cdot {}_nm_x^2 \, . \tag{8a}$$

As a moderate variation in the value of k has little effect on the value of ${}_nq_x$ except at the older ages, where most mortality tables follow Gompertz' Law fairly closely, and at the very young ages, which are generally dealt with by a different method, k may be safely taken as a constant throughout the table. Henderson states [6, p. 90] that for practically all mortality experiences $\log_{10} c$ lies between .035 and .045. Therefore, $k = \log_e c$ would fall between .080 and .104.

It is interesting to observe that Reed and Merrell, after finding equations (1) and (2) unsatisfactory at the older ages, suggested on empirical grounds the equation

$${}_nq_x = 1 - e^{-n \cdot {}_nm_x - an^3 \cdot {}_nm_x^2} \, , \tag{11}$$

a relation exactly equivalent to equation 8(a) if a is written for $k/12$. By fitting this curve to the thirty-three tables in Glover's 1910 series, they arrived at the value .008 for a. This would correspond to $k = .096$, a value well within the range given by Henderson. Reed and Merrell have published extensive tables of the function (11) with $a = .008$, both for $n = 5$ and for $n = 10$ and also for $n = 3$ over a limited range, intended for use in the age-interval two to five. With the aid of these tables a good computer can construct an entire abridged life table in less than two hours.

Although experiment would indicate that equation (6a) generally gives nearly as good results as equation (8a), the latter has certain theoretical advantages. In both cases ${}_nq_x = 0$ when ${}_nm_x = 0$. However, in equation (8a) ${}_nq_x$ approaches unity as it should do, when ${}_nm_x$ increases without limit, while in equation (6a) there is a point beyond which ${}_nq_x$ ceases to increase with ${}_nm_x$, and its limit is actually zero. ...

Reed and Merrell seem to have considered that, were it not for underenumeration, the expression (11) (with $a = .008$) should be applicable to ages zero and one as well as to the remainder of the life-span [3, p. 690]. Theoretical considerations would indicate that in a period of decreasing mortality $k = \log_e c$ (and therefore a in the Reed-Merrell formula) should be negative, and rough calculations based on recent United States Life Tables indicate for age zero a value of about $-.3$ for a instead of .008. The fact is, however, that for an age-interval of only one year, this adjustment is not of much consequence, and for such an interval equation (11) would be close enough for practical purposes. However, equation (1) or equation (2) would be slightly preferable, as the additional adjustment contained in equation (11) is in the wrong direction.

V. LIFE-TABLE POPULATION

The calculation of the values of l_x and ${}_nd_x = l_x - l_{x+n}$ from those of ${}_nq_x$ requires no explanation or comment. However, the computation of the ${}_nL_x$ column introduces some questions. Two distinct methods of obtaining ${}_nL_x$ have been suggested. The first [7] is based on the assumption that ${}_nm_x$ has the same value in the actual population and in the life-table population and employs the relation

$$ {}_nL_x = \frac{{}_nd_x}{{}_nm_x} . \tag{12} $$

which means, in practice, using an approximate integration formula, such as

$$ {}_nL_x = \frac{n}{2}\,(l_x + l_{x+n}) + \frac{n}{24}\,({}_nd_{x+n} - {}_nd_{x-n})\;. \tag{14} $$

This approximate method, although less direct and in theory less exact, generally produces superior results in actual practice. This is because the values of l_x obtained by the abridged process contain inaccuracies and irregularities, and it turns out that the effect on the value of ${}_nL_x$ in equation (13) of slight errors in l_x is less than the effect in equation (12) of the corresponding error in ${}_nd_x$ and ${}_nm_x$. A mathematical demonstration of this fact may be of interest.

From equation (13) we can write by the Theorem of Mean Value

$$ {}_nL_x = nl_{x+\theta}\;, $$

where θ is a number beween 0 and n. Therefore, if $d({}_nL_x)$ denotes the error in the value of ${}_nL_x$, we have

$$ \frac{d({}_nL_x)}{{}_nL_x} = \frac{d(l_{x\,+\theta})}{l_{x+\theta}}\;. $$

From this it appears that a given percentage of error in the l_x values will tend to produce about the same percentage of error in ${}_nL_x$. On the other hand, equation (12) gives

$$ \frac{d({}_nL_x)}{{}_nL_x} = \frac{d({}_nd_x)}{{}_nd_x} - \frac{d({}_nm_x)}{{}_nm_x}\;. $$

This indicates that, when equation (12) is used, a given percentage of error in either ${}_nd_x$ or ${}_nm_x$ tends to produce about the same percentage of error in ${}_nL_x$. However, it is obvious that errors of a given size in the values of l_x would tend to produce errors of the same absolute magnitude (and therefore much greater percentage errors) in ${}_nd_x$. Likewise, the values of ${}_nm_x$ are likely to contain greater relative errors than the l_x values. This explains why in practice the approximate integration formula gives better results.

BIBLIOGRAPHY

1. EDITORS of *JIA*. Note appended to the article "On a Short Method of Constructing an Abridged Mortality Table, by GEORGE KING, *JIA*, XLVIII, 301–3.

2. WOLFENDEN, H. H. *Population Statistics and Their Compilation.* ("Actuarial Studies," No. 3.) New York: Actuarial Society of America, 1925.

3. REED, L. J., and MERRELL, M. "A Short Method for Constructing an Abridged Life Table," *American Journal of Hygiene*, Vol. XXX, No. 2. Reprinted by the U.S. Bureau of the Census as *Vital Statistics—Special Reports*, Vol. IX, No. 54. All page references are to the Census publication.

4. DUBLIN, L. I., and LOTKA, A. J. *Length of Life.* New York: Ronald Press Co., 1936.

5. SPURGEON, E. F. *Life Contingencies.* London: Macmillan & Co., Ltd., 1938.

6. HENDERSON, R. *Mathematical Theory of Graduation.* ("Actuarial Studies," No. 4.) New York: Actuarial Society of America, 1938.

7. DOERING, C. R., and FORBES, A. L. "A Skeleton Life Table," *Proceedings of the National Academy of Sciences*, XXIV, 400–405.

8. GLOVER, J. W. *United States Life Tables, 1890, 1900, 1910, and 1901–1910.* Washington: Government Printing Office, 1921.

9. Life Tables for Natural Populations of Animals

EDWARD S. DEEVEY, Jr. (1947)

From *Quarterly Review of Biology* 22. Excerpts are from pages 283—284, 287—292, 294, 296—302, 312.

In selecting from Deevey's extensive review, we have tried to emphasize the different types of problems that arise in working with animal populations; our most serious omission is a detailed study of barnacles that examines crowding effects and mortality. Deevey begins his article with discussions of the life table and of different general survival patterns, which we also omit.

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Having gained some idea of the limits circumscribing his own mortality, man has turned to look at the other animals. In 1935 Pearl and Miner, in their discussion of the comparative mortality of lower organisms, attempted to formulate a general theory of mortality. They quickly gave up the attempt upon realizing that the *environmental* determinants of life duration can not, at least as yet, be disentangled from such *biological* determinants as genetic constitution and rate of living. They ended with a plea for "more observational data, carefully and critically collected for different species of animals and plants, that will follow through the life history from birth to death of each individual in a cohort of statistically respectable magnitude." Thus by implication Pearl and Miner appealed to the ecologists, who for the most part have been busy elsewhere. Accounts of the conceptions and methodology of life tables have not yet found their way into textbooks of ecology, and while field naturalists have devoted increasing attention to the dynamics of natural populations most of them have been content to leave the construction of life tables to the statisticians and laboratory ecologists.

This article, which is designed as an introduction to the subject rather than as a formal review, brings together from the ecological literature a mass of information bearing on the survival of animals in nature. This information has not heretofore been considered relevant by biometricians working with human populations, nor has it ever been considered in its context by ecologists. In collecting the material it was immediately obvious that it is still too early to formulate general theories. Serious deficiencies are only too apparent in the data. But the difficulties differ from case to case, and are therefore not insurmountable. Moreover, the bibliography will show that virtually all of this knowledge has been acquired in the twelve years since the appearance of the review by Pearl and Miner. By taking stock now, and by calling attention to gaps in our information, it is hoped that some guidance can be given to ecologists and others in the gathering of new material....

Ecological Life Tables

The field ecologist deals with populations which are by no means so elementary as those inside *Drosophila* bottles. Even the total size of the population of a species cannot be easily ascertained for an area large enough to be representative, and calculations of the birth rate and death rate are uncertain at best, largely owing to immigration and emigration. It is seldom indeed that the ecologist knows anything of the age structure of a natural population. In a few cases, growth rings on the scales or otoliths (fish) or horns (ungulates) make it possible to determine the age of an animal. Moore (1935) has shown that annual growth rings occur in the genital plates of sea-urchin tests, as they do in the shells of some molluscs. Moore checked the validity of the age determination by reference to the size-frequency distribution in his catches, and the separation of modal size classes in a population often affords a clue to age, particularly for younger age groups. The age of adult females can be determined in the case of certain mammals (whales, Wheeler, 1934, Laurie, 1937; seals, Bertram, 1940) by counting the corpora lutea in the ovaries. But for most animals it is possible to find out the ages of individuals only by marking them in some way.

Even when the age of a member of a natural population is known, it is not a simple matter to obtain accurate vital statistics. The source of greatest confusion lies in the impracticability of keeping the individuals under continuous observation. Migratory birds, for example, are easy to band as nestlings, but nearly impossible to find between fledging and the time they leave for winter quarters. Often they can not be found at all unless they return to the same area to breed, when they can be trapped in nest boxes. Their mortality between fledging and breeding can be calculated, but the calculation is rendered uncertain by the tendency of young birds not to return to their birthplaces as breeding adults.

As sources of data for the construction of life tables, the ecological information falls into three groups: (1) cases where the age at death (d_x) is directly observed for a large and reasonably random sample of the population; (2) cases where the survival (l_x) of a large cohort (born more or less simultaneously) is followed at fairly close intervals throughout its existence; (3) cases where the age structure is obtained from a sample, assumed to be a random sample of the population, and d_x is inferred from the shrinkage between successive age classes. It should be noticed that only the second sort of information is statistically respectable, since in so far as the breeding can safely be assumed to be simultaneous, it is comparable to that obtained from a *Drosophila* bottle. The first and third types can be used only if one is prepared to assume that the population is stable in time, so that the actual age distribution and the life table age distribution are identical. This assumption would certainly not be true of a human population; it may be approximately true for many natural populations of animals....

Age at Death Directly Observed

In the course of his careful investigation of the wolves of Mt. McKinley, Murie (1944) picked up the skulls of 608 Dall mountain sheep (*Ovis d. dalli*) which had died at some time previous to his visit, and an additional 221 skulls of sheep deceased during the four years he spent in the Park. The age of these sheep at death was determinable from the annual rings on the horns. "Time, which antiquates antiquities, and hath an art to make dust of all things, hath yet spared these minor monuments" (Sir Thomas Browne, *Urn Burial*). Most of the deaths presumably occurred directly as a result of predation by wolves. Many skulls showed evidence of a necrotic bone disease, but it is not possible to say whether death was due solely to the disease or whether the disease merely ensured death by predation.

The mean longevity of the later sample is significantly greater (7.83 years) that that of the earlier (7.09 years), but the interpretation of this fact is not clear. The form of the distribution of deaths is sensibly the same in the two samples. As the survival of the members of this population is astonishingly great, it seems best to be conservative, and attention has been focussed on the larger, earlier sample. Except for the "lamb" and "yearling" classes, which are doubtless underrepresented in the data owing to the perishability of their skulls, there is no reason to suppose that either group is anything but a fair sample of the total population, i. e., the probability of finding a skull is not likely to be affected by the age of its owner. A life table for the 608 sheep has accordingly been prepared (Table 1). The survivorship curve, plotted logarithmically in Fig. 2, is remarkably

Table 1. Life table for the Dall Mountain Sheep (Ovis d. dalli) based on the known age at death of 608 sheep dying before 1937 (both sexes combined)*. Mean length of life 7.09 years. Data from Murie (1944)

x	x'	d_x	l_x	$1000\ q_x$	e_x
Age *(years)*	Age as % Deviation from Mean Length of Life	Number Dying in Age Interval out of 1000 Born	Number Surviving at Beginning of Age Interval out of 1000 Born	Mortality Rate per Thousand Alive at Beginning of Age Interval	Expectation of Life, or Mean Life-Time Remaining to Those Attaining Age Interval *(years)*
0—0.5	−100	54	1000	54.0	7.06
0.5—1	−93.0	145	946	153.0	—
1—2	−85.9	12	801	15.0	7.7
2—3	−71.8	13	789	16.5	6.8
3—4	−57.7	12	776	15.5	5.9
4—5	−43.5	30	764	39.3	5.0
5—6	−29.5	46	734	62.6	4.2
6—7	−15.4	48	688	69.9	3.4
7—8	− 1.1	69	640	108.0	2.6
8—9	+13.0	132	571	231.0	1.9
9—10	+27.0	187	439	426.0	1.3
10—11	+41.0	156	252	619.0	0.9
11—12	+55.0	90	96	937.0	0.6
12—13	+69.0	3	6	500.0	1.2
13—14	+84.0	3	3	1000	0.7

* A small number of skulls without horns, but judged by their osteology to belong to sheep nine years old or older, have been apportioned *pro rata* among the older age classes.

"human" in showing two periods of relatively heavy mortality, very early and very late, with high and nearly constant survival ratios at intermediate ages.

The adult sheep have two principal methods of defense against wolves, their chief enemies: flight to higher elevations, where wolves can not pursue; and group action or herding. It is clear that these recourses confer a relative immunity to death by predation and that only the very young, which have not learned by experience, and the very old, which are too feeble to escape, suffer heavy losses. ...

The second case to be discussed is that of an aquatic invertebrate, the sessile rotifer *Floscularia conifera*. This species has been studied by Edmondson (1945) under conditions which are fully as natural as those enjoyed by Murie's mountain sheep. *Floscularia* lives attached to water plants, especially *Utricularia*, surrounded by a tube constructed by itself out of pellets of detritus. The tube is added to at the top continuously throughout life, and Edmondson was able to identify all the members of a population living in a pond by dusting the *Utricularia* plant with a suspension of powdered carmine. On subsequent visits the *Floscularia* present at the time of dusting were conspicuously marked by bands of carmine-stained pellets in the walls of their tubes, each band being surmounted by new construction of varying widths. Thus in one operation the stage was set for an analysis of growth, age, birth-plus-immigration, and death in a natural population. Among other spectacular results, Edmondson found that the expectation of life of solitary individuals was only half as great as that of members of colonies of two or more, and he presented separate life tables for each component of the population, calculated from the age at death. To facilitate comparison with other species, however, solitary and colonial individuals have been lumped together (for Edmondson's "Experiment 1") in the life table of Table 2.

Table 2. Life table for the sessile rotifer Floscularia conifera based on the known age at death of 50 rotifers, both solitary and colonial. Mean length of life 4.74 days. From Edmondson (1945), Experiment 1

x	x'	d_x	l_x	$1000\,q_x$	e_x
Age *(days)*	Age as % Deviation from Mean Length of Life	Number Dying in Age Interval out of 1000 Attaching	Number Surviving at Beginning of Age Interval out of 1000 Attaching	Mortality Rate per Thousand Alive at Beginning of Age Interval	Expectation of Life, or Mean Life Time Remaining to Those Attaining Age Interval *(days)*
0—1	−100	20	1000	20	4.76
1—2	−78.9	200	980	204	3.78
2—3	−57.8	60	780	77	3.70
3—4	−36.7	0	720	0	2.98
4—5	−15.6	300	720	416	1.97
5—6	+ 5.4	140	420	333	2.02
6—7	+26.7	60	280	214	1.79
7—8	+47.7	140	220	636	1.14
8—9	+68.8	40	80	500	1.25
9—10	+90.0	20	40	500	1.00
10—11	+111.0	20	20	1000	0.50

The survivorship curve (Fig. 2), like that of the Dall sheep, shows unexpectedly good survival. As Edmondson has pointed out, it is not so good as that of other rotifers reared in the laboratory under standard conditions *(Proales decipiens, P. sordida, Lecane inermis)*, but it is only a little less good....

The case of *Floscularia* is almost above reproach as an example of a life table obtained under natural conditions. It is, of course, open to the objection that only the age at death is known, and the age structure of the living animals must be assumed to be constant. Apart from this deficiency, it should also be realized that the origin of the life table is not at birth. The pelagic larval life of the rotifer, like the larval life of barnacles and insects, is omitted from consideration in such a table....

In his delightful book, *The Life of the Robin* (1943a) and in two admirable papers, Lack (1943b, c) has investigated the age at death of certain British birds,

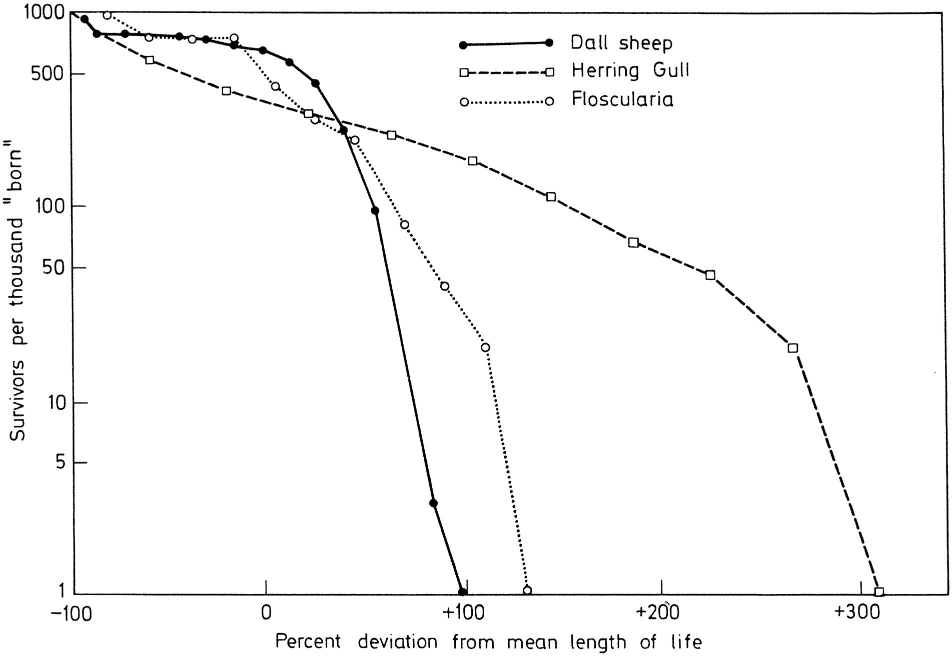

Fig. 2. Survivorship (l_x) Curves for the Dall Mountain Sheep, the Sessile Rotifer Floscularia conifera, and the Herring Gull, Age being Expressed as Percentage Deviation from the Mean Length of Life

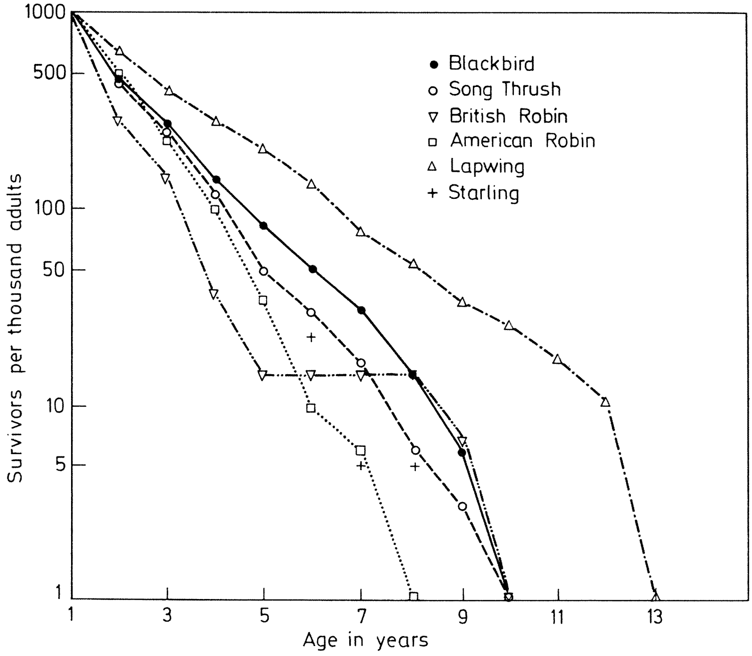

Fig. 3. Survivorship (l_x) Curves for the British Robin, Song Thrush, Blackbird, Starling, Lapwing and American Robin, Age being Expressed in Years

as obtained by recoveries of individuals banded as nestlings. Because banded nestlings are likely to be picked up near the banding stations or not at all, it is impossible to estimate the whole of the mortality in the first year of life with any accuracy, and Lack begins his life tables on August 1. The proportion of birds banded which are later recovered is small, ranging from 1.0 per cent for the robin to 18.4 per cent for the cormorant; but after August 1 of the first year it is considered that the ages at death of birds recovered are not likely to differ from the ages at death among the total population. The samples are small and of course become progressively smaller with increasing age, so that Lack does not regard the mortality rates and expectation of life as reliable beyond the fourth or fifth year.

Several of Lack's life tables are reproduced in Fig. 3....

The striking feature of these survivorship curves is their diagonal form. The mortality in the first year varies from 380 per thousand for the lapwing to 723 per thousand for the robin, but for a given species the mortality remains approximately constant throughout life, or at least for as long as the data are reliable....

Survivorship Directly Observed

The cases now to be discussed differ from the preceding in the character of the original observations. Instead of a fairly large sample of individuals about which little or nothing can be told except their age at death, we have a group of individuals known to have been born at a particular time and to have been present or absent at some later time. Their presence gives their survivorship, their absence implies death in the interval since they were last observed. This is the best sort of information to have, since it does not require the assumption that the age composition of the population is stable in time. Provided only that the season of birth is a small fraction of the age interval between successive observations, so that births can be assumed to be simultaneous, as in a *Drosophila* bottle, a horizontal life table can be directly constructed from the survivorship data. Unfortunately, most of the species which have been studied in this way have short spans of natural life, and when census data are obtained only once a year the number of points on the survivorship curve is too small to be satisfactory.

The best example of such observed survivorship comes from Hatton's work (1938) with the barnacle, *Balanus balanoides*. ... The case is very nearly ideal. The barnacle settles on rocks during a short time (two to six weeks) in early spring. Test areas were scraped clean one winter, and after new populations had settled, the survival of their members was followed at intervals of one to four months for three years. Barnacles which disappeared from the areas between observations were certainly dead, for emigration does not complicate the problem. Immigration, however, does present difficulties, though since it is confined to the attachment seasons of subsequent years it should be possible to control it in subsequent work. There is one further disadvantage in that the life tables necessarily start at metamorphosis, leaving out of account mortality during pelagic larval stages. A life table for a typical population of barnacles is presented in Table 6.

Table 6. Life table for a typical population of Balanus balanoides, based on the observed survival of adult barnacles settling on a cleaned rock surface in the spring of 1930. The population is that at Cité, (St. Malo, France), a moderately sheltered location, at Level III, at half-tide level. The initial settling density (2200 per 100 cm^2) is taken as the maximum density attained on May 15. Mean length of life 12.1 months. Data from Hatton (1938)

x	x'	d_x	l_x	$1000\ q_x$	e_x
*Age *(months)*	*Age as % Deviation from Mean Length of Life	Number Dying in Age Interval of 1000 Attaching	Number Surviving to Beginning of Age Interval out of 1000 Attaching	Mortality Rate per Thousand Alive at Beginning of Age Interval	Expectation of Further Life *(months)*
0—2	−100	90	1000	90	12.1
2—4	−83.5	100	910	110	11.3
4—6	−67.0	50	810	62	10.5
6—8	−50.4	60	760	79	9.1
8—10	−33.9	80	700	114	7.8
10—12	−17.4	160	620	258	6.7
12—14	− 0.9	80	460	174	6.7
14—16	+16.0	100	380	263	5.9
16—18	+32.2	50	280	179	5.7
18—20	+49.0	40	230	174	4.7
20—22	+65.4	100	190	526	2.4
22—24	+82.0	60	90	667	1.9
24—26	+98.8	20	30	667	1.8
26—28	+115.0	8	10	800	1.4
28—30	+132.0	2	2	1000	1.0

* Survivorship data given graphically by Hatton were smoothed by eye, and values at every other month were then read from the curve. The original observations were made at irregular intervals during three years.

The remaining examples suffer from more serious defects, and the data do not justify extended treatment. Green and Evans (1940) in their important study of the snowshoe rabbit *(Lepus americanus)* in Minnesota, followed the survival of marked individuals of several year classes, the total population present on the area and the number in each age-class being obtained by the mark-and-recapture method—also known as the "Lincoln index" (Jackson, 1939). Marking was done during most of the winter, and the annual census was made in February. It is perhaps unnecessarily, and certainly uncharitable, to point out two sources of error in this excellent and ingenious work. In the first place, when marked individuals are released into a population and later recaptured, the calculation of the total population from the fraction

$$\frac{\text{size of sample when recapturing}}{\text{number recaptured}} \times \text{number marked}$$

depends on two assumptions, neither of which is likely to be true in this case: that there is no mortality between marking and recapturing; and that the marked individuals disperse at random through the whole population. Secondly, the flow of vital events in this population was so rapid, very few rabbits more than three years old ever having been found, that observations made annually can give only a very rough idea of the life table.

The latter objection applies with equal force to the study of a pheasant population made by Leopold et al. (1943) in Wisconsin. The former objection, though doubtless it could be urged, has less validity here, since the population, as ascertained by trapping, was checked by census drives.

Nice's thoroughgoing work (1937) on the song sparrow *(Melospiza melodia)* included a consideration of the survival of banded birds from year to year. The number of individuals which could be kept under continuous observation was necessarily small, and to find a sample large enough to use as the basis of a life table, it is necessary to take the 144 males banded in the breeding season between 1928 and 1935. Unfortunately, some of these males were of unknown age when first banded. Even if one assumes, (and the assumption is not far from the truth) that all new males appearing are first-year birds born elsewhere, the survival ratios from year to year will be too low if any adult males were still alive but failed to return to the area. Evidently such emigration is of minor importance with adult male song sparrows. With adult females, however, it is so serious that Nice did not think it worth while to publish the data on their return. Clearly, work on the survival of migratory birds is full of uncertainties, though the same may be

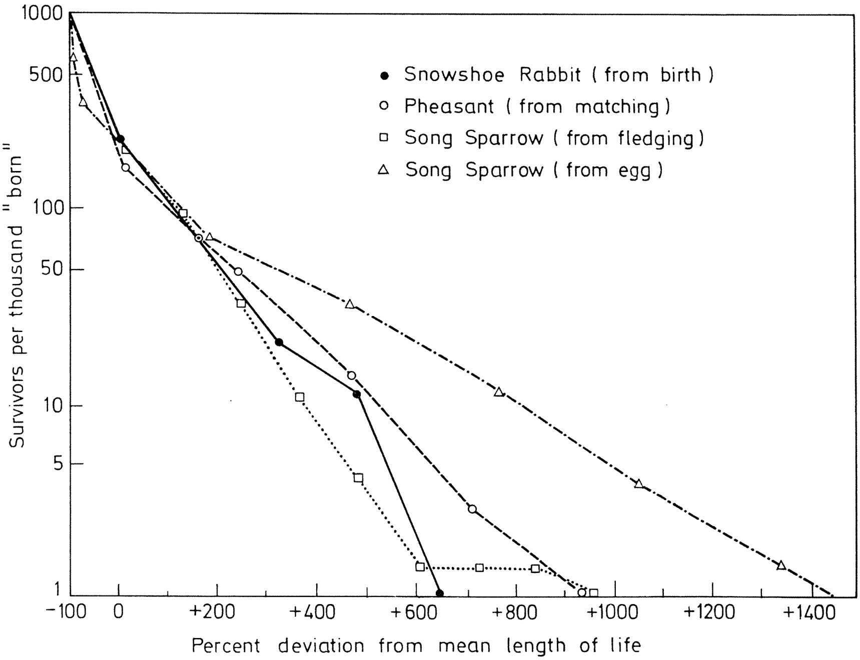

Fig. 5. Synthetic Survivorship (l_x) Curves for the Snowshoe Rabbit, the Pheasant, and the Song Sparrow, the Latter Calculated from Two Different Biological Ages. Data for different parts of the life spans are derived in different ways, as explained in the text.

said of resident species such as the wren-tit (Erickson, 1938) and the robin (Lack, 1943 a).

All of these cases, snowshoe rabbit, pheasant, and song sparrow have one defect in common. This is the necessity of calculating the survival between birth and the first year of adult life from other data than those given by banding. For the snowshoe rabbit, the initial strength of the year-class is calculated from the estimated breeding population present and its known fertility. Leopold et al., lacking observations of their own on the pheasant mortality between birth and the first census period, used the estimates given by Errington for pheasants in another state. Nice calculates the survival of fledged young song sparrows to their first breeding season, by assuming a stable population and combining the estimated mean length of life of adults with their average nesting success. These procedures, while perfectly defensible as approaches to the problem, are inadequate substitutes for direct observation.

The three sets of data, with all their uncertainties, have been used as bases for synthetic life tables, and the survivorship curves are presented in Fig. 5. ...

Age Structure Directly Observed

Ecological information of a third sort is available for a number of natural populations, principally of fishes and birds. In these cases the investigator has been able to determine how many individuals of each age are living in the population, and the age at death, though calculable from the shrinkage between successive age classes, has not been directly observed. This kind of information lends itself just as well as either of the others to the computation of life tables. As in the group where only the age at death is known, of course, it is necessary to assume the age composition to be unchanged with time. When this assumption is unreasonable, as it often is for fish populations, with their outrageous fluctuation in strength of year-classes, average age compositions obtained from several years' work can often be used. As it happens, however, all the life tables which fall in this third group are incomplete for one reason or another, and the data do not bear close comparison with such examples of natural life tables as those of the Dall sheep and the barnacle.

Kortlandt (1942) has recently given a very elaborate analysis of the Netherlands population of cormorants *(Phalacrocorax carbo sinensis)*. Birds banded as nestlings were later observed in their breeding and playing colonies, the numbers on the bands being read with the aid of a telescope. The age distribution of the banded birds being known in 1940 and 1941, it should be possible to infer the age distribution of the total population and from this to compute the annual mortality suffered by each year-class. A number of complicating conditions are present in this case, however, making direct calculation unreliable and necessitating a more circuitous approach: (1) the size of the Dutch cormorant population is not constant, but has been increasing by about 10 per cent per year, as estimated by counts of nests at the breeding colonies; (2) differences between the observed sex ratio among sexually mature birds and the sex ratio predicted on the basis of estimated mortality by sex and age class show clearly that there is some *band* *mortality;* that is, some birds either lose bands or die because of the band, making estimates of natural mortality too high by a factor of about 2; (3) it is not possible to infer the complete age structure from observations made at breeding colonies, since the one- and two-year-old birds occupy "colonies" elsewhere, returning to their birthplaces to breed no sooner than their third year.

In view of these difficulties, and others which need not be discussed here, Kortlandt's results must be regarded as schematic and preliminary only, and scarcely warrant the construction of a life table. His computations suggest that cormorants suffer mortality somewhat as follows: 17 per cent between fledging and the first May 1; 8 per cent in the first year; 6 per cent in the second year; and about 4 per cent per year in the third to twelfth years. These are astonishingly low figures for a natural population, but it must be remembered that the population is increasing rapidly....

The literature of fisheries biology is full of attempts to estimate the mortality of fishes, to distinguish natural mortality from rate of exploitation, and to determine the rate of exploitation which, given certain mortality rates and certain relationships between age and size, will steadily yield an optimum catch. These complex questions are fully discussed in the important works of Russell (1942), Thompson and Bell (1934), and Ricker (1940, 1944), and by various authors in the *Rapports et Procès-Verbaux of the International Council for the Study of the Sea*, Volume 110, 1939. Little of this information can be directly used for our purpose. The explanation is as simple as it is regrettable: although the age of a caught fish can be ascertained with more or less complete confidence, fishes of all ages can not be caught with equal facility. Inevitably the methods so skilfully developed for catching fish of desirable sizes will fail to catch fish of undesirable sizes. It is true that on various occasions the whole fish population of a lake has been removed by poisoning or drainage. The estimates given by Eschmeyer (1939) for the abundance of large-mouth black bass *(Huro salmoides)* in Howe Lake, Michigan, at the time of its poisoning in 1937, may be cited as a example:

Age	*Number*
0	18,374
0 (cannibals)	229
I	25
II	10
III	105
IV	7
V and older	9
Total	18,759

The implication of enormously greater mortality in the first year of life is plain from these figures, but such data can not be taken as they stand, partly because of very variable annual recruitment, and partly because young of the year were removed from the lake at various times for hatchery purposes....

This section may logically be concluded with a brief reference to the data for the fin whale, in which, as recent investigations have shown, the age of the female can be determined from the number of old corpora lutea in the ovary. By this method Wheeler (1934) arrived at the following as the age structure, observed over five seasons, 1926—1931, of the catch of female fin whales in the Antarctic:

Age	*Number caught*
3—4 years	130
5—6 years	95
7—8 years	72
9—10 years	53
11—12 years	37
13—14 years	28
15—16 years	10
17—18 years	4
19—20 years	1
21—22 years	1

The data imply (subject to the usual qualifications) a biennial mortality of about 26 per cent, increasing beyond the 15th year to much higher values. The author considers that the increased rate of loss with age is not real, but is due to failure of the older whales to return from their winter quarters in the north. This belief may or may not be well founded, but one suspects it to be predicated on the idea that mortality, at least when it is primarily due to exploitation, is constant among animals with respect to age. Edser (to whom the statistical analysis is credited) assumed, for the purpose of a rough calculation of the necessary rate of replacement, that the mortality between birth and breeding age is also 26 per cent. The improbability of this assumption may be surmised by reference to the life table for the Dall sheep (Table 1). Edser's calculation has the great merit of yielding a minimal estimate of the alarming exploitation being conducted by the whaling industry in the Antarctic. More realistic assumptions would darken the picture even more. In any case the data can not yet be cast into a life table....

Comparisons and Conclusions

Both in nature and in the laboratory, animals differ characteristically in their order of dying. When the mortality rate at all ages is constant, the survivorship (l_x) curve is diagonal on semi-logarithmic graph paper. Such a curve is found for many birds from adults stages onward; the mortality of adult birds is about 320 per thousand per hundred centiles of mean life span. If the constant age-specific mortality rate observed for the first few years of adult life is really maintained throughout life, the oldest bird in a cohort of 1000 lives 6.6 times as long as the average bird. Not all animals resemble birds in this respect, however, although many (e. g., fish) are assumed to do so. The Dall sheep, the rotifer, and possibly the barnacle are more like civilized man in that they seem to have evolved a mechanism for stretching the mean life span toward the maximum, so that the survivorship curve is convex. In these cases the maximum life span (among a sample of 1000) is only two or three times the mean. On the other hand there are undoubtedly species in which juvenile mortality is very heavy, but the few survivors to advanced ages die at reduced rates. This J-shaped or concave survivorship line, with the maximum longevity perhaps 15 or more times the mean, is presumed to characterize the oyster and other species, but it has not yet been formally recognized either in the laboratory or in nature. The closest approach to it, so far, is found when the survival of song sparrows is reckoned from the egg stage; but the mackerel will almost certainly provide an even better example.

Detailed comparisons between species cannot yet be made, partly because of the diverse statistical foundations of the life tables and partly because the data begin at different biological ages (birth, hatching, metamorphosis, sexual maturity, etc.) in the different cases. In all cases it is the youngest ages about which we know least, and ecologists should therefore concentrate their efforts on this segment of the life span of animals in nature.

List of Literature

Bertram, G. C. L. 1940. The biology of the Weddell and Crabeater Seals, with a study of the comparative behaviour of the Pinnipedia. *Brit. Mus. (Nat. Hist.): British Graham Land Expedition* 1934—37, *Sci. Rep.*, 1 (1): 139 pp., pls. 1—10.

Edmondson, W.T. 1945. Ecological studies of sessile Rotatoria, Part II. Dynamics of populations and social structures. *Ecol. Mon.*, 15: 141—172.

Erickson, M.M. 1938. Territory, annual cycle, and numbers in a population of wren-tits *(Chamaea fasciata)*. *Univ. Calif. Pub. Zool.*, 42 (5): 247—334, pls. 9—14.

Errington, P.L. 1945. Some contributions of a fifteen-year local study of the northern bobwhite to a knowledge of population phenomena. *Ecol. Mon.*, 15: 1—34.

Eschmeyer, R.W. 1939. Analysis of the complete fish population from Howe Lake, Crawford County, Michigan. *Papers Mich. Acad. Sci. Arts Lett.*, 24 (II): 117—137.

Green, R.G., and C.A. Evans. 1940. Studies on a population cycle of snowshoe hares on the Lake Alexander area. I. Gross annual censuses, 1932—1939. *J. Wildl. Man.*, 4: 220—238. II. Mortality according to age groups and seasons. Ibid., 4: 267—278. III. Effect of reproduction and mortality of young hares on the cycle. Ibid., 4: 347—358.

Hatton, H. 1938. Essais de bionomie explicative sur quelques espèces intercotidales d'algues et d'animaux. *Ann. Inst. Océanogr.*, 17: 241—348.

Jackson, C.H.N. 1939. The analysis of an animal population. *J. Anim. Ecol.*, 8: 238—246.

Kortlandt, A. 1942. Levensloop, samenstelling en structuur der Nederlandse aalscholver bevolking. *Ardea*, 31: 175—280.

Lack, D. 1943a. *The life of the robin.* 200 pp. H.F. & G. Witherby, London.

Lack, D. 1943b. The age of the blackbird. *Brit. Birds*, 36: 166—175.

Lack, D. 1943c. The age of some more British birds. *Brit. Birds*, 36: 193—197, 214—221.

Laurie, A. H. 1937. The age of female blue whales and the effect of whaling on the stock. *Discov. Rep.*, 15: 223—284.

Leopold, A., T. M. Sperry, W. S. Feeney, and J. S. Catenhusen. 1943. Population turnover on a Wisconsin pheasant refuge. *J. Wildl. Man.*, 7: 383—394.

Moore, H.B. 1935. A comparison of the biology of *Echinus esculentus* in different habitats. Part II. *J. Mar. Biol. Ass.*, n.s., 20: 109—128.

Murie, A. 1944. *The wolves of Mount McKinley.* (Fauna of the National Parks of the U.S., Fauna Series No. 5, xx + 238 pp.) U.S. Dept. Int., Nat. Park Service, Washington.

Nice, M.M. 1937. Studies on the life history of the song sparrow. Vol. I. A population study of the song sparrow. *Trans. Linn. Soc. N. Y.*, 4: vi + 247 pp.

Pearl, R. and J.R. Miner. 1935. Experimental studies on the duration of life. XIV. The comparative mortality of certain lower organisms. *Quart. Rev. Biol.*, 10: 60—79.

Ricker, W.E. 1940. Relation of "catch per unit effort" to abundance and rate of exploitation. *J. Fish. Res. Bd. Canada*, 5: 43—70.
Ricker, W.E. 1944. Further notes on fishing mortality and effort. *Copeia*, 1944: 23—44.
Russell, E.S. 1942. *The overfishing problem.* viii + 130 pp. Cambridge Univ. Press, Cambridge.
Thompson, W.F., and F.H. Bell. 1934. Biological statistics of the Pacific halibut fishery. (2) Effect of changes in intensity upon total yield and yield per unit of gear. *Rep. Intern. Fish. Comm.*, 8: 49 pp.
Wheeler, J.F.G. 1934. On the stock of whales at South Georgia. *Discov. Rep.*, 9: 351—372.

Part II
Stable Population Theory

In 1760 Leonard Euler introduced the concept of a stable age structure in which proportions in all age categories would remain fixed if mortality were constant and births increased exponentially over time. In his analysis, Euler also included a series of problems to demonstrate the usefulness of stable population theory for filling in gaps in population information. His paper was familiar to Milne (1815: his comments on Euler are included in paper 4 above), who recognized its theoretical interest but emphasized its restricted applicability. Milne's comments were appropriate for his time: the very limited empirical data that existed for Europe did not suggest stability, and did not indicate how abusive of the facts the assumption might be. [Some years later Milne applied Euler's theory in a communication to Robert Malthus (1970 (1830), pp. 228—229) demonstrating that three stable populations, one with a 25 year doubling time, one with a 50 year doubling time, and one stationary, would have substantially different crude death rates if they shared a common mortality schedule. Malthus used the analysis to show that crude death rates could not adequately measure the health of a population.] Euler's article and an earlier note to Johann Peter Süssmilch. who may have brought population mathematics to his attention, are presented here as papers 10 and 11.

The link between stable theory and real populations was very largely uncovered by Alfred Lotka (1907, 1922) and F. R. Sharpe and Lotka (1911), in works that form a singular achievement in demography. The three papers included here will indicate how Lotka's insight expanded from his realization that population could be represented as a renewal process displaying some stability, to his discovery that populations would nearly always stabilize, by predictable paths, if fertility and mortality were held constant.

The papers introduce the renewal equation

$$B(t) = G(t) + \int_{0}^{t} B(t-x)\,p(x)m(x)\,\mathrm{d}x$$

where $B(t)$ are births at time t, and are composed of births to the population alive at time zero [$G(t)$] and births to those born since; $B(t-x)$ being persons born $t-x$ years ago and, subject to their survival probability $p(x)$, currently age x; $m(x)$ being their chance of giving birth in the interval x to $x + \mathrm{d}x$. For $G(r) = 0$, substitution of an exponential birth function $B(t) = B_0\, e^{rt}$ provides the characteristic equation

$$\psi(r) = \int_0^\infty e^{-rx}\, p(x)\, m(x)\, \mathrm{d}x, = 1$$

whose single real root is the intrinsic growth rate of the population and whose other roots define the rate of convergence toward the stable age structure. [An examination of high order roots is provided in Coale (1972).]

The conjecture that populations stabilize was attributed by R. R. Kuczynski (1931 b, 1935) to L. von Bortkiewicz (1911), but this seems not to be the meaning of the relevant text. Bortkiewicz essentially followed Euler in examining relationships under stability, with the distinctions that he employed density functions corresponding to rectangular sections of the Lexis diagram, and provided applications to resolve current disputes in the literature. Except for a brief note on Euler, the approach to stability is not discussed. Lotka's debate with Kuczynski. which touches on other early contributions to stable theory, is the subject of paper 15 by Paul A. Samuelson.

Lotka's arguments were also subject to mathematical criticism and did not gain universal acceptance until a rigorous mathematical proof that in the main they held for both discrete and continuous cases was given by William Feller (1941), whose article is included here as paper 16. { **Ed. Note:** The mathematical object that plays a central role is known as the "renewal function". Building on the paper by Feller reprinted here, general conditions for convergence of the renewal function and so for the convergence of population age distributions to stable age distributions were established in Blackwell's Renewal Theorem proved in [5] along with work by Feller, Erdös, and Pollard described in [12]. }

Widespread application of the theory has come more recently, with the introduction of projection matrices and of modern computing equipment to facilitate the extraction of characteristic roots and vectors.

A more direct and considerably simpler method for calculating the intrinsic rate of growth than that developed by Lotka has been given by Ansley Coale (1957b) and is included here as paper 17. It is followed by a selection from R. A. Fisher (1958 (1930)) in which he develops the concept of Reproductive Value, complementing stable theory. It is found in Lotka [1939, equation (214)] as the term P, and in Feller [1941, equation (3.2)] as $\gamma(s)$, but without the intuitive meaning Fisher gives it.

In paper 19 we present an application of stable population theory by Coale (1957a). In the work he discusses the effects that changes in birth and death rates have on age distributions of real populations, among other points correcting the widespread misconception that a lowering of death rates necessarily results in an older population. The article also introduces the concept of weak ergodicity, an important contribution to stable theory that will be taken up in Part III in conjunction with projection matrices.

We conclude the chapter with two articles that attack a problem commonly overlooked in stable theory: the treatment of reproduction as a discontinuous process in work with very small populations. W.R. Thompson's (1931) pioneering analysis is included here as paper 20. To take into account chance fluctuations, a stochastic approach becomes more relevant to Thompson's small populations than the deterministic solution he gives, and readers should be aware that this limitation exists. It does not detract from Thompson's contribution: as he indicates, both the nature of the problem and of the solution change radically when numbers are small. (We present in the introduction to Thompson's paper a note on its matrix formulation which will relate it to the discussion of matrices in Part III. { **Ed. Note:** In the 1977 edition, the authors went on to say "The stochastic treatment of birth processes is taken up in chapters 6 and 7"; these parts are omitted in this edition. } The closing article by Lamont Cole (1954) clarifies Thompson and ties his work to stable theory. It includes as well a fascinating note on Fibonacci numbers and their relation to the Thompson problem.

{ **Ed. Note:** From the 1970s onward, stable population theory expanded into a subject with a rich variety of applications, many developed in the first, 1977 edition of *Applied Mathematical Demography* by Nathan Keyfitz, republished with substantial additions in a third edition by Keyfitz and Caswell [19]. Among the prominent applications are measures of population momentum under the "Keyfitz Scenario" in which fertility rates within a previously stable population suddenly drop to replacement levels at all ages by the same fixed factor with no change in mortality rates (see [19], pages 196–199). A decomposition of momentum into stable and non-stable contributions is given by Espenshade, Oligati, and Levin [11]. Biological applications extend ideas from stable population theory to populations structured both by age and life-cycle stage, as in [9].

Stable population theory applies to settings in which demographic rates do not depend on the size or age structure of the population. When fertility and mortality rates do respond to population through processes of demographic feedback, sustained cycles in births and age distributions over time may take the place of a stable state. Inspired by ideas of Richard Easterlin about mechanisms contributing to the baby booms that followed World War II, inherently non-linear demographic feedback models were developed by Lee [24] and Wachter [41] and others. Analysis of complex systems of feedbacks between population, environment, and economic varibles has gradually become feasible, largely employing computer-based simulations.

Further developments building on Stable Population Theory through partial differential equations are discussed in Part III. }

10. An Illustration of Population Growth

JOHANN PETER SÜSSMILCH (1761)

From *Die göttliche Ordnung*, Vol. 1, pp. 291—299. Berlin. Translated by Nathan Keyfitz.

The extract which follows assembles several fragments of Euler's work in stable theory that complement, and apparently precede, his better known 1760 article (paper 11 below). Euler here prepares for Süssmilch a population projection that begins with one couple of age 20 who give birth to an additional couple at ages 22, 24 and 26, and who die at age 40, later generations repeating the same fertility and mortality schedules. Euler demonstrates that after 300 years the population will be growing approximately geometrically and recognizes this as the limit of a recurrent series; i. e., he is working with a very general result.

The recurrent series is in this case a net maternity function expressed as a difference equation. We have:

$$B(t) = B(t-22) + B(t-24) + B(t-26)\,;$$

of which the solution is

$$\begin{aligned} B_0\lambda^t &= B_0\lambda^{t-22} + B_0\lambda^{t-24} + B_0\lambda^{t-26} \\ 1 &= \lambda^{26} - \lambda^4 - \lambda^2 \\ \lambda &\doteq 1.04696 \end{aligned}$$

where $B(t)$ represents births at time t, B_0 is a constant and $\lambda = e^r$, the annual intrinsic growth rate. From earlier work (1748) Euler recognized that the ratio between successive terms $B(t)$, $B(t+\alpha)$ in series of this form would approach a constant limit determined by the intrinsic rate of growth as $t \to \infty$; and both he and Joseph Louis Lagrange (1759) had worked with the conversion of the equations to polynomials and extraction of roots, learned in part from Abraham DeMoivre's *Miscellanea Analytica de Seriebus et Quadraturis* (1730, 72—83). A hint of this insight is contained in the closing paragraph of the present article. (Euler's note that the birth series can be generated by dividing out an algebraic fraction is also from DeMoivre. It reappears in Thompson (paper 20 below), in whose notation births at ages 22, 24, and 26 would be representable by the "Generation Law" $G = T^{22} + T^{24} + T^{26}$. The complete birth series is produced by dividing out the fraction $1/(1-G)$.)

Euler's mathematical contribution to Süssmilch goes no further and is very reminiscent of Fibonacci's rabbits (discussed in Cole, paper 21 below). It is as well the point at which Euler's important 1760 paper begins: from the net ma-

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ternity function an intrinsic rate of growth has been found; this rate, together with the complete mortality table, provides the stable age distribution and the intrinsic birth and death rates of that paper. E. J. Gumbel (1917) was first to see the full implication of the two papers.

Süssmilch's opening paragraph refers to an earlier table of population doublings (i. e., powers of 2) with imagined doubling times inserted; calculated, with some multiplication errors, by himself. It gives a total population after 300 years of $2^{23} = 8{,}338{,}608$ from an initial cohort of 2, a figure twice as great as in the Euler projection and one which corresponds to an annual growth rate of 5.3‰. The Euler projection runs to several pages and is shown here only in part.

To show that nothing impossible is contained in the preceding table I present another due to Professor Euler and prepared some years ago at my request. It is too elegant for me to omit. ...

Professor Euler assumes (1) at the outset there exists a married couple aged 20, (2) their descendents also always marry at the age of 20, (3) 6 children are born to each marriage. (This could certainly happen if couples of differing ages were precluded from marrying, if like marry like, and if all are able to marry at the correct time.) Also (4) variations must never occur; therefore twins will always be born, the first pair to each marriage coming in the 22nd, the next in the 24th and the third in the 26th year. It will be assumed (5) that all children survive, marry, and remain living until reaching age 40. (This is the average expectation of life on the whole: lives were of much longer duration and the average much higher before the Flood. If the fertility of the marriages appears too high it can be revised throughout.)

On these assumptions there will be only two people initially, 4 after 2 years, 6 after 4 years, 8 after 8 years. After this time no changes occur until the first two

Table of Growth According to the Conditions Assumed

Year	Number of Births	Births to Date	Deaths to Date	Number Alive
0	0	2	0	2
2	2	4	0	4
4	2	6	0	6
6	2	8	0	8
8	0	8	0	8
10	0	8	0	8
12	0	8	0	8
14	0	8	0	8
16	0	8	0	8
18	0	8	0	8
20	0	8	2	6
22	0	8	2	6
24	2	10	2	8
26	4	14	2	12
28	6	20	2	18
30	4	24	2	22
32	2	26	2	24
34	0	26	2	24
⋮				
280	71 632	1 679 344	280 484	1 398 860
282	122 112	1 791 456	328 610	1 462 846
284	178 036	1 969 492	379 908	1 589 584
286	260 362	2 229 854	428 068	1 801 786
288	342 310	2 572 164	467 934	2 104 230
290	403 268	2 975 432	497 348	2 478 084
292	426 034	3 401 466	517 874	2 883 592
294	404 348	3 805 814	534 572	3 271 242
296	346 570	4 152 384	555 274	3 597 110
298	273 884	4 426 268	589 506	3 836 752
300	214 370	4 640 638	646 684	3 993 954

children reach their 22nd year, which takes place after 24 years, when their first two children come into the world. Two years later this couple will produce 2 more, but the couple born in the 4th year will also produce 2 children; in the 28th year 6 children will arrive; in the 30th again only 4, and so forth. In order to grasp the nature of this initially irregular growth, the attached table shows in its second column the births of even numbered years. The third column shows the sum of all births to date, which would be the same as the number living if no one were to die.

But at age 40 all die, from which are derived the numbers of column 4. If these deaths are subtracted from the number born, the result is the number living each year, found in the 5th column.

One can see from this that after 24 years there is a trebling of the number alive, from which after 1000 or more years an astonishing increase must ensue. After 300 years the number living already runs to 4 million; if the trebling time is set at 25 years there will be 324 million people after 400 years, and after 450 years fully 3000 million, more than now live on the entire earth. The increase before the Flood, while not so great as this, was surely not an insignificant progression; so the earth must at that time have been more heavily populated than now....

Notice that although great unorderliness seems to rule in Euler's table, the numbers of births belong to a progression called a recurrent series [einem Geschlecht von Progreßionen, welche man *Series recurrentes* nennet] and which can be produced by dividing out an algebraic fraction. While these progressions initially appear irregular, if they are continued they finally change into a geometric progression; the initially perceived irregularities decrease with time until they finally almost entirely disappear.

11. A General Investigation into the Mortality and Multiplication of the Human Species

LEONARD EULER (1970 (1760))

Theoretical Population Biology 1: 307—314. Translated by Nathan and Beatrice Keyfitz.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_11,

TRANSLATOR'S NOTE

Euler's virtually unknown article, published by the Belgian Académie Royale des Sciences et Belles-Lettres in 1760, anticipates important parts of modern stable population theory for a one-sex population closed to migration. Its ideas have been published many times during the subsequent two centuries by writers who independently rediscovered them.

The application to which Euler oriented his argument was inference from incomplete data. Life tables were already in use for working out probabilities and expected values in relation to individuals; Euler saw them as a means of studying populations, provided the assumption of stability was appropriate. He follows a cohort of individuals born "en même tems" (sic) by means of the probability of survival from birth to age x that we know as l_x ; he supposed that l_x is available from some source other than the population under consideration, as contemporary users of model life tables do. The equation for the unknown rate of increase λ in paragraph 18, as well as the proposed method of solution, bear a resemblance to procedures now in use for inferring the rate of increase of a population.

If total births and deaths are known (but not by age), then Euler shows how assuming a life table permits the number of the population to be inferred, as well as the age distribution. Euler turned his stable population theory to filling the gaps in information of his own day: Western Europe had baptism and burial records before it had censuses. His model, used in only a slightly different fashion, would enable a present-day underdeveloped country with censuses and no vital statistics to infer its birth rate.

Euler's "hypothesis of mortality" means "life table," and is taken as unchanging in time and operating deterministically. The ratio λ of population increase per year is the same as e^r, if r is Lotka's intrinsic rate of natural increase. Though the cohort with which Euler deals is a group of children born at one moment, and hence projected by the life table function l_x , the alteration needed to work with cohorts evenly spread over the year is slight: We replace the probability of survival l_x by L_x/L_0 , where L_x is the integral $L_x = \int_0^1 l_{x+t}\, dt$.

Euler's annual births B occurring at one moment, while his deaths D are spread through the subsequent year, give a simple but unfamiliar expression for the ratio of increase: $\lambda = (P - D)/(P - B)$, where P is the population just after the annual births have occurred, and D the deaths in the succeeding twelve months.

The following list of the problems in stable population theory solved by Euler will serve as a table of contents of the important paragraphs of his work.

Paragraph	Given	Find
18	P, B, l_x	λ
19	λ, l_x	P/B
20	P, B, l_x	P_x
21	P, B, l_x	D
22	B, D, l_x	λ, P
24	B, D, l_x	D_x
26	P, B, D_x	l_x

where P is total population; P_x population by age; D is total deaths; D_x deaths by age; B is births for the given sex; l_x is the probability of surviving to age x; λ is the ratio of annual increase.

Euler uses "hommes" for the population and "enfants" for the births, which could be interpreted as meaning that the sexes are combined in his model. As he nowhere deals with age at childbearing, his argument would apply to the two sexes together, with a consolidated life table. Essentially a one-sex model seems to be intended, with males as the illustration, and this is the rendering of the translation that follows.

N. K.

1. The registrations of births and of deaths at each age which are published in various places every year give rise to many different questions on the mortality and the multiplication of mankind.

2. All of the general questions depend on two hypotheses. I shall call the first the hypothesis or law of mortality (i.e., life table), by which can be determined how many out of a certain number of men born at one time will be still alive at the end of each period of years. Here population increase does not enter the matter at all, and we go on to the second hypothesis, which I shall call that of multiplication: by how many the number of men is increased or diminished in the course of one year. The second hypothesis depends, of course, on the number of marriages and upon fertility, whereas the first is based on the vitality or life potential characteristic of man.

HYPOTHESIS OF MORTALITY

3. For the first hypothesis, let us suppose a number B of children born at the same time; the number of these still alive at the end of one year is Bl_1, at the end of two years Bl_2, of three years Bl_3,... . These show the decrease of the number of men born at one time; for each region and way of life they will have particular values. The numbers indicated by $l_1, l_2,...$, constitute a decreasing progression of fractions of which the greatest, l_1, is less than unity, and beyond 100 they disappear almost completely. For if of 100 million men not one reaches the age of 125, this means that l_{125} must be less than 1/100,000,000.

4. Having established for a certain place, by a sufficiently large number of observations, the values of the fractions $l_1, l_2,...$, one can resolve several questions which are commonly posed on the probability of human life. First, if the number of children born at the same time is equal to B, the expected number dying each year will be

From Age	To Age	Number of Deaths
0	1	$B-Bl_1$
1	2	Bl_1-Bl_2
2	3	Bl_2-Bl_3
–	–	—
–	–	—
–	–	—

And since of B children born Bl_x will be expected to be still alive after x years, the number of deaths before the end of x years must be equal to $B - Bl_x$.

5. Given a number of men of the same age, find how many of them will be expected to be alive after a certain number of years.

Suppose there are P men, all aged x years, and we want to know how many we can expect to be living after n years. We put $P = Bl_x$ to obtain $B = P/l_x$, where B is the number of male births, of whom P are still alive after x years. Of this number Bl_{x+n} will be the expected number still living $x + n$ years after their birth, and hence n years after the time in question. Then the number sought is equal to $P(l_{x+n}/l_x)$; that is, after n years we can expect there to be that many survivors of P men who are presently all aged x years.

Then of the P men all aged x years, the expected fraction who will die before the end of n years is $1 - l_{x+n}/l_x$.

6–14. (These paragraphs derive life table probabilities for individuals, along with the formula for a life annuity purchasable for the amount of money a. The annuity payable at the end of each year purchasable for a is shown to be

$$\frac{al_x}{l_{x+1}/(1+i) + l_{x+2}/(1+i)^2 + \cdots}$$

per year, where i is the rate of interest. If the annuitant is just born, and the annuity is to be deferred to start at age x, then the annual value purchasable for the amount a is

$$\frac{a}{l_x/(1+i)^x + l_{x+1}/(1+i)^{x+1} + \cdots}.$$

Euler then gives the l_x column by single years of age of a life table due to Keerseboom, and goes on to discuss births.)

15. Just as I assume that the regime of mortality remains ever the same, I shall assume a like constancy of fertility, so that the number of children born each year will always be proportional to the total number living. Let B be the number of children born in the course of one year, and λB the number of children born in the following year. Insofar as the ratio which changes B to λB continues to hold from any year to the next the number of births increases in the ratio 1 to λ. Consequently the births of the third year will be $\lambda^2 B$, of the fourth $\lambda^3 B, \dots$. Either the number of annual births will constitute a geometric progression, increasing or decreasing, or it will remain constant, according as $\lambda > 1$, or $\lambda < 1$, or $\lambda = 1$.

16. Suppose then that in a town or province the number of (boy) children born this year is equal to B, and the number born next year is equal to λB, and so on according to this progression:

	Number of Births
This year	B
After 1 year	λB
After 2 years	$\lambda^2 B$
$\vdots$	$\vdots$
After 99 years	$\lambda^{99} B$
After 100 years	$\lambda^{100} B$

If we suppose that after 100 years none of the men alive at this moment will still be living, all the men in existence after 100 years will be the survivors of the above births. Then, bringing the hypothesis of mortality to bear, we could determine the total number of men who will be alive after 100 years. Thus, since the number born in that year will be $\lambda^{100} B$, we will have the ratio of births to the total number living.

17. To make this clearer, let us see how many men will be alive after one hundred years from the births of the preceding years:

	Number of Births	Number Living After 100 Years
At present	B	Bl_{100}
After 1 year	λB	$\lambda B l_{99}$
After 2 years	$\lambda^2 B$	$\lambda^2 B l_{98}$
$\vdots$	$\vdots$	$\vdots$
After 99 years	$\lambda^{99} B$	$\lambda^{99} B l_1$
After 100 years	$\lambda^{100} B$	$\lambda^{100} B$

Thus the total number living after 100 years will be

$$\lambda^{100}B\left(1+\frac{l_1}{\lambda}+\frac{l_2}{\lambda^2}+\cdots+\frac{l_{100}}{\lambda^{100}}\right).$$

18. The terms of this series disappear finally, by virtue of the hypothesis of mortality, and since the total number living has a certain relationship to the number of births during the course of one year, the multiplication from one year to the next, which we have assumed to be λ, will reveal this relationship. For if (at the end of 100 years) the total number living is equal to P, and the number of children produced in the course of one year is B, we will have

$$P=B\left(1+\frac{l_1}{\lambda}+\frac{l_2}{\lambda^2}+\cdots+\frac{l_{100}}{\lambda^{100}}\right).$$

Thus, if we know the ratio P/B as well as the hypothesis of mortality, i.e. the values of the fractions $l_1, l_2, \ldots, l_{100}$, this equation will determine the ratio of multiplication λ from any year to the next. For each life table, if we calculate the quantity $1+l_1/\lambda+l_2/\lambda^2+\cdots$ for several values of λ, and set up a table of them, it will be easy to ascertain for the given ratio P/B, which expresses fertility, the annual increase λ in the number living.

19. Having this equation, it is indifferent whether we know the fertility P/B, or the multiplication λ, the one being determined by the other through the life table.

20. Given the hypotheses of mortality and fertility and the total number living, find how many there are at each age.

From P, B, and the life table we calculate the ratio of annual multiplication λ. From λ and paragraph 17 we see that there will be among the number P

B,	children just born;
Bl_1/λ,	aged one year;
Bl_2/λ^2,	aged 2 years;
Bl_3/λ^3,	aged 3 years;
$\vdots$	$\vdots$

and in general

$$Bl_a/\lambda^a, \qquad \text{aged } a \text{ years.}$$

The sum of all these numbers is equal to P.

21. The same things being given, find the number of persons who will die in one year.

Call P the number of persons living at present, including the number of children B born this year. The quotient P/B will determine the annual increase λ on a given life table. Then next year the number living will be λP, among whom the number just born will be λB; the others, of whom the number is $\lambda P - \lambda B$, are the survivors from the P persons of the previous year; from which it follows by subtraction that $(1 - \lambda) P + \lambda B$ of them have died. Thus if the number now living is P, then $D = (1 - \lambda) P + \lambda B$ of them will die in the course of one year.

22. Knowing both the number of births and of burials occurring in the course of one year, find the total number living, and their annual increase, for a given life table.

The preceding paragraph gave us

$$D = (1 - \lambda) P + \lambda B$$

or

$$P = \frac{\lambda B - D}{\lambda - 1}.$$

From the hypothesis of mortality

$$P = B\left(1 + \frac{l_1}{\lambda} + \frac{l_2}{\lambda^2} + \cdots + \frac{l_{100}}{\lambda^{100}}\right).$$

Substituting $P = (\lambda B - D)/(\lambda - 1)$ (and subtracting B from both sides), we obtain

$$\frac{B - D}{\lambda - 1} = B\left(\frac{l_1}{\lambda} + \frac{l_2}{\lambda^2} + \cdots + \frac{l_{100}}{\lambda^{100}}\right)$$

from which λ may be calculated. (Then from λ, B, and D the total population P can be found.)

24. Given the number of births and of burials in one year, find the number of each age among the deaths.

Call B the number of children born during a year, and D the number of deaths; from the preceding problem we have the number living P, along with the ratio of increase λ. Consider how many persons will be living at each age, this year as well as next year:

	This Year	Next Year
Just born	B	$B\lambda$
Aged 1 year	Bl_1/λ	Bl_1
Aged 2 years	Bl_2/λ^2	Bl_2/λ
Aged 3 years	Bl_3/λ^3	Bl_3/λ^2
⋮	⋮	⋮
Aged 100 years	Bl_{100}/λ^{100}	Bl_{100}/λ^{99}

Thus the deaths in the course of the year must number:

	Number of Deaths
Under one year	$B(1 - l_1)$
From 1 to 2 years	$B(l_1 - l_2)/\lambda$
From 2 to 3 years	$B(l_2 - l_3)/\lambda^2$
⋮	⋮
From 99 to 100 years	$B(l_{99} - l_{100})/\lambda^{99}$
Over 100 years	Bl_{100}/λ^{100}

26. Knowing the total number living P but not their ages, the number of births B, the number of deaths D, and the number of deaths at each age during the course of one year, find the law of mortality (i.e., calculate the life table).

We first find the annual multiplication $\lambda = (P - D)/(P - B)$. Next, for this year the number of deaths, from the preceding problem, must be

Under one year	$D_0 = B(1 - l_1)$
From 1 to 2 years	$D_1 = B(l_1 - l_2)/\lambda$
From 2 to 3 years	$D_2 = B(l_2 - l_3)/\lambda^2$
⋮	⋮

and hence we will find the fractions $l_1, l_2, l_3, \ldots$, which contain the law of mortality:

$$\begin{aligned}
l_1 &= 1 - D_0/B, \\
l_2 &= l_1 - \lambda D_1/B = 1 - \frac{D_0 + \lambda D_1}{B}, \\
l_3 &= l_2 - \lambda^2 D_2/B = 1 - \frac{D_0 + \lambda D_1 + \lambda^2 D_2}{B}, \\
l_4 &= l_3 - \lambda^3 D_3/B = 1 - \frac{D_0 + \lambda D_1 + \lambda^2 D_2 + \lambda^3 D_3}{B}, \\
&\vdots
\end{aligned}$$

28. I should point out once more that in the calculations which I have developed here I have assumed that the total number living in one place remains the same, or that it increases or decreases uniformly, so that I have had to exclude such extraordinary devastations as plague, wars, famine, as well as extraordinary increases like new colonies. It is well to choose a place where all those born remain in the region, and where strangers do not come in to live or die, which would upset the principles upon which I have based the preceding calculations. For places subject to such irregularities, it would be necessary to keep exact registrations of all those living and dying, and then, by following the principles which I have established here, one would be in a position to apply the same calculation. It always comes back to these two principles, that of mortality and that of fertility, which, once they have been established for a certain place, make it easy to resolve all the questions which one could propose on this subject, of which I am satisfied to have given an account of the principal ones.

12. Relation Between Birth Rates and Death Rates

ALFRED J. LOTKA (1907)

Science, N.S. 26: 21—22.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_12, © Springer-Verlag Berlin Heidelberg 2013

A short notice appeared on page 641 of Science, 1907, of a paper read by C.E. Woodruff before the American Association for the Advancement of Science, on the relation between birth rates and death rates, etc.

In this connection, it may be of interest to note that a mathematical expression can be obtained for the relation between the birth rate per head b and the death rate per head d, for the case where the general conditions in the community are constant, and the influence of emigration and immigration is negligible.

Comparison with some figures taken from actual observation shows that these at times approach very nearly the relation deduced on the assumptions indicated above.

I give here the development of the formula, and some figures obtained by calculation by its aid, together with the observed values, for comparison.

Let $c(a)$ be such a coefficient that out of the total number N_t of individuals in the community at time t, the number whose age lies between the values a and $(a+da)$ is given by $N_t c(a)da$.

Now the $N_t c(a)da$ individuals whose age at time t lies between the values a and $(a+da)$, are the survivors of the individuals born in time da at time $(t-a)$.

If we denote by $B_{(t-a)}$ the total birth rate at time $(t-a)$, and by $p(a)$ the probability at its birth, that any individual will reach age a, then the number of the above-mentioned survivors is evidently $B_{(t-a)}p(a)da$.

Hence:

$$N_t c(a)da = B_{(t-a)}p(a)da$$

$$c(a) = \frac{B_{(t-a)}}{N_t} p(a)$$

Now if general conditions in the community are constant, $c(a)$ will tend to assume a fixed form. A little reflection shows that then both N and B will increase in geometric progression with time,[1] at the same rate $r=(b-d)$. We may, therefore, write:

$$B_{(t-a)} = B_t e^{-ra}$$

$$c(a) = \frac{B_t}{N_t} e^{-ra} p(a)$$

$$= b e^{-ra} p(a) \tag{1}$$

Now from the nature of the coefficient $c(a)$ it follows that

$$\int_0^\infty c(a)da = 1$$

Substituting this in (1) we have:

$$\frac{1}{b} = \int_0^\infty e^{-ra} p(a)da \tag{2}$$

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[1] Compare M. Block, "Traité théorique et pratique de statistique," 1886, p. 209.

Equation (1) then gives the fixed age-distribution, while equation (2) (which may be expanded into a series if desired), gives the relation between b, the birth rate per head, and r, the rate of natural increase per head, and hence between b and d, since $r = b - d$.

Applying these formulae to material furnished by the Reports of the Registrar-General of Births, etc., in England and Wales, the following results were obtained:

England and Wales 1871—80 (Mean)

		Observed[2]	Calculated
Birth-rate per head	b	0.03546	0.0352
Death-rate per head	d	0.02139	0.0211
Excess	$(b-d)=r$	0.01407	(0.0141)

$p(a)$ from Supplement to 45th Ann. Rep. Reg. Gen. Births, etc., England and Wales, pp. vii and viii, assuming ratio:

$$\frac{\text{male births}}{\text{female births}} = 1.04.$$

Age Scale.—1,000 individuals, in age-groups of 5 and 10 years

$a_1\, a_2$	$1000 \int_{a_1}^{a_2} c(a)\,da$	
0— 5	136	138
5—10	120	116
10—15	107	106
15—20	97	97
20—25	89	87
25—35	147	148
35—45	113	116
45—55	86	87
55—65	59	59
65—75	33	33
75—∞	13	13

It will be seen that in the above example the values calculated for the age-scale and especially for b and d, show a good agreement with the observed values.

The above development admits of further extension. But this, as well as further numerical tests, must be reserved for a future occasion. In view of the recent note of the work by Major Woodruff, it appeared desirable to the writer to publish this preliminary note.

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[2] Mean b and d from 46th Ann. Rep. Reg. Gen. Births, etc., England and Wales, p. xxxi.

13. A Problem in Age-Distribution

F. R. SHARPE and ALFRED J. LOTKA (1911)

Philosophical Magazine, Series 6, Volume 21: 435—438.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_13,

The age-distribution in a population is more or less variable. Its possible fluctuations are not, however, unlimited. Certain age-distributions will practically never occur; and even if we were by arbitrary interference to impress some extremely unusual form upon the age-distribution of an isolated population, in time the "irregularities" would no doubt become smoothed over. It seems therefore that there must be a limiting "stable" type about which the actual distribution varies, and towards which it tends to return if through any agency disturbed therefrom. It was shown on a former occasion[1] how to calculate the "fixed" age-distribution, which, if once established, will (under constant conditions) maintain itself.

It remains to be determined whether this "fixed" form is also the "stable" distribution: that is to say, whether a given (isolated) population will spontaneously return to this "fixed" age-distribution after a small displacement therefrom.

To answer this question we will proceed first of all to establish the equations for a more general problem, which may be stated as follows:—

"Given the age-distribution in an isolated population at any instant of time, the 'life curve' (life table), the rate of procreation at every age in life, and the ratio of male to female births, to find the age-distribution at any subsequent instant."

1. Let the number of males whose ages at time t lie between the limits a and $a+da$ be $F(a,t)da$, where F is an unknown function of a and t.

Let $p(a)$ denote the probability[2] at birth that a male shall reach the age a, so that $p(0)=1$.

Further, let the male birth-rate (*i.e.* the total number of males born per unit of time) at time t be $B(t)$.

Now the $F(a,t)da$ males whose age at time t lies between a and $a+da$ are the survivors of the $B(t-a)da$ males born a units of time previously, during an interval of time da. Hence

$$F(a,t)da = B(t-a)p(a)da$$

$$F(a,t) = p(a)B(t-a)\,. \tag{1}$$

2. Let the number of male births per unit time at time t due to the $F(a,t)da$ males whose age lies between a and $a+da$ be $F(a,t)\beta(a)da$.

If γ is the age at which male reproduction ends, then evidently

$$B(t) = \int_0^\gamma F(a,t)\beta(a)da$$

$$= \int_0^\gamma B(t-a)p(a)\beta(a)da\,. \tag{2}$$

Now in the quite general case $\beta(a)$ will be a function of the age-distribution both of the males and females in the population, and also of the ratio of male births to female births.

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[1] A.J. Lotka, Am. Journ. Science, 1907, xxiv. pp. 199, 375; 'Science,' 1907, xxvi. p. 21.

[2] As read from the life table.

We are, however, primarily concerned with comparatively small displacements from the "fixed" age-distribution, and for such small displacements we may regard $\beta(a)$ and the ratio of male births to female births as independent of the age-distribution.

The integral equation (2) is then of the type dealt with by Hertz (*Math. Ann.* vol. lxv. p. 86). To solve it we must know the value of $B(t)$ from $t=0$ to $t=\gamma$, or, what is the same thing, the number of males at every age between 0 and γ at time γ. We may leave out of consideration the males above age γ at time γ, as they will soon die out. We then have by Hertz, *loc. cit.*,

$$B(t)=\sum_{h=1}^{h=\infty}\frac{\alpha_h^t\int_0^\gamma\left\{B(a)-\int_0^a\beta(a_1)p(a_1)B(a-a_1)\,da_1\right\}\alpha_h^{-a}\,da}{\int_0^\gamma a\,\beta(a)p(a)\alpha_h^{-a}\,da}, \tag{3}$$

where $\alpha_1,\alpha_2,\ldots$ are the roots of the equation for α;

$$1=\int_0^\gamma\beta(a)p(a)\alpha^{-a}\,da. \tag{4}$$

The formula (3) gives the value of $B(t)$ for $t>\gamma$, and the age-distribution then follows from

$$F(a,t)=p(a)B(t-a). \tag{1}$$

4. From the nature of the problem $p(a)$ and $\beta(a)$ are never negative. It follows that (4) has one and only one real root r, which is $\gtreqless 1$, according as

$$\int_0^\gamma\beta(a)p(a)\,da\gtreqless 1. \tag{5}$$

Any other root must have its real part less than r. For if $r_1(\cos\theta+i\sin\theta)$ is a root of (4),

$$1=\int_0^\gamma\frac{\beta(a)p(a)}{r_1^a}\cos a\theta\,da. \tag{6}$$

It follows that for large values of t the term with the real root r outweighs all other terms in (3) and $B(t)$ approaches the value

$$B(t)=A\,r^t. \tag{7}$$

The ultimate age-distribution is therefore given by

$$F(a,t)=A\,p(a)\,r^{t-a} \tag{8}$$

$$=A\,p(a)\,e^{r'(t-a)}. \tag{9}$$

Formula (9) expresses the "absolute" frequency of the several ages. To find the "relative" frequency $c(a, t)$ we must divide by the total number of male individuals.

$$c(a,t)=\frac{F(a,t)}{\int_0^\infty F(a,t)da}=\frac{A\,p(a)e^{r'(t-a)}}{A e^{r't}\int_0^\infty e^{-r'a}p(a)da}=\frac{p(a)e^{-r'a}}{\int_0^\infty e^{-r'a}p(a)da}$$

$$=be^{-r'a}p(a)\,, \tag*{(10)[3]}$$

where

$$\frac{1}{b}=\int_0^\infty e^{r'a}p(a)da\,. \tag*{(11)[3]}$$

The expression (10) no longer contains t, showing that the ultimate distribution is of "fixed" form. But it is also "stable;" for if we suppose any small displacement from this "fixed" distribution brought about in any way, say by temporary disturbance *of the otherwise constant conditions*, then we can regard the new distribution as an "initial" distribution to which the above development applies: that is to say, the population will ultimately return to the "fixed" age-distribution.

It may be noted that of course similar considerations apply to the females in the population. The appended table shows the age-distribution calculated according to formula (10) for England and Wales 1871—1880. The requisite data (including the life table) were taken from the Supplement to the 45th Annual Report of the Registrar General of Births, etc. The mean value of r' (mixed sexes) for that period was 0.01401, while the ratio of male births to female births was 1.0382.

It will be seen that at this period the observed age-distribution in England conformed quite closely to the calculated "stable" form.

Table

Age (Years)	Males		Females		Persons	
	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.
0— 5	139	139	136	132	138	136
5—10	118	123	115	117	116	120
10—15	107	110	104	104	106	107
15—20	97	99	95	95	96	97
20—25	88	87	87	91	87	89
25—35	150	144	148	149	149	147
35—45	116	112	116	115	116	113
45—55	86	84	88	87	87	86
55—65	57	59	62	61	59	59
65—75	30	31	35	35	33	33
75—∞	11	12	15	15	13	13

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[3] Compare Am. Journ. Science, xxiv. 1907, p. 201.

14. The Stability of the Normal Age Distribution

ALFRED J. LOTKA (1922)

Proceedings of the National Academy of Sciences 8: 339—345.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_14, © Springer-Verlag Berlin Heidelberg 2013

There is a unique age distribution which, in certain circumstances,[2] has the property of perpetuating itself when once set up in a population. This fact is easily established,[3] as is also the analytical form of this unique *fixed* or *normal* age distribution.

More difficult is the demonstration that this age distribution is *stable*, that a population will spontaneously revert to it after displacement therefrom, or will converge to it from an arbitrary initial age distribution. Such a demonstration has hitherto been offered only for the case of small displacements,[4] by a method making use of integral equations. The purpose of the present communication is to offer a proof of stability which employs only elementary analytical operations, and which is readily extended to cover also the case of large displacements. This method presents the further advantage that it is molded in more immediate and clearly recognizable relation to the physical causes that operate to bring about the normal age distribution.

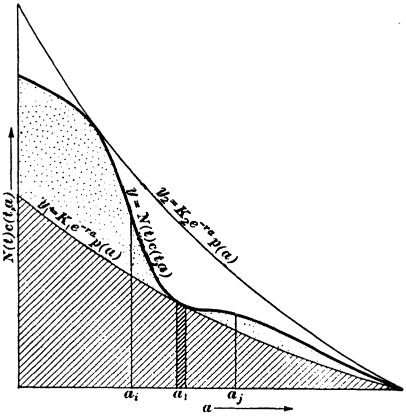

Fig. 1

Consider a population which, at time t has a given age distribution such as that represented by the heavily drawn curve in fig. 1, in which the abscissae represent ages a (in years, say), while the ordinates y are such that the area comprised between two ordinates erected at a_1 and a_2, respectively, represents the number of individuals between the ages a_1 and a_2.

If we denote by $N(t)$ the total population at time t, and if the ordinates of our curve are

$$y = N(t)\, c(t,a) \tag{1}$$

we have, evidently,

$$\int_{a_1}^{a_2} y\, da = N(t) \int_{a_1}^{a_2} c(t,a)da = N(t,a_1,a_2) \tag{2}$$

where $N(t,a_1,a_2)$ denotes the number of individuals living at time t and comprised within the age limits a_1 and a_2. We may speak of $c(t,a)$ as the coefficient of age distribution. It is, of course, in general a function of t, only in the special case of the fixed or self-perpetuating age distribution is $c(a)$ independent of the time.

Now, without assuming anything regarding the stability of the self-perpetuating age distribution, it is easy to show[2] that its form must be

$$c(a) = \frac{e^{-ra}p(a)}{\int_0^{\infty} e^{-ra}p(a)da} = be^{-ra}p(a) \tag{3}$$

where r is the real root of the equation

$$1 = \int_{a_i}^{a_j} e^{-ra}p(a)\beta(a)da \tag{4}$$

In this equation r is the natural rate of increase of the population, i.e., the difference $r = b - d$ between the birthrate per head b and the deathrate per head d, and $p(a)$ is the probability, at birth, that a random individual will live to age a (in other words, it is the principal function tabulated in life tables, and there commonly denoted by 1_x). The limits a_i and a_j of the integral are the lower and upper age limits of the reproductive period. The factor $\beta(a)$, which might be termed the procreation factor, or more briefly the birth factor, is the average number of births contributed per annum by a parent of age a. (In a population of mixed sexes it is, of course, immaterial, numerically, to what parent each birth is credited. It will simplify the reasoning, however, if we think of each birth as credited to the female parent only.)

The factor $\beta(a)$ will in general itself depend on the prevailing age distribution. This is most easily seen in the case of extremes, as for example in a population which should consist exclusively of males under one year of age and females over 45. But, except in such extreme cases, $\beta(a)$ will not vary greatly with changes in the age constitution of the population, and we shall first develop our argument on the supposition that $\beta(a)$ is independent of the age distribution. We shall then extend our reasoning to the more general case of $\beta(a)$ variable with $c(t,a)$.

Referring now again to fig. 1, let two auxiliary curves be drawn, a *minor tangent curve* and a *major tangent curve*

$$y_1 = K_1e^{-ra}p(a) \quad (5) \qquad y_2 = K_2e^{-ra}p(a) \quad (6)$$

the constants K_1, K_2 being so chosen that the minor tangent curve lies wholly beneath the given arbitrary curve, except where it is tangent thereto, while the major tangent curve lies wholly above the given curve, except where it is tangent thereto.

The given arbitrary curve representing the age constitution of the population at time t then lies wholly within the strip or area enclosed between the minor and the major tangent curves.

Now consider the state of affairs at some subsequent instant t'. Had the population at time t consisted solely of the individuals represented by the lightly shaded area in fig. 1, i.e. the area under the minor tangent curve, then at time t' the population would be represented by the lower curve of fig. 2, whose equation is

$$y'_1 = K'_1 e^{-ra} p(a) \tag{7}$$
$$= K_1 e^{r(t'-t)} e^{-ra} p(a) \tag{8}$$

For the age distribution (5) is of the fixed form (3), and therefore persists in (7); on the other hand, given such fixed age distribution, the population as a whole increases in geometric progression,[3] so that $K'_1 = K_1 e^{r(t'-t)}$. In point of fact, we have left out of reckoning that portion of the population which in fig. 1 is represented by the dotted area. Hence, in addition to the population under the lower curve of fig. 2, there will, at time t', be living a body of population which for our present purposes it is not necessary to determine numerically. We need only know that it is some positive number, so that the curve representing the actual population at time t' must lie wholly above or in contact with the curve (8).

By precisely similar reasoning it is readily shown that at time t' the actual curve lies wholly beneath or in contact with the curve

$$y'_2 = K_2 e^{r(t'-t)} e^{-ra} p(a) \tag{9}$$

Hence at time t' the actual curve lies wholly within the strip comprised between the two curves (8) and (9).

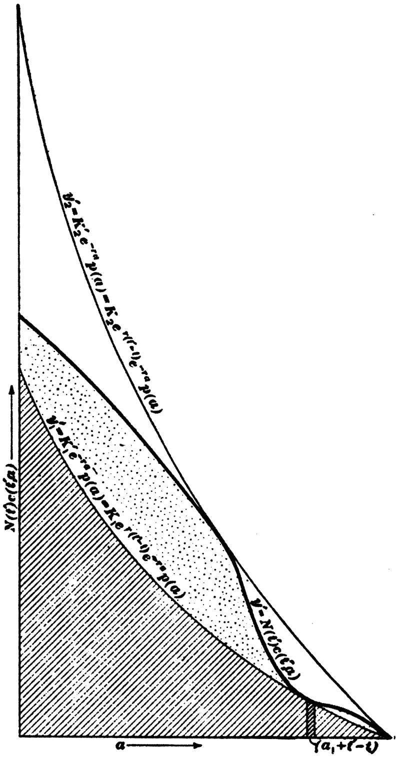

Fig. 2

The curves shown are intended to be interpreted only in a qualitative sense; so, for example, the increase in the ordinates in passing from fig. 1 to fig. 2 is very much exaggerated, to render it obvious to the eye.

Consider now an elementary strip, of width da, of the original population (shown heavily shaded in fig. 1), which at time t is in contact with the minor tangent curve. Let this contact occur at age a_1, so that $y_t(a_1) = K_1 e^{-ra_1} p(a_1)$. At time t' the survivors of the individuals comprised in this elementary strip will be of age $(a_1 + t' - t)$, so that they will then be represented by a strip of width da and of altitude

$$y_{t'}(a_1 + t' - t) = K'_1 e^{-ra_1} p(a_1 + t' - t)$$
$$= K_1 e^{r(t'-t)} e^{-r(a_1 + t' - t)} p(a_1 + t' - t)$$

From this it is seen that the elementary strip of population which at time t' contacts with the minor tangent curve (8) is built up of the survivors of the strip which at time t contacted with the minor tangent curve (5). In other words, if we follow up, by identity of individuals, the element of the population which at any instant contacts with the minor tangent curve, this element (so long as any part of it survives) continues in contact with that curve. (It must be remembered, however, that the tangent curve itself changes with time according to (5), (8).)

Similarly it follows that the element of population which at any instant contacts with the major tangent curve continues in this contact so long as any part of it survives.

And again, considering any element of the population which does not contact with the minor or the major tangent curves, but has its upper extremity at some point within the area enclosed between these curves, it can be shown by precisely similar reasoning that such element continues in such intermediate position.

Turning now from the consideration of the survivors of the original population, and taking in view the new population added by births since the time $t = t$, we note that if the original population had been that represented by the shaded area in fig. 1, i.e., by the area under the minor tangent curve, then the birthrate would at all times have been such as continuously to reproduce a population represented by the minor tangent curve (5), (8).

In point of fact, provided that contact with the minor tangent curve is not *continuous* over a range of ages equal to or greater than $a_j - a_i$, it is easily seen that the total birthrate

$$B = N \int_{a_i}^{a_j} c(a)\beta(a)da$$

is always greater (equality is here excluded) than that which would result from and in turn reproduce the age distribution represented by the minor tangent curve.

Similarly the total birthrate is always less than that which would result from a population and age distribution represented by the major tangent curve.

From this it is clear that, after the original population has once died out, the representative curve can never again contact with the original minor and major tangent curves, but must henceforth be separated from them everywhere by a finite margin (except, of course, where the several curves terminate upon the axis of a).

We may then begin afresh by drawing a new pair of tangent curves, lying within the original pair, and so on indefinitely, until the minor and major tangent curves coincide, and with them also coincides the actual curve of age distribution, which is then of the form

$$N(t)\, c(a) = Ke^{rt}e^{-ra}p(a)$$

which we recognize as the fixed age distribution (3), with

$$N(t) = Ke^{rt}\int_0^\infty e^{-ra}p(a)da$$

In this argument we have expressly excluded the case that the original age distribution curve contacts *continuously* with one of its tangent curves over a range greater than the reproductive period $(a_j - a_i) = A$. If such continuous contact occurs in the 0th generation over a range nA, where $a_j/A > n > 1$, then a simple reflection shows that in the next generation (the first) contact will occur over a range $(n - 1)A$, in the second generation over $(n - 2)A$, etc. A time is therefore reached (in practice very soon), when contact is over a period less than the period of reproduction A. After that the argument set forth above applies.

If the curve representing the original age distribution contacts with one of the tangent curves all the way from $a = 0$ to $a = a_j$, so that $a_j/A \leqq n$, then, of course, the fixed age distribution is practically established *ab initio*, or at any rate from the moment the original population above reproducing age a_j has died out.

It remains to consider the effect of variability in the form of $\beta(a)$ with changes in $c(t,a)$. Some such variability undoubtedly exists owing to the influence of the ages of the male and female constituents of the population upon the frequency of matings. We may, nevertheless, in this case also, define a minor and a major tangent curve (5), (6) in terms of the value of r given by equation (4); in order, however, to make this value determinate, it is now necessary to make some definite disposition regarding the form of $\beta(a)$, which is now variable. Merely for purposes of defining r, we shall suppose that the function $\beta(a)$ under the integral sign has that particular form which corresponds to the fixed age distribution.[5]

We cannot, however, now reason, as before, that the portion of the population represented by the shaded area in fig. 1 will, by itself, reproduce its own form of age distribution. For its constitution as to sex will in general differ from that of the "normal" population with self-perpetuating age distribution. We must therefore consider three different possibilities:

1. The shaded area alone will produce a population exceeding at all ages the normal or fixed type continuation (8) of the original shaded area. Should this be the case, then the argument presented with regard to the case of invariable $\beta(a)$ holds *a fortiori*, so far as the minor tangent curve is concerned.

2. The shaded area alone will produce a population deficient at some or all ages, as compared with the normal type continuation (8) of the original shaded area. In that case two alternatives present themselves:

a. The deficiency is more than counterbalanced by the additional population produced by that portion of the original population which is represented by the dotted area in fig. 1. In this case also the original argument, so far as the minor tangent curve is concerned, applies essentially as before.

b. After the contributions from all parts of the population have been taken into account, there remains an unbalanced deficiency short of the population defined by (8), in the population resulting from that originally present. In such case the argument presented on the assumption of invariable $\beta(a)$ fails, and the population may move away from, not towards the age distribution(3). Stability of the fixed age distribution may not extend to such displacements as this.

Similar reflections apply, *mutatis mutandis*, as regards the major tangent curve.

If conditions (1) or (2) prevail with respect to the minor, and corresponding conditions with respect to the major tangent curve, then we can argue as in the case of invariable $\beta(a)$, that after expiration of the existing generation a new pair of tangent curves can be drawn, which will lie within the pair defined by (8), (9). And if conditions (1) or (2) still persist with reference to the new tangent curves, the same process of closing in these curves at the expiration of the current generation can be repeated, and so on, as in the argument first presented. Now conditions (1) or (2) will thus continue to prevail for each new pair of tangents, if, as the minor and major tangents close up, $\beta(a)$ approaches a limiting form. In such case, therefore, the age distribution defined by (3), (4) is stable even for displacements of any magnitude, provided always that conditions (1) or (2) prevail, as indicated.

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[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 71.

[2] These circumstances are: (*a*) an invariable life curve (life table); (*b*) an invariable ratio of male to female births; (*c*) an invariable rate per head of procreation at each year of age, for any given sex-and-age distribution in the population.

[3] Lotka, *Amer. J. Sci.*, **26**, 1907 (21); *Ibid.*, **24** (199); *J. Wash. Acad. Sci.*, **3**, 1913 (241, 289).

[4] Sharpe, F. R., and Lotka, A. J., *Phil. Mag.*, April 1911, 435.

[5] A certain ambiguity is introduced here by the fact that, with $\beta(a)$ of variable form, equation (4) might have more than one real root for r. In practice, however, in a human population at any rate, probably only one such root exists, or has any effect upon the course of events.

15. Resolving a Historical Confusion in Population Analysis*

PAUL A. SAMUELSON (1976)

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Introduction

As Robert K. Merton (1973) has so well discussed, creative scientists are not immune to preoccupation with priorities: *their* priorities. Alfred J. Lotka (1880 to 1949), the father of self-renewal models in linear population analysis, was least of all an exception in this regard. A lone pioneer throughout much of his career,[1] with no cadre of graduate students and colleagues, he could naturally be expected to be prickly over failures to recognize and acknowledge his important contributions to demography.

The present brief note attempts to sort out a controversy that arose some 40 years ago between Lotka and another able self-made demographer, R. R. Kuczynski (1876—1947). The matter has an interest at two levels. Substantively, a reader of their polemics has still been left in ignorance of the true merits of the points being argued—whether Bortkiewicz (1911) and other writers had already established the asymptotic approach to a stable exponential equilibrium of a population subject to invariant (one-sex) age-specific mortality *and fertility*, an accomplishment properly attributed today primarily to Lotka. Psychologically, as an exercise in how new science gets itself done, the matter is also of some interest. Once emotions entered in, Lotka ceased to be his own best advocate and came gratuitously to denigrate legitimate theorems in demography that are of interest for their own sake. I shall briefly sort out the truths and misunderstandings of the discussion.

Act I

The story begins with Lotka (1929), a review of Kuczynski (1928), a book with an honorable role in the history of demography for its forceful pointing out an impending decline in European population levels, a decline shown to be concealed by swollen numbers of people of fertile age inherited from earlier generations of higher fertility. Kuczynski could have benefitted from the fundamental findings of Sharpe and Lotka (1911) and Lotka (1913, 1922). Perhaps he did not then know of the 1911 integral-equation finding; and perhaps he had neither the mathematical equipment nor the inclination to master it even were it called to

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his attention. For the purpose of discerning a shift from a growing to an ultimately declining population, Kuczynski could rely on a common-sense criterion,[2] the *net reproduction rate* (i.e., the expected number of female babies that will be born to a representative female baby who throughout her life will be subject to current age-specific mortality and fertility rates).

Lotka's review is favorable on Kuczynski's substantive thesis, but accuses him of not acknowledging that his analysis is taken from Dublin and Lotka (1925), a work said to receive only cursory acknowledgment by Kuczynski.

Kuczynski (1930) replies sturdily that the finding about swollen fertile-age numbers is in fact an ancient staple in the literature, even having been dealt with by Kuczynski in an 1897 book (when Lotka was only 17 and Kuczynski only 23); moreover, the concept of the net reproduction rate that Kuczynski relied on, Kuczynski points out, was put forth by his mentor, Richard Böckh, as far back as 1886.[3]

In his rebuttal to Kuczynski, Lotka (1930) had to scale down his accusations. But he does assert the incompleteness of the early work Kuczynski cited. And he makes the valid point that, although the net reproductive rate is an accurate indicator of whether the intrinsic rate of population growth is positive or negative, it only gives the rate of growth per ambiguous "generation", and *not* per *year*, so that it falls short of the full analysis of Sharpe and Lotka and of Dublin and Lotka. He concludes with the barb that, if he had known that the discussion of Dublin and Lotka (1925) on the net reproduction rate had been anticipated earlier, *he*, Lotka, would have felt obliged to acknowledge it—whatever Kuczynski's scholarly code might be.

This ends Act I of the drama. The antagonists are now sensitized to each other. In the years between 1930 and 1937, Lotka apparently became increasingly of the opinion that Kuczynski's publications were spreading misleading accounts of the true priorities. This finally culminated in two publications, Lotka (1937a, 1937b), which attempt to set the record straight and clarify the truth. To understand them, I first review some now familiar fundamentals. And then I formulate a number of distinguishable propositions or theorems that are needed to judge the various allegations.

Review of Fundamentals: Mortality Relations

Let $N(a,t)$ be the number of people (females presumably) of age a at time t, with total population number at t given by

$$(1) \qquad N(t)=\int_0^\infty N(a,t)\,da=\int_0^n N(a,t)\,da\,, \qquad a<n<\infty$$

where n is the maximum length of life. Let $p(a)$ be the fraction of new-born females surviving to age a. Then with births at t given as a prescribed function of time, $N(0,t)=B(t)$,

$$\begin{aligned} N(a,t) &= p(a)\,B(t-a) \\ &= \frac{p(a)}{p(a-\theta)}\,N(a-\theta,\,t-\theta), \qquad 0<\theta<a\,; \\ p(0)=1, &\quad p(n)=0, \quad p'(a)\leq 0, \end{aligned} \tag{2}$$

$$\partial N(a,t)/\partial a + \partial N(a,t)/\partial t = [p'(a)/p(a)]\,N(a,t)\,, \tag{3}$$

$$N(t) = \int_0^n p(a)\,B(t-a)\,da\,, \tag{4}$$

$$N'(t) = B(t) + \int_0^n p'(a)\,B(t-a)\,da\,. \tag{5}$$

A solution to (5), and under suitable specifications to (4), can be given by the "renewal function" so useful in industrial-equipment as well as actuarial analysis, $\Pi(t)$, as discussed in Feller (1941) and Lotka (1933, 1939a), namely by

$$B(t) = N'(t) + \int_{-\infty}^{t} \Pi(t-a)\,N'(a)\,da \tag{6}$$

where $\Pi(t-a)$ is the Volterra resolvent kernel to $p'(t-a)$, defined as the solution to

$$\begin{aligned} \Pi(t) &= -p'(t) - \int_0^t p'(t-a)\,\Pi(a)\,da \\ &= -p'(t) - \int_0^t \Pi(t-a)\,p'(a)\,da\,. \end{aligned} \tag{7}$$

One way of solving for $\Pi(t)$ is via its Laplace transform

$$\begin{aligned} \bar{\Pi}(\omega) &= \int_0^\infty \Pi(t)\,e^{-\omega t}\,dt \\ -\bar{p}'(\omega) &= \int_0^\infty -p'(t)\,e^{-\omega t}\,dt, \\ \bar{\Pi}(\omega) &= -\bar{p}'(\omega) - \bar{p}'(\omega)\,\bar{\pi}(\omega) \\ &= \frac{-\bar{p}'(\omega)}{1+\bar{p}'(\omega)}\,. \end{aligned} \tag{8}$$

Alternatively, writing symbolically,

$$\begin{aligned} f(t) = -p'(t), \qquad f(t)\cdot g(t) &= \int_0^t f(t-a)\,g(a)\,da \\ &= g(t)\cdot f(t)\,, \end{aligned}$$

$$\Pi(t) = f(t) + f(t)\cdot f(t) + f(t)\cdot f(t)\cdot f(t) + \cdots \tag{9}$$

which is a rapidly converging explicit solution, known to have the property

$$\lim_{t\to\infty} \Pi(t) = c_0 = 1/\int_0^n -ap'(a)\,da = 1/\int_0^n p(a)\,da > 0\,. \tag{10}$$

Also, with n finite, $\Pi(t)$ can be written as an infinite series of exponentials, of Hertz-Herglotz-Lotka type:

$$\Pi(t) \doteq c_0 + \sum_{-\infty}^{\infty} e^{m_j t} c_j \tag{11}$$

where m_j are the infinite number of complex roots of the transcendental equation

$$\int_0^n -p'(a)e^{-ma}\,da = 1\,. \tag{12}$$

The sole real root is $m_0 = 0$, and the real parts of all the conjugate complex roots are demonstrably negative

$$\begin{aligned} &m_j = \mu_j + i\upsilon_j\,, \qquad m_{-j} = \mu_j - i\upsilon_j\,,\\ &0 = m_0 > \mu_j \qquad (j = 1, 2, \ldots)\,. \end{aligned} \tag{13}$$

If any m_j root is multiple, the coefficients c_j and c_{-j} will be polynomials in t rather than constants.

It is worth noting that the non-real roots of (12) are also the roots of

$$\int_0^n p(a)e^{-ma}\,da = 0 \tag{12'}$$

as an integration by parts will verify.

Review: Fertility and Mortality Relations

So far nothing has been said about fertility rates. Writing the number of female births at time t to mothers of age a as $B(a, t)$, we define an invariant age-specific fertility function, $m(a)$, by

$$\begin{aligned} &m(a) \underset{t}{\equiv} B(a, t)/N(a, t) \geq 0\,,\\ &m(a) \equiv 0, \quad 0 < a < \alpha\,,\\ &m(a) > 0, \quad \alpha \leq a \leq \beta < n\,,\\ &m(a) \equiv 0, \quad \beta < a\,. \end{aligned} \tag{14}$$

A population self-propelled by invariant age-specific mortality and fertility functions, $p(a)$ and $m(a)$, and starting out from an initial non-negative age distribution, $N(0,a)$, will have births and numbers that forever after satisfy

$$B(t) = \int_{\alpha}^{\beta} m(a)p(a)B(t-a)\,da\,, \qquad t>n>\beta>\alpha>0, \tag{15}$$

$$= \int_{0}^{t} m(t-a)p(t-a)B(a)\,da + \int_{t}^{n} m(a)\frac{p(a)}{p(a-t)}N(a-t,0)\,da, \qquad 0<t<n,$$

$$= \int_{0}^{t} \phi(t-a)B(a)\,da + G(t)\,, \qquad t>0\,, \tag{16}$$

where

$$\phi(a) \equiv m(a)p(a)\,, \tag{17}$$

$$G(t) = \int_{t}^{n} m(a)\frac{p(a)}{p(a-t)}\,N(a-t,0)\,da \geq 0, \qquad 0<t<n$$

$$\equiv 0, \qquad n<t\,.$$

Just as $\Pi(t-a)$ was the resolvent kernel of $-p'(t-a)$ in (6)—(9), so is there a useful resolvent kernel to $\phi(t-a)$, which I write as $\phi^*(t-a)$, and which provides a solution to $B(t)$ in (16) and has the other listed verifiable properties:

$$B(t) = G(t) + \int_{0}^{t} \phi^*(t-a)G(a)\,da, \qquad t>0\,, \tag{18}$$

$$\phi^*(t) = \phi(t) + \int_{0}^{t} \phi(t-a)\phi^*(a)\,da \tag{19}$$

$$= \phi(t) + \int_{0}^{t} \phi^*(t-a)\phi(a)\,da$$

$$= \phi(t) + \phi(t)\cdot\phi(t) + \cdots$$

$$= Q_0 e^{r_0 t} + \sum_{-\infty}^{\infty} Q_j e^{r_j t}, \qquad Q_0>0\,,$$

where r_0 is the sole real root and r_j are the complex roots, necessarily infinite in number when β is a finite positive number, of the transcendental equation

$$\psi(r) = \int_{0}^{\infty} \phi(a)e^{-ra}\,da - 1\,, \tag{20}$$

$$r_0 > \text{real coefficients of } r_{\pm j} \qquad (j=1,2,\ldots)\,. \tag{21}$$

The $(Q_{\pm j})$ are constants or polynomials in t, and like Q_0 can be determined from the initial condition $N(a,0)$.

To determine whether $r_0 \gtreqless 0$, we can employ the useful Böckh (1886) net reproduction rate as a criterion, namely

$$(22) \qquad r_0 \gtreqless 0 \leftrightarrow \int_0^n m(a)p(a)\,da = \int_0^n \phi(a)\,da \gtreqless 1 .$$

As was done in (8), the Laplace transform may be used to solve for $\phi^*(t)$ in terms of known $\phi(t)$. Lotka (1928) gave the interpretation of $\phi^*(t)$ as the progeny of a Dirac pulse of new births at initial time zero:

$$(23) \qquad N(a,0) = \delta(a-0) ,$$

$$\int_0^n N(a,0)\,da = \int_0^n \delta(a-0)\,da = 1 .$$

From (19) and (20), or from more general analysis of Feller, one can prove

$$(24) \qquad \lim_{t\to\infty} \phi^*(t)e^{-r_0 t} = Q_0 = -1/\psi'(r_0) > 0 ,$$

$$(25) \qquad \lim_{t\to\infty} B(t)e^{-r_0 t} = b_0 > 0 ,$$

$$(26) \qquad \lim_{t\to\infty} N(a,t)e^{-r_0 t} = b_0 p(a)e^{-r_0 a} ,$$

$$(27) \qquad \lim_{t\to\infty} \frac{N(a,t)}{N(0,t)} = p(a)e^{-r_0 a} .$$

These two-dozen-odd equations are now standard in the demographic literature, as discussed in Lopez (1961), Keyfitz (1968), Coale (1972), J.H. Pollard (1973), and elsewhere. All of these relations owe much to Lotka; in particular there can be no doubt that equations (15)—(20), (25)—(27) are primarily due to him, so that any account which failed to indicate this fact is open to criticism by an impersonal jury, to say nothing of criticism by the injured scientist himself.

A Bouquet of Theorems

To illuminate the disputed issues, let me write down a number of relevant theorems. The list could be amplified, abbreviated, or arranged differently.

The first three theorems involve essentially nothing more than the actuarial assumption of age-specific mortality.

Theorem 1A. Suppose that, for a finite time interval, $t_0 - \delta < t < t_0 + \delta$

(i) $p(a)$ applies, so that from (3)

$$\partial[\ln N(a,t)]\,\partial a + \partial[\ln N(a,t)]/\partial t \equiv d[\ln p(a)]/da$$

(ii) the age distribution is stable, in the sense that

$$N(a,t)/N(0,t) \equiv c(a):$$

Then $N(0,t) = B(t) \equiv B(0)e^{ut}$, $t_0 - \delta < t < t_0 + \delta$, $u \gtreqless 0$

$$N(a,t)/N(0,t) \equiv p(a)e^{-ua},$$

$$b \equiv B(t)/N(t) = 1/\int_0^\infty p(a)e^{-ua}\,da,$$

$$d \equiv D(t)/N(t) = -\left[\int_0^\infty p'(a)e^{-ua}\,da\right]\Big/\int_0^\infty p(a)e^{-ua}\,da,$$

$$u \equiv b - d.$$

Bortkiewicz (1911) essentially states and proves Theorem 1A; however, it would be surprising if a thorough search did not turn it up in the earlier actuarial literature. With charity, and some charity is needed, Lotka (1907a, 1907b), in his maiden demographic papers, can be construed to have glimpsed the truth of 1A.

To prove 1A, using my notation rather than that of Bortkiewicz (1911), combine (i) and (ii) to derive

$$(28) \qquad 0 \equiv \partial \ln[N(a,t)/N(0,t)]/\partial t$$

$$\equiv d\ln[p(a)]/da - \partial \ln[N(a,t)]/\partial a - B'(t)/B(t),$$

$$(29) \qquad c(a) = p(a)e^{-ua},$$

$$u = B'(t)/B(t), \quad \text{a constant of any sign.}$$

No doubt Bortkiewicz (1911) and Lotka (1907a, 1907b) thought of δ as ∞ or as a large number; however, so long as $\delta > 0$, the theorem holds.

Theorem 1B. Suppose, for all positive time, $t \geq 0$,

(i) $p(a)$ applies, and
(ii) $B(t) = B(0)e^{ut}$, $u \gtreqless 0$, $t \geq 0$.

Then

$$N(a,t) \equiv p(a)B(t-a),$$

$$\equiv B(0)[p(a)e^{-ua}]e^{ut}, \qquad t \geq n,$$

$$N(t) = B(0)b^{-1}e^{ut}, \qquad b^{-1} = \int_0^\infty p(a)e^{-ua}\,da.$$

This almost trivial theorem, whose proof is direct from substitution, is in Euler (1760), paragraph 17; in perusing this celebrated early paper, I was surprised to find that Euler seems to have not gone beyond its fertility analysis, despite his earlier promising remarks that the multiplication of a population "depends, of course, on the number of marriages and upon fertility ...". However, in Euler's private communication to Süssmilch, reported in Süssmilch (1761) and discussed in my footnote 2, Euler did anticipate a case of the Sharpe-Lotka Theorem 2A about to be discussed.

Theorem 1C. Suppose, for *all* $t \geq 0$,

(i) $p(a)$ applies, and
(ii) $N(t)$ is of exponential growth

$$N(t) = N(0)e^{ut}, \qquad t \geq 0, \qquad u \gtreqless 0:$$

Then

$$\lim_{t\to\infty} \frac{N(a,t)}{N(0,t)} = p(a)e^{-ua},$$

$$\lim_{t\to\infty} B(t)e^{-ut} = bN(0), \qquad b^{-1} = \int_0 p(a)e^{-ua}da > 0.$$

If u were zero, and $N(a,0)$ were Dirac's $\delta(a-0)$, this would be the renewal equation for $\Pi(t) \equiv B(t)$. However, the theorem is true whatever the admissible specifications of $N(a,0)$, and it is far from absurd or trivial.

To prove the theorem, utilize (5) and (6). Thus, for $N(t) = e^{ut}$, $u > \mu_{\pm j}$,

$$\begin{aligned}
(30) \qquad N'(t) &\equiv u e^{ut} \\
&\equiv B(t) + \int_0^t p'(t)B(t-a)da - H(t), \\
H(t) &\equiv \int_t^n -\frac{p'(a)}{p(a)} N(a,t)da \geq 0, \qquad 0 < t < n \\
&\equiv \int_t^n \frac{p'(a)}{p(a)} N(a-t,0)da, \\
H(t) &\equiv 0, \qquad n < t.
\end{aligned}$$

Any general solution to (30) is known from the principle of superposition to be the sum of the special exponential solution proportional to e^{ut} plus $Y(t)$, the general solution to the following homogeneous integral equation:

$$\begin{aligned}
(31) \qquad Y(t) &= \int_0^n p(a)\, Y(t-a)da \\
&= \sum_{-\infty}^{\infty} h_j e^{m_j t}, \quad \text{where } m_j \text{ is as in (12').}
\end{aligned}$$

When the h's are tailored to admissible initial $H(t)$, the fact that all μ_j are negative guarantees that as t grows large, the non-exponential $Y(t)$ is damped down to zero. Q.E.D.

Remark: The fact that numbers in any age group can never be negative puts a restriction on how fast total $N(t)$ can fall exponentially: thus, if u in e^{ut} were more negative than the real part of some m_j in (13), μ_j, that would not be an admissible observed situation unless $B(t)$ was *already* exactly proportional to e^{ut}.

This serves as a reminder that one could strengthen the hypothesis in all three theorems to require their postulated conditions to hold for *all* time, $-\infty < t < \infty$. In that case, Theorems 1A and 1B have conclusions that hold for *all* time. And the conclusion of Theorem 1C *holds not merely asymptotically but for all t*, as in the following:

Theorem 1C′: Suppose, for $-\infty < t < \infty$,

(i) $p(a)$ applies,
(ii) $N(t) \equiv N(0)e^{ut}$, $-\infty < t < \infty$,
(iii) $N(a,t)$ must, of course, *always* be non-negative.

Then

$$N(a,t)/N(0,t) \equiv p(a)e^{-ut},$$

$$B(t) \underset{t}{\equiv} N(0)e^{ut} \Big/ \int_0^n p(a)e^{-ua}\,da,$$

$$N(a,0) \equiv \left[N(0) \Big/ \int_0^n p(a)e^{-ua}\,da \right] p(a)e^{-ua}.$$

For proof, recall that any *initial* deviation from the stable distribution would, when projected backward in the fashion of Samuelson (1976), generate negative numbers from the backward-anti-damped (!) oscillatory components implied to be present. This shows that there can be no such initial deviation.

The condition of non-negativity, (iii), is important, even if usually left implicit. Thus, Lotka (1931, 1939b) seems a bit casual in assuming that a particular formal solution for $B(t)$ is unique in $\int_0^\infty p(a)B(t-a)\,da =$ a prescribed $N(t)$ function. We can add to such a special solution terms like $\sum k_{\pm j}e^{m_j t}$ and have new *formal* solutions: however, with μ_j all negative in (12′)'s $m_{\pm j} = \mu_j \pm v_j$, these "appendages" would make $B(t)$ negative as $t \to -\infty$; and it is this property that Lotka should utilize in a cogent treatment.

All theorems up to now, 1A, 1B, 1C, or 1C′, have involved only age-specific mortality data as contained in the $p(a)$ survival function. They have not involved age-specific fertility data from an $m(a)$ function, all e^{ut} growth functions having been postulated or deduced from postulates *not* involving $m(a)$. They all belong to the pre-Sharpe-and-Lotka era of Euler, Bortkiewicz, and Lotka (1907a, 1907b).

The next set of theorems depend on both $m(a)$ and $p(a)$ belonging to the post-1911 age of Lotka.

The first theorem is the basic one of Sharpe and Lotka (1911), with Bernardelli (1941) and Leslie (1945) equivalences holding for discrete-time, discrete-age models.

Theorem 2A: Suppose, for $t \geq 0$,

(i) $p(a)$ applies,
(ii) age-specific fertility, $m(a)$, applies

$$B(t) = \int_{\alpha}^{\beta} m(a) N(a,t)\, da, \qquad t \geq 0$$

(iii) initial $N(a,0)$ is an arbitrarily given, non-negative, integrable function with some females not yet beyond the fertile ages

$$\int_{0}^{\beta} N(a,0)\, da > 0:$$

Then

$$\lim_{t \to \infty} B(t) e^{-r_0 t} = c_0 > 0\,,$$

$$\lim_{t \to \infty} N(t) e^{-r_0 t} = c_0 \int_{0}^{n} p(a) e^{-r_0 a}\, da\,,$$

$$\lim_{t \to \infty} \frac{N(a,t)}{N(0,t)} = p(a) e^{-r_0 a}\,,$$

$$\lim_{t \to \infty} N(a,t) e^{-r_0 t} = p(a) e^{-r_0 a} c_0\,,$$

$$\psi(r_0) = \int_{\alpha}^{\beta} m(a) p(a) e^{-r_0 a}\, da - 1\,, \qquad r_0 \text{ real}\,,$$

$$c_0 = \frac{\int_{0}^{\beta} e^{-r_0 a} G(a)\, da}{\int_{\alpha}^{\beta} a \phi(a) e^{-r a}\, da}\,; \qquad G(t) \text{ as defined in (17)}\,.$$

Warning: My $\psi(r)$ is often written in the demographic literature as $\psi(r) - 1$.

Two corollaries may be stated.

Corollary 2A. Depending upon whether the net reproduction rate, $\int_{\alpha}^{\beta} m(a) p(a)\, da$, is greater than, less than, or equal to unity, the population will ultimately grow, decay, or approach a constant level.

Corollary 2AA. The algebraic sign of r_0, the asymptotic or intrinsic rate of natural self-propelled increase, is determined by the algebraic sign of $\int_{\alpha}^{\beta} m(a)p(a)\,da - 1 = \psi(0)$, which represents the rate of growth per (ambiguous) "generation".

For $|r_0|$ not too large, good approximations are given to r_0 by r_0', r_0'', r_0''':

$$r_0' = -\psi(0)/\psi'(0) = (R_0 - 1)/R_1 = (R_0 - 1)/\mu R_0\,,$$

$$R_j = \int_{\alpha}^{\beta} a^j \phi(a)\,da, \qquad \mu = R_1/R_0, \qquad \text{average age of becoming a mother,}$$

$$\sigma^2 = (R_2/R_0) - \mu^2 = \text{variance of net fertility,}$$

$$r_0'' = \frac{\mu - \sqrt{\mu^2 - 2\sigma^2 \ln R_0}}{\sigma^2},$$

$$r_0''' = r_0' - \psi(r_0')\psi'(r_0')\,.$$

Corollary 2A is essentially due to Böckh (1886) who intuitively inferred it, and from whom Kuczynski (1928, 1932, 1935) derived his understanding.

Corollary 2AA was popularized by Dublin and Lotka (1925), but its essentials had already been stated in Lotka (1913). The first approximation given above can be derived from various series expansions of $\psi(r)$ and related functions, or as a Newton-Raphson approximation using an initial $r_0 = 0$; the final approximation given above involves a second Newton-Raphson whirl; the intermediate approximation has alternative derivations—from certain ratios of power expansions involving cumulants, or from fitting a Gaussian function to $m(a)p(a)$ (with its bizarre implication of some mothers who are not themselves conceived!). S. D. Wicksell, and others, have given alternative non-Gaussian graduations.

Lotka's formal proofs of 2A were made rigorous by Feller (1941); however, Lopez (1961) showed that the finiteness postulated for β guaranteed that the roots of $\psi(r) = 0$ were infinite in number and sufficed to provide an arbitrary $B(t)$ on the interval $(-\beta, 0)$ with its Fourier-like expansion, $\sum_{-\infty}^{\infty} c_j e^{r_j t}$.

A different theorem from 2A is provided by the following:

Theorem 2B: Suppose that, for *all* time, $-\infty < t < +\infty$,

(i) $p(a)$ applies,
(ii) $m(a)$ applies,
(iii) $N'(t)/N(t) \equiv$ a constant, $-\infty < t < \infty$.

Then, the asymptotic stable state of 2A holds already, and holds all the time

$$N(a,0)/N(0,0) \equiv p(a)e^{-r_0 a}$$

$$\underset{t}{\equiv} N(a,t)/N(0,t), \qquad -\infty < t < \infty\,,$$

$$B(t) \equiv B(0)e^{r_0 t}, \qquad N(t) \equiv \left[B(0) \int_0^n p(a)e^{-r_0 a}\,da \right] e^{r_0 t},$$

$$N(a,t) \equiv [B(0)p(a)e^{-r_0a}]e^{r_0t},$$

$$\psi(r_0) = \int_{\alpha}^{\beta} m(a)p(a)e^{-r_0a}\,da - 1 = 0.$$

Theorem 2B was stated very loosely in Lotka (1937a), and a purported proof of its more careful restatement was offered in Lotka (1937b).

A final theorem may be distinguished from 2A and 2B.

Theorem 2C: Suppose that, for a time period as long as the length of life, namely for $0 < t < \gamma \geq n$,

(i) $p(a)$ applies,
(ii) $m(a)$ applies,
(iii) $N'(t)/N(t) \equiv$ a constant.

Then

$$N(a,0) = B(0)p(a)e^{-r_0a}, \qquad \psi(r_0) = 0,$$

$$N(a,t) \equiv [B(0)p(a)e^{-r_0a}da]e^{r_0t}, \qquad t \geq 0,$$

$$N(a,t)/N(0,t) \equiv p(a)e^{-r_0a}, \qquad t \geq 0,$$

$$B(t) \equiv B(0)e^{r_0t}, \qquad N(t) \equiv \left[B(0)\int_0^n p(a)e^{-r_0a}da\right]e^{r_0t}, \qquad t \geq 0.$$

Remark: If the time interval in the hypothesis of 2C, λ, were permitted to be less than n, the conclusion that a stable age distribution already holds from the beginning could be shown by numerous counter-examples to be definitely false, contrary to what some of the remarks in Lotka (1937a) seem to me to suggest and thereby undermining his contention.

A proof of 2C would show that $N'(t)/N(t)$ would undergo a transient oscillation if any complex-root harmonic e^{r_jt} were present: since this is ruled out by hypothesis, the only admissible initial $N(a,0)$ is *already* in the stable configuration $p(a)e^{-r_0a}B(0)$.

Arbitrating the Quarrels

We are now armed to judge the litigants. Lotka (1937a) begins by alleging that Kuczynski's writings "probably" caused a recent monograph, German Statistical Office (1935), to credit to Bortkiewicz (1911) the first formulation of the Sharpe-Lotka results of my Theorem 2A and my Equation (20). The jury must agree that what Bortkiewicz did accomplish, namely Theorem 1A, cannot possibly be identified with Theorem 2A[4]. So Lotka's complaint, if accurate, is a serious one.

In support of his charge against Kuczynski, Lotka (1937a, p. 104) quotes the following passage from Kuczynski (1935, p. 226):

> Bortkiewicz had come to the conclusion that a population constantly subject to the same mortality and with a constant rate of increase must ultimately become stable, that is to say has a stable age composition, a stable birth rate and a stable death rate.

The reader will find this odd, since Kuczynski's quoted words can be construed as attributing to Bortkiewicz not Theorem 2A, but rather my Theorem 1C, which seems to be a perfectly valid theorem, albeit as far as I know not appearing explicitly in the pre-1976 literature.

Lotka goes on to remark that there is no $m(a)$ fertility function in Bortkiewicz, saying[5] in Lotka (1937a, p. 104, n. 2) that even "Kuczynski (1932, p. 43) admits this in his otherwise misleading history of the 'stable' population." Lotka goes on to argue that (1) Kuczynski has not correctly stated what Bortkiewicz (1911) did do—which Lotka correctly states was to *begin* with a stable age distribution and *to deduce from it* a balanced exponential growth rate proportional to my e^{ut} [not to be confused with my $e^{r_0 t}$ since u is *not* deduced from an $m(a)p(a)$ integral], and then to find a numerical estimate for u that made the $p(a)e^{-ua}$ theoretical distribution fit tolerably well the turn-of-the-century German data.

The jury must agree with Lotka's contention under (1), which is that Bortkiewicz (1911) proves my Theorem 1A. Kuczynski, like Homer, nodded. Lotka (1937a, p. 106) however goes a bit far in writing: "All this was old in the literature (which Kuczynski fails to point out)...", having already been done in Lotka (1907a, 1907b) with data for England and Wales. First, we might deem 1907 not to be long before 1911. Second, Lotka's 1907 utterances of Theorem 1A was, as noted already, loose at best and no semblance of an adequate proof was provided. Third, Bortkiewicz's work is thorough to the point of being tedious, much of it going back to the 1890's, while Lotka's maiden publications are brief and suggestive notes.

(2) Lotka (1937a, pp. 106—7) goes on seemingly to argue that what Kuczynski wrongly attributes to Bortkiewicz is in any case not worth doing, being "inherently absurd" in the light of Lotka's claim that, exponential growth in total self-propelled population *already* implies realization of the stable age distribution—as in Theorem 2B here. It escapes me why the truth of 2B should make it absurd for Bortkiewicz (if only he had done it!) to have formulated Theorem 2A (or, Kuczynski aside, for Bortkiewicz to have formulated Theorem 1C). Indeed, taken literally, Lotka seems to be cutting his own throat and that of Sharpe! Obscurely, Lotka (1937a, p. 106) argues that if Bortkiewicz had done what he is credited with doing, there would have been no need "for Kuczynski to go through the agonizing labor of testing the approach ... to the stable distribution in 15 pages of closely printed figures" of Kuczynski (1932c, p. 65).

This Lotka point seems dubious. The 1911 Sharpe-Lotka proof, or the 1941 proof of Feller of Theorem 2A is admittedly better than Kuczynski's numerical exercise, which (a) starts from an $m(a)p(a)$ such that $\int_0^n m(a)p(a)\,da=1$ and the population is in *stationary*, stable equilibrium, (b) suddenly lowers $m(a)$ by 10 per cent, and then (c) by laborious numerical projection depicts the 70 years of transient approach to a new exponential equilibrium with negative r_0. Still numerical exercises are frequently performed, and for a non-mathematician like Kuczynski this was a particularly valuable and insightful thing to do.

Lotka (1937a) concludes with his imperfect statement of the truth of Theorem 2B—perhaps not being clear in his own mind that, for the age distribution of a self-propelled system to necessarily *already* be in its stable form, the posited exponential growth for $N(t)$ would have had to have held forever, or at least as in my 2C for $n \approx 100$ years. Lotka (1937b), when it comes to provide a more careful statement of 2B and a proof, evidently assumes that his hypotheses hold *for all* time, $-\infty < t < \infty$; and even then I cannot follow the cogency of his proof.[6]

From my literal account of Lotka's 1937 papers, one can perhaps conclude that the plaintiff has not optimally pressed his case. But the jury cannot conclude from this that the defendant, Kuczynski, is without fault in Kuczynski (1935) or Kuczynski (1932). This requires special investigation, the results of which do not clear Kuczynski of fault in the matter.

Let me audit Kuczynski (1935), a much better-known work than Kuczynski (1932), one which purports to be a basic exposition. Around pp. 6—7, the author concentrates on the qualitative problem of intrinsic growth of population, or decay, properly pointing out that Böckh's net reproduction rate provides an appropriate answer. Fair enough. He goes on to document at length, p. 207, n. 2, how Böckh, Kuczynski, Hirschberg, Rahts had computed in the 1884—1912 period dozens of fertility tables for Germany, Sweden, Denmark, and France—so that Dublin and Lotka (1925) was something of a Johnny-come-lately. Fair enough as documentation of the Kuczynski (1930) reply to the Lotka (1929) review.

But it was quite misleading for Kuczynski to gloss over the difference between the rate of growth per annum, r_0, and the rate of growth per generation: Kuczynski (1935, p. 207) misleadingly says that, from his 1884 calculation of a Berlin fertility table, Böckh "... concluded that the real rate of increase of the Berlin population in 1879 was $\frac{2{,}172}{2{,}053} - 1 = 6$ per cent." The growth rate per year or decade is of course not 0.06, and Böckh could not give the correct number.

To conserve time and reduce tedium, I shall reproduce some further misleading passages from Kuczynski (1935, p. 224) with my bracketed editorial comments.

> "Will a population constantly subject to the same fertility and the same mortality ultimately become stable?
>
> The mathematical elements of this problem have for a long time [how long? and when before Sharpe and Lotka of 1911?] attracted the attention of both European [name one!] and American mathematicians.
>
> They [Lotka and who else?] have come to the conclusion that a population with a constant fertility and mortality will in fact ultimately become stable [yes, Lotka, Feller, Leslie, Lopez]. A comparatively easy approach to the computations necessary for ascertaining the age composition and the birth and death rates of the stable popula-

> tions is to be found in the report represented by Bortkiewicz to the 1911 congress of the International Statistical Institute [literally correct as written, but Kuczynski (1932, p. 41, n. 1) had already noted that Lotka (1907b) had already done this; and, in any case, this literally true assertion is a *non sequitur* in its seeming implication that Theorem 1A or 1C is Theorem 2A].

There soon follows the p. 226 passage on Bortkiewicz that Lotka quoted in protest, and which I showed to be ambiguous. Kuczynski (1935, p. 226) goes on to say:

> ... But one of his [Bortkiewicz's] assumptions, the stable rate of increase, was not and could not be based on the actual conditions presented by some specific statistical example [being *not* based on any $m(a)$ data, as Lotka pointed out was devastating to Kuczynski's apparent link up of Bortkiewicz with Theorem 2A], his findings, interesting as they were from a theoretical standpoint [misleading in that "theoretical" versus "statistical" is being confused with the Theorem 1A versus Theorem 2A issue], did not attract the attention of demographers.
>
> The attentions of demographers [Kuczynski at least] was indeed only aroused when 14 years later the American mathematician, Lotka, who for a long time had studied the theoretical properties of the stable population [which one? the 1A case? or the 2A case?] published with Dublin ... Their approach is highly mathematical and we shall confine ourselves here to showing how through Lotka's formulae the stable yearly rate of increase (r) may be derived from the net reproduction rate [and the first and second moments of $m(a)$].

A cross that mathematical pioneers in a subject must always bear is to have their pearls dismissed as theoretical and vaguely impractical; later, after capitulation, they receive their revenge.

It is true that in the rarer item, Kuczynski (1932), the author deals at greater length with Lotka's contributions, quoting from him copiously and correcting his numerical errors (but still writing the same misleading sentence about Bortkiewicz!). At one point, Kuczynski (1932, Appendix, pp. 62—3) makes this explicit acknowledgment:

> It goes without saying that we would not have devoted so much space [more than 17 pages!] to the presentation of the trend of Dublin and Lotka's argument and to the translation of their mathematical operations into simple arithmetic if we were not convinced that some of the methods which they apply are of great scientific value and if we did not feel the strong desire that those methods be applied in the future also by such statisticians as are unfamiliar with higher mathematics. We wish even to state expressly that the computation of the exact rate of increase in the stable population, as presented in this study of Dublin and Lotka, in our opinion, marks the only great progress that has been made in the methodology of measuring

> net reproduction since Boecke in 1886 puplished his first table of fertility. But just because we so emphatically recommend the application of those methods we feel obliged to show that the manner in which they themselves applied them to statistical data is inadequate.

This is merited if qualified praise, and all the greater the pity that a reader like myself can find it only in the Appendix of a work so rare that, in the end, I had to rely on a xerox from a copy in the Princeton Office of Population library, a copy which I judge from some marginal caligraphy, in comparison with some samples of Lotka's handwriting in my own possession, to have come from Lotka's own copy and which must have been among his books bequeathed to Princeton.

Before concluding, I ought to venture an opinion as to whether Kuczynski was being deliberately ungenerous to Lotka. At the conscious level, I think not. Kuczinski was always a plain spoken scholar, and such people never hesitate to point out the motes that they see in the eyes of others. In this regard he was not extreme: certainly, Bortkiewicz customarily meted out more trenchant criticisms than did Kuczynski, and R.A. Fisher's quill was dipped in stronger acid than Kuczynski's. Moreover, as I read and reread Kuczynski, both a third of a century ago when I developed a mild interest in the matter and recently in the preparation of this article, I sensed that he may really never have fully understood the magic of Lotka's Theorem 2A. He sensed that, if you knew r_0 you could derive the stable age distribution, and then from that you could compute from $p(a)$ alone the death rate; he may also have sensed the element of simultaneity involved, because only if you had happened to guess the right r_0 would your resulting death rate, when subtracted from the birth rate computable from $m(a)$, be consistent with the originally assumed r_0. But he shows signs of being unclear on the essential logic involved.[7] Nevertheless, I ought to point out explicitly that there is a way of reading his train of thought which makes his individual sentences about Bortkiewicz *literally true*, even though misleading in their context. Thus, Kuczynski, in the end seems to have considered the following procedure as optimal: First compute r_0 from the first few moments of the fertility table $m(a)p(a)$ (a result he ought to have clearly excluded Bortkiewicz from in favor of Lotka); second, use Bortkiewicz (or pre-Sharpe Lotka) to get the actuary's stable age distribution and the implied death rate (and, as a residual, the birth rate). This will be faster and digitally more accurate, Kuczynski decided, than computing out a Monte Carlo version of Lotka's Theorem 2A. But why did he not make this clear?

Perhaps at the unconscious level, Kuczynski was a bit grudging in his treatment of Lotka, writing passages, (1932, p. 65), like

> A good mathematician may be a poor statistician; a good statistician may be a poor mathematician. And since the author of this book, if anything, is a poor mathematician ...

A Gestalt psychologist trained in Vienna would expect the reader to complete the *chiasmus* by regarding Lotka as the poor statistician.

A sociologist of science like Merton would not be surprised to observe that controversy sours both contestants. Lotka (1925, p. 112, n. 7) praised Euler's early anticipations, writing, "An exceedingly interesting effort of early date to demonstrate the ultimate approach to geometric increase of the birth rate, independently of the initial conditions (e. g., starting with a single pair of parents) is to be found in L. Euler (1760)." This is generous praise—as I have argued even over-generous, if Lotka did not in 1925 know of Euler's private communication to Süssmilch (1761), and seemingly involving an error like that of Kuczynski's mistake in attributing to Bortkiewicz Theorem 2A rather than 1A—but understandable praise of a great scholar. However, once he has become alarmed for his own property rights, Lotka (1937a, p. 107) takes a shriller tone in defending his own originality, now writing: "... This must be abundantly clear to anyone who takes the trouble to examine the pertinent publications, from the first crude approach by Euler (based on highly unrealistic assumptions, and quite inapplicable to actual statistical data), ...".

As a final word, I ought to emphasize that the reason for now discussing this historical *contretemps* in detail has to do primarily with the need *to clarify the substance of the matter*. By no means was this controversy of unprecedented virulence: neither scholar ever stood in an extreme position with respect to temperament or emotion; both always conducted themselves with honor and dignity.

Moreover, in a sense Lotka has been the ultimate victor. It is he who is accorded full scholarly homage today. The danger is that, if anything, it is Kuczynski's commendable role in the development of the subject that will be lost. Thus, Lotka appears with a full page of references in the excellent Keyfitz (1968, pp. 424—5) bibliography; the works of Kuczynski escape notice. The one notice taken of Kuczynski in J. H. Pollard (1973, p. 82) is only in connection with his 1932 computation of a male NRR for France after World War I that exceeds unity whereas the female NRR falls short of it, a dramatic consequence of war casualties that, ironically, Lotka had adverted to in his original cited review of Kuczynski. Richard Böckh receives no citations in either text. Again, a Merton would understand how the brighter light drowns out the earlier light. By a process somewhat the opposite of Gresham's law the more polished mathematics seems to drive out the less certain experimenting with numerical data.

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the present effort, and I owe thanks to David P. Smith for translation of the Süssmilch (1761) account of Euler's important private communication. All interpretations must, of course, be on my own responsibility.

1. See Spengler (1968) on Lotka in *The International Encyclopedia of the Social Sciences;* also, the obituary notices of Dublin (1950) and Notestein (1950).

2. Edwin Cannan, a no-new-fangled-nonsense economist if there ever was one, showed by arithmetic projections of the absolute number of English births that ultimate U.K. population decline was likely. See Cannan (1895).

3. Cf. Kuczynski (1935, p. 207, n. 2) and his citation of Böckh (1886).

4. On the attainment of stability Bortkiewicz says only (1911, pp. 63, 69—70; the translator is unnamed):

> Three qualifying hypotheses underlie the following statements. It is assumed: 1) that in the population an unchanging order of deaths prevails; 2) that the current age distribution likewise is invariable, and 3) that no immigration or emigration takes place.
>
> Accordingly the "stationary" and the "stable" population appear in a sense, as the terms will be used here, almost as ideal types, to which reality never exactly corresponds, but to which it comes all the closer, the less significant the actual changes of the death order and the age distribution, and, relatively speaking, the smaller the immigration and emigration.
>
> The above hypotheses can be considered as the three characteristics which are held in common with the two concepts of the stationary and stable population. In addition, there comes a fourth characteristic: with the stationary population there is a continual constancy, whereas with the stable population a continual accretion in the total number of persons living.
>
> ... [T]he geometric progression as a standard norm for the growth of the population was established by L. Euler. He proceeded from one human couple, and let it as well as its offspring propagate from generation to generation according to certain invariable conditions. These conditions referred to age of marriage, the number of children begotten by each couple, and the number dying. Hence it resulted that the number of living at the end of every calendar year forms a line which, in its further course, approaches more and more a geometric progression.

The key second paragraph reads in the original German: „Demnach erscheinen die „stationäre“ und die „progressive“ Bevölkerung in dem Sinne, wie diese Termini hier gebraucht werden, gleichsam als Idealtypen, denen die Wirklichkeit niemals genau entspricht, denen sie aber um so näher kommt, je unerheblicher die tatsächlich vor sich gehenden Wandlungen der Absterbeordnung und der

Altersverteilung und je geringer, relativ genommen, die Zahlen der Ein- und Auswandernden sind.“

The statement suggests that real populations may *resemble* the stable ideal type, and not that they *approach* stability. Kuczynski may have given it the second meaning, taking the comment that the geometric is a standard norm as supportive evidence. (The illustration by Euler is from a personal communication to J. P. Süssmilch (1761, Vol. 1, pp. 291—299). Euler's example treats 1 female and 1 male as a "couple", and in effect postulates $B(t)=B(t-22)+B(t-24)+B(t-26)$, $D(t)=B(t-40)$, a Bernardelli-Leslie case that is "cyclic" because of an unfortunate choice of even numbers only: along with dominant $(\lambda^*)^t$ terms go also dominant $(-\lambda^*)^t$ terms and no strict approach to a stable age distribution at both odd and even ages.)

5. I follow Lotka (1937a) in giving the date 1932 to Kuczynski's *Fertility and Reproduction* of 1931; likewise Kuczynski (1928) gives 1886 for Böckh (1884), and I follow him in this (actually, the computation might date from 1886).

6. I am unable to discern the cogency of Lotka (1937b)'s purported proof that if $N(t)$ is observed to be in exponential growth (over *some* consecutive time periods? over all time-periods?—which is another thing), $N(a,t)$ is *already* in the stable configuration. The following discrete-time example meets the only hypotheses he purports to use in his demonstration—yet it is not "already in the stable distribution", and *after* $(t=0,1,2)$ its $N(t)$ ceases to grow like $N(0)(2)^t$! The example is based on $[p(0),p(1),p(2),p(3)]=[1,2^{-1},2^{-2},2^{-3}]$, $B(0)=1{,}032$, $[m(0),m(1),m(2),m(3)]=[2,0,0,0]$, with initial (and asymptotic!) increase of $N(t)$ like $(1+1)^t$, but lacking such exact exponential growth in the near future.

Period, t	Births, $N(0,t)$	$N(1,t)$	$N(2,t)$	$N(t,3)$	Total $N(t)$
0	1,032	88	192	48	1,360
1	2,064	516	44	96	$2\times 1{,}360$
2	4,128	1,032	258	22	$2^2\times 1{,}360$
3	8,256	2,064	516	129	$85+(2^3\times 1{,}360)$

The example was fabricated by perturbing an exact exponential solution to the self-propelled Sharpe-Lotka system through adding to a stable initial state, $[N(j,t)]=[1{,}024, 256, 64, 32]$, a non-exponential solution, $[8, -168, 128, 16]$, of higher harmonics *fitted* to the initial conditions that total $N(t)$ of the add-on *initially* vanish as for $t=(0,1,2)$. Only if Lotka postulates initial exponential growth of $N(t)$ over a long enough initial interval—the whole length of life, $n>\beta$—will it become impossible for me to find such a perturbing add-on. No hint of this appears in his purported proof. The demonstration goes off the tracks because he seems to confuse a functional-equation requirement of the type

$$f(a,t)/g(a,t) \quad \text{independent of } t \text{ for all } a$$

with his actual type

$$[f_1(t)+f_2(t)]/[g_1(t)+g_2(t)] \quad \text{independent of } t\,.$$

I must make clear that these last interpretations of mine have to be regarded as only tentative.

7. This surmise is corroborated by Kuczynski (1931, pp. 20, 32, 166): writing apparently simultaneously with his *Fertility and Reproduction*, the author eschews relations like those in (22) and Corollary 2AA, even though he already knows them, in favor of stable-state relations like

$$r \equiv N'(t)/N(t) \equiv B(t)/N(t) - D(t)/N(t),$$

$$r = \frac{\int_{\alpha}^{\beta} m(a)p(a)e^{-ra}\,da}{\int_{0}^{n} p(a)e^{-ra}\,da} - \frac{\int_{0}^{n} -p'(a)e^{-ra}\,da}{\int_{0}^{n} p(a)e^{-ra}\,da}$$

$$= \theta(r) = \frac{\psi(r)+1}{\bar{p}(r)} + \frac{r\bar{p}(r)-1}{\bar{p}(r)}$$

$$= \frac{\psi(r)}{\bar{p}(r)} + r$$

where real r is put in the indicated Laplace Transforms and where we solve for the unique r_0 root. Keyfitz (1968, p. 176) calculates r_0 iteratively from

$$r_0'' = \theta(r_0'), \quad r_0''' = \theta(r_0''), \ldots .$$

Even more rapid convergence would occur for the Newton-Raphson variant

$$r_0'' = r_0' - \theta(r_0')/\theta'(r_0').$$

References

Bernardelli, H.: Population Waves. Journal of the Burma Research Society **31**, Part I, 1—18 (1941).

Böckh, R.: Statistik des Jahres 1884. Statistisches Jahrbuch der Stadt Berlin, Volume 12 (Berlin) (1886).

Bortkiewicz, L. V.: Die Sterbeziffer und der Frauenüberschuß in der stationären und in der progressiven Bevölkerung. Bulletin de l'Institut International de Statistique **19**, 63—183 (1911).

Cannan, E.: The probability of a Cessation of the Growth of Population in England and Wales during the Next Century. Economic Journal **5**, 505—15 (1895).

Coale, A. J.: The Growth and Structure of Human Populations. A Mathematical Investigation. Princeton, N. J.: Princeton University Press 1972.

Dublin, L. I.: Alfred James Lotka, 1880—1949. J. Amer. Statistical Assoc. **45**, 138—9 (1950).

Dublin, L. I., Lotka, A. J.: On the True Rate of Natural Increase of a Population. J. Amer. Statistical Assoc. **20**, 305—39 (1925).

Euler, L.: Recherches générales sur la mortalité et la multiplication. Mémoires de l'Académie Royale des Sciences et Belles Lettres **16**, 144—64 (1760).

Feller, W.: On the Integral Equation of Renewal Theory. The Annals of Mathematical Statistics **12**, 243—67 (1941).

German Statistical Office: Neue Beiträge zum deutschen Bevölkerungsproblem (Berlin) (1935).
Keyfitz, N.: Introduction to the Mathematics of Population. Reading, Mass.: Addison-Wesley Publishing Co. 1968.
Kuczynski, R.R.: The Balance of Births and Deaths, Volume I, Western and Northern Europe. New York: Macmillan Co. for The Brookings Institution 1928.
The Balance of Births and Deaths, Volume II, Eastern and Southern Europe. Washington, D.C.: The Brookings Institution 1931.
Kuczynski, R.R.: A Reply to Dr. Lotka's Review of 'The Balance of Births and Deaths.' J. Amer. Statistical Assoc. **25**, 84—5 (1930).
Kuczynski, R.R.: Fertility and Reproduction. New York: Falcon Press 1931/32.
Kuczynski, R.R.: The Measurement of Population Growth. London: Sedgewick & Jackson, Ltd. 1935; reproduced by Gordon and Breach. New York: Science Publishers 1969.
Leslie, P.H.: On the Use of Matrices in Certain Population Mathematics. Biometrika **33**, 183—212 (1945).
Lopez, A.: Problems in Stable Population Theory. Princeton, N.J.: Office of Population Research 1961.
Lotka, A.J.: Relation between Birth Rates and Death Rates. Science, N.S. **26**, 21—22 (1907a).
Lotka, A.J.: Studies on the Mode of Growth of Material Aggregates. Amer. J. Science **24**, 199—216 (1970b).
Lotka, A.J.: A Natural Population Norm. J. Washington Acad. Sciences **3**, 241—48, 289—93 (1913).
Lotka, A.J.: The Stability of the Normal Age Distribution, Proc. Nat'l Acad. Sciences **7**, 339—45 (1922).
Lotka, A.J.: Elements of Physical Biology. Baltimore: Williams & Wilkins 1925; reproduced in posthumous edition with bibliography as: Elements of Mathematical Biology. New York: Dover Publications, Inc. 1956.
Lotka, A.J.: The Progeny of a Population Element. Amer. J. Hygiene **8**, 875—901 (1928).
Lotka, A.J.: Review of 'The Balance of Births and Deaths, Vol. I, Western and Northern Europe' by R. Kuczynski. New York: Macmillan, 1928; J. Amer. Statistical Assoc. **24**, 332—3 (1929).
Lotka, A.J.: Rejoinder to 'A Reply to Dr. Lotka's Review of The Balance of Births and Deaths by R. Kuczynski.' J. Amer. Statistical Assoc. **25**, 85—6 (1930).
Lotka, A.J.: The Structure of a Growing Population. Human Biology **3**, 459—93 (1931).
Lotka, A.J.: Notes: A Historical Error Corrected. Human Biology **9**, 104—7 (1937a).
Lotka, A.J.: Population Analysis: A Theorem Regarding the Stable Age Distribution. J. Washington Acad. Sciences **27**, 299—303 (1937b).
Lotka, A.J.: A Contribution to the Theory of Self-Renewing Aggregates, with Special Reference to Industrial Replacement. Annals of Mathematical Statistics **10**, 1—25 (1939a).
Merton, R.K.: The Sociology of Science. Theoretical and Empirical Investigations. Chicago: The University of Chicago Press 1973.
Notestein, F.W.: Alfred James Lotka: 1880—1949. Population Index **16**, 22—9 (1950), containing a Lotka bibliography of 114 items.
Pollard, J.H.: Mathematical Models for the Growth of Human Populations. Cambridge: Cambridge University Press 1973.
Samuelson, P.A.: Time Symmetry and Asymmetry in Population and Deterministic Dynamic Systems. J. Theoretical Population Biology **9**, 82—122 (1976).
Sharpe, F.R., Lotka, A.J.: A Problem in Age-Distribution. Philosophical Magazine **21**, 435—8 (1911).
Spengler, J.J.: Alfred J. Lotka. International Encyclopedia of the Social Sciences **9**. New York: Crowell Collier Macmillan 475—6 (1968).
Süssmilch, J.P.: Die göttliche Ordnung, Volume 1, Second Edition. Berlin: 1761.

16. On the Integral Equation of Renewal Theory

WILLIAM FELLER (1941)

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Feller's paper is a rigorous treatment of renewal theory, and to assist the reader his principal results are summarized below in demographic form and notation.

Equation (1.1) is the renewal equation

$$B(t)=G(t)+\int_0^t B(t-a)p(a)m(a)\,da\,, \tag{1.1 a}$$

$$=G(t)+\int_0^t B(t-a)\,\phi(a)\,da \tag{1.1 b}$$

where $B(t)$ is births at time t and is composed of births $G(t)$ to the parent population alive at time 0 plus, inside the integral, births to those born since; $B(t-a)$ being the number born a years ago, $p(a)$ their probability of surviving to age a, and $m(a)$ the probability of their giving birth to offspring in the interval a to $a+da$. In equation (1.1 b), $\phi(a)=p(a)m(a)$. Stieltjes integrals, which translate for discrete distributions as summations, are allowed by a comment following equation (1.3). The equation generalizes from births at time t to births in the interval 0 to t.

For Feller's example ii, equation (2.1) does not apply. We have instead

$$G(t)=\begin{cases}\displaystyle\int_0^{\beta-t} k(a)\frac{p(t+a)}{p(a)}m(t+a)\,da, & t<\beta\\ 0 & t\geqq\beta\end{cases} \tag{2.1 a}$$

where $k(a)$ is the parent population at ages a to $a+da$ at time 0, and other terms are as above. For $t\geqq\beta$ the parent population is no longer in the reproductive ages and equation (1.1) takes the homogeneous form

$$B(t)=\int_\alpha^\beta B(t-a)p(a)m(a)\,da\,, \tag{1.1 c}$$

where α is the youngest age at which $m(a)\neq 0$. Equation (2.3) is the net reproduction rate

$$\int_0^\infty f(t)\,dt=\int_\alpha^\beta p(a)m(a)\,da=R_0\,. \tag{2.3 a}$$

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For Theorems 1 and 2 we have:

$$F(t)=\int_0^t p(a)m(a)\,da\,, \tag{3.1 a}$$

which for $t \geqq \beta$ becomes R_0 as above. Feller's $G(t)$ will be the *integral* of equation (2.1 a), hence it is total births to the initial population over the interval 0 to t. Both are finite and are cumulative functions, hence non-decreasing. The Laplace Transforms (3.2) give their characteristic functions:

$$\varphi(s)=\phi^*(r)\ \left[\text{or } \psi(r)\right]=\int_\alpha^\beta e^{-rt}p(t)m(t)\,dt\,, \tag{3.2 a}$$

$$\gamma(s)=G^*(r)\ \left[\text{or } V(r)\right]=\int_0^\beta \int_0^{\beta-t} e^{-rt}k(x)\frac{p(t+x)}{p(x)}m(t+x)\,dx\,dt\,. \tag{3.2 b}$$

The first of these is the characteristic equation for r, the intrinsic growth rate; the second is the total reproductive value introduced by R.A. Fisher (1930, paper 18 below). Both integrals are convergent for $t \geqq \beta$ since $m(t)$ then becomes 0. Hence, by Theorems 1 and 2 the renewal equation has a unique solution.

Theorem 3 establishes stable properties of mean births $\bar{B}(t)$. We make the substitutions:

(4.2 a)	$a=R_0$,	the net reproduction rate.
(4.2 b)	$b=B$,	total births to the initial population.
(4.4 a)	$m=\mu$,	the mean age at childbearing in the stationary population.
(4.7 a)	$m_1=-\psi'(r)=A_r$,	the mean age at childbearing in the stable population.

All of these terms are finite, and therefore:

(i) For $R_0=1$ the population is stationary and mean births approach the limiting value B/μ;
(ii) For $R_0<1$ the *total* number of births accruing to the population after time 0 will be $B/(1-R_0)$;
(iii) For $R_0>1$ mean births when deflated by their intrinsic increase over time approach the limiting value B/A_r.

Theorem 4 establishes conditions under which births $B(t)$ as well as mean births $\bar{B}(t)$ stabilize. The theorem states that for $R_0=1$ and B finite as before, $B(t)$ stabilizes if: by equation (5.3) the fertile age distribution has a finite mean and variance; and by equation (5.4) births to the initial population eventually taper out, as by the population dying or ageing out of the reproductive years.

The *Remark* extends Theorem 4 to all R_0, establishing that under the conditions given by the theorem births at time t stabilize toward the limiting value $B(t) = Ve^{rt}/A_r$, where

$m_1 = m_1' = A_r$ as above (4.7 a),

$b' = V =$ total reproductive value as above (3.2 b).

The result has an intuitive meaning: V/A_r is the number of births at $t=0$ that would give rise over time to the same population as the parent generation's observed birth distribution. It is found as the backward projection of their births to time 0 at their intrinsic growth rate, adjusted for the delay to childbearing.

[In the *Remark*, note that the characteristic function $\psi(r)$ converges in the interval $0 > r > -\infty$ and that for the case $R_0 < 1$ the dominant root r_1 is also negative; hence the Lemma of Section 4 is applicable. For the case $R_0 = 1$, $V/A_r = B/\mu$. In the discrete case, which Feller does not complete, Theorem 4 holds, to an approximation for finite population size, if the fertility distribution $\phi(a)$ can be expressed as an irreducible and primitive projection matrix. These requirements are discussed in Parlett (1970, paper 29 below).]

Lotka's solution to the renewal equation enters in Theorem 6. He first substituted $B_0 e^{rt}$ for $B(t)$ and $B_0 e^{r(t-a)}$ for $B(t-a)$ in the homogeneous form of the equation, (1.1 c), by which it reduces to

$$\int_{\alpha}^{\beta} e^{-ra} p(a) m(a)\, da = 1 .$$

Given an infinite number of simple roots $r_1, r_2, \ldots$ he was able to assert that there existed constants Q such that

$$\text{(6.2 a)} \qquad B(t) = \int_{\alpha}^{\beta} B(t-a) p(a) m(a)\, da = \sum_i Q_i e^{r_i t} ;$$

and to identify the Q_i terms as $Q_i = V(r_i)/-\psi'(r_i)$, where these are the total reproductive value and mean age at childbearing associated with the i'th root of equations (3.2 b) and (4.7 a) (Lotka 1939, 64—67, 85—87).

Feller shows that the use of the Laplace Transform is both simpler than Lotka's solution and more general since it also applies where the characteristic equation has only a finite number of roots. By taking the transforms of the three terms of equation (1.1 b)

$$B(t) = G(t) + \int_0^t B(t-a)\, \phi(a)\, da ,$$

and noting that the last term is a convolution whose transform is the product of the transforms of the two functions, Feller comes down to

$$B^*(r) = G^*(r) + B^*(r)\, \phi^*(r) .$$

The solution becomes simply

$$B^*(r) = \frac{G^*(r)}{1-\phi^*(r)} = \frac{V(r)}{1-\psi(r)}, \tag{6.6 a}$$

where on the right we use the notation of equations (3.2 a) and (3.2 b).

By Theorem 6, when equation (6.6 a) admits the expansion

$$B^*(r) = \sum_i \frac{V(r_i)}{-\psi'(r_i)} \left[\frac{1}{r-r_i} \right], \tag{6.6 b}$$

and $\sum |V(r_i)/-\psi'(r_i)|$ is convergent, it has the inverse

$$B(t) = \sum_i \frac{V(r_i)}{-\psi'(r_i)} e^{r_i t} = \sum_i Q_i e^{r_i t} \tag{6.2 b}$$

given by Lotka for the case of infinite simple roots. In demographic work the conditions of the Theorem are always met.

Equation (6.10) extends the analysis to multiple roots, which introduce non-geometric components to growth and are not found in demographic work. Section 7 provides further comments on discrete solutions.

1. Introduction.

In this paper we consider the behavior of the solutions of the integral equation

$$u(t) = g(t) + \int_0^t u(t - x)f(x)\,dx, \tag{1.1}$$

where $f(t)$ and $g(t)$ are given non-negative functions.[1] This equation appears, under different forms, in population theory, the theory of industrial replacement and in the general theory of self-renewing aggregates, and a great number of papers have been written on the subject.[2] Unfortunately most of this literature is of a heuristic nature so that the precise conditions for the validity of different methods or statements are seldom known. This literature is, moreover, abundant in controversies and different conjectures which are sometimes supported or disproved by unnecessarily complicated examples. All this renders an orientation exceedingly difficult, and it may therefore be of interest to give a rigorous presentation of the theory. It will be seen that some of the previously announced results need modifications to become correct.

The existence of a solution $u(t)$ of (1.1) could be deduced directly from a well-known result of Paley and Wiener [21] on general integral equations of form (1.1).[3] However, the case of non-negative functions $f(t)$ and $g(t)$, with which we are here concerned, is much too simple to justify the deep methods used by Paley and Wiener in the general case. Under the present conditions, the existence of a solution can be proved in a simple way using properties of completely monotone functions, and this method has also the distinct advantage of showing some properties of the solutions, which otherwise would have to be proved separately. It will be seen in section 3 that the existence proof becomes most natural if equation (1.1) is slightly generalized. Introducing the summatory functions

$$U(t) = \int_0^t u(x)\,dx, \qquad F(t) = \int_0^t f(x)\,dx, \qquad G(t) = \int_0^t g(x)\,dx, \tag{1.2}$$

equation (1.1) can be rewritten in the form

$$U(t) = G(t) + \int_0^t U(t - x)\,dF(x). \tag{1.3}$$

However, (1.3) has a meaning even if $F(t)$ and $G(t)$ are not integrals, provided $F(t)$ is of bounded total variation and the integral is interpreted as a Stieltjes integral. Now for many practical applications (and even for numerical calculations) this generalized form of the integral equation seems to be the most

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[1] For the interpretation of the equation cf. section 2.

[2] Lotka's paper [8] contains a bibliography of 74 papers on our subject published before 1939. Yet it is stated that even this list "is not the result of an exhaustive search." At the end of the present paper the reader will find a list of 16 papers on (1.1) which have appeared during the two years since the publication of Lotka's paper.

[3] This has been remarked also by Hadwiger [3].

appropriate one and, as a matter of fact, it has sometimes been used in a more or less hidden form (e.g., if all individuals of the parent population are of the same age). Our existence theorem refers to this generalized equation.

We then turn to one of the main problems of the theory, namely the asymptotic behavior of $u(t)$ as $t \to \infty$. It is generally supposed that the solution $u(t)$ "in general" either behaves like an exponential function, or that it approaches in an oscillating manner a finite limit q; the latter case should arise if $\int_0^\infty f(t)\,dt = 1$, thus in particular in the cases of a stable population and of industrial replacement. However, special examples have been constructed to show that this is not always so.[4] In order to simplify the problem and to get more general conditions, we shall first (section 4) consider only the question of convergence in mean, that is to say, we shall study the asymptotic behavior not of $u(t)$ itself but of the mean value $u^*(t) = \frac{1}{t}\int_0^t u(x)\,dx$. The question can be solved completely using only the simplest Tauberian theorems for Laplace integrals. Of course, if $u(t) \to q$ then also $u^*(t) \to q$, but not conversely. The investigation of the precise asymptotic behavior of $u(t)$ is more delicate and requires more refined tools (section 5).

Most of section 6 is devoted to a study of Lotka's well-known method of expanding $u(t)$ into a series of oscillatory components, and it is hoped that this study will help clarify the true nature of this expansion. It will be seen that Lotka's method can be justified (with some necessary modifications) even in some cases for which it was not intended, e.g., if the characteristic equation has multiple or negative real roots, or if it has only a finite number of roots. On the other hand limitations of the method will also become apparent: thus it can occur in special cases that a formal application of the method will lead to a function $u(t)$ which apparently solves the given equation, whereas in reality it is the solution of quite a different equation.

Of course, most of the difficulties mentioned above arise only when the function $f(t)$ has an infinite tail. However, it is known that even computational considerations sometimes require the use of such curves, and, as matter of fact, exponential and Pearsonian curves have been used most frequently in connection with (1.1). It will be seen that even in these special cases customary methods may lead to incorrect results. Besides, our considerations show how much the solution $u(t)$ is influenced by the values of $f(t)$ for $t \to \infty$, and, accordingly, that extreme caution is needed in practice. The last section contains some simple remarks on the practical computation of the solution.

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[4] Cf. Hadwiger [2] and also Hadwiger, "Zur Berechnung der Erneuerungsfunktion nach einer Formel von V. A. Kostitzin," *Mitt. Verein. schweizerischer Versich.-Math.*, Vol. 34 (1937), pp. 37–43.

2. Generalities on equations (1.1) and (1.3). This section contains a few remarks on the meaning of our integral equation and on an alternative form under which it is encountered in the literature. A reader interested only in the abstract theory may pass immediately to section 3.

Equation (1.1) can be interpreted in various ways; the most important among them are the following two:

(i) In the theory of industrial replacement (as outlined in particular by Lotka), it is assumed that each individual dropping out is immediately replaced by a new member of zero age. $f(t)$ denotes the density of the probability at the moment of installment that an individual will drop out at age t. The function $g(t)$ is defined by

$$g(t) = \int_0^t \eta(x)f(t-x)\,dx, \tag{2.1}$$

where $\eta(x)$ represents the age distribution of the population at the moment $t = 0$ (so that the number of individuals of an age between x and $x + \delta x$ is $\eta(x)\delta x + o(\delta x)$). Obviously $g(t)$ then represents the rate of dropping out at time t of individuals belonging to the parent population. Finally, $u(t)$ denotes the rate of dropping out at time t of individuals of the total population. Now each individual dropping out at time t belongs either to the parent population, or it came to the population by the process of replacement at some moment $t - x$ $(0 < x < t)$, and hence $u(t)$ satisfies (1.1). It is worthwhile to note that in this case

$$\int_0^\infty f(t)\,dt = 1, \tag{2.2}$$

since $f(t)$ represents a density of probability.

(ii) In population theory $u(t)$ measures the rate of female births at time $t > 0$. The function $f(t)$ now represents the reproduction rate of females at age t (that is to say, the average number of female descendants born during $(t, t + \delta t)$ from a female of age t is $f(t)\delta t + o(\delta t)$). If $\eta(x)$ again stands for the age distribution of the parent population at $t = 0$, the function $g(t)$ of (2.1) will obviously measure the rate of production of females at time t by members of the parent population. Thus we are again led to (1.1), with the difference, however, that this time either of the inequalities

$$\int_0^\infty f(t)\,dt \lesseqqgtr 1 \tag{2.3}$$

may occur; the value of this integral shows the tendency of increase or decrease in the total population.

Theoretically speaking, $f(t)$ and $g(t)$ are two arbitrary non-negative functions. It is true that $g(t)$ is connected with $f(t)$ by (2.1); but, since the age distribution $\eta(x)$ is arbitrary, $g(t)$ can also be considered as an arbitrarily prescribed function.

It is hardly necessary to interpret the more general equation (1.3) in detail: it is the straightforward generalization of (1.1) to the case where the increase or decrease of the population is not necessarily a continuous process. This form of the equation is frequently better adapted to practical needs. Indeed, the functions $f(t)$ and $g(t)$ are usually determined from observations, so that only their mean values over some time units (years) are known. In such cases it is sometimes simpler to treat $f(t)$ and $g(t)$ as discontinuous functions, using equation (1.3) instead of (1.1). For some advantages of such a procedure see section 7. It may also be mentioned that the most frequently (if not the only) special case of (1.1) studied is that where $g(t) = f(t)$. Now it is apparent from (2.1) that this means that all members of the parent population are of zero age: in this case, however, there is no continuous age-distribution $\eta(x)$. Instead we have to use a discontinuous function $\eta(x)$ and write (2.1) in the form of a Stieltjes integral. Thus discontinuous functions and Stieltjes integrals present themselves automatically, though in a somewhat disguised form, even in the simplest cases.

At this point a remark may be inserted which will prove useful for a better understanding later on (section 6). In the current literature we are frequently confronted not with (1.1) but with

$$u(t) = \int_0^\infty u(t-x)f(x)\,dx, \tag{2.4}$$

together with the explanation that it is asked to find a solution of (2.4) which reduces, for $t < 0$, to a prescribed function $h(t)$. Now such a function, as is known, exists only under very exceptional conditions, and (2.4) is by no means equivalent to (1.1). The current argument can be boiled down to the following. Suppose first that the function $g(t)$ of (1.1) is given in the special form

$$g(t) = \int_t^\infty h(t-x)f(x)\,dx, \tag{2.5}$$

where $h(x)$ is a non-negative function defined for $x < 0$. Since the solution $u(t)$ of (1.1) has a meaning only for $t > 0$, we are free to *define* that $u(-t) = h(-t)$ for $t > 0$. This arbitrary definition, then, formally reduces (1.1) to (2.4). It should be noted, however, that this function $u(t)$ does not, in general, satisfy (2.4) for $t < 0$, for $h(t)$ was prescribed arbitrarily. Thus we are not, after all, concerned with (2.4) but with (1.1), which form of the equation is, by the way, the more general one for our purposes. If there really existed a solution of (2.4) which reduced to $h(t)$ for $t < 0$, we could of course define $g(t)$ by (2.5) and transform (2.4) into (1.1) by splitting the interval $(0, \infty)$ into the subintervals $(0, t)$ and (t, ∞). However, as was already mentioned, a solution of the required kind does not exist in general. It will also be seen (section 6) that the true nature of the different methods and the limits of their applicability can be understood only when the considerations are based on the proper equation (1.1) and not on (2.4).

3. Existence of solutions.

THEOREM 1. *Let $F(t)$ and $G(t)$ be two finite non-decreasing functions which are continuous to the right*[5]. *Suppose that*

$$F(0) = G(0) = 0, \tag{3.1}$$

and that the Laplace integrals[6]

$$\varphi(s) = \int_0^\infty e^{-st}\, dF(t), \qquad \gamma(s) = \int_0^\infty e^{-st}\, dG(t) \tag{3.2}$$

converge at least for $s > \sigma \geq 0$[7]. *In case that $\lim_{s\to\sigma+0} \varphi(s) > 1$, let $\sigma' > \sigma$ be the root*[8] *of the characteristic equation $\varphi(s) = 1$; in case $\lim_{s\to\sigma+0} \varphi(s) \leq 1$, put $\sigma' = \sigma$.*

Under these conditions there exists for $t > 0$ one and only one finite non-decreasing function $U(t)$ satisfying (1.3). *With this function the Laplace integral*

$$\omega(s) = \int_0^\infty e^{-st}\, dU(t) \tag{3.3}$$

converges for $s > \sigma'$, and

$$\omega(s) = \frac{\gamma(s)}{1 - \varphi(s)}. \tag{3.4}$$

PROOF: A trivial computation shows that for any finite non-decreasing solution $U(t)$ of (1.3) and any $T > 0$ we have

$$\int_0^T e^{-st}\, dU(t) = \int_0^T e^{-st}\, dG(t) + \int_0^T e^{-sx}\, dF(x) \int_0^{T-x} e^{-st}\, dU(t);$$

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[5] It is needless to emphasize that this restriction is imposed only to avoid trivial ambiguities.

[6] The integrals (3.2) should be interpreted as Lebesgue-Stieltjes integrals over *open* intervals; thus

$$\varphi(s) = \lim_{\epsilon\to+0} \int_\epsilon^\infty e^{-st}\, dF(t),$$

which implies that $\varphi(s) \to 0$ as $s \to \infty$. Alternatively it can be supposed that $F(t)$ and $G(t)$ have no discontinuities at $t = 0$. Continuity of $F(t)$ at $t = 0$ means that there is no reproduction at zero age. This assumption is most natural for our problem, but is by no means necessary. In order to investigate the case where $F(t)$ has a saltus $c > 0$ at $t = 0$, one should take the integrals (3.2) over the closed set $[0, \infty]$, so that

$$\varphi(s) = c + \lim_{\epsilon\to+0} \int_\epsilon^\infty e^{-st}\, dF(t).$$

It is readily seen that Theorem 1 and its proof remain valid if $0 < c < 1$. However, if $c > 1$, then (1.3) plainly has no solution $U(t)$. The continuity of $G(t)$ at $t = 0$ is of no importance and is not used in the sequel.

[7] The condition is formulated in this general way in view of later applications (cf., e.g., the lemma of section 4). In all cases of practical interest $\sigma = 0$.

[8] $\varphi(s)$ is, of course, monotonic for $s > \sigma$ and tends to zero as $s \to \infty$. In order to ensure the existence of a root of $\varphi(s) = 1$, it is sufficient to suppose that the saltus c of $F(t)$ at $t = 0$ is less than 1 (cf. footnote 6).

herein all terms are non-negative and hence by (3.2)

$$\int_0^T e^{-st}\,dU(t) \leq \gamma(s) + \varphi(s)\int_0^T e^{-st}\,dU(t).$$

Now $\varphi(s) < 1$ for $s > \sigma'$, and hence it is seen that the integral (3.3) exists for $s > \sigma'$ and satisfies (3.4). On the other hand it is well-known that the values of $\omega(s)$ for $s > \sigma'$ determine the corresponding function $U(t)$ uniquely, except for an additive constant, at all points of continuity. However, from (1.3) and (3.1) it follows that $U(0) = 0$ and, since by (1.3) $U(t)$ is continuous to the right, the monotone solution $U(t)$ of (1.3), if it exists, is determined uniquely.

To prove the existence of $U(t)$ consider a function $\omega(s)$ defined for $s > \sigma'$ by (3.4). It is clear from (3.2) that $\varphi(s)$ and $\gamma(s)$ are completely monotone functions, that is to say that $\varphi(s)$ and $\gamma(s)$ have, for $s > \sigma$, derivatives of all orders and that $(-1)^n\varphi^{(n)}(s) \geq 0$ and $(-1)^n\gamma^{(n)}(s) \geq 0$. We can therefore differentiate (3.4) any number of times, and it is seen that $\omega^{(n)}(s)$ is continuous for $s > \sigma'$. Now a simple inductive argument shows that $(-1)^n\omega^{(n)}(s)$ is a product of $\{1 - \varphi(s)\}^{-(n+1)}$ by a finite number of completely monotone functions. It follows that $(-1)^n\omega^{(n)}(s) \geq 0$, so that $\omega(s)$ is a completely monotone function, at least for $s > \sigma'$. Hence it follows from a well-known theorem of S. Bernstein and D. V. Widder[9] that there exists a non-decreasing function $U(t)$ such that (3.3) holds for $s > \sigma'$. Moreover, this function can obviously be so defined that $U(0) = 0$ and that it is continuous to the right. Using $U(t)$ let us form a new function

$$V(t) = \int_0^t U(t - x)\,dF(x). \tag{3.5}$$

$V(t)$ is clearly non-negative and non-decreasing. It is readily verified (and, of course, well-known) that

$$\psi(s) \equiv \int_0^\infty e^{-st}\,dV(t) = \omega(s)\varphi(s).$$

It follows, therefore, from (3.4) that $\psi(s) = \omega(s) - \gamma(s)$, and this implies, by the uniqueness theorem for Laplace transforms, that $V(t) = U(t) - G(t)$. Combining this result with (3.5) it is seen that $U(t)$ is a solution of (1.3).

THEOREM 2. *Suppose that* $f(t)$ *and* $g(t)$ *are measurable, non-negative and bounded in every finite interval* $0 \leq t \leq T$. *Let the integrals*

$$\varphi(s) = \int_0^\infty e^{-st}f(t)\,dt, \qquad \gamma(s) = \int_0^\infty e^{-st}g(t)\,dt \tag{3.6}$$

converge for $s > \sigma$. *Then there exists one and only one non-negative solution* $u(t)$

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[9] This theorem has been repeatedly proved by several authors; for a recent proof cf. Feller [19].

of (1.1) *which is bounded in every finite interval*[10]. *With this function the integral*

$$(3.7) \qquad \omega(s) = \int_0^\infty e^{-st} u(t)\, dt$$

converges at least for $s > \sigma'$, *where* $\sigma' = \sigma$ *if* $\lim\limits_{s \to \sigma+0} \varphi(s) \leq 1$, *and otherwise* $\sigma' > \sigma$ *is defined as the root of the characteristic equation* $\varphi(s) = 1$. *For* $s > \sigma'$ *equation* (3.4) *holds.*

If $f(t)$ *is continuous except, perhaps, at a finite number of points then* $u(t) - g(t)$ *is continuous.*

PROOF: Define $F(t)$ and $G(t)$ by (1.2). Under the present conditions these functions satisfy the conditions of Theorem 1, and hence (1.3) has a non-decreasing solution $U(t)$. Consider, then, an arbitrary interval $0 \leq t \leq T$ and suppose that in this interval $f(t) < M$ and $g(t) < M$. If $0 \leq t < t + h \leq T$ we have by (1.3)

$$\begin{aligned} 0 &\leq \frac{1}{h}\{U(t+h) - U(t)\} \\ &\equiv \frac{1}{h}\{G(t+h) - G(t)\} + \frac{1}{h}\int_t^{t+h} U(t+h-x) f(x)\, dx \\ &\qquad + \frac{1}{h}\int_0^t \{U(t+h-x) - U(t-x)\} f(x)\, dx \\ &\leq M + MU(T) + \frac{M}{h}\int_0^t \{U(t+h-x) - U(t-x)\}\, dx \\ &= M + MU(T) + \frac{M}{h}\int_t^{t+h} U(y)\, dy - \frac{M}{h}\int_0^h U(y)\, dy \\ &< M + 2MU(T). \end{aligned}$$

Thus $U(t)$ has bounded difference ratios and is therefore an integral. The derivative $U'(t)$ exists for almost all t and $0 \leq U'(t) \leq M$. Accordingly we can differentiate (1.3) formally, and since $U(0) = 0$ it follows that $u(t) = U'(t)$ satisfies (1.1) for almost all t. However, changing $u(t)$ on a set of measure zero does not affect the integral in (1.1), and since $g(t)$ is defined for all t it is seen that $u(t)$ can be defined, in a unique way, so as to satisfy (1.1) and obtain (1.3). Since the solution of (1.3) was uniquely determined it follows that the solution $u(t)$ is also unique. Obviously equations (3.7) and (3.3) define the same function $\omega(s)$, so that (3.4) holds, and (3.7) converges for $s > \sigma'$.

Finally, if $f(t)$ has only a finite number of jumps, the continuity of $u(t) - g(t)$ becomes evident upon writing (1.1) in the form

$$u(t) - g(t) = \int_0^t u(x) f(t-x)\, dx.$$

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[10] Without the assumptions of positiveness and boundedness this theorem reduces to a special case of a theorem by Paley and Wiener [21]; cf. section 1, p. 243.

4. **Asymptotic properties.** In this section we shall be concerned with the asymptotic behavior as $t \to \infty$ not of $u(t)$ itself but of the mean value $u^*(t) = \frac{1}{t}\int_0^t u(\tau)\,d\tau$. If $u(t)$ tends to the (not necessarily finite) limit C, then obviously also $u^*(t) \to C$, whereas the converse is not necessarily true. For the proof of the theorem we shall need the following obvious but useful

LEMMA: *If $u(t) \geq 0$ is a solution of* (1.1) *and if*

$$u_1(t) = e^{kt}u(t), \qquad f_1(t) = e^{kt}f(t), \qquad g_1(t) = e^{kt}g(t), \tag{4.1}$$

then $u_1(t)$ is a solution of

$$u_1(t) = g_1(t) + \int_0^t u_1(t-x)f_1(x)\,dx.$$

THEOREM 3: *Suppose that using the functions defined in Theorem 2 the integrals*

$$\int_0^\infty f(t)\,dt = a, \qquad \int_0^\infty g(t)\,dt = b, \tag{4.2}$$

are finite.

(i) *In order that*

$$u^*(t) = \frac{1}{t}\int_0^t u(\tau)\,d\tau \to C \tag{4.3}$$

as $t \to \infty$, where C is a positive constant, it is necessary and sufficient that $a = 1$, and that the moment,

$$\int_0^\infty t\,f(t)\,dt = m. \tag{4.4}$$

be finite. In this case

$$C = \frac{b}{m}. \tag{4.5}$$

(ii) *If $a < 1$ we have*

$$\int_0^\infty u(t)\,dt = \frac{b}{1-a}. \tag{4.6}$$

(iii) *If $a > 1$ let σ' be the positive root of the characteristic equation $\varphi(s) = 1$* (cf. (3.2)) *and put*[11]

$$\int_0^\infty e^{-\sigma' t}\,t\,f(t)\,dt = m_1. \tag{4.7}$$

Then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{-\sigma'\tau}u(\tau)\,d\tau = \frac{b}{m_1}. \tag{4.8}$$

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[11] (4.2) implies the finiteness of m_1.

REMARK: The case $a = 1$ corresponds in demography to a population of stationary size. In the theory of industrial replacement only the case $a = 1$ occurs; the moment m is the average lifetime of an individual. The case $a > 1$ corresponds in demography to a population in which the fertility is greater than the mortality. As is seen from (4.8), in this case the mean value of $u(t)$ increases exponentially. It is of special interest to note that in a population with $a < 1$ the integral (4.6) always converges.

PROOF: By (4.2) and (3.7)

$$\lim_{s\to+0} \phi(s) = a, \qquad \lim_{s\to+0} \gamma(s) = b. \tag{4.9}$$

If $a < 1$, it follows from (3.4) that $\lim \omega(s) = b/(1 - a)$ is finite. Since $u(t) \geq 0$ this obviously implies that (4.6) holds. This proves (ii).

If $a = 1$ and m is finite, it is readily seen that

$$\lim_{s\to+0} \frac{1 - \varphi(s)}{s} = m,$$

and hence by (3.4)

$$\lim_{s\to+0} s\omega(s) = \lim_{s\to+0} \gamma(s) \lim_{s\to+0} \frac{s}{1 - \varphi(s)} = \frac{b}{m}.$$

By a well-known Tauberian theorem for Laplace integrals of non-negative functions[12] it follows that $u^*(t) \to \frac{b}{m}$. Conversely, if (4.3) holds it is readily seen that[13]

$$\lim_{s\to+0} s\omega(s) = C,$$

which in turn implies by (3.4) and (4.9) that

$$\lim_{s\to+0} \frac{1 - \varphi(s)}{s} = \frac{b}{C}.$$

This obviously means that the moment (4.4) exists and equals b/C. This proves (i).

Finally case (iii) reduces immediately to (ii) using the above lemma with $k = -\sigma'$. This finishes the proof.

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[12] Cf. e.g. Doetsch [18], p. 208 or 210.

[13] Indeed, if (4.3) holds and if $U(t)$ is defined by (1.2), then there is a $M = M(\epsilon)$ such that $| U(t) - Ct | < M + \epsilon t$. Now

$$\varphi(s) = s \int_0^\infty e^{-st} U(t)\, dt,$$

and hence

$$s\varphi(s) - C = s^2 \int_0^\infty e^{-st}(U(t) - Ct)\, dt,$$

or

$$| s\varphi(s) - C | \leq s^2 \int_0^\infty e^{-st}(M + \epsilon t)\, dt = sM + \epsilon.$$

It may be remarked that the finiteness of the integrals (4.2) is by no means necessary for (4.3). ...

5. Closer study of asymptotic properties. In this section we shall deal almost exclusively with the most important special case, namely where

$$\int_0^\infty f(t)\,dt = 1. \tag{5.1}$$

The question has been much discussed whether in this case necessarily $u(t) \to C$ as $t \to \infty$, which statement, if true, would be a refinement of (4.3). Hadwiger [2] has constructed a rather complicated example to show that $u(t)$ does not necessarily approach a limit. Now this can also be seen directly and without any computations. Indeed, if $u(t) \to C$ and if (5.1) holds, then obviously

$$\lim_{t\to\infty} \int_0^t u(t - x)f(x)\,dx = C,$$

and hence it follows from (1.1) that $g(t) \to 0$. In order that $u(t) \to C$ it is therefore necessary that $g(t) \to 0$, and this proves the assertion. In Hadwiger's example $\limsup g(t) = \infty$, which makes his computations unnecessary.

It can be shown in a similar manner that not even the condition $g(t) \to 0$ is sufficient to ensure that $u(t) \to C$. Some restriction as to the total variation of $f(t)$ seems both necessary and natural (conditions on the existence of derivatives are not sufficient). In the following theorem we shall prove the convergence of $u(t)$ under a condition which is, though not strictly necessary, sufficiently wide to cover all cases of any possible practical interest.

THEOREM 4: *Suppose that with the functions $f(t)$ and $g(t)$ of Theorem 2*

$$\int_0^\infty f(t)\,dt = 1, \qquad \int_0^\infty g(t)\,dt = b < \infty. \tag{5.2}$$

Suppose moreover that there exists an integer $n \geqq 2$ such that the moments

$$m_k = \int_0^\infty t^k f(t)\,dt, \qquad k = 1, 2, \cdots, n, \tag{5.3}$$

are finite, and that the functions $f(t)$, $tf(t)$, $t^2f(t)$, $\cdots$, $t^{n-2}f(t)$ are of bounded total variation over $(0, \infty)$. Suppose finally that

$$\lim_{t\to\infty} t^{n-2}g(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} t^{n-2}\int_t^\infty g(x)\,dx = 0. \tag{5.4}$$

Then

$$\lim_{t\to\infty} u(t) = \frac{b}{m_1} \tag{5.5}$$

and

$$\lim_{t\to\infty} t^{n-2}\left\{u(t) - \frac{b}{m_1}\right\} = 0. \tag{5.6}$$

REMARK: As it was shown in section 4, the case where $\int_0^\infty f(t)\,dt > 1$ can readily be reduced to the above theorem by applying the lemma of section 4 with $k = \sigma'$, where σ' is the positive root of $\varphi(s) = 1$: it is only necessary to suppose that $e^{-\sigma' t}f(t)$ is of bounded total variation and that $e^{-\sigma' t}g(t) \to 0$. Obviously all moments of $e^{-\sigma' t}f(t)$ exist, so that the above theorem shows that $u_1(t) = e^{-\sigma' t}u(t)$ tends to the finite limit b'/m_1', where

$$b' = \int_0^\infty e^{-\sigma' t} g(t)\,dt, \qquad m_1' = \int_0^\infty e^{-\sigma' t} t f(t)\,dt.$$

Thus in this case and under the above assumptions $u(t) \sim \frac{b'}{m_1'} e^{\sigma' t}$, so that the renewal function increases exponentially as could be expected. If however

$$\int_0^\infty f(t)\,dt < 1,$$

$u(t)$ will in general *not* show an exponential character. If $f(t)$ is of bounded variation and has a finite moment of second order, and if $g(t) \to 0$, then it can be shown that $u(t) \to 0$. However, the lemma of section 4 can be applied only if the integral defining $\varphi(s)$ converges in some negative s-interval containing a value s' such that $\varphi(s') = 1$, and this is in general not the case.

PROOF: The proof of Theorem 4 will be based on a Tauberian theorem due to Haar[15]. With some specializations and obvious changes this theorem can be formulated as follows.

Suppose that $l(t)$ is, for $t \geq 0$, non-negative and continuous, and that the Laplace integral

$$\lambda(s) = \int_0^\infty e^{-st} l(t)\,dt \tag{5.7}$$

converges for $s > 0$. Consider $\lambda(s)$ as a function of the complex variable $s = x + iy$ and suppose that the following conditions are fulfilled:

(i) For $y \neq 0$ the function $\lambda(s)$ (which is always regular for $x > 0$) has continuous boundary values $\lambda(iy)$ as $x \to +0$, for $x \geq 0$ and $y \neq 0$

$$\lambda(s) = \frac{C}{s} + \psi(s), \tag{5.8}$$

where $\psi(iy)$ has finite derivatives $\psi'(iy), \cdots, \psi^{(r)}(iy)$ and $\psi^{(r)}(iy)$ is bounded in every finite interval;

(ii)
$$\int_{-\infty}^{+\infty} e^{ity}\lambda(x + iy)\,dy$$

converges for some fixed $x > 0$ uniformly with respect to $t \geq T > 0$;

(iii) $\lambda(x + iy) \to 0$ as $y \to \pm\infty$, uniformly with respect to $x \geq 0$;

(iv) $\lambda'(iy), \lambda''(iy), \cdots, \lambda^{(r)}(iy)$ tend to zero as $y \to \pm\infty$;

(v) The integrals

$$\int_{-\infty}^{y_1} e^{ity}\lambda^{(r)}(iy)\,dy \quad\text{and}\quad \int_{y_2}^{\infty} e^{ity}\lambda^{(r)}(iy)\,dy$$

(where $y_1 < 0$ and $y_2 > 0$ are fixed) converge uniformly with respect to $t \geq T > 0$.
Under these conditions

$$\lim t^r\{l(t) - C\} = 0. \tag{5.9}$$

Now the hypotheses of this theorem are too restrictive to be applied to the solution $u(t)$ of (1.1). We shall therefore replace (1.1) by the more special equation

$$v(t) = h(t) + \int_0^t v(t - x)f(x)\,dx, \tag{5.10}$$

where

$$h(t) = \int_0^t f(t - x)f(x)\,dx. \tag{5.11}$$

Plainly Theorem 2 can be applied to (5.10). It is also plain that $h(t)$ is bounded and non-negative and that (by (5.1))

$$\int_0^\infty h(t)\,dt = 1, \tag{5.12}$$

$$\chi(s) \equiv \int_0^\infty e^{-st}h(t)\,dt = \varphi^2(s). \tag{5.13}$$

Accordingly we have by Theorem 2

$$\zeta(s) \equiv \int_0^\infty e^{-st}v(t)\,dt = \frac{\varphi^2(s)}{1 - \varphi(s)}. \tag{5.14}$$

We shall first verify that $\zeta(s)$ satisfies the conditions of Haar's theorem with $r = n - 2$. For this purpose we write

$$f(t) = f_1(t) - f_2(t), \tag{5.15}$$

where $f_1(t)$ and $f_2(t)$ are non-decreasing and non-negative functions which are, by assumption, bounded:

$$0 \leq f_1(t) < M, \qquad 0 \leq f_2(t) < M. \tag{5.16}$$

(a) We show that $v(t)$ is continuous. Now by Theorem 2 the solution $v(t)$ of (5.10) is certainly continuous if $h(t)$ is continuous; however, that $h(t)$ is continuous follows directly from (5.11) and the fact that the functions

$$\int_0^t f_1(t - x)f(x)\,dx \quad\text{and}\quad \int_0^t f_2(t - x)f(x)\,dx$$

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[15] Haar [20] or Doetsch [18], p. 269.

are continuous.

(b) In view of (5.1) the function $\varphi(s)$ exists for $x = \Re(s) \geq 0$. Obviously $|\varphi(x + iy)| < 1$ for $x > 0$. Now

$$1 - \varphi(iy) = \int_0^\infty (1 - e^{-iyt})f(t)\,dt$$

$$= \int_0^\infty (1 - \cos yt)f(t)\,dt + i \int_0^\infty \sin yt\cdot f(t)\,dt,$$

and, since $1 - \cos yt \geq 0$ and $f(t) \geq 0$, the equality $\varphi(iy) = 1$ for $y \neq 0$ would imply that $f(t) = 0$ except on a set of measure zero. It is therefore seen that $\varphi(x + iy) \neq 1$ for all $x > 0$ and for $x = 0$, $y \neq 0$.

It follows furthermore from (5.3) that for $k = 1, \cdots, n$ and $x \geq 0$ the derivatives

$$\varphi^{(k)}(s) = \int_0^\infty (-t)^k e^{-st} f(t)\,dt$$

exist and that

$$\lim_{x\to+0} \varphi^{(k)}(x + iy) = \varphi^{(k)}(iy).$$

Finally, it is readily seen that in the neighborhood of $y = 0$ we have

$$\varphi(iy) = \int_0^\infty e^{-iyt} f(t)\,dt$$

$$= 1 - m_1 iy + \frac{m_2}{2}(iy)^2 - + \cdots \tag{5.17}$$

$$+ (-1)^{n-1}\frac{m_{n-1}}{(n-1)!}(iy)^{n-1} + O(|y|^n).$$

(c) From what was said under (b) it follows by (5.14) that $\zeta(s)$ is regular for $x > 0$, and that $\zeta(s), \zeta'(s), \cdots, \zeta^{(n)}(s)$ approach continuous boundary values as $s = x + iy$ approaches a point of the imaginary axis other than the origin. Now put

$$\psi(s) = \frac{\varphi^2(s)}{1 - \varphi(s)} - \frac{1}{m_1 s}, \tag{5.18}$$

so that by (5.14)

$$\zeta(s) = \frac{1}{m_1 s} + \psi(s). \tag{5.19}$$

For $x > 0$ and $x = 0$, $y \neq 0$ the function $\psi(x + iy)$ is obviously continuous; the derivatives $\psi'(iy), \cdots, \psi^{(n)}(iy)$ exist. To investigate the behavior of $\psi(iy)$ in the neighborhood of $y = 0$ put

$$P(y) = m_1 - \frac{m_2}{2}(iy) + - \cdots - (-1)^{n-1}\frac{m_{n-1}}{(n-1)!}(iy)^{n-2}. \tag{5.20}$$

By (5.17), (5.18) and (5.20)

$$\psi(iy) = \left[\frac{\{1 - iyP(y)\}^2}{P(y)} - \frac{1}{m_1}\right]\frac{1}{iy} + O(|y|^{n-2}). \tag{5.21}$$

Now the expression in brackets represents an analytic function of y which vanishes at $y = 0$. Hence $\psi(iy) = \mathfrak{P}(y) + O(|y|^{n-2})$, where $\mathfrak{P}(y)$ denotes a power series. It follows that the derivatives $\psi'(iy), \cdots, \psi^{(n-2)}(iy)$ exist for all real y (including $y = 0$) and are bounded for sufficiently small $|y|$: since they are continuous functions they are bounded in every finite interval.

(d). Next we show that there exists a constant $A > 0$ such that for sufficiently large $|y|$

$$(5.22) \qquad | \varphi(x + iy) | < \frac{A}{|y|}$$

uniformly in $x \geq 0$. By (5.15)

$$(5.23) \qquad \varphi(s) = \int_0^\infty \{\cos yt - i \sin yt\} e^{-xt} \{f_1(t) - f_2(t)\} \, dt.$$

Now $f_1(t)$ is non-decreasing and accordingly by the second mean-value theorem we have for any $T > 0$ and y

$$\int_0^T \cos yt \cdot f_1(t) \, dt = f_1(T) \int_\tau^T \cos yt \, dt = f_1(T) \frac{\sin Ty - \sin \tau y}{y},$$

where τ is some value between 0 and T (depending, of course, on y; at points of discontinuity, $f_1(T)$ should be replaced by $\lim_{t \to T-0} f_1(t)$). Hence by (5.16)

$$\left| \int_0^\infty \cos yt \cdot e^{-xt} \cdot f_1(t) \, dt \right| < \frac{2M}{|y|}.$$

Treating the other terms in (5.23) in a like manner, (5.22) follows.

Combining (5.22) with (5.14) it is seen that for sufficiently large $|y|$

$$| \zeta(s) | < \frac{2A^2}{y^2}$$

uniformly in $x \geq 0$. This shows that the assumptions (ii) and (iii) of Haar's theorem are satisfied for $\lambda(s) = \zeta(s)$. In order to prove that also conditions (iv) and (v) are satisfied it suffices to notice that the proof of (5.22) used only the fact that $f(t)$ is of bounded total variation. Now $\varphi^{(k)}(s)$ is the Laplace transform of $(-t)^k f(t)$, and, since $t^k f(t)$ is of bounded total variation for $k \leq n - 2$, it follows that

$$| \varphi^{(k)}(s) | = O(|y|^{-1}), \qquad k = 1, 2, \cdots, n - 2,$$

for sufficiently large $|y|$, uniformly in $x \geq 0$. Differentiating (5.14) k times it is also seen that

$$| \zeta^{(k)}(s) | = O(|y|^{-2}), \qquad k = 1, 2, \cdots, n - 2,$$

as $y \to +\infty$, uniformly with respect to $x \geq 0$.

This enumeration shows that $v(s) = l(t)$ and $\lambda(s) = \zeta(s)$ satisfy all hypotheses of Haar's theorem with $r = n - 2$ and $C = 1/m_1$. Hence

$$(5.24) \qquad \lim_{t \to \infty} t^{k-2} \left\{ v(t) - \frac{1}{m_1} \right\} = 0.$$

Returning now to (5.14) we get

$$\omega(s) = \gamma(s) + \gamma(s)\varphi(s) + \gamma(s)\zeta(s),$$

or, by the uniqueness property of Laplace integrals,

$$\begin{aligned} u(t) &= g(t) + \int_0^t g(x)f(t-x)\,dx + \int_0^t g(x)v(t-x)\,dx \\ &= g(t) + u_1(t) + u_2(t) \end{aligned} \tag{5.25}$$

(which relation can also be checked directly using (5.10)). Let us begin with the last term. We have by (5.2)

$$u_2(t) - \frac{b}{m_1} \equiv \int_0^t g(t-x)\left\{v(x) - \frac{1}{m_1}\right\}dx,$$

and hence

$$\begin{aligned} t^{n-2}\left|u_2(t) - \frac{b}{m_1}\right| \leq 2^{n-2} &\int_{t/2}^t g(t-x)x^{n-2}\left|v(x) - \frac{1}{m_1}\right|dx \\ &+ t^{n-2}\int_{t/2}^t g(y)\left|v(t-y) - \frac{1}{m_1}\right|dy. \end{aligned}$$

If t is sufficiently large we have by (5.24) in the first integral $x^{n-2}\left|v(x) - \frac{1}{m_1}\right| < \epsilon$.

In the second integral $v(t-y) - \frac{1}{m_1}$ is bounded, and hence by (5.4)

$$\lim_{t\to\infty} t^{n-2}\left|u_2(t) - \frac{b}{m_1}\right| = 0.$$

The same argument applies (even with some simplifications) also to the second term in (5.24); it follows that

$$\lim t^{n-2}u_1(t) = 0,$$

whilst $t^{n-2}g(t) \to 0$ by assumption (5.4). Now the assertion (5.6) of our theorem follows in view of (5.25) if the last three relationships are added. This finishes the proof of Theorem 4.

It seems that the solution $u(t)$ is generally supposed to oscillate around its limit b/m_1 as $t \to \infty$. It goes without saying that such a behavior is a priori more likely than a monotone character. It should, however, be noticed that there is no reason whatsoever to suppose that $u(t)$ *always* oscillates around its limit. Again no computation is necessary to see this, as shown by the following

EXAMPLE: Differentiating (1.1) formally we get

$$u'(t) = g'(t) + g(0)f(t) + \int_0^t u'(t-x)f(x)\,dx,$$

which shows that, if $g(t)$ and $f(t)$ are sufficiently regular, $u'(t)$ satisfies an integral equation of the same type as $u(t)$. Thus if

$$g'(t) + g(0)f(t) \geq 0$$

for all t, we shall have $u'(t) \geq 0$, and $u(t)$ is a monotone function. In particular, if $g'(t) + g(0)f(t) = 0$, then $u'(t) = 0$ and $u(t) = \text{const}$. For example, let $f(t) = g(t) = e^{-t}$. Then $\varphi(s) = \gamma(s) = 1/(s + 1)$ and hence $\omega(s) = 1/s$, which is the Laplace transform of $u(t) = 1$. It is also seen directly that $u(t) \equiv 1$ is the solution. We have however the following

THEOREM 5[16]: *If the functions $f(t)$ and $g(t)$ of Theorem 4 vanish identically for $t \geq T > 0$, then the solution $u(t)$ of (1.1) oscillates around its limit b/m as $t \to \infty$.*

PROOF: For $t \geq T$ equation (1.1) reduces to

$$u(t) = \int_{t-T}^{t} u(t - x)f(x)\,dx,$$

and since $\int_{t-T}^{t} f(x)\,dx = 1$ it follows that the maxima of $u(t)$ in the intervals $nT < t < (n + 1)T$ form, for sufficiently large integers n, a non-increasing sequence. Similarly the corresponding minima do not decrease. Since $u(t) \to b/m_1$, by Theorem 4, it follows that the minima do not exceed b/m_1 and the maxima are not smaller than b/m_1.

6. **On Lotka's method.** Probably the most widely used method for treating equation (1.1) in connection with problems of the renewal theory is Lotka's method. As a matter of fact this method consists of two independent parts. The first step aims at obtaining the exact solution of (1.1) in the form of a series of exponential terms (this is achieved by an adaptation of a method which was used by P. Herz and Herglotz for other purposes. The second part of Lotka's theory consists of devices for a convenient approximative computation of the first few terms of the series. While restricting ourselves formally to Lotka's theory, it will be seen that some of the following remarks apply equally to other methods.

Lotka's method rests essentially on the fundamental assumption that the characteristic equation

$$\varphi(s) = 1 \tag{6.1}$$

has infinitely many distinct simple[17] roots $s_0, s_1, \cdots$, and that the solution $u(t)$ of (1.1) can be expanded into a series

$$u(t) = \sum_k A_k e^{s_k t} \tag{6.2}$$

where the A_k are complex constants. The argument usually rests on an assumed completeness-property of the roots. Thus, starting from (2.4) it is required that

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[16] Under some slight additional hypotheses and with quite different methods this theorem was proved by Richter [16].

[17] Hadwiger [3] objected to the assumption that all roots of (6.1) be simple. The modifications which are necessary to cover the case of multiple roots also will be indicated below.

(6.2) reduces to $h(t)$ for $t < 0$; in other words, that an arbitrarily prescribed function $h(x)$ be, for $x < 0$, respresentable in the form

$$h(x) = \sum_k A_k e^{s_k x} \qquad (x < 0). \tag{6.3}$$

In practice we are, of course, usually not concerned with $h(t)$ but with $g(t)$ (cf. (2.5)), and according to Lotka's theory the coefficients A_k of the solution (6.2) of (1.1) can be computed directly from $g(t)$ in a way similar to the computation of the Fourier coefficients.

Lotka's method is known to lead to correct results in many cases and also to have distinct computational merits. On the other hand it seems to require a safer justification, since its fundamental assumptions are rarely realized. Thus clearly an arbitrary function $h(x)$ cannot be represented in the form (6.3): to see this it suffices to note that (6.1) frequently has only a finite number of roots (cf. also below). It should also be noted that, the series (6.3) having regularity properties as are assumed in Lotka's theory, any function representable in the form (6.3) is necessarily a solution of the integral equation (2.4), whereas the theory requires us to construct a solution $u(t)$ which reduces to an *arbitrarily* prescribed function $h(t)$ for $t < 0$, (which frequently is an empirical function, determined by observations). Nevertheless, it is possible to give sound foundations to Lotka's method so that it can be used (with some essential limitations and modifications) sometimes even in cases for which it originally was not intended. For this purpose it turns out to be necessary that all considerations be based on the more general equation (1.1), instead of (2.4) (cf. also section 2).

Before proceeding it is necessary to make clear *what is really meant by a root of* (6.1). The function $\varphi(s)$ is defined by (3.2), and the integral will in general converge only for s-values situated in the half-plane $\Re(s) > \sigma$. Usually only roots situated in this half-plane are considered . It is also argued that $\varphi(s)$ is, for real s, a monotone function, so that (6.1) has at most one real root: accordingly the terms of (6.2) are called "oscillatory components." However, the function $\varphi(s)$ can usually be defined by analytic continuation even outside the half-plane $\Re(s) > \sigma$, and, if this is done, (6.1) will in general also have roots in the half-plane $\Re(s) < \sigma$. It will be seen in the sequel that these roots play exactly the same role for the solution $u(t)$ as the other ones, and that the applicability of Lotka's method depends on the behavior of $\varphi(s)$ in the entire complex s-plane. ...

From now on we shall consistently denote by $\varphi(s)$ the function defined by the integral (3.4) and by the usual process of analytic continuation; accordingly we shall take into consideration *all* roots of (6.1). The main limitation of Lotka's theory can then be formulated in the following way: Lotka's method depends only on the function $g(t)$ and on the roots of (6.1). Now two different functions $f(t)$ can lead to characteristic equations having the same roots. Lotka's method would be applicable to both only if the corresponding two integral equations (1.1) had the same solution $u(t)$. This, however, is not necessarily the case. Thus, if Lotka's method is applied, and if all computations are correctly performed, and if the resulting series for $u(t)$ converges uniformly, there is no possibility of telling which equation is really satisfied by the resulting $u(t)$:

it can happen that one has unwittingly solved some unknown equation of type (1.1) which, by chance, leads to a characteristic equation having the same roots as the characteristic equation of the integral equation with which one was really concerned. ...

These preparatory remarks enable us to formulate rigorous conditions for the existence of an expansion of type (6.2). The following theorem shows the limits of Lotka's method, but at the same time it also represents an extension of it. In the formulation of the theorem we have considered only the case of absolute convergence of (6.2). This was done to avoid complications lacking any practical significance whatsoever. The conditions can, of course, be relaxed along customary lines.

THEOREM 6: *In order that the solution $u(t)$ of Theorem 2 be representable in form (6.2), where the series converges absolutely for $t \geq 0$ and where the s_k denote the roots of the characteristic equation*[21] *(6.1), it is necessary and sufficient that the Laplace transform $\omega(s)$ admit an expansion*

$$\omega(s) \equiv \frac{\gamma(s)}{1-\varphi(s)} = \sum \frac{A_k}{s-s_k} \tag{6.6}$$

and that $\Sigma | A_k |$ converges absolutely. The coefficients A_k are determined by

$$A_k = -\frac{\gamma(s_k)}{\varphi'(s_k)}. \tag{6.7}$$

In particular, it is necessary that $\omega(s)$ be a one-valued function.

PROOF: All roots s_k of (6.1) satisfy the inequality $\Re(s_k) \leq \sigma'$, where σ' was defined in Theorem 2. It is therefore readily seen that in case $\Sigma | A_k |$ converges, the Laplace transform of (6.2) can be computed for sufficiently large positive s-values by termwise integration so that (6.6) certainly holds for sufficiently large positive s. Now with $\Sigma | A_k |$ converging, (6.6) defines $\omega(s)$ uniquely for all complex s (with singularities at the points s_k and the points of accumulation of s_k, if any). Since the analytic continuation is unique, it follows that (6.6) holds for all s. The series $\Sigma | A_k |$ must, of course, converge if (6.2) is to converge absolutely for $t = 0$, and this proves the necessity of our condition. Conversely, if $\omega(s) = \dfrac{\gamma(s)}{1-\varphi(s)}$ is given by (6.6), and if $\Sigma | A_k |$ converges, then $\omega(s)$ is the Laplace transform of a function $u(t)$ defined by (6.2). Since the Laplace transform is unique, $u(t)$ is the solution of (1.1) by Theorem 2. The series (6.2) converges absolutely for $t \geq 0$ since $| A_k e^{s_k t} | \leq | A_k | e^{\sigma' t}$. Finally (6.7) follows directly from (6.6).

It is interesting to compare (6.7) with formulas (50) and (56) of Lotka's paper [8]. Lotka considers the special case $g(t) = f(t)$; in this case $\gamma(s_k) = \varphi(s_k) = 1$, and (6.7) reduces to $A_k = -\dfrac{1}{\varphi'(s_k)}$. If s_k lies in the domain of con-

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[21] The number of roots may be finite or infinite. It should also be noted that it is not required that $s_k \to \infty$. If the s_k have a point of accumulation, $\omega(s)$ will have an essential singularity. That this actually can happen can be shown by examples.

vergence of the integral $\varphi(s) = \int_0^\infty e^{-st} f(t)\, dt$, that is, if $\Re(s_k) \geq \sigma$ then

$$(6.8) \qquad \frac{1}{A_k} = \int_0^\infty e^{-st} t f(t)\, dt,$$

in accordance with Lotka's result. However, (6.8) becomes meaningless for the roots with $\Re(s_k) < \sigma$, whereas (6.7) is applicable in all cases.

Theorem 6 can easily be generalized to the case where the *characteristic equation has multiple roots.* The expansion (6.6) (which reduces to the customary expansion into partial fractions whenever $\omega(s)$ is meromorphic) is to be replaced by

$$(6.9) \qquad \omega(s) = \sum_k \left\{ \frac{A_k^{(1)}}{s - s_k} + \frac{A_k^{(2)}}{(s - s_k)^2} + \cdots + \frac{A_k^{(m_k)}}{(s - s_k)^{m_k}} \right\},$$

where m_k is the multiplicity of the root s_k . This leads us formally to an expansion

$$(6.10) \qquad u(t) = \sum_k e^{s_k t} \left\{ A_k^{(1)} + A_k^{(2)} \frac{t}{1!} + \cdots + A_k^{(m_k)} \frac{t^{m_k - 1}}{(m_k - 1)!} \right\},$$

which now replaces (6.2). Generalizing Theorem 6 it is easy to formulate some simple conditions under which (6.1) will really represent a solution of (1.1). Other conditions which ensure that (6.9) is the transform of (6.10) are known from the general theory of Laplace transforms; such conditions usually use only function-theoretical properties of (6.9) and are applicable in particular when $\omega(s)$ is meromorphic. We mention in particular a theorem of Churchill [17] which can be used for our purposes.

7. **On the practical computation of the solution.** There are at hand two main methods for the practical computation of the solution of (1.1). One of them has been developed by Lotka and consists of an approximate computation of a few coefficients in the series (6.2). The other method uses an expansion

$$(7.1) \qquad u(t) = \sum_{n=0}^{\infty} u_n(t),$$

where $u_n(t)$ represents the contribution of the nth "generation" and is defined by x

$$(7.2) \qquad u_0(t) = g(t), \qquad u_{n+1}(t) = \int_0^t u_n(t - x) f(x)\, dx.$$

Now the Laplace transform of $u_{n+1}(t)$ is $\gamma(s)\varphi^n(s)$, and hence (7.2) corresponds to the expansion

$$(7.3) \qquad \omega(s) = \frac{\gamma(s)}{1 - \varphi(s)} = \gamma(s) \sum_{n=0}^{\infty} \varphi^n(s).$$

In practice the functions $g(t)$ and $f(t)$ are usually not known exactly. Frequently their values are obtained from some statistical material, so that only their integrals over some time units, e.g. years, are actually known or, in other words, only the values

$$f_n = \frac{1}{\delta}\int_{n\delta}^{(n+1)\delta} f(t)\,dt, \qquad g_n = \frac{1}{\delta}\int_{n\delta}^{(n+1)\delta} g(t)\,dt, \tag{7.4}$$

are given, where $\delta > 0$ is a given constant. Ordinarily in such cases some theoretical forms (e.g. Pearson curves) are fitted to the empirical data and equation (1.1) is solved with these theoretical functions. Now such a procedure is sometimes not only very troublesome, but also somewhat arbitrary. Consider for example the limit of $u(t)$ as $t \to \infty$; this asymptotic value is the main point of interest of the theory and all practical computations. However, as has been shown above, this limit depends only on the moments of the first two orders of $f(t)$ and $g(t)$, and, unless the fitting is done by the method of moments, the resulting value will depend on the special procedure of fitting. Accordingly it will sometimes happen that it is of advantage to use the empirical material as it is, and this can, at least in principle, always be done.

If only the values (7.4) are used it is natural to consider $f(t)$ and $g(t)$ as step-functions defined by

$$\left.\begin{aligned} f(t) &= f_n, \\ g(t) &= g_n, \end{aligned}\right\} \quad \text{for} \quad n\delta \le t < (n+1)\delta. \tag{7.5}$$

In practice only a finite number among the f_n and g_n will be different from zero: accordingly the Laplace transforms $\gamma(s)$ and $\varphi(s)$ reduce to trigonometrical polynomials, so that the analytic study of $\omega(s) = \dfrac{\gamma(s)}{1 - \varphi(s)}$ becomes particularly simple. Lotka's method can be applied directly in this case.

For a convenient computation of (7.1) it is better to return to the more general equation (1.3), instead of (1.1). The summatory functions $F(t)$ and $G(t)$ should not be defined by (1.2) in this case, but simply by

$$F(t) = \sum_{n=0}^{\left[\frac{t}{\delta}\right]} f_n, \qquad G(t) = \sum_{n=0}^{\left[\frac{t}{\delta}\right]} g_n. \tag{7.6}$$

It is readily seen that the solution $U(t)$ of (1.3) can be written in the form $U(t) = \sum\limits_{n=0}^{\infty} U_n(t)$, where

$$U_0(t) = G(t), \qquad U_{n+1}(t) = \int_0^t U_n(t - x)\,dF(x);$$

in our case $U_n(t)$ will again be a step-function with jumps at the points $k\delta$, the corresponding saltus being

$$u_0^{(k)} = g_k, \qquad u_{n+1}^{(k)} = \sum_{r=0}^{k} u_n^{(k-r)} f_r.$$

Thus we arrive at exactly the same result as would have been obtained if the integrals (7.2) had been computed, starting from (7.4), by the ordinary methods for numerical integration of tabulated functions. It is of interest to note that this method of approximate evaluation of the integrals (7.2) leads to the *exact values of the renewal function* of a population where all changes occur in a discontinuous way at the end of time intervals of length δ in such a way that each change equals the mean value of the changes of the given population over the corresponding time interval.

REFERENCES

I. Papers on the integral equation of renewal theory

Note: Lotka's paper [8] contains a list of 74 papers on the subject published before 1939. The following list is to bring Lotka's list up to June 1941; however no claims to completeness are made.

[1] A. W. BROWN, A note on the use of a Pearson type III function in renewal theory," *Annals of Math. Stat.* Vol. 11 (1940), pp. 448–453.

[2] H. HADWIGER, "Zur Frage des Beharrungszustandes bei kontinuierlich sich erneuernden Gesamtheiten," *Archiv f. mathem. Wirtschafts- und Sozialforschung*, Vol. 5 (1939), pp. 32–34.

[3] H. HADWIGER, "Über die Integralgleichung der Bevölkerungstheorie," *Mitteilungen Verein. schweizer Versicherungsmathematiker* (*Bull. Assoc. Actuaires suisses*), Vol. 38 (1939), pp. 1–14.

[4] H. HADWIGER, "Eine analytische Reproduktionsfunktion für biologische Gesamtheiten," *Skand. Aktuarietidskrift* (1940), pp. 101–113.

[5] H. HADWIGER, "Natürliche Ausscheidefunktionen für Gesamtheiten und die Lösung der Erneuerungsgleichung," *Mitteilungen Verein. Schweiz. Versich.-Math.*, Vol. 40 (1940), pp. 31–39.

[6] H. HADWIGER and W. RUCHTI, "Über eine spezielle Klasse analytischer Geburtenfunktionen," *Metron*, Vol. 13 (1939), No. 4, pp. 17–26.

[7] A. LINDER, "Die Vermehrungsrate der stabilen Bevölkerung," *Archiv f. mathem. Wirtschafts- und Sozialforschung*, Vol. 4 (1938), pp. 136–156.

[8] A. LOTKA, "A contribution to the theory of self-renewing aggregates, with special reference to industrial replacement," *Annals of Math. Stat.*, Vol. 10 (1939), pp. 1–25.

[9] A. LOTKA, "On an integral equation in population analysis," *Annals of Math. Stat.*, Vol. 10 (1939), pp. 144–161.

[10] A. LOTKA, "Théorie analytique des associations biologiques II," *Actualités Scientifiques No. 780*, Paris, 1939.

[11] A. LOTKA, "The theory of industrial replacement," *Skand. Aktuarietidskrift* (1940), pp. 1–14.

[12] A. LOTKA, "Sur une équation intégrale de l'analyse démographique et industrielle," *Mitt. Verein. Schweiz. Versich.-Math.*, Vol. 40 (1940), pp. 1–16.

[13] H. MÜNZNER, "Die Erneuerung von Gesamtheiten," *Archiv f. math. Wirtschafts- u. Sozialforschung*, Vol. 4 (1938).

[14] G. A. D. PREINREICH, "The theory of industrial replacement," *Skand. Aktuarietidskrift* (1939), pp. 1-19.

[15] E. C. RHODES, "Population mathematics, I, II, III," *Roy. Stat. Soc. Jour.*, Vol. 103 (1940), pp. 61-89, 218-245, 362-387.

[16] H. RICHTER, "Die Konvergenz der Erneuerungsfunktion," *Blätter f. Versicherungsmathematik*, Vol. 5 (1940), pp. 21–35.

[16a] H. HADWIGER, "Über eine Funktionalgleichung der Bevölkerungstheorie und eine spezielle Klasse analytischer Lösungen," *Bl. f. Versicherungsmathematik*, Vol. 5 (1941), pp. 181-188.

[16b] G. A. D. PRIENREICH, "The present status of renewal theory," Waverly Press, Baltimore (1940).

II. Other papers quoted

[17] R. V. Churchill, "The inversion of the Laplace transformation by a direct expansion in series and its application to boundary-value problems," *Math. Zeits.*, Vol. 42 (1937), pp. 567–579.

[18] G. Doetsch, *Theorie und Anwendung der Laplace Transformation.* J. Springer, Berlin, 1937.

[19] W. Feller, "Completely monotone functions and sequences," *Duke Math. Jour.*, Vol. 5 (1939), pp. 661–674.

[20] A. Haar, "Über asymptotische Entwicklungen von Funktionen," *Math. Ann.*, Vol. 96 (1927), pp. 69–107.

[21] R. E. A. C. Paley and N. Wiener, "Notes on the theory and application of Fourier transforms, VII. On the Volterra equation," *Amer. Math. Soc. Trans.*, Vol. 35 (1933), pp. 785–791.

17. A New Method for Calculating Lotka's *r*— The Intrinsic Rate of Growth in a Stable Population

ANSLEY J. COALE (1957)

Population Studies 11: 92—94.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_17,

The determination of the intrinsic rate of increase in a stable population involves finding the real root r_o of the following integral equation:

$$\int_0^{\omega} e^{-ra}p(a)m(a)da = 1 \tag{1}$$

where r is the intrinsic rate of growth, $p(a)$ is the probability of surviving to age a, and $m(a)$ is the probability of giving birth to a female child at age a.

Lotka uses the first three terms of a Taylor expansion to obtain a quadratic equation in r yielding a very close approximation to the exact root.

The equation is:[2]

$$\frac{1}{2}\beta r^2 + \alpha r - \log_e R_0 = 0 \tag{2}$$

$$\text{where } \alpha = \frac{R_1}{R_0}$$

$$\beta = \alpha^2 - \frac{R_2}{R_0}$$

$$\text{and } R_n = \int_0^{\omega} a^n p(a)m(a)da$$

The calculation required is fairly laborious. Typically one calculates R_0, R_1, and R_2 by dividing the childbearing age interval into five-year groups and approximating the integral expression for R_n by a finite sum. Three such sums must be calculated, and then the quadratic equation (2) solved.

Lotka points out that the correctness of his solution can be verified by substituting the value of r in Equation (1), and approximating the integral by forming a summation by five-year age groups. The summation should, of course, have a value of unity. This method of verification suggests a method of finding a root for (1) that is somewhat less laborious and at least as precise.

In brief, the simpler computation involves substituting a rough approximation for r in Equation (1), ascertaining how much the resulting integral *diverges* from unity, and using the amount of divergence to determine how the approximate value of r must be adjusted.

The rough approximation can be obtained from the relation

$$R_0 = e^{rT} \tag{3}$$

(where T is the mean length of generation), and the fact that T normally takes on only a limited range of values, with an average of about 29 years. An assumption that the value of T is 29 years gives a first approximation of r–as $\left(\frac{\log_e R_0}{29}\right)$

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[1] The author proposed in an earlier note a still simpler method of calculating approximate intrinsic rates. This earlier method produced an approximation that was adequate for most purposes but not nearly as close as that produced by the procedure here outlined. *Cf.* Coale, A. J., "The Calculation of Approximate Intrinsic Rates", *Population Index*, vol. xxi, April 1955, pp. 94–97.

[2] Dublin, Louis I., and Lotka, Alfred J., "On the True Rate of Natural Increase", *Journal of the American Statistical Association*, vol. xx, no. 151, September 1925.

that may be designated r_1. Substituting r_1 in (1) yields an integral slightly larger or smaller than unity—smaller if r_1 is greater than the correct value of r.

$$\int_0^{\omega} e^{-r_1 a}\, p(a)m(a)da = 1 + \delta \tag{4}$$

The problem is to infer from δ—the divergence of the integral from unity—the difference between r_1 and the root r_0 of (1). Again following Lotka, let $y = \int_0^{\omega} e^{-ra}\, p(a)m(a)da$, then

$$\frac{dy}{dr} = -\int_0^{\omega} a\, e^{-ra}\, p(a)m(a)da \tag{5}$$

$= -Ay$, where A is the average age of childbearing in the stable population.

But since $y = 1$ when $r = r_0$ (the root of Equation (1)),

$$\left(\frac{dy}{dr}\right)_{r=r_0} = -A \tag{6}$$

Hence for values of r differing only slightly from r_0, the integral in (1) will differ from unity by an amount $-A\,\Delta r$, where Δr is the error in r. Conversely, the indicated adjustment in r in order to yield a value of unity for the integral is

$$\Delta r = -\frac{\delta}{A} \tag{7}$$

The mean age of childbearing (A) differs from the mean length of generation (T) by an amount that depends on r,[1] but that in no case exceeds 0·6 of a year, or about 2% of T. Hence 29 years is also a good approximation for A.

Thus r can be estimated by adding $\frac{\delta}{29}$ (with due regard for the sign of δ) to the first approximation, r_1. Or, by taking note of the fact that the error in assuming that A equals 29 years is nearly the same as in assuming T is 29 years, and that the relative error in the latter is $\frac{\Delta T}{T} \cong -\frac{\Delta r}{r}$, a more exact determination is obtained by the following expression :

$$r = r_1 + \frac{\delta}{29 - \frac{\delta}{r_1}} \tag{8}$$

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[1] The difference between A and T is (neglecting higher powers of r) $\frac{1}{2}\beta r$, with β defined as $\left(\frac{R_1}{R_0}\right)^2 - \frac{R_2}{R_0}$ (*cf.* page 92). β lies between about 35 and 50 years, with a typical value of 42. If r is as great as 3%, T and A will differ by about 0·6 years. (See Dublin and Lotka, *op. cit.*)

As a check of the validity of the technique here described, it was used to recompute a number of values of r already computed by Lotka's procedure. The agreement is excellent. In fact, if the mean length of generation is within about a year of 29 years (as it almost always is), (8) should usually give a closer approximation than the algebraic solution of (2). However, when the intrinsic rate of increase for mid-nineteenth century Sweden is computed, Lotka's procedure yields a value of 9·65 per thousand and (8) gives a value of 9·68. In this instance, the mean length of generation is about 32·3 years, which is unusually long. The value of r that brings the integral in (1) within one part in ten thousand of unity is 9·655 per thousand. But even in this instance (where r_1 is a relatively poor first approximation) the technique outlined here gives an error of only 2·5 per 100,000 in r, or of less than three-tenths of 1% of the correct value.

This method of computation has two virtues to recommend it—it is less laborious than the solution of the quadratic equation while achieving about the same precision, and it has the pedagogical advantage of staying close to fundamental ideas of stable population theory.

Equation (1) follows from the fact that the stable age distribution is

$$c\,(a)da = be^{-ra}\,p(a)da \qquad (9)$$

where $c(a)da$ is the fraction of the population in the interval between a and $a+d\,a$, and b is the intrinsic birth rate; and the further fact that the number of births per unit of time must equal the product of the number of women at each childbearing age and the age-specific fertility rate for each age; or

$$b = \frac{B}{N} = \int_0^{\omega} c\,(a)m(a)da \qquad (10)$$

Equation (3) relates the rate of increase *per generation* (the net reproduction rate) in the stable population to the annual rate of increase. No further concepts (such as Thielean semi-invariants) need be introduced.

18. The Fundamental Theorem of Natural Selection

R.A. FISHER (1958 (1930))

From *The Genetical Theory of Natural Selection*, pp. 25—30. New York: Dover.

The total reproductive value, due to Fisher, is of great importance in stable population theory. It is defined as

$$V=\int_{0}^{\infty} P(x)\,v(x)\,dx$$

$$=\int_{t=0}^{\beta}\int_{x=0}^{\beta-t} e^{-rt}P(x)\frac{l(t+x)}{l(x)}m(t+x)\,dx\,dt\,,$$

where $P(x)$ is the observed population between ages x and $x+dx$ at time 0, $v(x)$ is the reproductive value, and standard notation r, $m(x)$ is used in place of Fisher's m and b_x for the intrinsic growth rate and probability of giving birth in the age interval x to $x+dx$, respectively. The fraction $l(t+x)/l(x)$ is the probability that an individual age x at time 0 survives t years to his $(t+x)$th birthday. In words, V is the backward projection of fertility accruing to an observed population to find the size of a birth cohort that would be reproductively equivalent to it. Its application is shown in Feller (1941, paper 16 above).

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The Malthusian parameter of population increase

If we combine the two tables giving the rates of death and reproduction, we may, still speaking in terms of human populations, at once calculate the expectation of offspring of the newly-born child. For the expectation of offspring in each element of age dx is $l_x b_x dx$, and the sum of these elements over the whole of life will be the total expectation of offspring. In mathematical terms this is

$$\int_0^\infty l_x b_x dx,$$

where the integral is extended from zero, at birth, to infinity, to cover every possible age at which reproduction might conceivably take place. If at any age reproduction ceases absolutely, b_x will thereafter be zero and so give automatically the effect of a terminating integral.

The expectation of offspring determines whether in the population concerned the reproductive rates are more or less than sufficient to balance the existing death rates. If its value is less than unity the reproductive rates are insufficient to maintain a stationary population, in the sense that any population which constantly maintained the death and reproduction rates in question would, apart from temporary fluctuations, certainly ultimately decline in numbers at a calculable rate. Equally, if it is greater than unity, the population biologically speaking is more than holding its own, although the actual number of heads to be counted may be temporarily decreasing.

This consequence will appear most clearly in its quantitative aspect if we note that corresponding to any system of rates of death and reproduction, there is only one possible constitution of the population in respect of age, which will remain unchanged under the action of this system. For if the age distribution remains unchanged the relative rate of increase or decrease of numbers at all ages must be the same; let us represent the relative rate of increase by m; which will also represent a decrease if m is negative. Then, owing to the constant rates of reproduction, the rate at which births are occurring at any epoch will increase proportionately to e^{mt}. At any particular epoch, for which we may take $t=0$, the rate at which births were occurring x years ago will be proportional to e^{-mx}, and this is the rate at which births were occurring at the time persons now of age x were being born. The number of persons in the infinitesimal age interval

dx will therefore be $e^{-mx}l_x dx$, for of those born only the fraction l_x survive to this age. The age distribution is therefore determinate if the number m is uniquely determined. But knowing the numbers living at each age, and the reproductive rates at each age, the rate at which births are now occurring can be calculated, and this can be equated to the known rate of births appropriate to $t=0$. In fact, the contribution to the total rate, of persons in the age interval dx, must be $e^{-mx}l_x b_x dx$, and the aggregate for all ages must be

$$\int_0^\infty e^{-mx}l_x b_x dx,$$

which, when equated to unity, supplies an equation for m, of which one and only one real solution exists. Since e^{-mx} is less than unity for all values of x, if m is positive, and is greater than unity for all values of x, if m is negative, it is evident that the value of m, which reduces the integral above expressed to unity, must be positive if the expectation of offspring exceeds unity, and must be negative if it falls short of unity.

The number m which satisfies this equation is thus implicit in any given system of rates of death and reproduction, and measures the relative rate of increase or decrease of a population when in the steady state appropriate to any such system. In view of the emphasis laid by Malthus upon the 'law of geometric increase' m may appropriately be termed the Malthusian parameter of population increase. It evidently supplies in its negative values an equally good measure of population decrease, and so covers cases to which, in respect of mankind, Malthus paid too little attention.

In view of the close analogy between the growth of a population supposed to follow the law of geometric increase, and the growth of capital invested at compound interest, it is worth noting that if we regard the birth of a child as the loaning to him of a life, and the birth of his offspring as a subsequent repayment of the debt, the method by which m is calculated shows that it is equivalent to answering the question—At what rate of interest are the repayments the just equivalent of the loan? For the unit investment has an expectation of a return $l_x b_x dx$ in the time interval dx, and the present value of this repayment, if m is the rate of interest, is $e^{-mx}l_x b_x dx$; consequently the Malthusian parameter of population increase is the rate of interest at which the present value of the births of offspring to be expected is equal to unity at the date of birth of their parent. The actual values of the parameter of population increase, even in sparsely populated dominions, do not, however, seem to approach in magnitude the rates of interest earned by money, and negative rates of interest are, I suppose, unknown to commerce.

Reproductive value

The analogy with money does, however, make clear the argument for another simple application of the combined death and reproduction rates. We may ask, not only about the newly born, but about persons of any chosen age, what is the present value of their future offspring; and if present value is calculated at the rate determined as before, the question has the definite meaning—To what extent will persons of this age, on the average, contribute to the ancestry of future generations? The question is one of some interest, since the direct action of Natural Selection must be proportional to this contribution. There will also, no doubt, be indirect effects in cases in which an animal favours or impedes the survival or reproduction of its relatives; as a suckling mother assists the survival of her child, as in mankind a mother past bearing may greatly promote the reproduction of her children, as a foetus and in less measure a sucking child inhibits conception, and most strikingly of all as in the services of neuter insects to their queen. Nevertheless such indirect effects will in very many cases be unimportant compared to the effects of personal reproduction, and by the analogy of compound interest the present value of the future offspring of persons aged x is easily seen to be

$$v_x = \frac{e^{mx}}{l_x}\int_x^\infty e^{-mt} l_t b_t dt.$$

Each age group may in this way be assigned its appropriate reproductive value. Fig. 2 shows the reproductive value of women according to age as calculated from the rates of death and reproduction current in the Commonwealth of Australia about 1911. The Malthusian parameter was at that time positive, and as judged from female rates was nearly equivalent to $1\frac{1}{4}$ per cent. compound interest; the rate would be lower for the men, and for both sexes taken together, owing to the excess of men in immigration. The reproductive value,

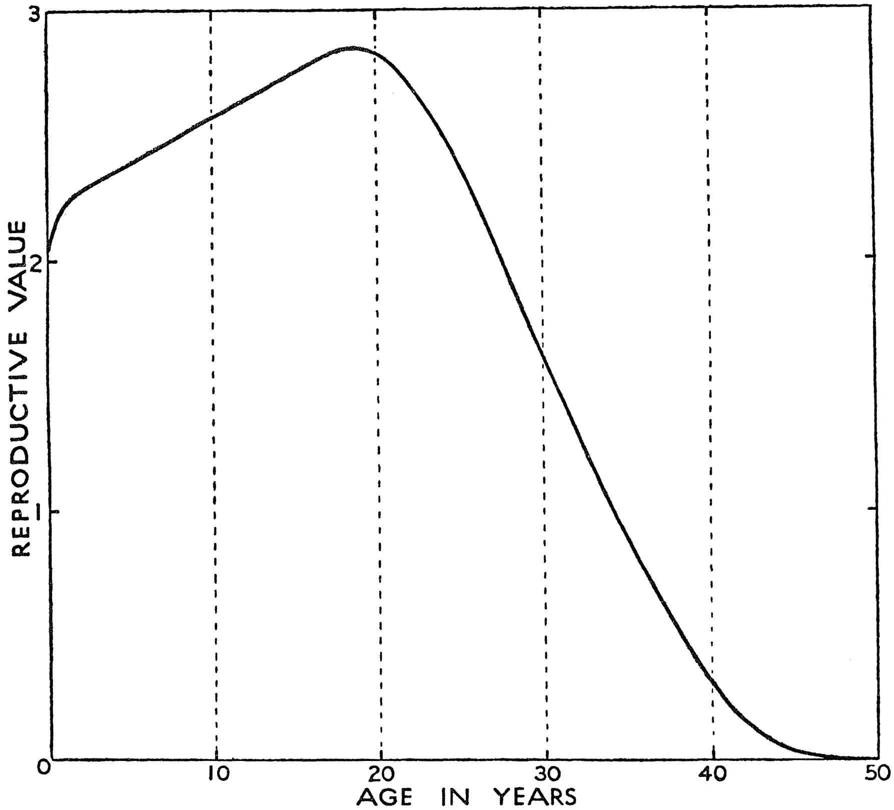

FIG. 2. Reproductive value of Australian women.
The reproductive value for female persons calculated from the birth- and death-rates current in the Commonwealth of Australia about 1911. The Malthusian parameter is $+0{\cdot}01231$ per annum.

which of course is not to be confused with the reproductive rate, reaches its maximum at about 18½, in spite of the delay in reproduction caused by civilized marriage customs; indeed it would have been as early as 16, were it not that a positive rate of interest gives higher value to the immediate prospect of progeny of an older woman, compared to the more remote children of a young girl....

A property that well illustrates the significance of the method of valuation, by which, instead of counting all individuals as of equal value in respect of future population, persons of each age are assigned an appropriate value v_x, is that, whatever may be the age constitution of a population, its total reproductive value will increase or decrease according to the correct Malthusian rate m, whereas counting all heads as equal this is only true in the theoretical case in which the population is in its steady state. For suppose the number of persons in the age interval dx is $n_x dx$; the value of each element of the population will be $n_x v_x dx$; in respect of each such group there will be a gain in value by reproduction at the rate of $n_x b_x v_0 dx$, a loss by death of $n_x \mu_x v_x dx$, and a loss by depreciation of $-n_x dv_x$, or in all

$$n_x\{(b_x v_0 - \mu_x v_x)\,dx + dv_x\},$$

but by differentiating the equation by which v_x is defined, it appears that

$$\frac{1}{v_x}\frac{dv_x}{dx} + \frac{1}{l_x}\frac{dl_x}{dx} - m = \frac{-l_x b_x e^{-mx}}{\frac{v_x}{v_0} l_x e^{-mx}} = -\frac{b_x v_0}{v_x},$$

or that

$$dv_x - \mu_x v_x dx + b_x v_0 dx = m v_x dx.$$

Consequently the rate of increase in the total value of the population is m times its actual total value, irrespective of its constitution in respect of age. A comparison of the total values of the population at two census epochs thus shows, after allowance for migration, the genuine biological increase or decrease of the population, which may be entirely obscured or reversed by the crude comparison of the number of heads. The population of Great Britain, for example, must have commenced to decrease biologically at some date obscured by the war, between 1911 and 1921, but the census of 1921 showed a nominal increase of some millions, and that of 1931 will, doubtless in less degree, certainly indicate a further spurious period of increase, due to the accumulation of persons at ages at which their reproductive value is negligible.

19. How the Age Distribution of a Human Population is Determined

ANSLEY J. COALE (1957)

Cold Spring Harbor Symposia on Quantitative Biology 22: 83—88.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_19,

This analysis of age distribution will be restricted to closed populations of human females; restricted to closed populations because to consider the effects of migration would be unduly complicated; restricted to human populations because the author is a demographer rather than a biologist; and restricted to females because differences in age composition and in age at parenthood between the sexes make a combined treatment awkward. If data were equally available, a completely similar analysis could be made for males, with only one major reservation: The special effects of war mortality on males of military age would require special analysis.

I. STABLE POPULATIONS

The forces affecting the shape of the age distribution are most readily visualized in the special case of an unchanging distribution. Lotka (Dublin and Lotka, 1925; Lotka, 1939) has shown that constant mortality and fertility schedules will ultimately produce a constant age distribution, and a constant rate of growth. This constant age distribution can be calculated as the product of a factor representing relative number of survivors to each age from birth (the life table) and a factor reflecting the continuously growing (or shrinking) number of births. Specifically:

$$c(a) = be^{-ra}p(a) \tag{1}$$

where $c(a)$ is the fraction at any given age a, b is the birth rate (which Lotka shows is constant under the specified conditions) r is the growth rate, and $p(a)$ is the fraction who survive from birth to age a. No matter what the initial age distribution (provided the childbearing ages are amply represented) a constant age schedule of fertility ($m(a)$) and probability of surviving ($p(a)$) eventually establish the distribution given in equation (1). Actually, a period of 50 to 100 years is adequate to produce a very close approximation to the stable population.

The basis for equation (1) can be simply explained. The birth rate (b) is the proportion at age zero (set $a = 0$, and the other factors are each unity). The term e^{-ra} relates the size at birth of the cohort now aged a to the current birth cohort, and $p(a)$ allows for the attrition of mortality.

Since $b = \dfrac{1}{\int_0^\omega e^{-ra}p(a)\,da,}$ (where ω is the highest age attained) the stable age distribution is wholly determined by the growth rate r, and the survivorship function $p(a)$. The value of r in turn can be calculated from the second fundamental equation of stable population theory.

$$\int_0^\omega e^{-ra}p(a)m(a)\,da = 1 \tag{2}$$

The real root of this integral equation is the stable growth rate, while the complex roots determine how the stable population is approached from arbitrary initial conditions.

We now turn to the role of fertility and mortality in determining the shape of the stable age distribution. The role of mortality will be described first.

Mortality and the stable age distribution

Two rather surprising conclusions emerge when the effect of mortality schedules on the stable age distribution is examined:

(a) The effect of alternative mortality schedules is relatively minor. Roughly similar age distributions result from a given fertility schedule in conjunction with a very high mortality life table on the one hand, or a very low mortality life table on the other. Figure 1 illustrates this point. The life table of Sweden for 1860, with a life expectancy of 45.4 years, and that for 1946 to 50, with a life expectancy of 71.6 years produce about the same stable age distributions when combined with the same fertility schedule.

(b) Among life tables reflecting experience so far recorded, it is nearly universally true that a more favorable mortality schedule—with a higher life expectancy—will yield a stable population with a higher proportion under 15. In a vast majority of contrasting life tables, the lower mortality life table will produce a distribution with a lower average age; and at least half the time, lower mortality will produce a smaller fraction over 65.

These conclusions are partly analytical and partly empirical. A full analysis of the contrasting stable age distributions associated with different mortality schedules is somewhat laborious (Coale, 1956). A simple qualitative argument is enough to dispel the common belief that lower mortality inevitably means an older population. If in one life table the probability of surviving for a year at each age exceeded the corresponding probability in another life table by a fixed proportion (for example, 1%) the two life tables would produce precisely the same age distributions. The tendency toward more survivors with advancing age would be exactly offset by a faster rate of growth that tends to make each cohort *smaller* than the next younger.

This point can be proved by assuming a sudden shift to a life table with a one per cent higher probability of surviving at each age. The year following this change, there would be one per cent more one-year olds surviving from birth, one per cent more two-year olds, etc. Improved survivorship would produce one per cent more persons at every age above 0. But since the increase would yield one per cent more persons at every childbearing age, with constant fertility there would be one per cent more births—persons at age 0—as well. The whole population would be one per cent larger; and the age distribution would be unaffected.

The common view that lower mortality means an older population takes account of only part of the effect of lower death rates. It is immediately clear that lower death rates produce more old people. However, lower death rates also produce more parents, more births, and more children. Whether the dominant effect on the stable age distribution of a lower level of mortality is to enlarge the upper end of the age distribution through higher survivorship, or to tilt the age distribution more steeply through more rapid growth depends on the relative age pattern of mortality in the two life tables (Coale, 1956).

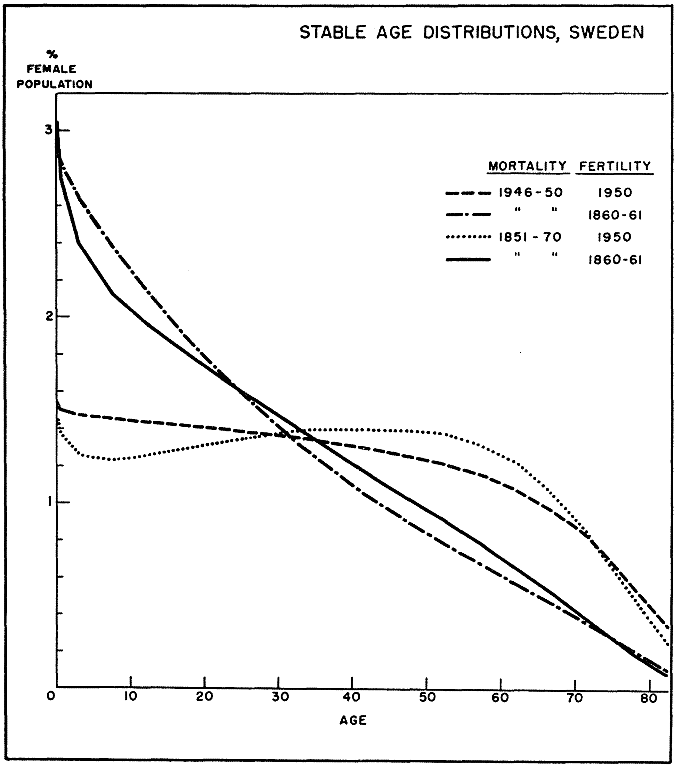

Figure 1.

A particularly illuminating way of comparing the age pattern of mortality is to compute the proportionate difference in the probability of surviving for one year at each age. Let the probability of surviving from age a to $a + 1$ be $\pi'(a)$ in the better life table, and $\pi(a)$ in the higher mortality table, and consider the ratio $\frac{\pi' - \pi}{\pi}(a)$. We have already shown that a constant value of $\frac{\pi' - \pi}{\pi}(a)$ implies the same age distributions. We can subtract the minimum value of $\frac{\pi' - \pi}{\pi}$ from the value at other ages—the subtracted portion is equivalent to no difference in mortality—and consider only the residual. A large excess of $\frac{\pi' - \pi}{\pi}$ above the minimum at ages below childbearing means there is a large difference in the *growth rates* of the two stable distributions. Moreover, since a large excess $\frac{\pi' - \pi}{\pi}$ at the youngest ages implies more survivors at *all* subsequent ages, the *proportion* of survivors above—say—age 50 in the better life table would not be much larger than in the poorer one. In short, an excess of $\frac{\pi' - \pi}{\pi}$ above the minimum at young ages means that the better mortality schedule tends to have a *younger* age distribution.

On the other hand, an above-the-minimum $\frac{\pi' - \pi}{\pi}(a)$ at ages over 50 implies no difference in the long term growth rate (since the reproductive ages are not involved) but does mean more old age survivors in the better life table. The net result is a higher fraction at ages above 50.

The effects of three special instances of percent difference in the probability of surviving have been described in a non-rigorous fashion. These particular patterns have been emphasized because differences among the great majority of recorded life tables can be approximated as the sum of three components—a minimum per cent difference in the probability of surviving from age 5 to 50, above minimum differences below age 5, and above minimum differences above age 50 (see Fig. 2). The general pattern of $\frac{\pi' - \pi}{\pi}(a)$ is roughly U-shaped, declining from a high at age zero to near its minimum by age 5. It is relatively constant until age 50 or 60 where it frequently but not universally rises. The central portion (from 5 to 50) contributes nothing to the differences in stable age distribution, serving only to diminish the effect of the differences represented by the two legs of the U, and to raise the growth rate of the population.

The two legs of the U-shaped pattern work more or less in opposition in causing differences in

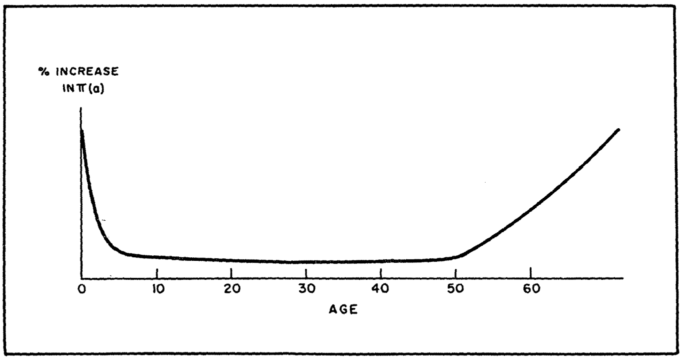

FIGURE 2.

age distribution. If there were no right leg, the age distribution with the better life table could be approximated at all ages except those under 5 by a sort of pivoting of the higher mortality distribution on its average age, with increased fractions at less than the average age, and reduced fractions above. If there were no left leg, rising per cent differences in survivorship over age 50 would mean higher fractions above 50, and lower fractions below. When the two legs operate together, the net effect is to reduce the differences each would cause alone. If the left leg is large, and the right leg is small, the effect of growth rates will dominate; and the low mortality population may have smaller proportions at all ages above the average. This domination is typical when a life table of high mortality is compared with a moderate or low mortality table. However, when mortality of 20 years ago in the most advanced areas is compared with current mortality, the right leg—improved survivorship at the older ages—takes on more importance. There is not much room for a left leg in a comparison of future life tables with the best current ones. A two or three per cent further improvement in the probability of surviving to age 5 in Sweden would raise this probability to one.

There is no logical necessity that mortality differences should produce only minor differences in stable age distributions. This result arises from the age structure of differences among life tables observed to date. However, a contributing factor in the typical U-shaped age structure of differences in the probability of surviving is that large differences can last only when there is room for them. Differences of the order of those observed for age 0 could not occur at those ages where even with high general mortality levels the risk of death is moderate.

Fertility levels and the stable age distribution

The role of fertility in shaping the stable age distribution is at once simpler than that of mortality, and quantitatively much more important, when account is taken of the range of mortality and fertility observed in the world. The simplicity and the quantitative importance have a common source: the fact that differences in fertility operate in a single direction in affecting the relative shape of the age distribution. While higher probabilities of surviving simultaneously flatten out $p(a)$ in equation (1) and make e^{-ra} steeper, higher probabilities of giving birth affect only the growth rate—making it larger, of course, and making the exponential factor in equation (1) contribute to a more rapid taper in the distribution.

If a high fertility stable age distribution is compared with a low fertility distribution with the same mortality, the two will be found to intersect at the mean of the average ages. The higher fertility population will have higher proportions at ages below the average, of course.

Figure 1 makes it clear that fertility differences can produce profoundly different stable age distributions. This fact together with the relatively slight influence of mortality on the stable age distribution means that a schedule of fertility by age is sufficient to give at least a fair approximation to the stable age distribution even if mortality rates by age are not known. One would simply use whatever female life table was lying around in calculating the stable population. It must be admitted that there is a much better chance of a close fit if some hint about infant mortality is visible.

One final comment on a common sense basis for understanding the powerful influence of fertility on the stable age distribution. The general fertility rate establishes the ratio of the area in the age distribution from ages 17 to 44 to the zero ordinate of the distribution. If fertility is twice as high, this ratio must be cut in half irrespective of mortality. The exact inverse relation of this ratio to fertility is a consequence of the fact that the zero ordinate is proportional to births, and the area from 17 to 44 to the number of women of childbearing age. Thus the fertility level clamps a vice on the relation of the beginning of the distribution to an area near the middle.

Several straightforward conclusions emerge from this consideration of stable age distributions:

(1) Sustained high fertility (average completed size of family, 6 or more children) produces a young population with a median age well below 25 years, more than 40 per cent of the population under 15, and no more than three or four per cent over 65.

(2) Sustained low fertility (average completed size of family below 2.5 children) pro-

duces an old population with a median age above 35, no more than 20 per cent of the population under 14, and at least 15 per cent over 65.

(3) In general, the approximate form of the age distribution is determined by the level of fertility. The level of mortality has more or less second order effects on the distribution. The general quality of these effects has been to date that low mortality yields a slightly larger fraction at ages up to at least 15 but as high as the average age, and somewhat smaller fractions either at all higher ages or until age 50 or higher. More often than not, sustained low mortality yields a slightly lower average age.

II. VARYING FERTILITY AND MORTALITY AND THE AGE DISTRIBUTION

We turn now to a brief consideration of how an age distribution is determined under a regime of continuously changing mortality and fertility rates.

The general equations corresponding to equation (1) are:

(3) $n(a, t) = B(t - a)p(a, t)$ where $B(t - a)$ is the number of births a years before time t, $n(a, t)$ is the number of persons at age a at time t, and $p(a, t)$ is the proportion of those born at $(t - a)$ who survive to achieve age a at time t, and

$$(4)\quad c(a, t) = \frac{n(a, t)}{\int_0^\omega n(a, t)\, da}\, .$$

The equation corresponding to (2) is:

(5) $B(t) = \int_0^\omega n(a, t)m(a, t)\, da$ where $m(a, t)$ is the probability of bearing a female child at age a and time t.

These equations of course do not lead to Lotka's tidy solutions since we here permit mortality and fertility to vary with time. In fact, they do not even give much clue to the role of fertility and mortality in determining age distributions, since equation (3) requires us to know $B(t - a)$, and equation (5) tells us that $B(t - a)$ depends on the age distribution at time $(t - a)$ as well as on fertility rates at time $(t - a)$. It would appear that to account for the present age distribution, one needs to know:

(a) an age distribution at some past date, and

(b) schedules of mortality and fertility since that date.

However, as the date of the past age distribution is made more remote, its form makes less and less difference to the shape of the current age distribution. It seems intuitively plausible, in fact, that if the course of fertility and mortality were known since $t - \infty$, the proportionate age distribution would be wholly determined at time t no matter what the distribution at $t - \infty$, assuming of course that the initial distribution was not one—for example with no one under 50—headed for extinction. Lotka (1939) shows this statement to hold in the special case of endlessly unchanging fertility and mortality. But the same factors that cause the transient effects of a particular initial age distribution to disappear from the stable population would also operate for *any* observed time path of fertility and mortality rate. After a a suitably long period the effect of an initial age distribution is swamped by the cumulative effect of the time pattern of vital rates. To put the point more concretely, any age distribution with persons at every age could be assumed for 100 years ago in place of the actual distribution. If such an assumed population were projected to the present with observed fertility and mortality rates, the proportionate age distribution would differ negligibly from the actual. In short, the age distribution of a closed population is determined by the mortality and fertility rates of recent history.

Another result from stable population analysis is suggestive for our general case—the conclusion that mortality differences have only second order effects on the age distribution. An increase or decrease in mortality tends to decrease or increase all cohorts, implying a small effect of mortality on the immediate age distribution as well as the stable. The short run transient age distribution effect of a mortality change may differ somewhat in form from the long run effects as expressed by stable age distributions, but the magnitude should be small in both instances.

We shall proceed with the provisional hypothesis that changes over time in mortality schedules do not have major effects on the age distribution. If this hypothesis is valid, a current age distribution could be closely approximated by calculating what the distribution would have been had observed fertility rates and *unchanging* mortality rates prevailed for the last 80 or 90 years. I have tried such a calculation using Swedish data. The results are shown in Figure 3. The proportionate distribution based on mortality unchanged for 90 years indeed does come close to the actual distribution. Moreover, the differences between the census population and the hypothetical are nearly identical to the differences between two stable populations with the same fertility, one based on an 1860 Swedish life table, and the other on a life table for 1946 to 50 (compare Fig. 1). The principal effect of projecting with unchanged mortality is to produce smaller fractions at ages under 25 and over 70. The reason is that since 1860 there have been disproportionate improvements in the probability of surviving in infancy and in the older ages. If the projection from 1860 is calculated with mortality unchanging at 1946 to 50 levels, the result is a distribution virtually indistinguishable from the census distribution. If in the 90 years before 1950 mortality had remained as high as in 1860 the result would have been a much smaller female population with an age distribution similar to the actual; if in these 90 years mortality had always been as low as in 1950, the result would have been a much *larger* female population with an age distribution nearly identical to the actual.

Figure 3 also shows the distribution that would have resulted with observed mortality risks during the 90 years before 1950, and with fertility assumed constant at 1860 levels. This figure makes clear what is the major determinant of an age distribution—the course of fertility. I wish I were now able to present a short, simple explanation of the effects of fertility on the age distribution. It is clear that fertility determines the age distribution, but an attempt to explain the relationship precisely soon runs afoul of major complications.

The basic difficulty is that when high fertility produces a brief series of unusually large cohorts, for example, these cohorts not only exceed their

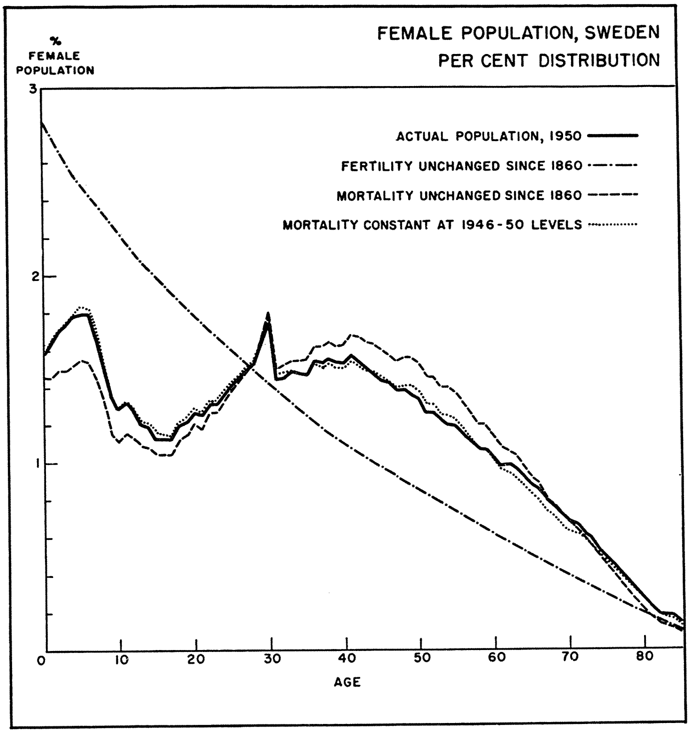

FIGURE 3.

neighbors through their lifetime (this is a simple effect), but also when they pass through the childbearing ages they produce a diffused "echo" of larger birth cohorts. It is easy to give instructions about how to compute these consequences, but impossible to describe them simply in terms of the resultant age distribution.

We can make these observations about the effects of fertility on age distribution:

(1) Current and recent high fertility produce a younger population than would low fertility, and *vice versa.*

(2) Transitory waves of unusually high or low fertility create humps and hollows that move out through the age distribution as the cohorts move through life. In fact all of the notches and knobs in the Swedish female age distribution of 1950 can be traced back to unusual birth crops.

(3) A long period of high fertility, or a period of rising fertility creates a section of the age distribution that tapers rapidly with age. Conversely a long period of low fertility, or a period of falling fertility creates a relatively flat (or even rising) section of the age distribution.

(4) When a cohort of unusual size reaches the childbearing ages, it sets up an attenuated and flattened out "echo" in the number of births. It is this second generation effect that makes the analysis of the relation of the age distribution to the course of fertility so complex.

Finally I turn to one or two practical or at least worldly observations arising from this anslysis. First, the rising fraction of the aged in western countries has not resulted from lowered death rates but almost wholly from a long history of declining fertility. However, future improvements in mortality may make this statement obsolete. Second, improved mortality will not reduce the "burden of dependency" imposed on low income areas because of their age distributions. To the contrary, mortality reduction can be expected to lead to a somewhat higher fraction in dependent ages. If the burden is to be reduced, fertility must be lowered. Third, the remarkable variation in fertility in the U. S. during the past 20 years has produced a very irregular age distribution. The succession of variously sized cohorts will doubtless have interesting implications in the next half century. Thus in the next 20 years the number of persons 20 to 24 is due to increase by at least 75 per cent, while the number 40 to 44 can be expected to decline by about three per cent.

REFERENCES

COALE, ANSLEY, J., 1956, The effects of changes in mortality and fertility on age composition. Milbank Mem. Fund Quart. *34:* 79–114.

DUBLIN, LOUIS I., AND LOTKA, ALFRED J., 1925, On the true rate of natural increase. J. Amer. Statis. Assoc. *20:* 305–327.

LORIMER, FRANK, 1951, Dynamics of age structure in a population with initially high fertility and mortality. Population Bull. U. N. *1:* 31–41.

LOTKA, ALFRED J., 1939, Théorie analytique des associations biologiques. Deuxième partie: Analyse démographique avec application particulière à l'espèce humaine. Paris, Hermann et Cie.

20. On the Reproduction of Organisms with Overlapping Generations

W. R. THOMPSON (1931)

From *Bulletin of Entomological Research* 22. Excerpts are from pages 147—152, 154—160, 163, 168—170.

We omit the final section of Thompson's paper, in which he shows the effects on population size of (a) lengthening the pre-reproductive interval, (b) lengthening the period of reproduction, (c) increasing the number of offspring produced per day, and (d) compacting the reproductive period while leaving the number of offspring constant. The effects can be summarized in the approximate equality, due to Lotka (1939, pp. 70—71)

$$r=\frac{\mu-\sqrt{\mu^2-2\sigma^2\ln R_0}}{\sigma^2}=\frac{\ln R_0}{\mu-\dfrac{\sigma^2 r}{2}}$$

where r is the intrinsic rate of growth, R_0 the net reproduction rate, and μ and σ^2 the mean and variance of the net maternity function. Where the net maternity function takes the form of a normal distribution the equalities are exact.

Much of Thompson's work can be translated to matrix form, where his generation law G becomes the first row of a projection matrix whose sub-diagonal terms (survival) are unity. Thus, the generation law: $G=(aT^1+bT^2+cT^3+dT^4)$, becomes the projection matrix $\boldsymbol{M}$:

$$\underset{\boldsymbol{M}}{\begin{array}{c} \begin{matrix} T^1 & T^2 & T^3 & T^4 \end{matrix} \\ \begin{bmatrix} a & b & c & d \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}} \underset{\boldsymbol{B}^{(0)}}{\begin{bmatrix} B(T^0) \\ 0 \\ 0 \\ 0 \end{bmatrix}} = \underset{\boldsymbol{B}^{(1)}}{\begin{bmatrix} B(T^1) \\ B(T^0) \\ 0 \\ 0 \end{bmatrix}}$$

which we show post multiplied by the initial forebears $B(T^0)$ to yields births $B(T^1)$ and total population $\boldsymbol{B}^{(1)}$ at time 1. (Mortality can be subsumed in the generation law or in the subdiagonals, hence the form shown here is quite general.) Matrix formulation facilitates the calculation of characteristic roots and vectors but was not yet adopted when Thompson wrote. It does not lend itself as conveniently to the separation of births by generation or to an appreciation of generation overlap, which constitute important contributions of Thompson's work.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_20,

INTRODUCTION.

In the great majority of organisms, the reproducing individual does not engender all of its progeny simultaneously. Between the beginning and the end of the reproductive period there is usually a certain interval of time, during which the production of offspring continues in a manner depending, on the one hand, upon the specific characteristics of the organism, and on the other, upon the conditions under which it lives. From this simple fact follow some interesting consequences.

If the successive generations are separated by a long interval of time and conditions during this interval are of such a nature that the organism is inactive and in the condition of obligatory dormancy, known as diapause, then, in spite of the increase of the population, the generations will remain separate and distinct, and a comparison of the total populations in the successive reproductive periods will enable us to determine the rate of increase.

On the other hand, if the interval between generations is short and favourable to activity and the organism does not pass through a period of obligatory dormancy at this time, then, as the species increases, the successive generations will come to overlap, and the amount of overlapping will increase progressively as time goes on.

This is primarily due to the fact that when there is reproduction over a period of time-intervals, in a typical individual of the species, and conditions remain constant, then the number of time-intervals during which births occur will increase from generation to generation. In this way there will be produced a lengthening of the reproductive period for the whole mass of individuals ; and this will cause a diminution of the intervals between stages in the same generations, and between generations, so that individuals belonging to different stages of development, and to different generations, will be found simultaneously in the field.

On account of these facts the calculation of the number of progeny and of the number of individuals in successive stages, existing at any given moment, presents serious difficulties, which make both the practical and the theoretical treatment of the subject laborious. Some years ago I began to study the matter and succeeded in getting the process of calculation into a form in which the required data could be obtained, but the method of calculation was extremely tedious. The results obtained were accordingly submitted, through the kindness of Professor M. Greenwood, of the Department of Epidemiology and Vital Statistics of the London School of Hygiene and Tropical Medicine, to Mr. H. E. Soper. Within a few days I received from Mr. Soper a communication, in which he explained and developed in detail an extremely elegant method of dealing with the problem. It is this method which I have utilised in this paper. I have added such explanations as seem to me necessary in order to make the process of reasoning clear to those who, like myself, have little knowledge of mathematics, and have worked out a series of examples to show how the method may be employed in entomological problems. The credit for the solution utilised is entirely due to Mr. Soper, and I am glad to have this occasion of thanking him for the kind and patient interest he has shown in the problems submitted to him. ...

The first part of the paper is devoted to an exposition of the problem ; the second part to the methods which can be used in dealing with it ; and the third part to an application of these methods. The mathematical theory of the method is given by Mr. Soper in the Appendix.

The object of the paper is, in the first place, to enable the entomologist to investigate the natural control of insects with reproduction of this type ; and, in the second place, to show how variation in the length of the pre-reproductive and reproductive periods, and in the production of offspring, affect the increase in population. When the length of life and of the pre-reproductive and reproductive periods have been determined, and the average daily production of progeny is known, we can calculate from the formulae given the daily births or the daily population of individuals of all ages, when no causes of mortality intervene. By comparing the figures with those obtained in an actual experiment, we can then determine when causes of mortality begin to operate and get some idea as to the nature and effect of the controlling factors which come into play at any given date. Organisms with overlapping generations are particularly suitable for work of this kind, because of their rapidity of reproduction, but the complex way in which their populations increase has, hitherto, constituted a very serious obstacle to their study. The formulae developed by Mr. Soper enormously simplify the problem and constitute, I believe, a very important contribution to our methods of study of the process of natural control.

I.

Let us suppose, to begin with, that we have an *initial organism* or stem-mother, producing d offspring per day during a certain period, which we may call k days for the moment; and let us suppose that the reproducing individual dies on the day it has produced its last offspring. Then, in the first generation, if we put $k=4$, for the sake of simplicity, we shall have births occurring as follows:—

Day ...	1	2	3	4	5	6	7	8	9	10
G.I ...	d	d	d	d						

Now suppose that both the organisms produced in the first generation and the environmental conditions under which they live remain identical. The group of organisms produced on the second day will reproduce a day later than those produced on Day 1; the group produced on the third day, two days later than those produced on Day 1, and so on. The second generation, produced by the reproduction of the groups occurring on the successive days, will thus be as follows, since each individual of the group of d produces d offspring:—

	DAY	1	2	3	4	5	6	7	8	9	10
G.II ...	1	d^2	d^2	d^2	d^2						
G.II ...	2		d^2	d^2	d^2	d^2					
G.II ...	3			d^2	d^2	d^2	d^2				
G.II ...	4				d^2	d^2	d^2	d^2			
Total		$1d^2$ ·	$2d^2$ ·	$3d^2$ ·	$4d^2$ ·	$3d^2$ ·	$2d^2$ ·	$1d^2$			

In the next generation the process will obviously continue in the same way; during the first reproductive period $1d^2$ individuals will produce d offspring per day; during the second period $2d^2$ will produce d each per day; during the third period $3d^2$ will produce d per day, and so on—the result being as follows:—

	DAY	1	2	3	4	5	6	7	8	9	10
G.III ...	1	d^3	d^3	d^3	d^3						
	2		$2d^3$	$2d^3$	$2d^3$	$2d^3$					
	3			$3d^3$	$3d^3$	$3d^3$	$3d^3$				
	4				$4d^3$	$4d^3$	$4d^3$	$4d^3$			
	5					$3d^3$	$3d^3$	$3d^3$	$3d^3$		
	6						$2d^3$	$2d^3$	$2d^3$	$2d^3$	
	7							d^3	d^3	d^3	d^3
Total		$1d^3$ ·	$3d^3$ ·	$6d^3$ ·	$10d^3$ ·	$12d^3$ ·	$12d^3$ ·	$10d^3$ ·	$6d^3$ ·	$3d^3$ ·	$1d^3$

When we compare the series of births in the successive generations the first thing that we notice is that the number of days (or time-intervals) on which births occur becomes greater in each generation. Thus, as an organism increases in numbers, the length of time during which births occur in the field will also increase, from generation to generation, and so will the length of time during which the successive developmental stages are found, without there being, necessarily, any deviation whatever from the specific reproductive habits, or any change in the environmental conditions; or, in other words, any real irregularity in the course of events. ...

Furthermore, since the number of time-intervals during which births occur increase from generation to generation, it is also evident that, as time goes on, the interval between the end of one generation and the beginning of another will become smaller and smaller, and then disappear, after which the successive generations will overlap. The overlap will become greater and greater as the generations succeed one another. Finally, the number of generations overlapping in a given time-interval will also become greater, though the rate of increase is not a smooth and regular one.

Thus, suppose the pre-reproductive period of a species is of two days (including day of birth) and the reproductive period also two days, each individual producing d births per day, the succession of events will be as follows:—

The first and second generations do not overlap, but the second and third overlap on one day, the third and fourth on two days, the fourth and fifth on three days, the fifth and sixth on four days, the nth and $(n+1)$th on $(n-1)$ days. On the first four days individuals of only one of the two generations produced are found on any one day; on the fifth day individuals of G.II and G.III co-exist; on the seventh and eighth days individuals of G.III and G.IV, on the ninth and tenth days individuals of G.IV and G.V; on the eleventh day individuals of G.IV, V, and VI, and on the seventeenth day individuals of G.VI, VII, VIII, and IX.

II.

It will be evident from what has preceded that the direct calculation of the progeny produced day by day in the successive generations, to say nothing of the total number of progeny produced by all generations taken together, or, to take the most difficult case of all, of the total population, including both the newly born and the older individuals, is a very complex matter. In order to deal with the problem it is, therefore, necessary to simplify the operation.

It is, of course, possible to represent the increase in population by using the compound interest formula, according to which the number of individuals existing at any given moment is given by the equation

$$N_1 = N_0 \epsilon^{Kt}$$

Where N_1 is the population at the moment considered, N_0 the initial population, t the number of time-intervals (hours, days, years, etc.), ϵ the incommensurable number 2·71828 . . ., and K a constant depending upon the particular case considered. This formula is, however, not satisfactory for experimental work. In the first place, the constant K cannot be determined until the growth of the population under certain definite conditions has been studied during a considerable period; in the second place, no intelligible significance can be attached to the constant after its value has been determined; in the third place, the growth of the population is considered in this formula to be at every moment proportional to the size of the population, which is not true except with large numbers and over long periods, and cannot be safely taken as a basis for the examination of experimental data. Thus, using the figures for births in the last example given in this paper, we obtain values from K ranging from 0·652 to 0·693. The former value is valid when the population reaches a high level, but under such conditions accurate analytical experimental work is not possible. The number of individuals at the end of five time-intervals might be anything from 208 to 255. It is therefore necessary to deal with the reproductive process as a discontinuous phenomenon. ...

Suppose the ancestor or stem-mother is born on T^0, and that we represent the successive days or "time-intervals" on which reproduction takes place by T^1, T^2, T^3, T^4, . . . meaning the first, second, third, fourth, etc., days, and suppose one offspring is produced each day. Suppose the reproductive period is one of four days, and the pre-reproductive period is of one day, including the day of birth, then the first generation, $= G_1$ may be written—

$$G_1 = 1T^1 + 1T^2 + 1T^3 + 1T^4.$$

Now the one individual born on T^1 will begin to reproduce on T^2 and from thenceforward during four days; the one individual born on T^2 will begin to reproduce on T^3 and from thenceforward during four days; and similarly, for the individuals born on T^3 and T^4, which begin reproducing on T^4 and T^5 respectively. Thus, writing the series of births for G^2 we have—

$$\begin{array}{lllllllll}
G_2 = & 1T^2 & +1T^3 & +1T^4 & +1T^5 & & & \\
& & 1T^3 & +1T^4 & +1T^5 & +1T^6 & & \\
& & & 1T^4 & +1T^5 & +1T^6 & +1T^7 & \\
& & & & 1T^5 & +1T^6 & +1T^7 & +1T^8 \\
\hline
& 1T^2 & +2T^3 & +3T^4 & +4T^5 & +3T^6 & +2T^7 & +1T^8 \\
\hline
\end{array}$$

which is—

$$= (T^1 + T^2 + T^3 + T^4)^2.$$

In G^3, the one individual produced on T^2 will reproduce on four succeeding days, and similarly, for the two produced on T^3, the three produced on T^4 and so on; and if these series of births are written out and added as before, we obtain—

$$G_3 = 1T^3 + 3T^4 + 6T^5 + 10T^6 + 12T^7 + 12T^8 + 10T^9 + 6T^{10} + 3T^{11} + 1T^{12}$$
$$= (T^1 + T^2 + T^3 + T^4)^3.$$

It is clear that, in general, we have—

$$G_n = (T^1 + T^2 + T^3 + T^4)^n,$$

and that by multiplying this out, or, as the mathematicians say, "expanding it," we shall obtain a series of T's multipled by numbers, or coefficients in which the index of T designates the *day* and the coefficient gives the number of births on that day in the generation considered.

From this simple case all others may readily be derived without any alteration in the reasoning.

Thus, suppose the pre-reproductive period is of three days, including the day of birth, and the reproductive life is of four days, as before. If the original ancestor or stem-mother was born on T^0 (an assumption made in the first case), then its first offspring would be produced, not on T^1 but on T^3, and the first generation would be written—

$$G_1 = (1T^3 + 1T^4 + 1T^5 + 1T^6);$$

then the individual produced on T^3 would begin reproducing on T^6, and we should obtain, by proceeding as before, a series for the second generation equivalent to—

$$(1T^3 + 1T^4 + 1T^5 + 1T^6)^2.$$

Finally, if, instead of one birth on each day, we assume d on each day, we should obtain—

$$G_1 = d(T^1 + T^2 + T^3 + T^4);$$
$$G_2 = d^2(T^1 + T^2 + T^3 + T^4)^2;$$
$$\cdots\cdots\cdots$$
$$G_n = d^n(T^1 + T^2 + T^3 + T^4)^n;$$

while if the number of births varied from day to day, so that we had—

$G^1 = (aT^1 + bT^2 + cT^3 + dT^4)$; we shall have

$$\cdots\cdots\cdots\cdots$$
$$G_n = (aT^1 + bT^2 + cT^3 + dT^4)^n.$$

Suppose that on one or more days in the reproductive period no progeny are produced. This is allowed for simply by omitting these days. Thus, in the last series mentioned, let $b=0$. The formula for the nth generation will then be :—

$$G_n=(aT^1+cT^3+dT^4)^n. \ \ldots$$

What has preceded covers fully the series of births (or production of offspring) in each generation considered separately ; but it does not give us any information as to the overlapping of the generations, nor does it enable us to calculate directly the *total* number of births produced by the individuals of all the different generations which happen to be reproducing on any given day ; and that individuals belonging to different generations will be producing simultaneously we have already seen....

If, taking the successive generations in the example already studied, we write them down and add, we have—

$$G_1+G_2+G_3=(T^1+T^2+T^3+T^4)+(T^1+T^2+T^3+T^4)^2$$
$$+(T^1+T^2+T^3+T^4)^3.$$

It is obvious, from what has preceded, that by expanding these expressions and adding up all the T^1, T^2, T^3 .... T^{12} we shall get the number of births produced by all the individuals of all the generations existing simultaneously on any given day, up to and including the third generation.

If we designate—

$$T^1+T^2+T^3+T^4 \text{ by } G$$

so that we have

$$G_1=G$$
$$G_2=G^2$$
$$\ldots\ldots\ldots$$
$$G_n=G^n,$$

we can write

$$G_1+G_2+G_3=G+G^2+G^3$$
$$=\frac{G(1-G^3)}{1-G}$$

by the usual rule for the summation of a geometrical series.

But this formula does not represent the *total* number of individuals produced up to the day on which G_3 ends (T^{12}), because by that time generations G_4 to G_{12} inclusive will already have begun.

Let us, therefore, take the sum of $G_1+G_2+G_3+$...........to n generations. We have, including the original ancestor or stem-mother—

$$1+G_1+G_2+G_3+\ldots\ldots\ldots+G_n=1+G+G^2+G^3+\ldots\ldots\ldots+G^n$$
$$=\frac{(1-G^n)}{(1-G)}.$$

We have also $G=T^1+T^2+T^3+T^4$, which may be written—

$$\frac{T-T^5}{1-T}$$

as division will prove.

Substituting, we have—

$$1+G_1+\ldots\ldots\ldots+G_n=\frac{1-\left\{\dfrac{T^1-T^5}{1-T}\right\}^n}{1-\left\{\dfrac{T^1-T^5}{1-T}\right\}}$$

$$=\frac{(1-T)\ (1-\{T^1+T^2+T^3+T^4\}^n)}{1-2T+T^5}$$

$$=\frac{(1-T)\ (1-T^n-\text{higher powers of } T)}{1-2T+T^5}$$

$$=\frac{(1-T-T^n-\text{higher powers of } T).}{1-2T+T^5}$$

Now here, exactly as in the case of—

$$\left\{\frac{T-T^5}{1-T}\right\}.$$

we have only to divide the numerator by the denomination to get a series in which the index of T will designate the day and the coefficient of T the total number of births for that day.

If we suppose n is infinitely large, then T^n and the powers of T following it may be neglected, and the expression becomes—

$$\frac{(1-T)}{(1-2T+T^5)}$$

which is the same thing as

$$1+G+G^2+G^3+\dots\dots\dots+G^n$$

summed to infinity as a diminishing series

$$=\frac{1}{1-G}.$$

When we substitute—

$$\left\{\frac{T-T^5}{1-T}\right\}$$

for G.

The same method of procedure, up to this point, may obviously be used whatever the initial value of T, whatever the number of terms in the series for G_1, and whatever the daily birth rate, provided it is constant and regular.

Thus we have—

$$G_1=(T^3+T^4+T^5+T^6+T^7)\text{ ; this can be written}$$

$$G_1=T^3\frac{(1-T^5)}{1-T}=\frac{T^3-T^8}{1-T};$$

so that $G^1+G^2+G^3+\dots\dots\dots\dots$

$$=\frac{1-T}{1-T-T^3+T^8}$$

If we have—

$$G_1=(dT^3+dT^4+dT^5+dT^6+dT^7)=\frac{d(T^3-T^8)}{(1-T)}$$

the formula will be—

$$1+G_1+G_2+G_3+\dots\dots\dots\text{ etc.}=\frac{1-T}{(1-T-dT^3+dT^8)}$$

If the number of births on the successive days of the period of reproduction is not the same, suppose these births to be a, b, c, d, as before. We then have—

$$G=aT^1+bT^2+cT^3+dT^4$$

$$G_2=(aT^1+bT^2+cT^3+dT^4)^2=G^2\text{ etc., and}$$

$$1+G_1+G_2+\dots\dots\dots=\frac{1}{1-G}$$

$$=\frac{1}{1-(aT^1+bT^2+cT^3+dT^4)}$$

which will be valid even when there are no births on one or more days of the reproductive period, or, in other words, when one or more of the coefficients of T are made equal to 0. ...

It is evident that if the organism continues to live after it has ceased to reproduce, this will still be valid, so that if the life is of l days, the series of births must be multiplied by—

$$\frac{(T^1-T^l)}{(1-T)}$$

to get the population series. Thus, if the formula of the births series is—

$$\frac{1-T}{1-2T+T^5}$$

the formula for the population series will be—

$$\frac{1-T^l}{1-2T+T^5}$$

The rule for finding the successive members of the series is precisely the same as before, after the term T^l has been obtained.

The series given by the formulae includes, of course, only the individuals which are capable of reproducing themselves, *i.e.*, only the females; to obtain the total number of individuals, each member of the series must be multiplied by a figure representing the ratio of total population to females, *e.g.*, by 2, if the numbers of males and females in the progeny are equal, on the average; by 4, if there are three times as many males as females, etc.

Provided we know the length of the reproductive period we can readily obtain from observations giving either the series of daily births, or the series giving the total number of individuals produced up to and including each day, the formula for individual reproduction.

It has already been shown that if the generation formula is represented by G, the series of daily births is given by the formula—

$$\frac{1}{1-G}.$$

Thus, if the series be represented by S, we have—

$$S = \frac{1}{1-G}, \text{ whence}$$

$$G = 1-\frac{1}{S}.$$

Thus, suppose the series of daily births, due to an individual reproductive period of four days, is—

$$1+1T^1+3T^2+8T^3+21T^4+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,$$

we obtain for the generation formula, by dividing the above series into 1 and subtracting the result from 1 :—

$$G=1T^1+2T^2+3T^3+4T^4.$$

APPENDIX.

PROBLEMS OF SIMPLE PROPAGATION WHOSE SOLUTION MAY BE FACILITATED BY THE USE OF OBJECTIVE SYMBOLS.

By H. E. SOPER,

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The problem of finding the resultant at any time, from given independent incoming births, or outgoing, *i.e.*, destroyed, births of the operation of a given law of reproduction and the converse problems of how to find the independent birth happenings, knowing the resultant births and the law, and how to find the law knowing the independent birth happenings and resultant births, and many such problems, are susceptible of analysis by a process, akin to that known as "generating functions" (akin also to chemical formulation), but which is more aptly described as one that employs objective symbols to indicate and array the differing characters of the things enumerated.

In the present instance we enumerate " events " ; and the " time of occurrence " is the characteristic that we desire to set against any event. Let then T be the symbol, whose power or index has the duty of indicating the number of units of time (which unit we shall call a day for convenience although any other period may be chosen), from any zero time, at which the event happens. Hence T^0 (or 1), T^1 (or T), T^2, ... T^t, ... indicate zero time and times 1, 2, ... t, ... days after zero and an array or formula as,

$$F_0=2+5T+11T^2+25T^3+\dots, \qquad \dots \quad \dots \quad \dots \quad (i)$$

will be read as " 2 events at zero time, 5 events one day after, 11 events two days after, 25 events three days after zero ... ."

Suppose the event is a birth and suppose each event gives rise to other like events, that is births, at known intervals, then the law of reproduction can also be expressed in symbols and, for instance, the law,

$$G=5T^3+10T^5+3T^8, \qquad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (ii)$$

would indicate 5 births 3 days after, 10 births 5 days after, and 3 births 8 days after zero, which can be the birth of the reproducing organism.

In this case it will be clear that if F_0 stand for a births formula as (*i*) and G for a generation formula as (*ii*), then

$$F^1=F_0 \times G \qquad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (iii)$$

will formulate, in the time symbol T, the births F_1 in the first generation arising from the original births F_0.

Here it may be pointed out that T is an objective symbol only, without the quantitative connotation of such a symbol as t in algebra, and that other large letters as F and G stand for arrays of the symbol T such as (*i*) or (*ii*). We interpret $11T^2$ in statistics in much the same way as we should interpret $11H^2$ in chemistry, but in statistics the convention leads to a useful calculus, based upon such algebraic renderings as (*iii*).

The following elementary propositions may now be enunciated.

Proposition I.—*The independent or original incoming births being* F_0 *and the generation law being* G *to find the births in the first and each succeeding generation and the total births.*

The original births are ... ... F_0.
The 1st generations births are ... ... F_0G.
The 2nd „ „ „ ... ... F_0G^2.

....

The r-1th „ „ „ ... ... F_0G^{r-1}.
The total births to the r-1th generation are ... $F_0 \dfrac{1-G^r}{1-G}$.
The total births to unlimited generations are ... $\dfrac{F_0}{1-G}$.*

Proposition II.—*To find the population under these conditions when each birth survives* s *days or unit intervals.*

A birth at zero time gives rise to a population 1 in the succeeding interval and 1 in each succeeding interval to s intervals. This succession may be represented by

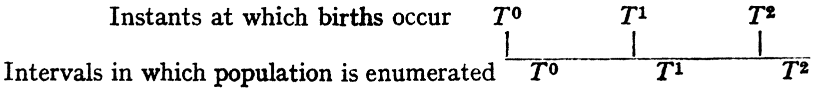
$$1+T+T^2+\dots+T^{s-1}=\frac{1-T^s}{1-T}.$$

---

* We say that G^r is negligible when $r=\infty$, not because $G<1$, but because we do not concern ourselves with the exact enumeration of events at infinite time.

Hence from births F formulated as occurring at successive *instants of time* 0 1 2 ...... there results a population in the succeeding *intervals of time*,† which we may call the 0 1 2 ... intervals, formulated as,

$$F\frac{1-T^{s}}{1-T}$$



Proposition III.—*To find the independent incoming births* F_0 *that must be postulated to give rise to the observed total births* F *under the known generation law* G.

Since, by Proposition I, $F = \frac{F_0}{1-G}$,

it follows that, $F_0 = F\,(1-G)$.

Proposition IV.—*To find the generation law, given the independent incoming or originating births* F_0 *and the observed total births* F.

Since, by Proposition I, $F = \frac{F_0}{1-G}$,

it follows that, $G = 1-\frac{F_0}{F}$.

Proposition V.—*A series of events being expressed as an array* F *in the symbol* T, *to find the slope of the curve and its inflexion and to find the successive sums or totals, these also being expressed in array form.*

The slope or backward first difference is $(1-T)F$.

The inflexion or backward 2nd difference is $(1-T)^2F$.

The sum to (and including) the instant is $\frac{F}{1-T}$.

The second, third, etc., sums are $\frac{F}{(1-T)^2}$, $\frac{F}{(1-T)^3}$, etc.

The succeeding items follow from the first, and the first is obvious since if

$$F=a+bT+cT^2+ \;......$$

then

$$(1-T)F=a+(b\text{-}a)T+(c\text{-}b)T^2+ \;......$$

Many other propositions could be stated, but the process is so elementary that the student will be able, with the help of the above examples and using similar algebraical renditions and transformations, to evolve many desired results.

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† It would perhaps have accorded better with our common conceptions had populations been put at instants and births in intervening intervals of time.

21. The Population Consequences of Life History Phenomena

LAMONT C. COLE (1954)

From *Quarterly Review of Biology* 29. Excerpts are from pages 107—115.

The part of Cole's article included here outlines the relationship of Thompson's work (1931, paper 20 above) to stable population theory. Omitted sections discuss the relationships between the rate of increase, numbers and spacing of offspring, and age structure, with their implications for species survival.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_21,

Quite recently a number of ecologists have recognized the importance of a knowledge of the value of r for non-human populations and have computed its value for various species by employing empirical values of age-specific birth rates and survivorship (Leslie and Ranson, 1940; Birch, 1948; Leslie and Park, 1949; Mendes, 1949; Evans and Smith, 1952). While Chapman's term "biotic potential" would seem to have ecological merit as the name for this parameter r it has been variously called by Lotka the "true," the "incipient," the "inherent," and the "intrinsic" rate of increase, and by Fisher (1930) the "Malthusian parameter" of population increase. Probably for the sake of stabilizing nomenclature it is advisable to follow the majority of recent writers and refer to r as "the intrinsic rate of natural increase."

In the works of Dublin and Lotka (1925), Kuczynski (1932), and Rhodes (1940) on human populations and in the papers mentioned above dealing with other species, the value of r has typically been determined by some application of three fundamental equations developed by Lotka (1907a, b; Sharpe and Lotka, 1911). He showed that if the age-specific fecundity (b_x) and survivorship (l_x) remained constant, the population would in time assume a fixed or "stable" age distribution such that in any interval of age from x to $x + dx$ there would be a fixed proportion (c_x) of the population. Once this stable age distribution is established the population would grow exponentially according to our formula (1′) and with a birth rate per head, β. Then the following equations relate these quantities:

$$\int_0^\infty e^{-rx} l_x b_x \, dx = 1 \qquad (3)$$

and

$$\int_0^\infty e^{-rx} l_x \, dx = 1/\beta \qquad (4)$$

$$\beta e^{-rx} l_x = c_x. \qquad (5)$$

While the use of formulas (3), (4), and (5) to compute the value of r often presents practical difficulties owing to the difficulty of approximating the functions l_x and b_x by a mathematical function, and also because the equations usually must be solved by iterative methods, it may fairly be stated that Lotka's pioneer work establishing these relationships provided the methods for interpreting the relationships between life history features and their population consequences.

However, the exceedingly important ecological questions of what potential advantages might be realized if a species were to alter its life history features have remained largely unexplored. Doubtless, as already noted, this is largely to be explained by a certain suspicion felt by biologists toward analyses such as those of Lotka, which seem to involve assumptions very remote from the realities of life histories as observed in the field and laboratories. A particularly pertinent statement of this point of view is that of Thompson (1931), who recognized the great practical need for methods of computing the rate of increase of natural populations of insects adhering to particular life history patterns but who insisted that the reproductive process must be dealt with as a discontinuous phenomenon rather than as a compound interest phenomenon such as that of formula (1′). His methods of computation were designed to give the exact number of individuals living in any particular time period and, while he recognized that the population growth can be expressed in an exponential form such as (1′), he rejected its use on these grounds:

> "In the first place, the constant (r) cannot be determined until the growth of the population under certain definite conditions has been studied during a considerable period; in the second place, no intelligible significance can be attached to the constant after its value has been determined; in the third place, the growth of the population is considered in this formula to be at every moment proportional to the size of the population, which is not true except with large numbers and over long periods and cannot be safely taken as a basis for the examination of experimental data."

In the following sections of the present paper an effort will be made to reconcile these two divergent points of view and to show under what conditions Thompson's "discontinuous" approach and the continuous methods lead to identical results. Practical methods of computation can be founded on either scheme, and there are circumstances where one or the other offers distinct advantages. It is hoped that a theoretical approach to population phenomena proceeding from exact computational methods will clarify the meaning of some of the approximations made in deriving equations such as (3), (4), and (5) by continuous methods, and will stimulate students of ecology to a greater interest in the population consequences of life history phenomena.

Before proceeding to a discussion of potential population growth, one point which has sometimes caused confusion should be mentioned. This concerns the sex ratio and the relative proportions of different age classes in the growing population. Once stated, it is obvious that if a population is always growing, as are the populations in the models used for determining potential population growth, then each age and sex class must ultimately come to grow at exactly the same rate as every other class. If this were not the case the disproportion between any two classes would come to exceed all bounds; the fastest growing class would continue indefinitely to make up a larger and larger proportion of the total population. It is thus intuitively recognizable that with fixed life history features there must ultimately be a fixed sex ratio and a stable age distribution. In discussing potential population growth it is often convenient to confine our attention to females or even to a restricted age class, such as the annual births, while recognizing that the ultimate growth rate for such a restricted population segment must be identical to the rate for the entire population.

SIMPLEST CASES OF POPULATION GROWTH

Non-overlapping generations

The simplest possible cases of population growth from the mathematical point of view are those in which reproduction takes place once in a lifetime and the parent organisms disappear by the time the new generation comes on the scene, so that there is no overlapping of generations. This situation occurs in the many plants and animals which are annuals, in those bacteria, unicellular algae, and protozoa where reproduction takes place by fission of one individual to form two or more daughter individuals, and in certain other forms. Thus in the century plants (*Agave*) the plant dies upon producing seeds at an age of four years or more, the Pacific salmon (*Oncorhynchus*) dies after spawning, which occurs at an age of two to eight years (two years in the pink salmon *O. gorbuscha*), and cicadas breed at the end of a long developmental period which lasts from two years (*Tibicen*) to 17 years in *Magicicada*. For many other insects with prolonged developmental stages such as neuropterans and mayflies potential population growth may be considered on the assumption that generations do not overlap.

In these cases, perhaps most typically illustrated in the case of annuals, the population living in any year or other time interval is simply the number of births which occurred at the beginning of that interval. Starting with one individual which is replaced by b offspring each of which repeats the life history pattern of the parent, the population will grow in successive time intervals according to the series: $1, b, b^2, b^3, b^4, \cdots b^x$. Hence the number of "births," say B_x, at the beginning of any time interval, T_x, is simply b^x which is identical with the population, P_x, in that interval of time. If the population starts from an initial number P_0 we have:

$$P_x = P_0 b^x \tag{6}$$

which is obviously identical with the exponential formula (1′), $P_x = Ae^{rx}$, where the constant A is precisely P_0, the initial population size, and $r = \ln b$; the intrinsic rate of increase is equal to the natural logarithm of the litter size.

If litter size varies among the reproductive individuals, with each litter size being characteristic of a fixed proportion of each generation, it is precisely correct to use the average litter size, say $\bar{b}$, in the computations, so that we have $r = \ln \bar{b}$. Furthermore, if not all of the offspring are viable, but only some proportion, say l_1, survive to reproduce, we shall have exactly $r = \ln \bar{b}^{l_1}$. Thus, mortality and variations in litter size do not complicate the interpretation of population growth in cases where the generations do not overlap. On the other hand, even in species which reproduce only once, if the generation length is not the same for all individuals, this will lead to overlapping generations, and the simple considerations which led to formula (6) will no longer apply. In other words, we can use an average figure for the litter size b but not for the generation length x. It will be shown in the next section, however, that the more general formula (1′) is still applicable.

In these simplest cases the assumption of a geometric progression as the potential form of population growth is obviously correct, and numerous authors have computed the fantastic numbers of offspring which could potentially result from such reproduction. For example, according to Thompson (1942), Linnaeus (1740?) pointed out that if only two seeds of an annual plant grew to maturity per year, a single individual could give rise to a million offspring in 20 years. (In all editions available to the present writer this interesting essay of Linnaeus' is dated 1743, and the number of offspring at the end of twenty years is stated by the curious and erroneous figure 91,296.)

That is, $P_{20} = 2^{20} = e^{20 \ln 2} = 1{,}048{,}576$. Additional examples are given by Chapman (1935, p. 148).

Formulas (1′) or (6) may, of course, also be used in an inverse manner to obtain the rate of multiplication when the rate of population growth is known. For the example given by Molisch (1938, p. 25), referring to diatoms reproducing by binary fission where the average population was observed to increase by a factor of 1.2 per day, we have $1.2 = e^{x \ln 2}$, where x is the number of generations per day. Solving for $1/x$, the length of a generation, we obtain $1/x = \frac{\ln 2}{\ln 1.2} = \frac{.69315}{.18232} = 3.8$ days.

Overlapping generations

Interest in computing the number of offspring which would be produced by a species adhering to a constant reproductive schedule dates back at least to Leonardo Pisano (=Fibonacci) who, in the year 1202, attempted to reintroduce into Europe the study of algebra, which had been neglected since the fall of Rome. One of the problems in his *Liber Abbaci* (pp. 283–84 of the 1857 edition) concerns a man who placed a pair of rabbits in an enclosure in order to discover how many pairs of rabbits would be produced in a year. Assuming that each pair of rabbits produces another pair in the first month and then reproduces once more, giving rise to a second pair of offspring in the second month, and assuming no mortality, Fibonacci showed that the number of pairs in each month would correspond to the series

$$1, 2, 3, 5, 8, 13, 21, 34, 55, \text{etc.},$$

where each number is the sum of the two preceding numbers. These "Fibonacci numbers" have a rather celebrated history in mathematics, biology, and art (Archibold, 1918; Thompson, 1942; Pierce, 1951) but our present concern with them is merely as a very early attempt to compute potential population growth.

Fibonacci derived his series simply by following through in words all of the population changes occurring from month to month. One with sufficient patience could, of course, apply the same procedure to more complicated cases and could introduce additional variables such as deductions for mortality. In fact, Sadler (1830, Book III) did make such computations for human populations. He was interested in discovering at what ages persons would have to marry and how often they would have to reproduce to give some of the rates of population doubling which had been postulated by Malthus (1798). To accomplish this, Sadler apparently employed the amazingly tedious procedure of constructing numerous tables corresponding to different assumptions until he found one which approximated the desired rate of doubling.

Although we must admire Sadler's diligence, anyone who undertakes such computations will find that it is not difficult to devise various ways of systematizing the procedure which will greatly reduce the labor of computation. By far the best of these methods known to the present writer is that of Thompson (1931), which was originally suggested to him by H. E. Soper.

In the Soper-Thompson approach a "generation law" (G) is written embodying the fixed life history features which it is desired to consider. The symbol T^x stands for the x^{th} interval of time, and a generation law such as $G = 2T^1 + 2T^2$ would be read as "two offspring produced in the first time interval and two offspring produced in the second time interval." This particular generation law might, for example, be roughly applicable to some bird such as a cliff swallow, where a female produces about four eggs per year. Concentrating our attention on the female part of the population, we might wish to compute the rate of population growth which would result if each female had two female offspring upon attaining the age of one year and had two more female offspring at the age of two years. The fundamental feature of the Thompson method is the fact that the expression:

$$\frac{1}{1 - G} \tag{7}$$

is a generating function which gives the series of births occurring in successive time intervals. In the algebraic division the indices of the terms T^1, T^2, etc., are treated as ordinary exponents and the number of births occurring in any time interval T^x is simply the coefficient of T^x in the expansion of expression (7). Thus, for our example where $G = 2T^1 + 2T^2$ we obtain:

$$\frac{1}{1 - 2T^1 - 2T^2} = 1 + 2T^1 + 6T^2 + 16T^3 + 44T^4 + 120T^5 + 328T^6 + \cdots,$$

showing that one original female birth gives rise to 328 female offspring in the sixth year. The series could be continued indefinitely to obtain the number of births any number of years hence. However, in practice it is not necessary to continue the division. In the above series the coefficient of each term is simply twice the sum of the coefficients of the two preceding terms; hence the generation law gives us the rule for extending the series. $G = 2T^1 + 2T^2$ instructs us to obtain each new term of the series by taking twice the preceding term plus twice the second term back. In the case of the Fibonacci numbers we would have $G = T^1 + T^2$, telling us at once that each new term is the sum of the two preceding it.

From the birth series we can easily obtain the series enumerating the total population. If each individual lives for λ years, the total population in T^x will be the sum of λ consecutive terms in the expansion of the generating function. Multiplying formula (7) by the length of life expressed in the form $1 + T^1 + T^2 + T^3 + \cdots + T^{\lambda-1}$ will give the population series. In our above example if we assume that each individual lives for three years, although, as before, it only reproduces in the first two, we obtain for the population

$$\frac{1 + T^1 + T^2}{1 - 2T^1 - 2T^2} = 1 + 3T^1 + 9T^2 + 24T^3 + 66T^4 + 180T^5 + 492T^6 + \cdots,$$

a series which still obeys the rule $G = 2T^1 + 2T^2$.

Thompson's method for obtaining the exact number of births and members of the population in successive time intervals is very general. As in the case of non-overlapping generations, the coefficients in the generation law may refer to average values for the age-specific fecundity. Also the length of the time intervals upon which the computations are based can be made arbitrarily short, so that it is easy to take into account variations in the age at which reproduction occurs. For the above example, time could have been measured in six-month periods rather than years so that the generation law would become $G = 2T^2 + 2T^4$, with the same results already obtained.

Furthermore, the factor of mortality can easily be included in the computations. For example, suppose that we wish to determine the rate of population growth for a species where the females have two female offspring when they reach the age of one, two more when they reach the age of two, and two more when they reach the age of three. Neglecting mortality, this would give us the generation law $G = 2T^1 + 2T^2 + 2T^3$. If we were further interested in the case where not all of the offspring survive for three years, the coefficients in the generation law need only be multiplied by the corresponding survivorship values. For example, if one-half of the individuals die between the ages of one and two, and one half of the remainder die before reaching the age of three we would have $l_1 = 1$, $l_2 = \frac{1}{2}$, $l_3 = \frac{1}{4}$, and the above generation law would be revised to $G = 2T^1 + T^2 + \frac{1}{2}T^3$. The future births per original individual would then be

$$\frac{1}{1 - G} = 1 + 2T^1 + 5T^2 + 25/2T^3 + 31T^4 + 151/2T^5 + \cdots.$$

Very generally, if the first reproduction for a species occurs at some age α and the last reproduction occurs at some age ω, and letting b_x and l_x represent respectively the age-specific fecundity and survivorship, we may write the generation law as:

$$G = l_\alpha b_\alpha T^\alpha + l_{\alpha+1} b_{\alpha+1} T^{\alpha+1} + \cdots + l_\omega b_\omega T^\omega = \sum_{x=\alpha}^{\omega} l_x b_x T^x. \tag{8}$$

Therefore, in the Thompson method we have a compact system of computation for obtaining the exact number of births and the exact population size at any future time, assuming that the significant life history features (α, ω, l_x, and b_x) do not change.

Not all of the possible applications of Thompson's method have been indicated above. For example, formula (7) may be used in an inverse manner so that it is theoretically possible to work back from a tabulation of births or population counts made in successive time intervals and discover the underlying generation law. Formulas (7) and (8), together with the procedure of multiplying the birth series by the length of life expressed as a sum of T^x values, provide the nucleus of the system and offer the possibility of analyzing the potential population consequences of essentially any life-history phenomena. The system has the merit of treating the biological units and events as discontinuous variates, which, in fact, they almost always are. The members of populations are typically discrete units, and an event such as reproduction typically occurs at a point in time with no spreading out or overlapping between successive litters. While survivorship, l_x, as a population quantity, is most realistically regarded as continuously changing in time, the product $l_x b_x$ which enters our computations by way of formula (8) is typically discontinuous because of the discontinuous nature of b_x.

It is quite obvious that equations of continuous variation such as (1′) are often much more convenient for purposes of computation than the series of values obtained by expanding (7). This is especially true in dealing with the life histories of species which have long reproductive lives. In writing a generation law for man by (8) we should have to take α at least as small as 15 years and ω at least as great as 40 years, since for the population as a whole reproduction occurs well outside of these extremes and it would certainly be unrealistic to regard b_x as negligibly small anywhere between these limits. Thus there would be at least 25 terms in our generation law, and the computations would be extremely tedious. By selecting special cases for study it is sometimes possible greatly to simplify the procedures. For example, if one is interested in the case where there is no mortality during the reproductive span of life and where the litter size is a constant, say b, the expression for the generation law (8) can be simplified to:

$$G = bT^{\alpha} + bT^{\alpha+1} + \cdots bT^{\omega} = \frac{bT^{\alpha} - bT^{\omega+1}}{1 - T}.$$

Since one can also write the length of life as

$$1 + T^1 + T^2 + \cdots + T^{\lambda-1} = \frac{1 - T^{\lambda}}{1 - T},$$

the generating function for the total population simplifies to

$$\frac{1 - T^{\lambda}}{1 - T - bT^{\alpha} + bT^{\omega+1}}.$$

This last formula is much more convenient for computations than one containing 25 terms or so in the denominator, but it applies only to a very special case and is much less convenient than formula (1′). Consequently, great interest attaches to these questions: can (1′) be used as a substitute for (7)? (i.e., does Thompson's method lead to a geometric progression?) and, if it is so used, can the constants, particularly r, be interpreted in terms of life-history features?

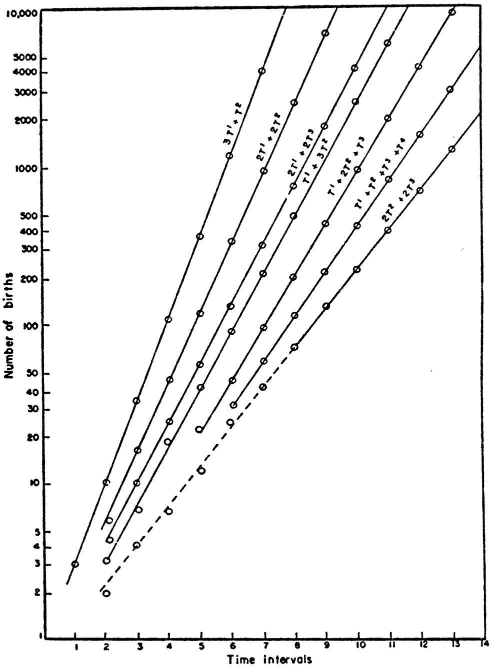

FIG. 1. EXACT VALUES OF POPULATION GROWTH IN TERMS OF BIRTHS PER UNIT TIME UNDER SEVERAL GENERATION LAWS, WHEN EACH FEMALE HAS A TOTAL OF FOUR FEMALE OFFSPRING

In each case it is assumed that a single female exists at time zero and produces her four progeny on or before her fourth birthday. The plotted points represent exact values as determined by Thompson's method. To the extent that the points for any generation law fall on a straight line in this logarithmic plot, they can be represented by the exponential growth formula (1′), and the slope of each line is a measure of the intrinsic rate of natural increase (r).

THE GENERALIZATION OF THOMPSON'S METHOD

Fig. 1 shows the exact values, as determined by Thompson's method, of the birth series arising from several generation laws (life-history patterns) which have in common the feature that in each case every female produces a total of four female offspring in her lifetime and completes her reproductive life by the age of four "years." The number of births is plotted on a logarithmic scale, hence if it can be represented by formula (1′), $P = Ae^{rx}$ or, logarithmically, $\ln P = \ln A + rx$, the points should fall on a straight line with slope proportional to r. It is apparent from Fig. 1 that after the first few time intervals the points in each case are well represented by a straight line. Therefore, except in the very early stages, formula (1′) does give a good representation of potential population growth. The question remains, however, as to whether we can meet Thompson's objection to (1′) and attach any intelligible significance to the constants of the formula. From Fig. 1 it is obvious that the lines do not, if projected back to time 0, indicate exactly the single individual with which we started. Thus, in these cases the constant A cannot be precisely P_0 as was the case with non-overlapping generations.

Before proceeding to interpret the constants of formula (1′) for the case of overlapping generations it will be well to notice one feature of Fig. 1 which is of biological interest. In the literature of natural history one frequently encounters references to the number of offspring which a female can produce per lifetime, with the implication that this is a significant life-history feature. The same implication is common in the literature dealing with various aspects of human biology, where great emphasis is placed on the analysis of total family size. From Fig. 1 it will be seen that this datum may be less significant from the standpoint of contributions to future population than is the age schedule upon which these offspring are produced. Each life history shown in Fig. 1 represents a total production of four offspring within four years of birth, but the resulting rates of potential population growth are very different for the different schedules. It is clear that the cases of most rapid population growth are associated with a greater concentration of reproduction into the early life of the mother. This is intuitively reasonable because we are here dealing with a compound interest phenomenon and should expect greater yield in cases where "interest" begins to accumulate early. However, the writer feels that this phenomenon is too frequently overlooked in biological studies, possibly because of the difficulty of interpreting the phenomenon quantitatively.

In seeking to reconcile the continuous and discontinuous approaches to potential population growth, let us first note that Thompson's discontinuous method corresponds to an equation of finite differences. We have seen above that the generation law gives us a rule for indefinitely extending the series representing the population size or the number of births in successive time intervals by adding together some of the preceding terms multiplied by appropriate constants. If we let $f_{(x)}$ represent the coefficient of T^x in the expansion of the generating function (7) and, for brevity, write in (8) $V_x = l_x b_x$, then our population series obeys the rule:

$$f_{(x)} = V_\alpha f_{(x-\alpha)} + V_{\alpha+1} f_{(x-\alpha-1)} + \cdots + V_\omega f_{(x-\omega)}, \qquad (9)$$

which may be written in the alternative form,

$$f_{(x+\omega)} - V_\alpha f_{(x+\omega-\alpha)} - V_{\alpha+1} f_{(x+\omega-\alpha-1)} - \cdots - V_\omega f_{(x)}, = 0. \qquad (10)$$

Thus for our "cliff swallow" example, where we had $G = 2T^1 + 2T^2$ we have

$$f_{(x)} = 2f_{(x-1)} + 2f_{(-2)} \text{ or,}$$
$$f_{(x+2)} - 2f_{(x+1)} - 2f_{(x)} = 0.$$

Formula (10) represents the simplest and best understood type of difference equation, a homogeneous linear difference equation with constant coefficients. It is outside the scope of the present paper to discuss the theory of such equations, which has been given, for example, by Jordan (1950). By the nature of our problem as summarized in formula (9), all of our V_x values are either equal to zero or are positive real numbers and all of the signs of the coefficients in (9) are positive: features which considerably simplify generalizations. By virtue of these facts it can be shown that there is always a "characteristic" algebraic equation corresponding to (10). This is obtained by writing ρ^x for $f_{(x)}$ and dividing through by the ρ value of smallest index. This gives

$$\rho^\omega - V_\alpha \rho^{\omega-\alpha} - V_{\alpha+1}\rho^{\omega-\alpha-1} \ldots - V_\omega = 0 \qquad (11)$$

an algebraic equation which has the roots ρ_1, ρ_2, etc.

The general solution of the corresponding difference equation (10) is

$$f_{(x)} = C_1\rho_1^x + C_2\rho_2^x + \cdots + C_n\rho_n^x \qquad (12)$$

where the C's are constants to be determined by the initial conditions of the problem. Formula (12) is precisely equivalent to Thompson's method and is a general expression for the number of births or the population size in any future time interval.

As an example we may consider the case where $G = 2T^1 + 2T^2$. The difference equation, as already noted, is $f_{(x+2)} - 2f_{(x+1)} - 2f_{(x)} = 0$ and the characteristic algebraic equation is $\rho^2 - 2\rho - 2 = 0$ which is a quadratic equation with the roots $\rho_1 = 1 + \sqrt{3}$, and $\rho_2 = 1 - \sqrt{3}$. Hence the general solution is $f_{(x)} = C_1(1+\sqrt{3})^x + C_2(1-\sqrt{3})^x$. To determine the constants C_1 and C_2 we look at the beginning a of the seriesnd note that we have $f_{(0)} = 1$ and $f_{(1)} = 2$. Substituting these values in the general solution we obtain $C_1 = \frac{\sqrt{3}+1}{2\sqrt{3}}$ and $C_2 = \frac{\sqrt{3}-1}{2\sqrt{3}}$. Therefore, the general expression for the number of births in time interval T^x is

$$f_{(x)} = \frac{\sqrt{3}+1}{2\sqrt{3}}(1+\sqrt{3})^x + \frac{\sqrt{3}-1}{2\sqrt{3}}(1-\sqrt{3})^x$$

which can be simplified to $f_{(x)} = \frac{\rho_1^{x+1} - \rho_2^{x+1}}{\sqrt{3}} = \rho_1^x + \rho_1^{x-1}\rho_2 + \cdots + \rho_2^x$.

In order to have the difference equation (12) correspond to the equation of exponential growth (1′), the ratio between populations in successive time intervals must assume a constant value giving

$$\frac{f_{(x+1)}}{f_{(x)}} = \mathrm{e}^r. \tag{13}$$

By the nature of our problem, as already noted, the potential population growth is always positive, so that any limit approached by the ratio $\frac{f_{(x+1)}}{f_{(x)}}$ must be a positive real number.

It is beyond the scope of the present paper to discuss the conditions, for difference equations in general, under which this ratio does approach as a limit the largest real root of the characteristic algebraic equation. (See, for example, Milne-Thompson, 1933, chap. 17). Dunkel (1925) refers to the homogeneous equation with real constant coefficients corresponding to our formulas (10) and (11). The algebraic equation (11) has a single positive root which cannot be exceeded in absolute value by any other root, real or complex. Using (12) to express the ratio between successive terms, we have

$$\frac{f_{(x+1)}}{f_{(x)}} = \frac{C_1\rho_1^{x+1} + C_2\rho_2^{x+1} + \cdots + C_n\rho_n^{x+1}}{C_1\rho_1^x + C_2\rho_2^x + \cdots + C_n\rho_n^x}. \tag{14}$$

If we let ρ_1 represent the root of (11) of greatest absolute value and divide both numerator and denominator of (14) by $C_1\rho_1^x$ we obtain

$$\frac{f_{(x+1)}}{f_{(x)}} = \rho_1\left[\frac{1+\frac{C_2}{C_1}\left(\frac{\rho_2}{\rho_1}\right)^{x+1} + \frac{C_3}{C_1}\left(\frac{\rho_3}{\rho_1}\right)^{x+1} + \cdots + \frac{C_n}{C_1}\left(\frac{\rho_n}{\rho_1}\right)^{x+1}}{1+\frac{C_2}{C_1}\left(\frac{\rho_2}{\rho_1}\right)^{x} + \frac{C_3}{C_1}\left(\frac{\rho_3}{\rho_1}\right)^{x} + \cdots + \frac{C_n}{C_1}\left(\frac{\rho_n}{\rho_1}\right)^{x}}\right]. \tag{15}$$

The expressions in parentheses are all less than unity, on the assumption that ρ_1 is the largest root, and the entire expression in brackets approaches unity as x increases. Consequently we have, for x large

$$\frac{f_{(x+1)}}{f_{(x)}} \sim \rho_1 \sim \mathrm{e}^r. \tag{16}$$

This then explains the shape of the potential birth and population series as illustrated in Fig. 1. In the very early stages population growth is irregular, because the expressions in (12) and (15) involving the negative and complex roots of (11) are still large enough to exert an appreciable influence. As x increases, the influence of these other roots becomes negligible and the population grows exponentially, conforming to (16). In considering potential population growth we are concerned with the ultimate influence of life-history features, and the equation of geometric progression or compound interest does actually represent the form of potential population growth. We are interested only in the single positive root of (11) for the purpose of determining the constant r, and this can readily be computed with any desired degree of precision by elementary algebraic methods.

Having established the relationship of formula (13) or (16), it is easy to reconcile Thompson's discontinuous approach to population growth with Lotka's continuous approach, as exemplified by formulas (3), (4), and (5).

Employing formula (9) we may write the ratio between populations in successive time intervals as

$$\frac{(f_{x+1})}{f_{(x)}} = V_\alpha\frac{f_{(x-\alpha+1)}}{f_{(x)}} + V_{\alpha+1}\frac{f_{(x-\alpha)}}{f_{(x)}} + \cdots + V_\omega\frac{f_{(x-\omega+1)}}{f_{(x)}}.$$

Substituting the relationship given by (13), this becomes

$$\mathrm{e}^r = V_\alpha\mathrm{e}^{-r(\alpha-1)} + V_{\alpha+1}\mathrm{e}^{-r\alpha} + \cdots + V_\omega\mathrm{e}^{-r(\omega-1)}, \text{or}$$

$$1 = V_\alpha\mathrm{e}^{-r\alpha} + V_{\alpha+1}\mathrm{e}^{-r(\alpha+1)} + \cdots + V_\omega\mathrm{e}^{-r\omega}.$$

Replacing V_x by its equivalent, $l_x b_x$, this is

$$1 = \sum_{x=\alpha}^{\omega} e^{-rx} l_x b_x. \qquad (17)$$

Formula (17) is the precise equivalent in terms of finite time intervals of Lotka's equation (3) for infinitesimal time intervals. In Lotka's equation, as in (17), the limits of integration in practice are α and ω since b_x is zero outside of these limits. Formula (17) was in fact employed by Birch (1948) as an approximation to (3) in his method of determining r for an insect population. The only approximation involved in our derivation of (17) is the excellent one expressed by formula (13); otherwise the formula corresponds to Thompson's exact computational methods. It is hoped that recognition of this fact will make some of the approaches of population mathematics appear more realistic from the biological point of view.

Formulas (4) and (5), originally due to Lotka, are also immediately derivable from the relationship (13). In any time interval, T_x, we may say that the population members aged 0 to 1 are simply the births in that interval, say B_x. The population members aged 1 to 2 are the survivors of the births in the previous interval, that is $l_1 B_{x-1}$, or employing (13), $l_1 B_x e^{-r}$. Quite generally, the population members aged between z and $z + 1$ are the survivors from the birth z intervals previous, or $l_z B_x e^{-rz}$. If λ is the extreme length of life for any population members ($l_{\lambda+1} = 0$) we have for the total population

$$P_x = B_x(1 + l_1 e^{-r} + l_2 e^{-2r} + \cdots + l_\lambda e^{-rz} = B_x \sum_{x=0}^{\lambda} e^{-rx} l_x.$$

The birth rate per individual, β, is B_x/P_x, therefore,

$$1/\beta = \sum_{0}^{\lambda} e^{-rx} l_x \qquad (18)$$

which is the equivalent in finite time intervals of Lotka's equation (4). Also the proportion, c_z, of the population in the age range z to $z + 1$ is $\dfrac{l_z B_x e^{-rz}}{P_x}$ which is simply,

$$c_z = \beta e^{-rz} l_z. \qquad (5)$$

LIST OF LITERATURE

Archibold, R. C. 1918. A Fibonacci series. *Amer. math. Monthly*, 25: 235–238.

Birch, L. C. 1948. The intrinsic rate of natural increase of an insect population. *J. Anim. Ecol.*, 17: 15–26.

Chapman, R. N. 1938. *Animal Ecology*. McGraw-Hill, New York.

Dublin, L. I., and A. J. Lotka. 1925. On the true rate of natural increase as exemplified by the population of the United States, 1920. *J. Amer. statist. Ass.*, 20: 305–339.

Dunkel, O. 1925. Solutions of a probability difference equation. *Amer. math. Monthly*, 32: 354–370.

Evans, F. C., and F. E. Smith. 1952. The intrinsic rate of natural increase for the human louse, *Pediculus humanus* L. *Amer. Nat.*, 86: 299–310.

Fibonacci, L. (1857). *Liber Abbaci di Leonardo Pisano publicati da Baldasarre Boncompagni.* Tipografia delle Scienze mathematiche e fisiche, Roma.

Fisher, R. A. 1930. *The Genetical Theory of Natural Selection.* Oxford Univ. Press, London.

Jordan, C. 1950. *Calculus of Finite Differences*, 2nd ed. Chelsea, New York.

Kuczynski, R. R. 1932. *Fertility and Reproduction.* Falcon Press, New York.

Leslie, P. H., and R. M. Ranson. 1940. The mortality, fertility, and rate of natural increase of the vole (*Microtus agrestis*) as observed in the laboratory. *J. Anim. Ecol.*, 9: 27–52.

——, and T. Park. 1949. The intrinsic rate of natural increase of *Tribolium castaneum* Herbst. *Ecology*, 30: 469–477.

Linnaeus, C. 1743. Oratio de telluris habitabilis. In *Amoenitates Academicae seu Dissertationes Variae....* Editio tertia curante J. C. D. Schrebero, 1787, Vol. 2: 430–457. J. J. Palm, Erlangae.

Lotka, A. J. 1907a. Relation between birth rates and death rates. *Science*, 26: 21–22.

——. 1907b. Studies on the mode of growth of material aggregates. *Amer. J. Sci.*, 24: 119–216.

Malthus, T. R. 1798. *An Essay on the Principle of Population as it Affects the Future Improvement of Society, with Remarks on the Speculations of Mr. Godwin, M. Condorcet, and Other Writers.* Printed for J. Johnson in St. Paul's Churchyard, London.

Mendes, L. O. 1949. Determinação do potencial biotico da "broca do café"—*Hypothenemus hampei* (Ferr.)—e consideracões sôbre o crescimento de sua população. *Ann. Acad. bras. Sci.*, 21: 275–290.

Milne-Thompson, L. M. 1933. *The Calculus of Finite Differences.* Macmillan, London.

MOLISCH, H. 1938. *The Longevity of Plants.* E. H. Fulling, Lancaster Press, Lancaster, Pa.

PIERCE, J. C. 1951. The Fibonacci series. *Sci. Monthly*, 73: 224–228.

RHODES, E. C. 1940. Population mathematics. *J. R. statist. Soc.*, 103: 61–89, 218–245, 362–387.

SADLER, M. T. 1830. *The Law of Population: A Treatise, in Six Books; in Disproof of the Superfecundity of Human Beings, and Developing the Real Principle of their Increase.* Murray, London.

SHARPE, F. R., and A. J. LOTKA. 1911. A problem in age distribution. *Phil. Mag.*, 21: 435–438.

THOMPSON, D'ARCY W. 1942. *On Growth and Form*, new ed. Cambridge Univ. Press, Cambridge.

THOMPSON, W. R. 1931. On the reproduction of organisms with overlapping generations. *Bull. ent. Res.*, 22: 147–172.

Part III

Attempts at Prediction and the Theory They Stimulated

The tools needed for population projections, age-specific fertility and mortality rates, were in existence by the early part of the 19th century. They remained unused almost to the 20th. Edwin Cannan's (1895) article, paper 22, which begins Part III, both established life table rates as the necessary base for competent projections and demonstrated that if care were taken the rates could reasonably be adjusted in anticipation of future changes.

Independently of Cannan's work and later contributions by A. L. Bowley (1924, 1926), P. K. Whelpton (1936) also developed component population projections. Whelpton in his projections for the United States divided total population into ethnic groupings recognized by the census, handling each separately, and considered immigration as an independent component: apart from fertility he followed procedures he and W. S. Thompson had previously worked out (Whelpton 1928: Thompson and Whelpton 1933). The estimates were almost immediately overtaken by events, a war that was not widely expected in 1936 and a postwar baby boom not expected even when it hit. Yet for the few years that the future was orderly it accorded with Whelpton's calculations, and the clear logic of separating age groups has never since been at issue. { **Ed. Note:** Portions of Whelpton's 1936 article were included in the 1977 edition but omitted here. } Mathematical treatment of component-based projection followed shortly afterward with the introduction of matrix methods.

The use of matrices in population projections was first suggested by Harro Bernardelli (1941), paper 23, who noted from the sizes of Burmese birth cohorts that birth peaks occurred at intervals of a generation and that these might carry important economic and social implications, a question examined more recently by Richard Easterlin (1966) for its possible relationship to Kuznets cycles. From a projection matrix formed by holding fertility and mortality constant Bernardelli isolated the characteristic roots of the net maternity function for the discrete case, identifying the dominant root as the intrinsic growth rate of the population and relating the other roots to oscillations in the birth rate. By mischance, Bernardelli utilized a matrix concentrating fertility into a single age category. This gives rise to the exceptional result, which he took to be a general case and drew several spurious conclusions from, that oscillations in the birth rate and hence in the age structure might be maintained or amplified over time rather than becoming damped.

In a paper that followed Bernardelli's address by three months, E. G. Lewis (1942) presented a better but still essentially preliminary exploration of the projection matrix. This was followed by P. H. Leslie's (1945) definitive article, paper 24. Leslie develops the use of matrices fully, with attention to the several types of information they make accessible and their relationship to the earlier theory developed by Lotka. The present widespread use of projection matrices is an extension of his work. A generalized inverse for the Leslie matrix to permit limited backward projection is given by Greville and Keyfitz (1974).

{ **Ed. Note:** Results from the use of generalized inverses are, however, usually quite unstable. When estimates of aggregate counts of births and deaths over earlier historical years are available and certain conditions are satisfied, the quite different technique of "inverse projection" is now available for working out estimates of age structure backward in time. See Ronald Lee's original paper [25], the collection [3], and mathematical results on ergodicity by Wachter [39]. }

We follow Leslie's article with an extension due to Leon Tabah (1968), in which he shows how matrices may be applied to migration and labor force projections as well as to overall population change. { **Ed. Note:** The paper by Tabah is omitted from the present edition. } The method is also explained in Rogers (1968, pp. 6–15).

{ **Ed. Note:** In 1977 Smith and Keyfitz concluded their introduction to Part III with the following paragraph. As explained below, our knowledge of the history has since changed. }

From the development of the projection matrix we turn to the concepts of strong and weak ergodicity, whose initial proofs were developed through matrix formulation but which otherwise are a continuation of the discussion in Part II. A paper { in the 1977 edition, omitted here, } extracts from Alvaro Lopez (1961) a statement of ergodicity. The concluding { omitted } article by Beresford Parlett (1970) compactly summarizes matrix development of ergodicity and general stable population theory. The reader might return from Parlett to the opening article of the chapter to appreciate the impressive achievements in discrete stable theory and determination of characteristic roots and vectors that have followed from Cannan's introduction of simple but correct principles for population projection.

{ **Ed. Note:** In demography ergodic theorems are theorems that assert that the age structure of a population projected into the future becomes independent of the initial age structure. An ergodic theorem for population processes with time-varying rates was actually proved by Henry T. J. Norton [30] and published in the *Proceedings of the London Mathematical Society in 1928.* Unaware of this result, at the urging of Ansley Coale, Alvaro Lopez reproved a version of Norton's theorem. Golubitsky, Keeler, and Rothschild [15] pointed out that the theorem follows from a famous lemma of Birkhoff [4], pages 381–390. A compact elementary proof of this kind is given by Le Bras [23], Chapter 8.

The term "weak ergodicity" is often used by demographers for cases where rates vary over time, with "strong ergodicity" used for cases where rates are assumed constant over time. This terminology is less than ideal, since the weak ergodic theorem implies the strong ergodic theorem.

When rates are constant over time, the age structure of a population generally converges to a stable state independent of the starting age structure, as featured in Part II. Seen from the perspective of renewal theory, this convergence involves a process of averaging, set forth in section 7 of Feller's (1941) paper, paper 16, and in the textbook by Feller [12], pages 181 to 189. A version for demographers is given by Arthur [1].

The age-specific rates that go into population projections may also be taken to vary randomly over time in accordance with some stochastic process. Ergodic theorems in such a setting are proved in a series of papers by Joel E. Cohen (see [8]) and by Le Bras [21]. More wide-ranging theory for time-varying environments is found in Tuljapurkar [36].

Matrix projection methods for demographers are covered extensively in the third edition of Keyfitz' *Applied Mathematical Demography* by Keyfitz and Caswell [19].

When time and age are treated as continuous variables and age-specific rates of fertility and mortality for a closed one-sex population are allowed to vary over time, population dynamics can be described by a partial differential equation which goes back at least to a 1926 paper of McKendrick [27], a pioneer of epidemiology. Boundary conditions along the time axis are supplied by the Renewal Equation. Population as a function of age and time is represented by a surface over the Lexis Plane called a Lexis Surface. The partial derivative with respect to time of the logarithm of the height of this surface is the continuous counterpart of an age-group growth rate. In 1982 Preston and Coale [32] developed theory for age-group growth rates, often called "variable r", leading to an identity reminiscent of Lotka's Equation, as well as to applications exploiting the robustness of age-group growth rates in the face of common kinds of census coverage errors. Arthur and Vaupel [2] elucidated connections to the properties of general paths on the Lexis Surface. Applications to population momentum were explored and clarified by Preston [31], Wachter [40], and Schoen [34]. Further developments relating both to prediction and to parametrization are mentioned in Part IV. }

22. The Probability of a Cessation of the Growth of Population in England and Wales during the Next Century

EDWIN CANNAN (1895)

From *The Economic Journal* 5. Excerpts are from pages 505—514.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_22, © Springer-Verlag Berlin Heidelberg 2013

Many things about which we are habitually obliged to form estimates are of a much more speculative character than the growth of population, and the estimates which we do form about them in many cases actually depend on our estimate of the probable growth of population. There is not a builder nor a town council in the country that is not obliged to prophesy every month what the growth of population in a particular district is likely to be, and it was the speculations of the Metropolitan Water Commission as to the population of London in 1931 that inspired me to make this contribution to the literature of the subject. The real question is not whether we shall abstain altogether from estimating the future growth of population, but whether we shall be content with estimates which have been formed without adequate consideration of all the data available, and can be shown to be founded on a wrong principle.

The generally accepted principle is that of 'as the increase has been in the past so it will be in the future.' This is susceptible of more than one interpretation. All we really know of the increase in the past is contained in the totals obtained by the censuses. A schoolboy whose arithmetic is described as 'V. G.' would probably boldly disregard all the intermediate censuses and divide the total increase of 20,109,989 between 1801 and 1891 by ninety years. The quotient of 223,444 he would call 'the average annual increase,' and say that in order to get the probable increase of population in a certain number of years after 1891 you must simply multiply 223,444 by that number of years. ...

A person with a slight smattering of statistics will probably say that the schoolboy's method is quite wrong, and that in arguing from the past to the future, you ought to consider not the number which has been added to the population, but the factor by which the population has been multiplied. You ought to consider not the absolute amount, but the rate of the increase. But which rate? For there is a different rate in every one of the nine decades. Are we to take the rate from 1801 to 1891, disregarding the intermediate censuses, or are we to somehow deduce some-

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[1] This article contains the substance of a paper read before Section F of the British Association at Ipswich.

thing which may be called an average rate from the nine different rates? The Metropolitan Water Commission groped for an answer to this question and found none. A very able mathematician to whom I gave the figures of the ten censuses tells me that they are so irregular that no law of increase of the smallest value can be deduced from them. The Registrar-General's method in forming what are called the 'official estimates' cuts the knot by disregarding all the decades except the last. The only conceivable argument in favour of this course is the allegation that the immediate past being nearer the future than the long past, affords a better basis for estimating future probability. By itself this would justify the *reductio ad absurdum*, 'if the last ten years are better than the last ninety, the last year must be better than the last ten,' but 'temporary fluctuations' have of course to be considered, and it may perhaps be asserted that ten years is sufficient, while twenty years is more than sufficient, to give an average unaffected by such fluctuations.

Which rate of multiplication be taken, however, does not very much matter, since any rate that can possibly be deduced from the census figures gives ridiculous results. Carried back into the past the assumption of a uniform rate of increase equal to that which prevailed in the decade 1881 to 1891 necessitates the supposition that in A.D. 525 there were scarcely two persons in England, and carried forward into the future it gives a population of a thousand millions in less than 300 years. After 350 years more there would be only just room for the people to lie down on the ground. Yet the rate of increase in the decade 1881—90 is the lowest of the nine.

These periods, it may be urged, are long. But applied to short periods the assumption of a uniform rate of increase equal to that shown by the last decade leads to just as absurd results as when it is applied to long periods. As is well known, the Registrar-General, whose annual reports reveal no acquaintance with the very accurate statistics of immigration and emigration now collected by the Board of Trade, ignores his own figures of births and deaths, on the ground that in estimating a population immigration and emigration cannot be disregarded. The official estimates of the population in the middle of each of the following ten years which he forms after each census on the assumption that the rate of increase shown by it will continue, are allowed to stand in each of his annual reports till the next census, and would, we must suppose, be allowed to stand if a plague swept away half the people. When the next census is taken they are, of course, found to be wrong. [At the last census the error amounted to 701,843 at the end of the decade, while Dr. Longstaff had in the pages of this Journal estimated the population within 10,251.[1]] They are then corrected on the assumption that throughout the decade population annually increased at the rate observed during the whole decade. Now whenever the rate of increase in a decade is less than in the preceding decade, these 'corrected' figures, on which the 'corrected' birth, marriage, and death-rates are calculated, present a very odd and most unnatural series of increases. Thus they represent the increase for 1871–2 to have been 307,901, and make the annual increase grow gradually till in 1879–80 it amounted to 342,799. Then comes the census year, which stands by itself, as three-quarters of it lie in one decade and the other quarter in another; it is allotted an increase of 331,854. After this the new period begins in earnest with an increase of only 288,800 in 1881–2; and the rising process starts again and continues till the increase of 1890–1 amounts to 318,912. In short, the increase grows steadily till it reaches the census of 1881, then it comes down with a crash only to begin immediately to rise again. It must always happen under this system that the addition to the population in the last half of a decade is greater than the addition in the first half, even when the addition in each decade is less than the addition in the preceding decade. The truth is that every estimate of population, past, present, and to come, ought to be founded on a consideration of the factors on which the growth or decline o. population is dependent—births, deaths, immigration, and emigration. The number of births and deaths, and of immigrants and emigrants, is now so well known, that if two government departments, the Board of Trade and the General Registry Office, would only recognise each other's existence, the population at the present time, or at any point in recent years, could be given within ten thousand of the actual number.

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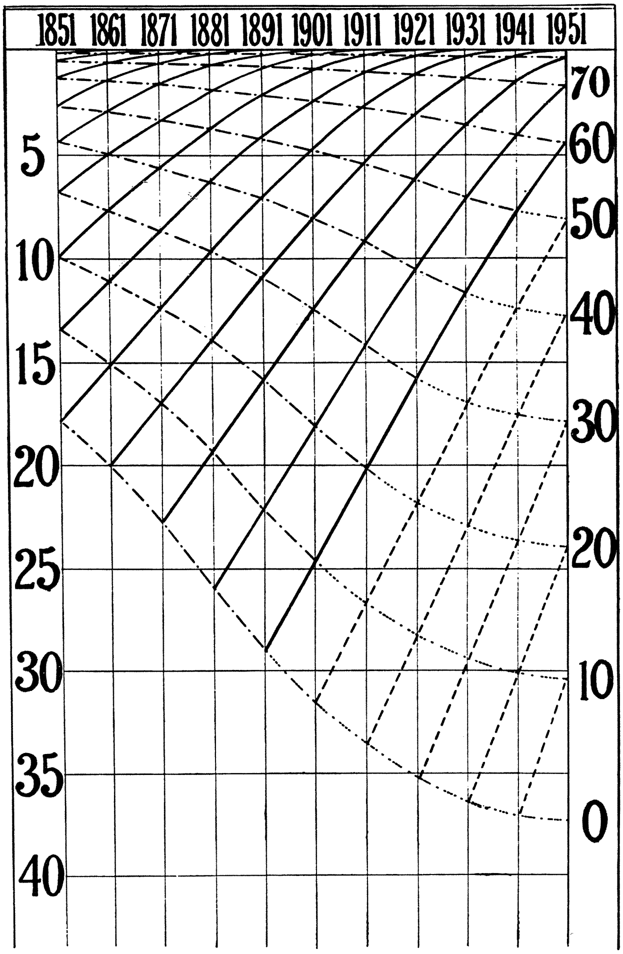
[1] ECONOMIC JOURNAL, vol. i., p. 382.

Population of England and Wales, 1851–1951.



The population in each census year is measured down from the top, the figures on the left indicating the number of millions. The lines sloping downwards from left to right divide the population at each census into persons over 80 ; persons between 70 and 80 ; between 60 and 70 ; and so on down to between 0 and 10. The ages are shown by the figures on the right. The lines sloping upwards from left to right divide the persons born in each decade from those born before and after it. They thus show graphically how each generation becomes smaller as it passes from under ten years of age at one census to over ten but under twenty at the next, and then to over twenty but under thirty, and so on.

In estimating future population the most important data we have to rely upon are the ages of the people as taken at recent censuses. These are shown measured down from the top on the first five perpendicular lines of the diagram opposite. The lines sloping downwards from left to right divide the total population according to each ten years of age from eighty downwards, so that the top layer of persons includes all over eighty years of age, the next all between seventy and eighty,

the next all between sixty and seventy, and so on down to the children under ten. Now as the number of immigrants at any particular decade of age is probably always exceeded by the number of emigrants of that age, the people who are between ten and twenty at one census are (with the exception that some immigrants are substituted for some emigrants) the survivors in England of those who were under ten at the preceding census, and similarly with each ten years of age. So if we call the persons born between one census and the next a generation, the lines sloping upwards from left to right divide each generation from the next, and show how each generation becomes smaller and smaller, till it finally becomes extinct on the death of some centenarian.

Now, provided that the rate of mortality and loss by migration remain the same at each age period, it is possible to calculate with absolute accuracy from the observed decrease of a generation between any two censuses, how much the next generation will decrease when it arrives at the same age. For example, if it be known that everyone over 100 in 1881 died before 1891, it may be confidently assumed that everyone over 100 in 1891 will die before 1901. If it be known that the people between fifty and sixty in 1881 lost about 27 per cent. of their numbers before 1891, it may be taken for granted that the people between fifty and sixty in 1891 will lose the same proportion between 1891 and 1901. Nothing is requisite to get the total population over ten at the next census except to work out by proportion sums the population in each ten years of age and add up the results.

Before, however, we can apply this method, it is necessary to inquire whether we are justified in assuming that the rate of mortality and loss by migration will not change. As to the future of mortality and loss by migration every man has a right to his own opinion, but no one can expect to prove anything. We may therefore fairly treat it as an even chance whether the rate of loss is likely to increase or diminish. In the past a decrease of the rate of loss by mortality has been counteracted by an increase in the rate of loss by migration, so that estimates in which this method has been used have been surprisingly accurate. ...

The question is, therefore, what is the probable number of births? Here we border on considerations of a more speculative character, but we have still something statistical to rely upon.

The fact that it is the custom to calculate birth-rates as a rate per thousand on the whole population makes it natural to say that the number of births depends in the first place upon the population, so that if the population increases rapidly the births will increase rapidly. The continuous lines of the diagram show that the population over certain ages will increase rapidly for many decades, so that we might consequently expect a considerable increase in the number of births. But of course, as a matter of fact the increase of population over and under certain ages has obviously no tendency to increase the number of births. The number of old women and children may be doubled without making it the least more likely that the births will increase until the children have grown to marriageable age. The number of births is more likely indeed to be somewhat repressed, since the old people and children form a burden upon the shoulders of those in the prime of life. It is consequently much more true to say that the number of births depends in the first place upon the number of men and women between certain ages. For practical purposes the ages of twenty and forty are sufficiently near the mark, and they are much more convenient than the ages for each sex which would have to be taken if perfect accuracy were required. Now from 1853 to 1876 the number of births, after allowing for some deficiency in registration, increased rapidly and was almost uniformly just about 12 per cent. on the number of persons between twenty and forty. From 1876 onwards the number of births has been almost stationary, and the rate per cent. on the persons between twenty and forty has consequently been rapidly declining. In ten years it had fallen to 11 per cent.; by 1890 it had further fallen to 10 per cent.; in 1891 it went up to about 10·4 per cent.; in the next two years it was 10 per cent.; in 1894 it descended to 9·8 per cent.

Now if the future births were estimated at this rate on the number of persons between twenty and forty we should have no grounds for expecting a cessation of the growth of population, though the growth would be at a much less rate than heretofore. But the statistics make it probable that the birth-rate on persons between twenty and forty will continue to fall. It must be remembered that the effect of the births having been nearly stationary for twenty years will be to change considerably the age distribution of the group of people between twenty and forty. A larger proportion of them will be at the higher ages. It seems at first sight paradoxical to say that the persons between twenty and forty can be older at one time than another, but it is really quite simple. The persons between 0 and 100 years old obviously do not average fifty years, and in just the same way the persons between twenty and forty do not average thirty. What the exact average age will be depends chiefly on the variation in the number of births between twenty and forty years before. As the number of persons born rose rapidly year by year before 1876, the number of persons becoming twenty years of age in each year has been rising rapidly and will continue to rise rapidly till 1896. After that it will still continue to rise because there has been a reduction of juvenile mortality, but the rise will be far less rapid. The consequence will be that while in 1891 30 per cent. of the twenty to forty group were under twenty-five, in 1911 not more than 27⅓ per cent. will be so.

This increase in the average age of persons in the prime of life, or, to put the same thing in other words, this diminution in the increase of the number of persons reaching marriageable age in each year, must tend somewhat to reduce the birth-rate. How much effect should be attributed to it would not be very difficult to calculate if the necessary statistics of the ages of parents were forthcoming, but unfortunately none such exist.

If, however, we take into account not only the diminution to be expected from this cause, but also the diminution to be expected from the working of the enormously strong economic and social forces which have brought about the diminution of the last twenty years, it seems a very moderate hypothesis to suppose that the rate of births on the number of persons between twenty and forty may fall to a little below 9 per cent. by 1901 and to the neighbourhood of 8 per cent. by 1911. This, with a further slight diminution to a little below 8 per cent. by 1921 is all that is necessary in order to keep the number of births stationary at the level of 1881–90. Adopting then this hypothesis, I have continued the lines of the diagram so as to show what the future of the population will be if the rate of mortality and loss by migration at each age, and also the absolute number of births, remain the same as in 1881–90. It will be seen that the increase of population, large at first, becomes less and less, till it is trifling in 1941–51. It would continue, but always growing less and less, till about 1995, when the last survivor of the period before 1891 would disappear, and the population would then stand at its maximum of 37,376,000.

I have no desire to stake my reputation as a prophet on the growth of population following exactly the line shown in the diagram, and ceasing to increase in 1991. I am only prepared to assert confidently that the line shown is a much more probable one than that which might be laid down by the ‘ official ’ method, and which would shoot through the bottom of the diagram between 1921–31 and encircle the globe before the diagram was widened very many yards. Whether the cessation of the growth of population is reached, as I personally should expect, before 1991, or afterwards, it must be reached at last, and if it is reached without any violent changes in mortality, migration, or natality, it will necessarily be reached by a curve of increase closely approximating to that laid down in the diagram. The value of the diagram lies not in its prediction of a maximum population of thirty-seven millions, but in the fact that it shows how a cessation of growth may be reached within no very long period without any violent or unnatural changes.

23. Population Waves

HARRO BERNARDELLI (1941)

From *Journal of the Burma Research Society* 31, Part 1. Excerpts are from pages 1—4, 6—7.

We omit Bernardelli's discussions of economic cycles and his further comments on cycles in human and animal populations, which are of uneven quality. The appendix referred to in the text is also omitted: it introduces a projection matrix and indicates the derivation of its characteristic roots.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_23,

1. In thinking of population growth usually the notion is applied—consciously or unconsciously—that such growth takes place along a smooth curve: either in the form of Malthus' geometric progression, the interest law, or according to Verhulst-Pearl's logistic equation, or along such curves as have been worked out recently by Dr. Kuczynski and his collaborators in order to trace the decline of the European race. Practical statisticians, no doubt, realise that populations actually show considerable deviations from such models, but they have tended on the whole to consider such "irregularities" as "random variations" which are best smoothed out.

Having worked on a variety of problems which all involve assumptions about the manner in which the numbers of a population change, it appears to me now rather doubtful that the preference given to smooth curves is justified even as a first approximation. Smoothing always means introducing a definite hypothesis about the nature of the causes that determine the value of the variable under consideration; in eliminating the irregular oscillations typical for a population graph, we assume, in fact, that there are two sets of causes operating: a fundamental, basic set of causes which, left alone, would produce the picture of our smooth curve, and a second set—more irregular and haphazard in occurrence and intensity—which accounts for the superimposed aberrations.

The justification of the procedure of smoothing thus depends essentially on the validity of the hypothesis that the basic causes will manifest themselves smoothly. If it can be shown that this is not the case, that the basic causes themselves are bound, in fact, to generate oscillations, then the operation of smoothing will lose its rationale. Its application will merely blur the picture, and possibly even block the way for a correct understanding of the causal effects which we wish to elucidate.

2. The oscillations of which I am talking can easily be demonstrated in any Census Report. The following table, taken from the Burma Census 1931, gives the age-distribution of 10,000 persons of each sex for the indigenous population

I

Age Group	Males 1931	1921	1911	1901	Females 1931	1921	1911	1901
0— 5	1397	1265	1354	1433	1414	1286	1372	1442
5—10	1282	1271	1355	1301	1236	1260	1344	1277
10—15	1186	1223	1236	1143	1134	1146	1140	1046
15—20	916	1002	899	877	1002	1082	966	961
20—25	886	862	775	828	997	941	864	917
25—30	838	777	760	828	848	801	792	850
30—35	776	722	767	799	737	687	727	756
35—40	604	600	656	626	560	539	584	552
40—45	532	571	566	539	504	558	552	521
45—50	412	446	409	407	397	406	374	369
50—55	361	410	373	383	358	418	384	395
55—60	283	252	241	244	278	251	242	241
60—65	239	274	278	—	229	264	278	—
65—70	130	133	133	—	130	126	128	—

of this country[1]. The figures have been compiled for the indigenous inhabitants only, because they reveal best the pure effects of population growth, and are not, in any way, affected by migration.

The table shows a remarkable instability of the age-structure. Its characteristic features can be studied most effectively by averaging for the four censuses the frequencies in each age-group and recording for each census the deviations of the actual frequencies from their mean. The result is shown in Fig. 1.

It will be observed that there was in 1901 an abnormally high frequency of children in the age-group 0—5 which subsequently produced a wave of survivors in the later census records. Similarly the low frequency for the age-groups 10—20 in 1901 can be traced as a depression in the waves for the following three decades. The waves for females (shown in the Census Report 1931 page 73ff.) run exactly parallel. It is obvious that the new wave crest of children which is seen to develop in 1931 originates from the survivors of the wave of female children in 1901.

These waves have a sufficiently large amplitude to exert a significant influence on the economic and social conditions of the country. This became painfully evident during the Great Depression, in 1929 and the succeeding years. There was during those years an unusually large proportion of indigenous males in the age-group 15—30 which forms, of course, the most active element of a population from which the new entries into the labour market are drawn (See Section ABC of the 1921 and DEF of the 1931 wave). With agriculture disorganised by the impact of the unprecedented depression, a large number of these persons—who normally would have found accommodation in agricultural pursuits were forced to look for openings in the industrial labour market which up to that time had been the undisputed domain of the Indian immigrant[2]. ...

This unfortunate constellation of a large wave of indigenous males looking for employment, and of an industrial labour market which offered no chances of employment, undoubtedly contributed to the series of unprecedented riots, and the intense growth of racial animosity, which—busily fanned by interested parties—has never since subsided, and which is likely now to produce the usual crop of obnoxious restrictionism, although—among other things—the labour market is long again in perfect adjustment, and although at the present, and during the next years, the entry of indigenous people into the labour market will be by no means brisk (See the depression GHD of the 1931 wave which is relevant for the present phase).

3. Naturally the question arises, how do such waves originate? A little reflection shows that population cycles easily can be produced synthetically. Take a species, say a beetle, which lives three years only, and which propagates in its third year of life. Let the survival rate of the first age-group be $\frac{1}{2}$, of the second $\frac{1}{3}$, and assume that each female in the age 2—3 produces, in the average, 6 new living females. (These values are, of course, entirely arbitrary.) Then a population of these beetles which, we may assume, has in a certain year 1 a thousand females

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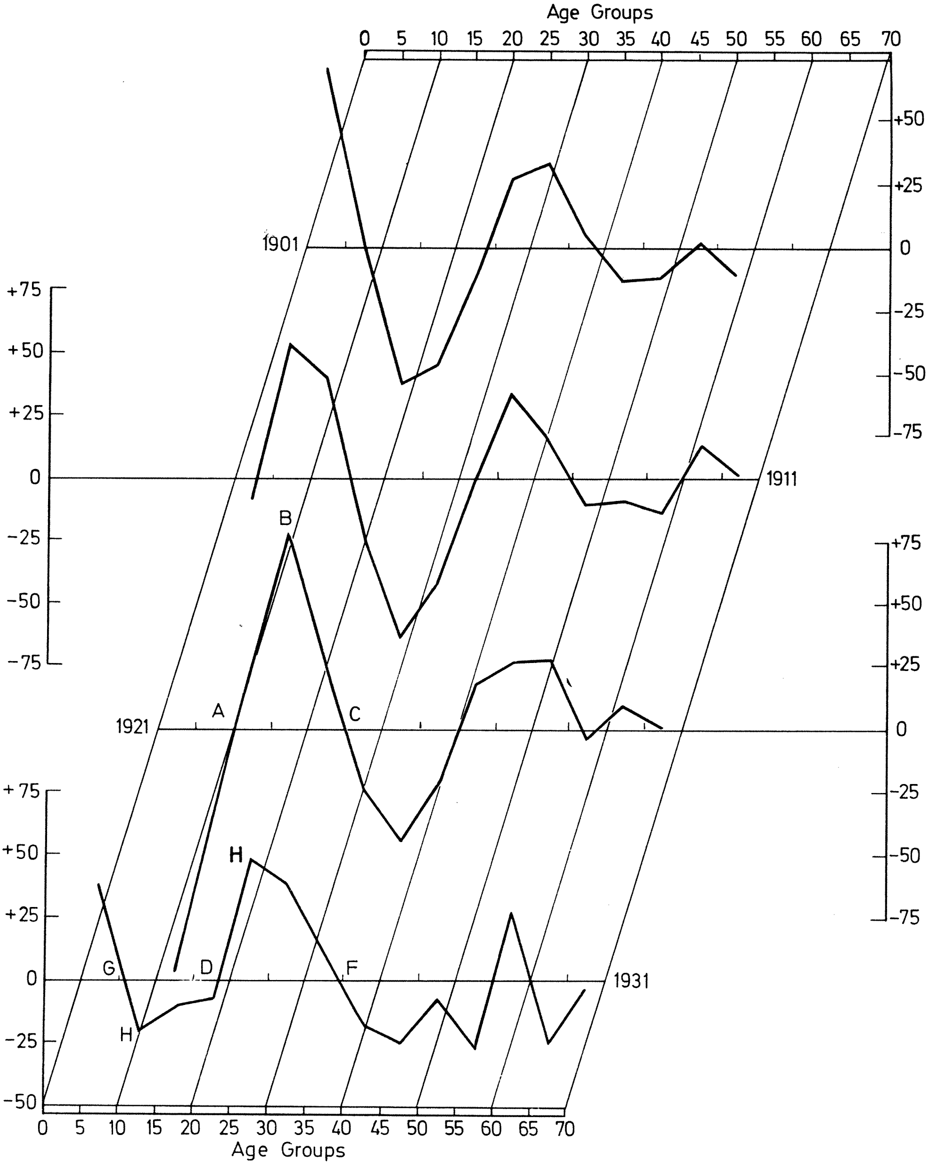
[1] Census of India, 1931, Vol. XI, Burma, Part I, Rept. by J. J. Bennison, Rangoon 1933, pp. 73ff. See also Census of India, 1921, Vol. X, Burma, Part I, Rept. by S. G. Grantham, Rangoon 1923, p. 121ff.

[2] "The economic depression"—says Bennison l.c. p. 77—"has caused a great deal of unemployment, and this has apparently been aggravated by the unusually large proportion of males between 20 and 30."



Age Distribution of indigenous Males since 1901

Fig. 1

in each age-group—males are irrelevant in matters of reproduction—will produce waves in the following manner:—

II

Age Group	1	2	3	4
0—1	1000	6000	1999.9	1000
1—2	1000	500	3000	1000
2—3	1000	333.3	166.6	1000
Total	3000	6833.3	5166	3000

So long as the rates of fertility and mortality remain constant, this development will, of course, continue indefinitely in regular three-year cycles. The population will oscillate round a stationary level, as the Net Reproduction Rate R is here assumed to be $=1$.

What determines the amplitude of these waves? Without going into the Mathematics of the process, it can be seen by means of simple Arithmetic, that the amplitudes depend on the assumptions made with regard to the initial distribution of females in the age-groups. If one takes in the year 1 the numbers 1000, 600, 200 respectively, it is easily seen that the amplitudes of the resulting waves become much smaller: and with the numbers 1000, 500, 167 instead, the waves disappear altogether.

For a mathematical analysis interested readers are referred to the Appendix where a system of equations is given from which all the relevant features of the process can be deduced with greatest ease. Here only the following observations will be used:

I. For any given set of fertility and mortality conditions there exists always one characteristic age distribution (1000:500:167 in the above example) which does not give rise to disequilibrium waves; but whenever the population deviates from this equilibrium structure, waves will be set up with amplitudes that depend entirely on the degree of these deviations.

II. It is not necessary that the population should be stationary, as in the above example; waves will also occur, if the Net Reproduction Rate $R>1$, indicating a growing population, or if $R<1$, so that the population eventually dies out.

III. It is not necessary that the waves repeat themselves year after year with the same amplitude. The occurrence of dampening, or of magnifying factors depends entirely on the given fertility and mortality rates.

IV. Similarly the wave length depends entirely on those rates, so that any change of the fertility and mortality conditions will produce waves of a different character. Wave length as well as the degree of dampening, and the amplitudes will be modified, if the situation changes.

24. On the Use of Matrices in Certain Population Mathematics

P. H. LESLIE (1945)

From *Biometrika* 33. Excerpts are from pages 183—185, 187—193, 199—202.

Leslie's work, rather than that of his predecessors Bernardelli and Lewis, is most commonly cited in the widespread literature using matrices, largely for the reason that Leslie worked out the mathematics and the application with great thoroughness. Some of his elaboration was designed to save arithmetic—for example his transformation of the projection matrix into an equivalent form with unity in the subdiagonal positions. Such devices, like a considerable part of classical numerical analysis, are unnecessary in a computer era.

The case of double roots seemed to require attention, for Leslie could not then know that no case of double roots would ever arise with real data. In the present excerpt we omit this, as well as the spectral decomposition of the matrix, which has not found extensive application.

Among the latent roots or eigen values and the corresponding stable vectors, as Leslie points out, demographic and biological interest is confined to three. The first, of largest absolute value, is positive and represents the ratio of population at the end of a cycle to that at the beginning, when the process has been operating for a considerable period; it provides the component of geometric increase. The second and third roots produce waves of diminishing amplitude having the length of the generation, usually 25 to 30 years. These waves measure the echo effect—after a baby boom they provide for a smaller boom a generation later—on the condition that the age-specific rates of birth and death remain constant.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_24,

CONTENTS

1. Introduction

If we are given the age distribution of a population on a certain date, we may require to know the age distribution of the survivors and descendants of the original population at successive intervals of time, supposing that these individuals are subject to some given age-specific rates of fertility and mortality. In order to simplify the problem as much as possible, it will be assumed that the age-specific rates remain constant over a period of time, and the female population alone will be considered. The initial age distribution may be entirely arbitrary; thus, for instance, it might consist of a group of females confined to only one of the age classes.

The method of computing the female population in one unit's time, given any arbitrary age distribution at time t, may be expressed in the form of $m+1$ linear equations, where m to $m+1$ is the last age group considered in the complete life table distribution, and when the same unit of age is adopted as that of time. If

n_{xt} = the number of females alive in the age group x to $x+1$ at time t,

P_x = the probability that a female aged x to $x+1$ at time t will be alive in the age group $x+1$ to $x+2$ at time $t+1$,

F_x = the number of daughters born in the interval t to $t+1$ per female alive aged x to $x+1$ at time t, who will be alive in the age group 0–1 at time $t+1$,

then, working from an origin of time, the age distribution at the end of one unit's interval will be given by

$$\begin{aligned}
\sum_{x=0}^{m} F_x n_{x0} &= n_{01} \\
P_0 n_{00} &= n_{11} \\
P_1 n_{10} &= n_{21} \\
P_2 n_{20} &= n_{31} \\
\vdots \quad &\quad \vdots \\
P_{m-1} n_{m-1,0} &= n_{m1}
\end{aligned}$$

or, employing matrix notation, $Mn_0 = n_1$, where n_0 and n_1 are column vectors giving the age distribution at $t = 0$ and 1 respectively, and the matrix

$$M = \begin{bmatrix}
F_0 & F_1 & F_2 & \cdots & . & F_k & F_{k+1} & \cdots & F_{m-1} & F_m \\
P_0 & . & . & \cdots & . & . & . & \cdots & . & . \\
. & P_1 & . & \cdots & . & . & . & \cdots & . & . \\
. & . & P_2 & \cdots & . & . & . & \cdots & . & . \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
. & . & . & \cdots & P_{k-1} & . & . & \cdots & . & . \\
. & . & . & \cdots & . & P_k & . & \cdots & . & . \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
. & . & . & \cdots & . & . & . & \cdots & P_{m-1} & .
\end{bmatrix} \quad 0 < P_x < 1;\ F_x \geqq 0.$$

This matrix is square and consists of $m+1$ rows and $m+1$ columns. All the elements are zero, except those in the first row and in the subdiagonal immediately below the principal diagonal. The P_x figures all lie between 0 and 1, while the F_x figures are by definition necessarily positive quantities. Some of the latter, however, may be zero, their number and position depending on the reproductive biology of the species we happen to be considering in any particular case, and on the relative span of the pre- and post-reproductive ages. If $F_m = 0$, the matrix M is singular, since the determinant $|M| = 0$.

Since $Mn_0 = n_1$, and $Mn_1 = M^2_{n_0} = n_2$, etc., the age distribution at time t may be found by pre-multiplying the column vector $\{n_{00} n_{10} n_{20} \dots n_{m0}\}$, i.e. the age distribution at $t = 0$, by the matrix M^t. Moreover, it will be seen that with the help of the jth column of M^t the age distribution and number of the survivors and descendants of the $n_{j-1,0}$ individuals, who were alive at $t = 0$, can readily be calculated. Thus, $n_{j-1,0}$ times the sum of the elements in the jth column of M^t gives the number of living individuals contributed to the total population at time t by this particular age group.

2. DERIVATION OF THE MATRIX ELEMENTS

The basic data, from which the numerical elements of this matrix may be derived, are given usually in the form of a life table and a table of age specific fertility rates. To take the P_x figures first; if at $t = 0$ there are n_{x0} females alive in the age group x to $x+1$, the survivors of these will form the $x+1$ to $x+2$ age group in one unit's time, and thus $P_x n_{x0} = n_{x+1,1}$. Then it is usually assumed (e.g. Charles, 1938, p. 79; Glass, 1940, p. 464) that

$$P_x = \frac{L_{x+1}}{L_x},$$

where

$$L_x = \int_x^{x+1} l_x \, dx,$$

or the number alive in the age group x to $x+1$ in the stationary or life table age distribution. This method of computing the survivors in one unit's time would be exact if the distribution of those alive within a particular age group was the same as in the life-table distribution.

The F_x figures are more troublesome, and in the numerical example which will be given later they were obtained from the basic maternal frequency figures (m_x = the number of live daughters born per unit of time to a female aged x to $x+1$) by an argument which ran as follows. Consider the n_{x0} females alive at $t = 0$ in the age group x to $x+1$, and let us suppose that they are concentrated at the midpoint of the group, $x+\frac{1}{2}$. During the interval of time 0–1 some of these individuals are dying off, and at $t = 1$ the $n_{x+1,1}$ survivors can be regarded as concentrated at the age $x+1\frac{1}{2}$. Although these deaths are taking place continuously, we may assume them all to occur around $t = \frac{1}{2}$, so that at this latter time the number of females alive in the age group we are considering changes abruptly from n_{x0} to $n_{x+1,1} = P_x n_{x0}$. Then during the time interval 0–$\frac{1}{2}$ these n_{x0} females will have been exposed to the risk of bearing daughters, and the number of the latter they will have given birth to per female alive will be given by the maternal frequency figure for the ages $x+\frac{1}{2}$ to $x+1$. This figure may be obtained by interpolating in the integral curve of the m_x values, and thus expressing the latter in $\frac{1}{2}$ units of age throughout the reproductive span instead of in single units. The daughters born during the interval of time 0–$\frac{1}{2}$ will be aged $\frac{1}{2}$–1 at $t = 1$, the number of them surviving at this time being determined approximately by multiplying the appropriate $m_{x+\frac{1}{2}}$ figure by the factor $2\int_{\frac{1}{2}}^{1} l_x dx$ according to the given life table. Similarly, each of the $P_x n_{x0}$ females during the interval of time $\frac{1}{2}$–1 give birth to $m_{x+1-x+1\frac{1}{2}}$ daughters, the survivors of which form part of the 0–$\frac{1}{2}$ age group at $t = 1$. The survivorship factor is in this case taken to be $2\int_0^{\frac{1}{2}} l_x dx$.

Combining these two steps together we obtain a series of F_x figures, which may be defined as the number of daughters alive in the age group 0–1 at $t = 1$ per female alive in the age group x to $x+1$ at $t = 0$. Putting

$$k_1 = 2\int_0^{\frac{1}{2}} l_x dx, \quad k_2 = 2\int_{\frac{1}{2}}^1 l_x dx,$$

then

$$F_x = (k_2 m_{x+\frac{1}{2}-x+1} + k_1 P_x m_{x+1-x+1\frac{1}{2}}),$$

and

$$\sum_{x=0}^{m} F_x n_{x0} = n_{01},$$

the total number of daughters alive aged 0–1 at $t = 1$. ...

4. PROPERTIES OF THE BASIC MATRIX

The matrix M is square and of order $m+1$; it is not necessary, however, in what follows to consider this matrix as a whole. For, if $x = k$ is the last age group within which reproduction occurs, F_k is the last F_x figure which is not equal to zero. Then, if the matrix be partitioned symmetrically at this point,

$$M = \begin{bmatrix} A & . \\ B & C \end{bmatrix}.$$

The submatrix A is square; B is of order $(m-k) \times (k+1)$; C again is square consisting of $m-k$ rows and columns, the only numerical elements being in the subdiagonal immediately below the principal diagonal. The remaining submatrix is of order $(k+1) \times (m-k)$ and consists only of zero elements. Then in forming the series of matrices M^2, M^3, M^4, etc.,

$$M^t = \begin{bmatrix} A^t & . \\ f(ABC) & C^t \end{bmatrix}.$$

The submatrix C is, however, of such a type that $C^{m-k} = 0$, so that M^t, $t \geqq m-k$, will have all its last $m-k$ columns consisting of zero elements. This is merely an expression of the obvious fact that individuals alive in the post-reproductive ages contribute nothing to the population after they themselves are dead. It is the submatrix A which is principally of interest, and in the mathematical discussion which follows, attention is focused almost entirely on it and on age distributions confined to the prereproductive and reproductive age groups.

The matrix A is of order $(k+1) \times (k+1)$, where $x = k$ is the last age group in which reproduction occurs, and written in full,

$$A = \begin{bmatrix} F_0 & F_1 & F_2 & F_3 & \cdots & F_{k-1} & F_k \\ P_0 & . & . & . & \cdots & . & . \\ . & P_1 & . & . & \cdots & . & . \\ . & . & P_2 & . & \cdots & . & . \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ . & . & . & . & \cdots & P_{k-1} & . \end{bmatrix}.$$

This matrix is non-singular, since the determinant $|A| = (-1)^{k+2}(P_0 P_1 P_2 \ldots P_{k-1} F_k)$. There exists, therefore, a reciprocal matrix of the form

$$A^{-1} = \begin{bmatrix} . & P_0^{-1} & . & . & \cdots & . \\ . & . & P_1^{-1} & . & \cdots & . \\ . & . & . & P_2^{-1} & \cdots & . \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ . & . & . & . & \cdots & P_{k-1}^{-1} \\ F_k^{-1} & -(P_0 F_k)^{-1} F_0 & -(P_1 F_k)^{-1} F_1 & -(P_2 F_k)^{-1} F_2 & \cdots & -(P_{k-1} F_k)^{-1} F_{k-1} \end{bmatrix}.$$

Thus, given an initial age distribution n_{x0} ($x = 0, 1, 2, 3, \ldots, k$) at $t = 0$, in addition to the forward series of operations An_0, $A^2 n_0$, $A^3 n_0$, ..., etc., there is also a backward series $A^{-1} n_0$,

$A^{-2}n_0$, $A^{-3}n_0$, ..., etc. There is, however, a fundamental difference between these; for, whereas the forward series can be carried on for as long as we like, given any initial age distribution, the backward series can only be performed so long as n_{xt} remains $\geqq 0$, since a negative number of individuals in an age group is meaningless. Apart from this limitation, it is possible to foresee that the reciprocal matrix might be of some use in the solution of certain types of problem.

5. TRANSFORMATION OF THE CO-ORDINATE SYSTEM

Hitherto an age distribution n_{xt} has been regarded as a matrix consisting of a single column of elements. For simplicity in notation, this column vector will now be termed the vector ξ and different ξ's will be distinguished by different subscripts (ξ_a, ξ_x, etc.). We may picture an age distribution as a vector having a certain magnitude and related to a definite direction in a vector space, the space of the ξ's. The different age distributions which may arise in the case of any particular population will be assumed to be ξ's all radiating from a common origin. The numerical elements of a ξ vector are thus taken to be the co-ordinates of a point in multi-dimensional space referred to a general Cartesian co-ordinate system, in which the reference axes may make any angles with one another. At this point in the argument another type of vector will be introduced, which in matrix notation will be written as a row vector, and which will be termed the vector η. There is an intimate relationship between this new type and the old, for, associated with each vector ξ_a, there is a uniquely determined vector η_a, and vice versa. The inner or scalar product, $\eta_a \xi_a$, is the square of the length of the vector ξ_a. Either we may picture each of these vectors as associated with a different kind of vector space, the space of the ξ's and the dual space of the η's, which are not entirely disconnected but related in a special way; or, alternatively, we may regard them as two different kinds of vector associated with the same vector space. The relationship between η and ξ is precisely the same as that between covariant and contravariant vectors in differential geometry.

If we pass from our original co-ordinate system to a new frame of reference, and the variables η and ξ undergo the non-singular linear transformations,

$$\eta = \phi H, \quad \xi = H^{-1}\psi, \quad |H| \neq 0,$$

it can be seen that since the variables are contragredient, $\eta\xi = \phi\psi$, so that the square of the length of a vector remains invariant. Moreover, since the result of operating on a vector ξ_a with the matrix A is, in general, another vector ξ_b, where ξ_a and ξ_b are both referred to the original co-ordinate system, it follows that in the new frame of reference which is defined by the linear transformations given above, the relationship

$$A\xi_a = \xi_b$$

becomes

$$HAH^{-1}\psi_a = \psi_b,$$

or

$$B\psi_a = \psi_b.$$

Thus, in the new frame of reference the matrix $B = HAH^{-1}$ operating on the vector ψ_a is equivalent to the matrix A operating on the vector ξ_a in the original frame.

It is convenient, for the purposes of studying the matrix A and of performing any numerical computations with it, to transform the variables η and ξ in the above way, choosing the matrix H so as to make $B = HAH^{-1}$ as simple as possible. For $B^t = (HAH^{-1})^t = HA^tH^{-1}$, and since A is non-singular, by the reversal law, $(HAH^{-1})^{-1} = HA^{-1}H^{-1}$. Thus, if $f(A)$ is a rational integral function of A, $f(B) = f(HAH^{-1}) = Hf(A)H^{-1}$; and the properties of matrix functions $f(A)$ can be studied by means of the simpler forms $f(B)$. Moreover, the matrices A and B have the same characteristic equation and, therefore, the same latent roots. For $B - \lambda I = H(A - \lambda I)H^{-1}$ and, forming the determinants of both sides,

$$|B - \lambda I| = |H|\,|A - \lambda I|\,|H|^{-1},$$

so that the characteristic equation is

$$|A-\lambda I| = |B-\lambda I| = 0.$$

If, in the present case, the transforming matrix is taken to be

$$H = \begin{bmatrix} (P_0P_1P_2\dots P_{k-1}) & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & (P_1P_2P_3\dots P_{k-1}) & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & (P_2P_3\dots P_{k-1}) & \cdots & \cdot & \cdot & \cdot \\ \hdotsfor{7} \\ \cdot & \cdot & \cdot & \cdots & (P_{k-2}P_{k-1}) & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & P_{k-1} & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 1 \end{bmatrix}$$

in which, it is to be noted, the only numerical elements lie in the principal diagonal and are derived entirely from the life table, then

$$B = HAH^{-1} = \begin{bmatrix} F_0 & P_0F_1 & P_0P_1F_2 & P_0P_1P_2F_3 & \cdots & (P_0P_1P_2\dots P_{k-1})F_k \\ 1 & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdots & \cdot \\ \hdotsfor{6} \\ \cdot & \cdot & \cdot & \cdot & \cdots\ 1 & \cdot \end{bmatrix}.$$

Comparing this matrix B with the original form A, it can be seen that the latter has been simplified to the extent that the original P_x figures in the principal subdiagonal are now replaced by a series of units, and the matrix A has been reduced to the rational canonical form $B = HAH^{-1}$ (see Turnbull & Aitken, 1932, chap. v). In this way any computations with the matrix A are made easier, and we may work henceforward in terms of ϕ and ψ vectors together with the matrix B, instead of with the original η and ξ vectors, and the matrix A. Any results obtained in this new system of co-ordinates may be transformed back again to the original system whenever necessary. It is evident that by suitably enlarging H the original matrix M may be transformed in a similar way.

This linear transformation of the original co-ordinate system is equivalent biologically to the transformation of the original population we were considering into a new and completely imaginary type which, although intimately connected with the old, has certain quite different properties. Thus, it can be seen from the transformed matrix B that the individuals in this new population, instead of dying off according to age as the original ones did, live until the whole span of life is completed, when they all die simultaneously. This is indicated by the P_x figures being now all equal to unity; an individual alive in the age group x to $x+1$ at $t = 0$ is certain of being alive at $t = 1$, excepting in the last age group of all where none of the individuals will be alive in one unit's time. Accompanying this somewhat radical change in the life table, there is a compensatory adjustment made in the rates of fertility so that the new population has the same inherent power of natural increase (r) as that of the old. This follows from the fact that the latent roots of the matrices A and B are the same, and, as will be shown later, the dominant latent root is closely related to the value of r obtained by the usual methods of computation. Insomuch as the transformation is reversible and $A = H^{-1}BH$, it can be seen that by changing H we could transform the canonical form B, if we wished, into another matrix in which the P_x subdiagonal might be a specified set of figures derived from some other form of life table. But, for our present purposes, the canonical form B, in which all the P_x figures are units, offers advantages over any other matrix of a similar type owing to the greater ease with which it can be handled. ...

7. THE STABLE AGE DISTRIBUTION

The result of operating on an age distributon ψ_x with the matrix B is, in general, a different distribution ψ_y. But, in the special case when the relation between the two distributions is such that

$$B\psi_a = \lambda\psi_a,$$

where λ is an algebraic number, then ψ_a may be said to be a stable age distribution appropriate to the matrix B. For the sake of brevity it will be referred to as a stable ψ. Similarly for initial row vectors, if

$$\phi_a B = \lambda\phi_a,$$

then ϕ_a is said to be a stable ϕ.

The matrix equation defining a stable ψ may be written as $k+1$ linear equations, of which the ith is

$$\sum_{j=1}^{k+1} b_{ij}n_j - \lambda n_i = 0,$$

where n_i $(i = 1, 2, \ldots, k+1)$ are the co-ordinates of the stable ψ, and b_{ij} the element in the ith row and jth column of B. Eliminating the n_i from this system of equations, we obtain the characteristic equation of B, namely,

$$|B - \lambda I| = 0;$$

and, expanding this determinant in powers of λ, we have in the present case,

$$\lambda^{k+1} - F_0\lambda^k - P_0F_1\lambda^{k-1} - P_0P_1F_2\lambda^{k-2} - \ldots - (P_0P_1\ldots P_{k-2})F_{k-1}\lambda - (P_0P_1\ldots P_{k-1})F_k = 0.$$

The $k+1$ roots λ_a of this equation are the latent roots of B, and corresponding to each distinct λ_a there is a pair of stable vectors, ϕ_a and ψ_a, determined except for an arbitrary scalar factor.

Once a latent root λ_a has been determined, it is a comparatively simple matter to find the appropriate stable ψ_a and ϕ_a vectors. Thus, it is easily shown that the stable ψ_a is the column vector $\{\lambda_a^k\lambda_a^{k-1}\lambda_a^{k-2}\ldots\lambda_a 1\}$. A short method of estimating ϕ_a is the following. Suppose, to take a simple case, that

$$B = \begin{bmatrix} a & b & c & d \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

and let y_x $(x = 1, 2, 3, 4)$ be the elements of the stable ϕ_a appropriate to the root λ_a. Then

$$\begin{aligned} \phi_a B &= [ay_1 + y_2 \quad by_1 + y_3 \quad cy_1 + y_4 \quad dy_1] \\ &= [\lambda_a y_1 \quad \lambda_a y_2 \quad \lambda_a y_3 \quad \lambda_a y_4]. \end{aligned}$$

By equating similar elements and putting $y_1 = 1$, $y_4 = d/\lambda_a$, $y_3 = \dfrac{c+y_4}{\lambda_a}$, etc., it is easy to see how the required row vector can be built up. Having in this way obtained the stable ψ and ϕ vectors for the matrix B, they may be transformed to the appropriate stable ξ and η for the matrix A by means of the relations

$$\eta = \phi H, \quad \xi = H^{-1}\psi.$$

The characteristic equation of the matrix B, when expanded, is of degree $k+1$ in λ, and once B has been obtained this equation can immediately be written down, since the numerical coefficients of λ^k, λ^{k-1}, λ^{k-2}, etc., are merely the elements of the first row taken with a negative sign. Since there is only one change of sign in this equation, only one of the latent roots will be real and positive. Excluding the rather special case when the first row of B has only a single non-zero element, and taking the more usual type of matrix which will be met with, namely, that for a species breeding continuously over a large proportion of its total life span, it will be found that the modulus of this root (λ_1) is greater than any of the others,

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \ldots > |\lambda_{k+1}|,$$

the remaining roots being either negative or complex.

This dominant latent root λ_1, which will be $\gtreqless 1$ according as to whether the sum of the elements in the first row of B is $\gtreqless 1$, is the one which is principally of interest. Since it is real and positive, it is the only root which will give rise to a stable ψ or ξ vector consisting of real and positive elements. It is this stable ξ_1 associated with the dominant root λ_1 which is ordinarily referred to as the stable age distribution appropriate to the given age specific rates of fertility and mortality. Since

$$A^t\xi_1 = \lambda_1^t \xi_1,$$

it can be seen that the latent root λ_1 of the matrix A and the value of r obtained in the usual way from

$$\int_0^\infty e^{-rx} l_x m_x dx = 1,$$

are related by

$$\log_e \lambda_1 = r. \quad \ldots$$

8. PROPERTIES OF THE STABLE VECTORS

Before proceeding further it is necessary to mention briefly the reasons why the methods given above for the computation of the stable ψ and ϕ vectors were adopted, apart from their simplicity in practice. If the $k+1$ distinct roots of the characteristic equation are known, we may form a set of $k+1$ matrices $f(\lambda_a)$ by inserting in turn the numerical value of each root in the matrix $[B-\lambda_a I]$. The adjoint of $f(\lambda_a)$ is

$$F(\lambda_a) = \prod_{b \neq a} [B-\lambda_b I] \quad \text{and} \quad f(\lambda_a)F(\lambda_a) = 0.$$

It may be shown that the stable ψ_a appropriate to the root λ_a can be taken proportional to any column, and the stable ϕ_a proportional to any row of the matrix $F(\lambda_a)$ (see e.g. Frazer, Duncan & Collar, 1938, chap. III). Moreover, $F(\lambda_a)$ is a matrix product of the type $\psi\phi$, where the ψ vector is given by the first column and the ϕ vector by the last row of $F(\lambda_a)$, each divided by the square root of the element in the bottom left-hand corner; and the trace of the matrix is equal to the scalar product $\phi\psi$. Now $[B-\lambda_a I]$ is a square matrix of order $k+1$ with only zero elements below and to the left of the principal subdiagonal, which itself consists of units. The product of k such matrices, which gives $F(\lambda_a)$, will have therefore a unit in the bottom left-hand corner. Since the stable ϕ_a and ψ_a vectors obtained by the methods suggested in § 7 have respectively their first and last elements $= 1$, it follows that

$$\psi_a\phi_a = F(\lambda_a), \quad \phi_a\psi_a = \text{trace}\, F(\lambda_a).$$

The stable vectors may now be normalized. If the scalar product, $\phi_a\psi_a = z^2$, say, then

$$\frac{\phi_a}{|z|}\frac{\psi_a}{|z|} = 1.$$

From now on it will be assumed that the stable vectors appropriate to each of the latent roots have been normalized in this way.

These vectors have the following important properties:

(1) The $k+1$ stable ψ are linearly independent. There is thus no such relationship, with non-zero coefficients c, as

$$c_1\psi_1 + c_2\psi_2 + c_3\psi_3 + \ldots + c_{k+1}\psi_{k+1} = 0.$$

(2) The scalar product of a stable ψ, ψ_a with the associated vector of another stable ψ, ψ_b is zero, i.e.

$$\phi_b\psi_a = 0 \quad (a \neq b).$$

The normalized stable ψ thus form a set of $k+1$ independent and mutually orthogonal vectors of unit length.

(3) Any arbitrary ψ—ψ_x say—can be expanded in terms of the stable ψ, thus

$$\psi_x = c_1\psi_1 + c_2\psi_2 + c_3\psi_3 + \ldots + c_{k+1}\psi_{k+1},$$

where the coefficients c may be either real or complex. Similarly an arbitrary vector ϕ_x can be expanded in terms of the stable ϕ. ...

13. THE APPROACH TO THE STABLE AGE DISTRIBUTION

A stable age distribution appropriate to the matrix B has been defined mathematically by the equation

$$B\psi = \lambda\psi,$$

and it has already been shown that since only one latent root of B is real and positive, only one of the stable ψ will consist of real and positive elements. But, in addition to this Malthusian age distribution, it is also of some interest to inquire whether any significance can be attached to the remaining stable ψ associated with the negative and complex roots of the characteristic equation.

Any age distribution ψ_x, the elements of which are necessarily $\geqq 0$, may be expressed as a vector of deviates from the stable ψ_1 associated with the dominant latent root, and we may therefore write the expansion of ψ_x in terms of the stable ψ as

$$(\psi_x - c_1\psi_1) = c_2\psi_2 + c_3\psi_3 + \ldots + c_{k+1}\psi_{k+1} = \psi_d,$$

where the coefficients c are given by the vector $c = U\psi_x$. Thus, the way in which a particular type of age distribution will approach the stable form may be studied by means of the vector ψ_d.

Among the terms occurring in the right-hand side of this expression there will be, corresponding to each negative root, a single term $c_a\psi_a$ which will consist of real elements alternately positive and negative in sign. (Even if the normalized ψ_a is imaginary this term will consist of real numbers, since in this case c_a will also become imaginary.) Moreover, corresponding to every pair of complex roots there will be a pair of terms $(c_m\psi_m + c_n\psi_n)$ which taken together will also give a single vector with real elements. This follows from the fact that c_m is the conjugate complex of c_n owing to the way in which the matrix U is constructed. Then, apart from the scalar c_1 which must necessarily be > 0, some of the remaining coefficients $c_2, c_3, \ldots, c_{k+1}$ in the expansion of ψ_d may be zero. The first and most obvious case is when they are all zero, and the age distribution ψ_x is therefore already of the stable form. But, if either

$$\psi_d = c_a\psi_a,$$

where ψ_a corresponds to a negative latent root, or

$$\psi_d = c_m\psi_m + c_n\psi_n,$$

where ψ_m and ψ_n are associated with a conjugate pair of complex roots, then it follows that the age distribution ψ_x will, as time goes on, approach the stable form in a particular way defined by either

$$B^t\psi_d = c_a\lambda^t\psi_a \quad \text{or} \quad B^t\psi_d = c_m\lambda^t\psi_m + c_n\bar{\lambda}^t\psi_n,$$

in which λ^t for a pair of complex roots $u + iv$ with modulus r may be written in the form of $r^t(\cos\theta t \pm i\sin\theta t)$. Thus, the negative and complex latent roots of B serve to determine a number of age distributions which are of some interest owing to the fact that they will approach the Malthusian form in what may be termed a stable fashion.

Since $|\lambda_1| > |\lambda_2| > |\lambda_3| > \ldots > |\lambda_{k+1}|$, the vector of deviates ψ_d will tend towards zero as $t \to \infty$ whenever $\lambda_1 \lesseqgtr 1$. Thus, in the case of a stationary population, any ψ_x will converge to the stable form of age distribution. But if $\lambda_1 > 1$, there is a possibility of one or more of the remaining roots having a modulus $\geqq 1$, e.g. $|\lambda_2| \geqq 1$. In the latter case there may be certain age distributions with $c_2 \neq 0$ for which the amplitude of the deviations from the stable form tend either to increase ($|\lambda_2| > 1$), or to remain constant ($|\lambda_2| = 1$). From the practical point of view, however, we may still say that a population with such an age distribution approaches or becomes approximately equal to the stable population, since λ_1^t is much greater than λ_2^t when t is large.

14. SPECIAL CASE OF THE MATRIX WITH ONLY A SINGLE NON-ZERO F_x ELEMENT

The interesting case of the matrix A having only a single non-zero element in the first row has been illustrated in a numerical example by Bernardelli (1941).* This author has also used a matrix notation in the mathematical appendix to his paper, and the form of his basic matrix is the same as that referred to here as M or A. It is not clear, however, from the definitions which he gives whether he regards the elements in the first row of his matrix as being constituted by the maternal frequency figures (m_x) themselves, or by a series of values similar to those defined here as the F_x figures. He refers to them merely as the specific fertility rates for female births.

In discussing the causes of population waves, Bernardelli describes a hypothetical species, such as a beetle, which lives for only three years and which propagates in the third year of life. He assumes, for the sake of argument, that—to employ the terminology used here—$P_0 = \frac{1}{2}$ and $P_1 = \frac{1}{3}$, and that 'each female in the age 2–3 produces, on the average, 6 new living females'. Assuming for the moment that he is here defining a F_x figure, we may write this system of mortality and fertility rates as

$$A = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}, \quad B = HAH^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The characteristic equation expanded in terms of λ is $\lambda^3 - 1 = 0$; and the latent roots are therefore 1, $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, all three being of equal modulus. The matrix A has the interesting properties

$$A^2 = A^{-1}, \quad A^3 = I,$$

so that any initial age distribution repeats itself regularly every three years. Thus, as Bernardelli shows, a population of 3000 females distributed equally among the three age groups becomes a total population of 6833 at $t = 1$; of 5166 at $t = 2$; and again 3000 distributed equally among the age groups at $t = 3$. Unless a population has already an initial age distribution in the ratio of $\{6:3:1\}$, no approach will be made to the stable form associated with the real latent root, and the vector of deviates ξ_d will continue to oscillate with a stable amplitude, which will in part depend on the form of the initial distribution. Although this numerical example refers specifically to a stationary population, it is evident that a similar type of argument may be developed in the case when $|\lambda| > 1$ and $A^3 = \lambda^3 I$.

We have assumed here that his definition of the fertility rate refers to a F_x figure. But, if we were to interpret the words quoted above as referring to a maternal frequency figure, namely that every female alive between the ages 2–3 produces on the average 6 daughters per annum, then the results become entirely different. For, deriving the appropriate F_x figures by the method described in § 2, the matrix is now

$$A = \begin{bmatrix} 0 & 1 & 3 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}, \quad B = HAH^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and the latent roots are 1, $-\frac{1}{2} \pm \frac{1}{2} i$. The modulus of the pair of complex roots is $1/\sqrt{2}$, which is < 1, so that every age distribution will now converge to the stable form associated with the

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* At this point I should like to acknowledge the gift of a reprint of this paper, which was received by the Bureau of Animal Population at a time when I was in the middle of this work, and when I was just beginning to appreciate the interesting results which could be obtained from the use of matrices and vectors: also a personal communication from Dr Bernardelli, received early in 1942, at a time when it was difficult to reply owing to the developments of the war situation in Burma. Although the problems we were immediately interested in differed somewhat, this paper did much to stimulate the theoretical development given here, and it is with great pleasure that I acknowledge the debt which I owe to him.

real root. Thus, to take the same example as before, 3000 females distributed equally among the age groups will tend towards a total population of 4000 distributed in the ratio of $\{6:3:1\}$, and it was found that this age distribution would be achieved at approximately $t = 23$. During the approach to this stable form periodic waves are apparent both in the age distribution and in the total number of individuals, but these oscillations are now damped, in contrast with the results obtained with the first type of matrix.

This simple illustration serves to emphasize the importance which must be attached to the way in which the basic data are defined and to the marked difference which exists between what are termed here the m_x and F_x figures. Nevertheless, apart from the question of the precise way in which the definition of the fertility rates is to be interpreted in this example, the first type of matrix with only a single element in the first row does correspond to the reproductive biology of certain species. Thus, in the case of many insect types the individuals pass the major portion of their life span in various immature phases and end their lives in a short and highly concentrated spell of breeding. The properties of this matrix suggest that any stability of age structure will be exceptional in a population of this type, and that even if the matrix remains constant we should expect quite violent oscillations to occur in the total number of individuals.

15. NUMERICAL COMPARISON WITH THE USUAL METHODS OF COMPUTATION

From the practical point of view it will not always be necessary to estimate the actual values of all the stable vectors and of the associated matrices which are based on them. Naturally, much will depend on the type of information which is required in any particular case. In order to compute, for instance, the matrices U, Q and G, it is necessary first of all to determine all the latent roots of the basic matrix. The ease with which these may be found depends very greatly upon the order of the matrix. Thus, in the numerical example for the brown rat used previously in § 3, the unit of age and time is one month and the resulting square matrix A is of order 21. To determine all the 21 roots of the characteristic equation would be a formidable undertaking. It might be sufficient in this case to estimate the positive real root and the stable vector associated with it. On the other hand, it is possible to reduce the size of the matrix by taking a larger unit of age, and in some types of problem, where extreme accuracy is not essential, a unit say three times as great might be equally satisfactory, which would reduce the matrix for the rat population to the order of 7×7. It is not too difficult to find all the roots of a seventh degree equation by means of the root-squaring method (Whittaker & Robinson, 1932, p. 106). But the reduction of the matrix in this way will generally lead to a value of the positive real root which is not the same as that obtained from the larger matrix, and it is therefore necessary to see by how much these values may differ owing to the adoption of a larger unit of time.

Another important point which must be considered is the following. By expressing the age specific fertility and mortality rates in the form of a matrix and regarding an age distribution as a vector, an element of discontinuity is introduced into what is ordinarily taken to be a continuous system. Instead of the differential and integral calculus, matrix algebra is used, a step which leads to a great economy in the use of symbols and consequently to equations which are more easily handled. Moreover, many quite complicated arithmetical problems can be solved with great ease by manipulating the matrix which represents the given system of age specific rates. But the question then arises whether these advantages may not be offset by a greater degree of inaccuracy in the results as compared with those obtained from the previous methods of computation. It is not easy, however, to settle this point satisfactorily. In the way the usual equations of population mathematics are solved, a similar element of discontinuity is introduced by the use of age grouping. Thus, in the case of a human population, if we were estimating the inherent rate of increase in the ordinary way, we should not expect to obtain the same value of r from the data grouped in five year intervals of age as that from the data grouped in one year intervals. The estimates of the seminvariants would not be precisely the same in both cases. Nevertheless, the estimate from the data grouped in five year intervals is usually considered to be sufficiently accurate for all ordinary purposes, and there is little doubt that if we merely require the inherent rate of increase and the stable age distribution, these methods of computation are perfectly satisfactory when applied to human data.

REFERENCES

Bernardelli, H. (1941). Population waves. *J. Burma Res. Soc.* **31**, no. 1, 1–18.

Charles, Enid (1938). The effect of present trends in fertility and mortality upon the future population of Great Britain and upon its age composition. Chap. II in *Political Arithmetic*, ed. Lancelot Hogben. London: Allen and Unwin.

Frazer, R. A., Duncan, W. J. & Collar, A. R. (1938). *Elementary Matrices*. Camb. Univ. Press.

Glass, D. V. (1940). *Population Policies and Movements in Europe*. Oxford: Clarendon Press.

Turnbull, H. W. & Aitken, A. C. (1932). *An Introduction to the Theory of Canonical Matrices*. London and Glasgow: Blackie and Son.

Whittaker, E. T. & Robinson, G. (1932). *The Calculus of Observations*, 2nd ed. London and Glasgow: Blackie and Son.

Part IV
Parameterization and Curve Fitting

An early attempt to describe demographic observations in a mathematical formula was contributed by Abraham DeMoivre, whose hypothesis (1725. p.4) "consists in supposing that the number of lives existing at any age is proportional to the number of years intercepted between the age given and the extremity of old age," i.e.

$$\ell_x = \ell_0 \left(1 - \frac{x}{\omega}\right),$$

from which we have for the force of mortality

$$\mu_x = \frac{1}{\omega - x}\ .$$

DeMoivre chose the upper limit $\omega = 86$ for its quite good fit to Halley's (1693) life table, and qualified his analysis by setting a lower age limit of 12. His assumption that ℓ_x, is a linear function of age was used for short age intervals by Joshua Milne (1815) to develop his q_x formula and it remains in use in that context. As an approximation to the full life table it has long been obsolete. DeMoivre (1733) was also first to develop the normal distribution. discussed below in connection with fertility.

We begin this chapter with an article by Benjamin Gompertz (1825), paper 25, who used the better mortality statistics of his time to suggest the expressions

$$\begin{aligned} \mu_x &= Bc^x, \\ \ell_x &= \ell_0 g^{(c^x - 1)}, \end{aligned}$$

where B, c, and g are constants; in effect a law that mortality increases exponentially with age. Gompertz was innovative in that he reasoned from a law of mortality in finding ℓ_x, where a century earlier DeMoivre had not attempted to justify his implied mortality function. Gompertz is as much remembered for having reasoned well: his Law is a better description of the upper ages than DeMoivre's hypothesis that populations collapse towards a specific age limit. From Gompertz are derived Farr's (1864) and indirectly Reed and Merrell's (1939) excellent approximations to ${}_nq_x$.

An improvement on Gompertz' Law was contributed by William Makeham (1860, 1867), whose 1867 article is included here as paper 26. In a careful examination of causes of death (as they were then listed), Makeham found that overall mortality levels could be better represented if a constant term were added to μ_x to account for causes of mortality not dependent on age, a possibility Gompertz also had noted. This gives

$$\mu_x = A + B\,c^x,$$

$$\ell_x = \ell_0\, s^x\, g^{(c^x-1)}.$$

For ages beyond childhood and youth Makeham's Law combines intuitive plausibility and a close fit to observed mortality.

Other equations have been suggested that give a slightly better fit to mortality rates overall but have spurious inflection points or are otherwise not intuitive. For these the reader may consult Hugh M. Wolfenden (1942, pp. 79—85; 1954, pp. 164—167). A further contribution to the interpretation of Gompertz' and Makeham's Laws and their generalization will be found in Brillinger (1961).

The lack of a satisfactory analytic expression for mortality that includes both infancy and adulthood has driven workers in the field to the alternative approach provided by model life tables, which average observed mortality schedules by regression, usually on life expectancy at birth or at age ten where the strong impact of infant mortality is not felt. An early attempt was published by the United Nations (1955); but by far the best known is that of Ansley Coale and Paul Demeny (1966). From their examination of historical mortality patterns in countries having reliable data, Coale and Demeny were able to isolate four general patterns and to construct regressions of the forms

$${}_nq_x = a + b\,\overset{\circ}{e}_{10},$$

$$\ln {}_nq_x = a + b\,\overset{\circ}{e}_{10}$$

for the separate age groups within each of the four families of tables. Their blending of the regressions and general methodology are given as paper 27. How the model tables are applied to countries with poor demographic data is discussed in a paper from *United Nations Manual IV* (1967) which Coale and Demeny prepared { omitted here }. An application of factor analysis to the problem of sorting out existing life tables and making the models will be found in Ledermann and Breas (1959).

William Brass (Brass and Coale 1968) has suggested an ingenious combination of analytic curves and model tables that provides greater flexibility in table use, which we include here as paper 28. Brass takes a selected model table as the standard, say $\ell_x^{(s)}$ and adjusts it to an observed set of rates by a regression of the form

$$\ln[(1 - \ell_x)/\ell_x)] = \alpha + \beta \ \ln[(1 - \ell_x^{(s)}/\ell_x^{(s)}].$$

The equation preserves the extreme values $\ell_0 = 1$, $\ell_\omega = 0$, and allows model tables to be used in cases where observed rates are incomplete and not quite of the (usually European) pattern the models reflect. The Brass technique also has much potential value for forecasting, since in many instances the constants α and β show fairly clear trends over time (Brass 1974, pp. 546—551).

The early attention and degree of success achieved in mapping and analyzing mortality patterns contrasts sharply with the very meager efforts that were made to understand marriage and fertility. Most of the 19th century elapsed between Nicander's publication of fertility rates and their use by Milne (1815), and Richard Böckh's (1886, p. 30) introduction of the Net Reproduction Rate, defined as the integral

$$R_0 = \int_\alpha^\beta p(a)\, m(a)\, \mathrm{d}a,$$

where $p(a)$ is the probability that an individual survives from birth to age a and $m(a)$ is the expected number of offspring of the same sex born to him (her) in the interval a to $a + \mathrm{d}a$. The integration is across all fertile ages and hence gives expected progeny to an individual just born. [Richard R. Kuczynski, at one time a student of Böckh's and an assistant in the Berlin statistical office, in several of his works drew attention to the implications of the rate for population survival and growth; e.g. (Kuczynski 1928, pp. 41—42): "The pertinent question is not: is there an excess of births over deaths? but rather: are natality and mortality such that a generation which would be permanently subject to them would during its lifetimes, that is until it has died out, produce sufficient children to replace that generation? If, for instance, 1,000 newly born produce in the course of their lives exactly 1,000 children, the population after the death of the older 1,000 will remain unaltered ... and if natality and mortality remain permanently the same, the population will always exactly hold its own. If more than 1,000 children are produced by a generation of 1,000 newly born, the population will increase: if less than 1,000 are produced, the population will decrease and finally die out."] Neither the net reproduction rate nor the simpler age-specific fertility rates were in use in England when Cannan carried out his population projections in 1895.

With Lotka's contributions to stable population theory good approximations to the net maternity function and related measures became immediately valuable, and several attempts to fit equations to them were made. The earliest employ Pearson distributions (Elderton and Johnson 1969, pp. 35—109; Kendall and Stuart 1969, Vol. 1, pp. 148—154), which are solutions to the differential equation

$$\frac{\mathrm{d}\ln f(x)}{\mathrm{d}x}=\frac{(x-a)}{b_0+b_1x+b_2x^2},$$

most but not all having a single mode at $x = a$ and making smooth contact with the x-axis at the extremities. They include the normal distribution, investigated by Dublin and Lotka (1925); the Pearson Type I by S. J. Pretorius (1930) and Lotka (1933); and Pearson Types I and III by S.D. Wicksell (1931). (For the normal, $b_1 = b_2 = 0$; for the Type III, $b_2 = 0$.)

The normal, Type III and an exponential introduced by Hadwiger (1940) are analyzed in detail in Keyfitz (1968, pp. 141—169). We include here as paper 29 Wicksell's remarks about the normal and Pearson curves.

None of the equations we mention has been defined to express either marriage patterns or birth interval and family size distributions, and the most recent work in the field has been directed toward providing this essential base. A large measure of success has been achieved by Ansley Coale (Coale 1971; Coale and McNeil 1972; Coale and Trussell 1974), in work directed toward the development of model nuptiality and fertility tables. The work combines four elements. Drawing on evidence from a number of populations, Coale (1971) found that the age-specific risk of marriage among those who ever marry could be closely approximated by a double exponential (Gompertz) distribution, differing between populations in origin (age at which marriages essentially begin) and the relative intensity of the marriage process once underway. For fertility by duration of marriage, Coale suggested taking as a standard that of a natural fertility population (i.e., one in which birth limitation is not intentionally practiced), reducible by an exponential decay function to represent fertility patterns imposed by family planning practice.

In later work extending Griffith Feeney's (1972) suggestion that the age-specific probability of marriage could be represented as the convolution of a normal distribution to represent age at eligibility with an exponential distribution representing delays between eligibility and marriage, Coale and McNeil found that as further exponentials are introduced the distribution takes as its limiting value a Makeham curve of the form (Coale and McNeil 1972, p. 744)

$$\bar{g}(x)=\frac{\lambda}{\Gamma(\alpha/\lambda)}\exp\{-\alpha(x-\mu)-\exp[-\lambda(x-\mu)]\},$$

where Γ is the gamma function, $\mu = a + \frac{\Gamma'(\alpha/\lambda)}{\lambda\Gamma(\alpha/\lambda)}$, and a is the mean of $\bar{g}(x)$.

The distribution has the important properties that it closely matches the probability distribution $g(x)$ generated by the Coale risk function and can be approximated to a high degree of accuracy using 1 to 3 exponentials. The distribution is thus both empirical and intuitive. The process of fitting the curve is greatly simplified by taking advantage of the similarities between different populations to establish a standard distribution. This distribution, fitted by two constants, and the distribution for fertility by duration of marriage, fitted by one constant, form the base for the Coale and Trussell (1974) model fertility tables. Paper 30 is from this article.

[For another model expressing age at first marriage. also with an intuitive base, see Hernes (1972). The various models about equally reflect observed marriage patterns.]

The future growth of populations was first treated mathematically by Pierre-François Verhulst (1838), in response to Malthus' argument that populations would tend to grow exponentially until checked by resource limits. By implication, long range projections might be formed without detailed reference to current age structure, fertility or mortality, an assumption compatible with the lack of fertility information in Verhulst's time. His suggestion, paper 31, was to represent population growth by the logistic

$$r_t = b\left(1 - \frac{P_t}{k}\right),$$

$$P_t = \frac{k}{1 + e^{a-bt}},$$

which makes the intrinsic growth rate r_t a linearly decreasing function of population size P_t, k being the asymptotic population. The equation was independently applied to U.S. population growth by Raymond Pearl and Lowell Reed (1920), whose article here follows Verhulst's paper 31. Pearl and Reed's satisfaction with the logistic was not warranted when they wrote; in particular they failed to notice that the good fit of their equation neglected extraordinary contributions of technology, immigration and territorial expansion to U.S. population growth, making the correspondence of r_t and P_t rather more fortuitous than imperative. (As it turned out, the curve was 0.3% under the 1930 census but over by 3.5% at the next count. The higher figure translates as a 57% overestimate of intercensal growth. It may be compared with Whelpton's fine componant projection for 1940, 0.2% above the census, and his 6.0% and 49% underestimates for 1950 population and 1940—1950 intercensal growth respectively. In this case the better method was not a better guarantee of success.) Pearl and Reed toward the end of the paper take care in suggesting limitations of the logistic, and the reader will find there some of the reasons why they should not have used it; others were provided by Lancelot Hogben (1931, pp. 176—184) and William Feller (1940).

In another work that deserves mention, Thomas Edmonds (1852) applied Gompertz' Law to population growth by defining the rate of population increase as:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = B\,c^t,$$

with the distinction that $c < 1$ in order that the rate of growth would decline rather than increase over time. (In Verhulst, this corresponds to the damping factor $\phi(p)$.) Edmonds' equations are those later reapplied to mortality by Farr (1864), and he is almost certainly Farr's immediate source. The equations have not been remembered in population projections.

An examination by Otis Dudley Duncan (1958) { omitted here } treats human spatial measurement and efforts that have been made to fit curves to city size distributions, using either Felix Auerbach's (1913) rank-size rule or Mario Saibante's (1928) power function. These analyses confront measurement problems more complex than we have treated elsewhere in the book. { **Ed. Note:** A treatment of spatial demography, rank-size rules, and power laws from the perspective of early Twentyfirst Century demography is found in [23], chapters 13 to 15. }

Readers should be aware that the equations presented here are fitted to real data in different ways: DeMoivre's and those given by Duncan are solved by regression; the early fertility functions by moments; and Gompertz and Verhulst used simultaneous equations with selected data points. [Here, note that we might not follow the same approach. Curves in which constants enter non-linearly are accessible to least squares fitting by computer, using iterative methods; neither moment nor selected-point fittings are of equivalent quality.] { **Ed. Note:** Maximum likelihood estimation has replaced these specialized fitting procedures over the course of time. }

The method of least squares was first suggested by Carl Friedrich Gauss in personal communications and by Adrien Marie Legendre (1805) in print; fitting by moments is due principally to Pearson (1893, 1948).

{ **Ed. Note:** Parametric models for lifetables like the one in paper 28 generally known as the Brass Relational Logit model are useful for population forecasting. Trends in parameter values can be extrapolated into the future using econometric time-series methods which yield measures of uncertainty as well as of central tendency. Ronald D. Lee and Larry Carter [26] introduced methods for creating such a family of models from a matrix of historical rates of mortality arrayed by age and time using the matrix technique known as the singular value decomposition. The factor varying with time supplies a parameter which can often be treated for the purpose of forecasting as mimicking a random walk with drift. Lee-Carter forecasts have given impetus to the larger field of stochastic forecasting, in which Bayesian statistical approaches, as in [13], also play a prominent role. }

25. On the Nature of the Function Expressive of the Law of Human Mortality

BENJAMIN GOMPERTZ (1825)

From *Philosophical Transactions* 27. Excerpts are from pages 513—519.

We omit Gompertz' method of fitting, which the reader will find in Makeham, below. Also omitted are Gompertz' mortality tables, and his discussion of life expectancy and annuities under his Law. The hyperbolic logarithms in Art. 5 are the natural logs; we are not able to follow completely his integration, which is by Newton's method of fluxions.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_25, © Springer-Verlag Berlin Heidelberg 2013

Article 1. In continuation of Art. 2. of my paper on the valuation of life contingencies, published in the Philosophical Transactions of this learned Society, in which I observed the near agreement with a geometrical series for a short period of time, which must pervade the series which expresses the number of living at ages in arithmetical progression, proceeding by small intervals of time, whatever the law of mortality may be, provided the intervals be not greater than certain limits: I now call the reader's attention to a law observable in the tables of mortality, for equal intervals of long periods; and adopting the notation of my former paper, considering L_x to express the number of living at the age x, and using λ for the characteristic of the common logarithm; that is, denoting by $\lambda(L_x)$ the common logarithm of the number of persons living at the age of x, whatever x may be, I observe that if $\lambda(L_n)-\lambda(L_{n+m})$, $\lambda(L_{n+m})-\lambda(L_{n+2m})$ $\lambda(L_{n+2m})-\lambda(L_{n+3m})$, etc. be all the same; that is to say, if the differences of the logarithms of the living at the ages $n, n+m$; $n+m, n+2m$; $n+2m, n+3m$; etc. be constant, then will the numbers of living corresponding to those ages form a geometrical progression; this being the fundamental principle of logarithms.

Art. 2. This law of geometrical progression pervades, in an approximate degree, large portions of different tables of mortality; during which portions the number of persons living at a series of ages in arithmetical progression, will be nearly in geometrical progression; thus, if we refer to the mortality of Deparcieux, in Mr. Baily's life annuities, we shall have the logarithm of the living at the ages 15, 25, 35, 45, and 55 respectively, 2.9285; 2.88874; 2.84136; 2.79379; 2.72099, for $\lambda(L_{15})$; $\lambda(L_{25})$; $\lambda(L_{35})$; etc. and we find $\lambda(L_{25})-\lambda(L_{35})=.04738$, $\lambda(L_{35})-\lambda(L_{45})=.04757$, and consequently these being nearly equal (and considering that for small portions of time the geometrical progression takes place very nearly) we observe that in those tables the numbers of living in each yearly increase of age are from 25 to 45 nearly, in geometrical progression. If we refer to Mr. Milne's table of Carlisle, we shall find that according to that table of mortality, the number of living at each successive year, from 92 up to 99, forms very nearly a geometrical progression, whose common ratio is $\frac{3}{4}$; thus setting out with 75 for the number of living at 92, and diminishing continually by $\frac{1}{4}$, we have to the nearest integer 75, 56, 42, 32, 24, 18, 13, 10, for the living at the respective ages 92, 93, 94, 95, 96, 97, 98, 99, which in no part differs from the table by $\frac{1}{37}$th part of the living at 92. ...

Such a law of mortality would indeed make it appear that there was no positive limit to a person's age; but it would be easy, even in the case of the hypothesis, to show that a very limited age might be assumed to which it would be extremely improbable that any one should have been known to attain.

For if the mortality were, from the age of 92, such that $\frac{1}{4}$ of the persons living at the commencement of each year were to die during that year, which I have observed is nearly the mortality given in the Carlisle tables between the ages 92 and 99,[1] it would be above one million to one that out of three millions of persons, whom history might name to have reached the age of 92, not one would have

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[1] If from the Northampton tables we take the numbers of living at the age of 88 to be 83, and diminish continually by $\frac{1}{4}$ for the living, at each successive age, we should have at the ages 88, 89, 90, 91, 92, the number of living 83; 61.3; 45.9; 34.4; 25.8; almost the same as in the Northampton table.

attained to the age of 192, notwithstanding the value of life annuities of all ages above 92 would be of the same value. And though the limit to the possible duration of life is a subject not likely ever to be determined, even should it exist, still it appears interesting to dwell on a consequence which would follow, should the mortality of old age be as above described. For, it would follow that the non-appearance on the page of history of a single circumstance of a person having arrived at a certain limited age, would not be the least proof of a limit of the age of man; and further, that neither profane history nor modern experience could contradict the possibility of the great age of the patriarchs of the scripture. And that if any argument can be adduced to prove the necessary termination of life, it does not appear likely that the materials for such can in strict logic be gathered from the relation of history, not even should we be enabled to prove (which is extremely likely to be the state of nature) that beyond a certain period the life of man is continually becoming worse.

Art. 4. It is possible that death may be the consequence of two generally co-existing causes; the one, chance, without previous disposition to death or deterioration; the other, a deterioration, or an increased inability to withstand destruction. If, for instance, there be a number of diseases to which the young and old were equally liable, and likewise which should be equally destructive whether the patient be young or old, it is evident that the deaths among the young and old by such diseases would be exactly in proportion of the number of young to the old; provided those numbers were sufficiently great for chance to have its play; and the intensity of mortality might then be said to be constant; and were there no other diseases but such as those, life of all ages would be of equal value, and the number of living and dying from a certain number living at a given earlier age, would decrease in geometrical progression, as the age increased by equal intervals of time; but if mankind be continually gaining seeds of indisposition, or in other words, an increased liability to death (which appears not to be an unlikely supposition with respect to a great part of life, though the contrary appears to take place at certain periods) it would follow that the number of living out of a given number of persons at a given age, at equal successive increments of age, would decrease in a greater ratio than the geometrical progression, and then the chances against the knowledge of any one having arrived to certain defined terms of old age might increase in a much faster progression, notwithstanding there might still be no limit to the age of man.

Art. 5. If the average exhaustions of a man's power to avoid death were such that at the end of equal infinitely small intervals of time, he lost equal portions of his remaining power to oppose destruction which he had at the commencement of those intervals, then at the age x his power to avoid death, or the intensity of his mortality might be denoted by aq^x, a and q being constant quantities; and if L_x be the number of living at the age x, we shall have $a L_x \times q^x \cdot \dot{x}$ for the fluxion of the number of deaths $= -(L_x)^{\cdot}; \therefore abq^x = -\frac{\dot{L}_x}{L_x}$, $\therefore abq^x = -$hyp.log of $b \times$ hyp.log of L_x, and putting the common logarithm of $\frac{1}{b} \times$ square of the hyperbolic logarithm of $10 = c$, we have $c \cdot q^x =$ common logarithm of $\frac{L_x}{d}$; d being a constant quantity, and therefore L_x or the number of persons living at the age of $x = d \cdot g^{q^x}$; g being put for the number whose common logarithm is c. The reader should be aware that I mean g^{q^x} to represent g raised to the power q^x and not g^q raised to the x power; which latter I should have expressed by $(g^q)^x$, and which would evidently be equal to g^{qx}. I take this opportunity to make this observation, as algebraists are sometimes not sufficiently precise in their notation of exponentials.

This equation between the number of the living, and the age, becomes deserving of attention, not in consequence of its hypothetical deduction, which in fact is congruous with many natural effects, as for instance, the exhaustions of the receiver of an air pump by strokes repeated at equal intervals of time, but it is deserving of attention, because it appears corroborated during a long portion of life by experience; as I derive the same equation from various published tables of mortality during a long period of man's life, which experience therefore proves that the hypothesis approximates to the law of mortality during the same portion of life; and in fact the hypothesis itself was derived from an analysis of the experience here alluded to.

26. On the Law of Mortality

WILLIAM M. MAKEHAM (1867)

From *Journal of the Institute of Actuaries* 13. Excerpts are from pages 335—340, 348.

We omit Makeham's opening comments on life table functions, some of the mortality tables he examined, and his appendix, in which he develops some of his formulas. Makeham's F_x is the force of mortality.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_26,

The formula for F_x, according to Mr. Gompertz's theory, is Bq^x. For this I propose to substitute $A+Bq^x$, where A is the sum of certain partial forces which we assume to be, in the aggregate, of equal amount at all ages. The quantity Bq^x may also consist of the aggregate of several forces of a similar nature. So that we may put

$$F_x=(a+a'+a''+\ldots)+(b+b'+b''+\ldots)q^x,$$

where $a+a'+a''+\ldots=A$, and $b+b'+b''+\ldots=B$.

I do not profess to be able to separate the whole category of diseases into the two classes specified—viz., diseases depending for their intensity solely upon the gradual diminution of the vital power, and those which depend upon other causes, the nature of which we do not at present understand. I apprehend that medical science is not sufficiently advanced to render such a desideratum possible of attainment at present. I propose only at present to show that there are certain diseases—and those too of a well-defined and strictly homogeneous character—which follow Mr. Gompertz's law far more closely than the aggregate mortality from all diseases taken together. I shall then have given sufficient reason for the substitution of the form $Bq^x+\phi(x)$ for the force of mortality in lieu of Bq^x: the proof that the terms included in $\phi(x)$ form, *in the aggregate*, a constant quantity, I shall leave until we come to the examination of data more satisfactory than the returns of population and the public registers of deaths.

The two following tables are taken from the supplement before referred to. They give, first, the number of annual deaths (from all causes) to 1,000,000 living; and secondly, the number of annual deaths from certain specified causes to the same number living. The causes of death, as well as the ages for which they are given, have of course been selected as the most favourable exponents of the law of geometrical progression; but is will be observed that the former embrace all the principal vital organs of the body, and the latter include the whole of the period from early manhood to the confines of extreme old age.

The column headed "total force of mortality" should form a geometrical progression if Gompertz's law were applicable thereto. That it does not, however, form such a progression, is evident by inspection; the rate of increase in the earlier terms being less than 50 per cent., and gradually increasing until it exceeds 100 per cent. A similar result is found in all the known tables when the law is applied to the *total* force of mortality, the remedy for which (in constructing mortality tables by Mr. Gompertz's formula) is usually sought in a change of the constants of the formula after certain intervals. It is this gradual but constant variation of the rate of increase *in one direction*, and the fact of its being uniformly found in *all* tables, that show unmistakeably that if the law itself be true, its application stands imperatively in need of some modification.

The modification which I have suggested, viz., there are certain partial forces of mortality (how many I do not pretend to say) which increase in intensity with the age in a constant geometrical ratio, while there are also certain other partial forces which do not so increase, may be tested by an examination of the six columns which follow that of the *total* force above referred to. The tendency to a geometrical progression is more or less apparent in all of them; the average rate

Male Life, 1851—60

Ages	Total Force of Mortality	Partial Forces of Mortality					
		Lungs	Heart	Kidneys	Stomach and Liver	Brain	Sum of five preceding Columns
25—34	9,574	772	514	174	464	638	2,562
35—44	12,481	1,524	1,002	292	890	1,180	4,888
45—54	17,956	3,092	1,898	471	1,664	1,990	9,115
55—64	30,855	6,616	4,130	937	3,032	4,097	18,812
65—74	65,332	13,416	8,714	2,453	4,837	9,831	39,251

Female Life, 1851—60

Ages	Total Force of Mortality	Partial Forces of Mortality					
		Lungs	Heart	Kidneys	Stomach and Liver	Brain	Sum of five preceding Columns
25—34	9,925	582	603	109	570	532	2,395
35—44	12,147	1,049	1,118	151	937	872	4,127
45—54	15,198	2,062	2,064	212	1,608	1,681	7,627
55—64	27,007	5,027	4,558	317	2,967	3,818	16,687
65—74	58,656	11,016	8,916	485	4,692	8,905	34,014

of increase being such that the force of mortality somewhat more than doubles itself in 10 years.

It should be observed that, in addition to the diseases of the particular organs specified, other diseases of a kindred nature are also included under each of the above five partial forces. Possibly if more detailed information were accessible, we might be able to trace the geometrical character during a still more extended period of life. This, at least, I find to be the case in reference to one particular disease, viz., bronchitis, which in the preceding tables is included in the class of "lung diseases." Now it so happens that the deaths from bronchitis alone, for a long series of years, are given in the 26th Annual Report of the Registrar-General, from which the materials for the following table are taken. The number living is supposed to be 100,000, instead of 1,000,000 as in the two preceding tables.

In the preceding examination of the results of the Registrar-General's returns of deaths, I have confined myself to the object of proving that Gompertz's law is traced much more distinctly in the deaths arising from certain specified diseases, than in the deaths arising from all causes together. If I have succeeded in this object (and I think it can scarcely be denied that I *have* succeeded), I have justified the introduction of an additional term in the formula representing the total force of mortality; but I have as yet brought forward nothing to show that such additional term is a constant in respect of the age, and varying only with the peculiar characteristics which distinguish different sets of observations from each other.

Ages	1848 to 1854 (7 Years)		1855 to 1857 (3 Years)		1858 to 1863 (6 Years)	
	Males	Females	Males	Females	Males	Females
15—25	8	9	9	9	9	9
25—35	17	16	21	22	22	21
35—45	42	34	55	45	59	50
45—55	107	85	133	112	151	126
55—65	259	218	333	316	379	351
65—75	589	525	801	697	876	834
75—85	1,027	906	1,463	1,325	1,614	1,479

The several observations, however, which I now proceed to examine, if they do not enable us (like the former) to test particular terms of the function referred to, yet they will nevertheless afford a very satisfactory criterion of the complete expression. Not only, therefore, do they form by themselves (on account of their unquestionable accuracy and trustworthiness) ample evidence of the truth of the supposed law of mortality, but they also supply the deficiency, above adverted to, in the preceding investigation, as regards the requisite proof of the constancy of the term representing the aggregate of the remaining partial forces of mortality.

Commencing with the very valuable observations on the "Peerage Families" (both sexes), I find, by dividing the entire period of life into intervals of 14 years—neglecting, however, the first—the following results:—

$$\begin{array}{lll}
\log L_{14}=0.99034 & & \\
 & -0.05068 & \\
\log L_{28}=0.93966 & & -0.00716 \\
 & -0.05784 & \\
\log L_{42}=0.88182 & & -0.02559 \\
 & -0.08343 & \\
\log L_{56}=0.79839 & & -0.11395 \\
 & -0.19738 & \\
\log L_{70}=0.60101 & & -0.41273 \\
 & -0.61011 & \\
\log L_{84}=\bar{1}.99090 & &
\end{array}$$

The tendency to a geometrical progression in the four terms of the second order of differences is sufficiently apparent. In order, however, to show this more distinctly, I have devised the following method of correcting the series $\log L_x$ so that the four terms in question shall form a perfect geometrical progression.

If the series consist of *five* terms, and consequently the second order of differences of *three*, the latter may be converted into a pure geometrical progression by substituting for the original series another of the following form, viz.,

$$\log L_0+p, \log L_n-p, \log L_{2n}+p, \log L_{3n}-p, \log L_{4n}+p,$$

where p is derived from the equation

$$4p = \frac{(\Delta_n^2)^2-\Delta_0^2\times\Delta_{2n}^2}{\Delta_0^2+2\Delta_n^2+\Delta_{2n}^2}.$$ [1]

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[1] The differences are those of the function $\log L_x$.

This method, it is true, changes the value of the radix of the table, but I see no necessity for making a distinction between that and other terms of the series; for in comparing the terms of the *altered* with those of the *original* series, the object is to ascertain their bearing with respect to the original series *generally*, and not to any one term in particular. Secondly, by the method adopted, the first differences (which are the logarithms of the probabilities of living n years) are increased or diminished by an uniform quantity; whereas by omitting the correction in $\log \mathrm{L}_0$, the first term of the first order of differences would be increased or diminished by one-half of the quantity introduced into the remaining terms. Lastly, the equation for p would be of the second order, instead of the simple one given above.

Again, if the series consist of *six* terms—in which case there will be *four* terms in the second order of differences—the required effect may be produced by substituting for $\log \mathrm{L}_x$ the series

$$\log \mathrm{L}_0+(v-w),\ log\ L_n-(v-w),\ \log \mathrm{L}_{2n}+v,\ \log \mathrm{L}_{3n}-v,$$
$$\log \mathrm{L}_{4n}+(v+w),\ \log \mathrm{L}_{5n}-(v+w),$$

v and w being determined from the equations

$$2w=\frac{\mathrm{AC}-\mathrm{B}^2}{\mathrm{A}+2\mathrm{B}+\mathrm{C}} \quad \text{and} \quad 8v=\frac{\mathrm{A'C'}-\mathrm{B'}^2}{\mathrm{A'}+2\mathrm{B'}+\mathrm{C'}},$$

where

$$\mathrm{A}=\Delta_0^2+\Delta_n^2, \quad \mathrm{B}=\Delta_n^2+\Delta_{2n}^2, \quad \mathrm{C}=\Delta_{2n}^2+\Delta_{3n}^2,$$

and

$$\mathrm{A'}=\Delta_0^3+4w, \quad \mathrm{B'}=\Delta_n^3, \quad \mathrm{C'}=\Delta_{2n}^3-4w.$$

Here again, by involving the corrections symmetrically, we obtain for the unknown quantities simple instead of complicated quadratic equations.

Applying these formulae to the [preceding series], we have $2w=0.002651$, and $8v=0.007725$. The transformed series therefore becomes—

$$\begin{array}{lllll}
\log \mathrm{L}'_{14}=0.989980 & & & & \\
 & -0.049960 & & & \\
\log \mathrm{L}'_{28}=0.940020 & & -0.007274 & \log=\bar{3}.86177 & \\
 & -0.057234 & & & 0.58737 \\
\log \mathrm{L}'_{42}=0.882786 & & -0.028128 & \log=\bar{2}.44914 & \\
 & -0.085362 & & & 0.58733 \\
\log \mathrm{L}'_{56}=0.797424 & & -0.108760 & \log=\bar{1}.03647 & \\
 & -0.194122 & & & 0.58737 \\
\log \mathrm{L}'_{70}=0.603302 & & -0.420572 & \log=\bar{1}.62384 & \\
 & -0.614694 & & & \\
\log \mathrm{L}'_{84}=\bar{1}.988608 & & & &
\end{array}$$

The logarithms of the third series, and their differences, show that the transformed series fulfils the required conditions.

I have now to show that this result has been attained without a greater alteration of the original series than is warranted by the probable errors of the latter. In the following table the first column contains the age, the second the natural numbers corresponding to the original series $\log \mathrm{L}_x$, the third gives the decrement (deduced from the original data) of the year immediately following, while the fourth and fifth contain respectively the transformed series (denoted by L'_x)[2] and the amount by which it differs from the original series in the second column.

x	L_x	D_x	L'_x	$L_x - L'_x$
14	9780.0	41.4	9771.9	+ 8.1
28	8702.8	69.8	8710.0	− 7.2
42	7617.6	78.3	7634.6	−17.0
56	6286.2	150.3	6272.3	+13.9
70	3990.3	220.3	4011.5	−21.2
84	979.3	152.2	974.1	+ 5.2
		712.3		72.6

Comparing columns 3 and 5 together, term by term, we find that in one instance only (viz. at age 42) does the alteration made in the numbers living exceed one-fifth [0.2171] of the corresponding yearly decrement; while from the sums of the same columns it appears that the *average* alteration is little more than one-tenth of the *average* decrement. We may, therefore, say the limit of the variation of the two series (cols. 2 and 4) is about one-fifth of a year. ...

We have seen that the expression for the number living at any given age in a normally-constituted increasing population—in which the yearly births as well as the yearly deaths (and consequently also the excess of the former over the latter) are proportional to the existing population—is of the same form as that representing the numbers living at the given age in a stationary population, and also the numbers living in a table of mortality. But the function $L_x v^x$ is also of the same form as the latter, for L_x contains a factor s^x which combines with that introduced by the interest of money. Hence it follows that to determine the number living between given ages in a population normally constituted— whether increasing or stationary—as well as the expectation of life and the value of annuities, the summation of a function of one form only (viz., $dg^{q^x} s^x$) is required.

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[2] Hitherto the accent has been used to distinguish the "partial" from the "total" forces of mortality, but as we have now done with this branch of the subject, no confusion will be caused by using it to denote (as it will be used henceforth) the *corrected* values of the function to which it is applied.

27. Calculation of Model Tables

ANSLEY J. COALE and PAUL DEMENY (1966)

From *Regional Model Life Tables and Stable Populations*, Chapter 2, pp. 11—14, 20, 23—26. Princeton: Princeton University Press.

We have omitted sections of the chapter on the correlation of ${}_nq_x$ to ${}_nq_y$ and $\mathring{e}_{10}$, which was high in all cases, and on the calculation of stable populations. The regression coefficients for the ${}_nq_x$ on $\mathring{e}_{10}$ and the separation factors k, α are also left out here, since these are less important than the methods underlying their derivation.

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CALCULATION OF FOUR FAMILIES OF MODEL LIFE TABLES

The model tables presented here are based on a tendency that we noted for the life tables based on accurate data to cluster around four different lines, representing distinct age patterns of mortality in certain geographical regions.

The four families of life tables were isolated as the result of working with a preliminary one-parameter family of model tables designed to represent the entire collection of 326 life tables. The following method was used to construct the preliminary model tables: all of the q_x values were ordered, from highest to lowest, at each age. The values were then plotted as a function of order for each age, and occasional erratic fluctuations removed by hand smoothing. Preliminary model tables were then formed by putting together mortality rates with the same rank. This system of construction has the virtue of simplicity, and of symmetrical treatment of mortality rates at all ages. That is to say, no particular rate or rates are singled out as the basis for estimating others. The usefulness of the preliminary set of tables is questionable, however, because the 326 tables are so widely scattered about the "line" these model tables formed, as we could see by examining two-dimensional scatter diagrams.

The next step was to examine deviations of individual life tables from the age pattern of mortality in the preliminary model tables. The pattern of deviations was measured by the difference between q_x in the given table and q_x in a model life table with the same general level of mortality. The "comparison" model table was selected by first noting the expectation of life at birth in the model life tables with the same ${}_nq_x$ as the given life table, at $x = 0, 1, 5, \ldots, 75$. The median model table (the table with the median e_0°) was the "comparison" table. The graph showing ${}_nq_x - ({}_nq_x)_{\text{model}}$ displays, for ready visual comprehension, the age pattern of mortality relative to the age pattern in the model tables, in a way that does not depend on the over-all level of mortality. Graphs of this sort were constructed for all 326 life tables. The following general observations emerged from an examination of the patterns:

1. Some life tables have large variations from the model patterns. Large deviations were especially frequent under age 10 and over age 60.
2. The biggest deviations were found in life tables where the quality of the underlying data is suspect: life tables for Western Europe prior to 1850; certain Eastern European life tables—for Russia in 1897 and 1926, for Bulgaria and Greece; and life tables in underdeveloped countries in Asia, Africa, and Latin America. In most of these tables, age-reporting in the censuses, and doubtless in the mortality register, is inaccurate. The completeness of death registration—especially for infants—is uncertain.
3. Where data are known to be accurate, deviations are usually moderate. Statistics that are comparatively free of age misstatement and omission are found in some European countries since 1870, in 20th-century Canada, the United States, Australia, and New Zealand. Unfortunately, these areas have a rather narrow range of cultural diversity and doubtless represent a narrow sample of age patterns of mortality. It is of special interest to note, then, that the life tables of Taiwan and the more recent Japanese life tables—both based on accurate data—conform as well as do European tables to the preliminary model tables.

Two features of the deviations found within life tables based on accurate data led to the construction of four separate families of model tables: (a) The fact that the pattern of deviations is often similar among life tables expressing the mortality of the same population at different times, and (b) the fact that several groups of geographically linked populations exhibited similar patterns of deviations.

Figure 1 shows patterns of deviation in certain Scandinavian life tables; in tables from Germany, Austria, Czechoslovakia, Poland, and North Italy; and in tables from South Italy, Spain, and Portugal. Within each group the similarity of deviations is easily visible, although from group to group the patterns differ.

On the basis of an examination of the patterns of deviations in the 326 life tables for each sex, correlation matrices were calculated for various groups of life tables—zero-order correlations among 19 variables: log ${}_nq_x$ at various ages ($x = 0, 1, 5, 10, \ldots, 75$), and expectation of life at age 0 and age 10. We experimented with nine principal sets of correlations for each sex: (1) tables before 1870; (2) tables for Russia and certain Balkan areas; (3) tables for selected Central European areas; (4) tables for Scandinavian countries; (5) tables for Spain, Portugal, and Southern Italy; (6) tables for Switzerland; (7) tables for countries with reliable data not included in (3), (4), (5), or (6); (8) tables reflecting mortality when there is an unusually high incidence of tuberculosis; and (9) modern tables based on relatively inaccurate data—primarily from Asia, Africa, and Latin America.

The nature of the life tables underlying each of the "regional" model tables is as follows:

1. *Tables underlying "East" model tables.* The life tables of Austria, Germany (including tables in 1878 and the 1890's for Bavaria and Prussia), Czechoslovakia, North and Central Italy, Hungary, and Poland show deviations from the preliminary model life tables characterized by high mortality rates in infancy, and increasingly high rates over age 50 (Figure 1, central panel). Switzerland's life tables show deviations very similar to this group until 1920, although the early Swiss life tables have a less conspicuous positive deviation in infancy. After 1920, the Swiss life tables show zero or negative deviations in infant mortality. Hungarian life tables exhibit substantial deviations in an age pattern indicating an extraordinary incidence of tuberculosis. Inclusion of the Hungarian life tables lowers the correlation coefficients from age 5 to 35, but has little other effect, and they were omitted from the calculation of the "East" model tables, as were the Swiss life tables. The tables in the "East" group include 13 from Germany (of which 3 are from Prussia or Bavaria), 5 from Austria, 3 from Poland, 4 from Czechoslovakia, and 6 from North or Central Italy.

2. *Tables underlying "North" model tables.* The life tables of Norway, Sweden until 1920, and Iceland deviate from the preliminary model tables in having low infant mortality rates, and rates that are lower than the model rates by an increasing margin at ages beyond 45 or 50. Later Swedish life tables do not have this characteristic pattern (Figure 1, left panel). In the life tables of all three "North" countries from 1890 or 1900 to 1940, there are deviations in the mortality rates from age 5 to 35 or 40 indicating the effect of an unusual incidence of tuberculosis. Model tables in-

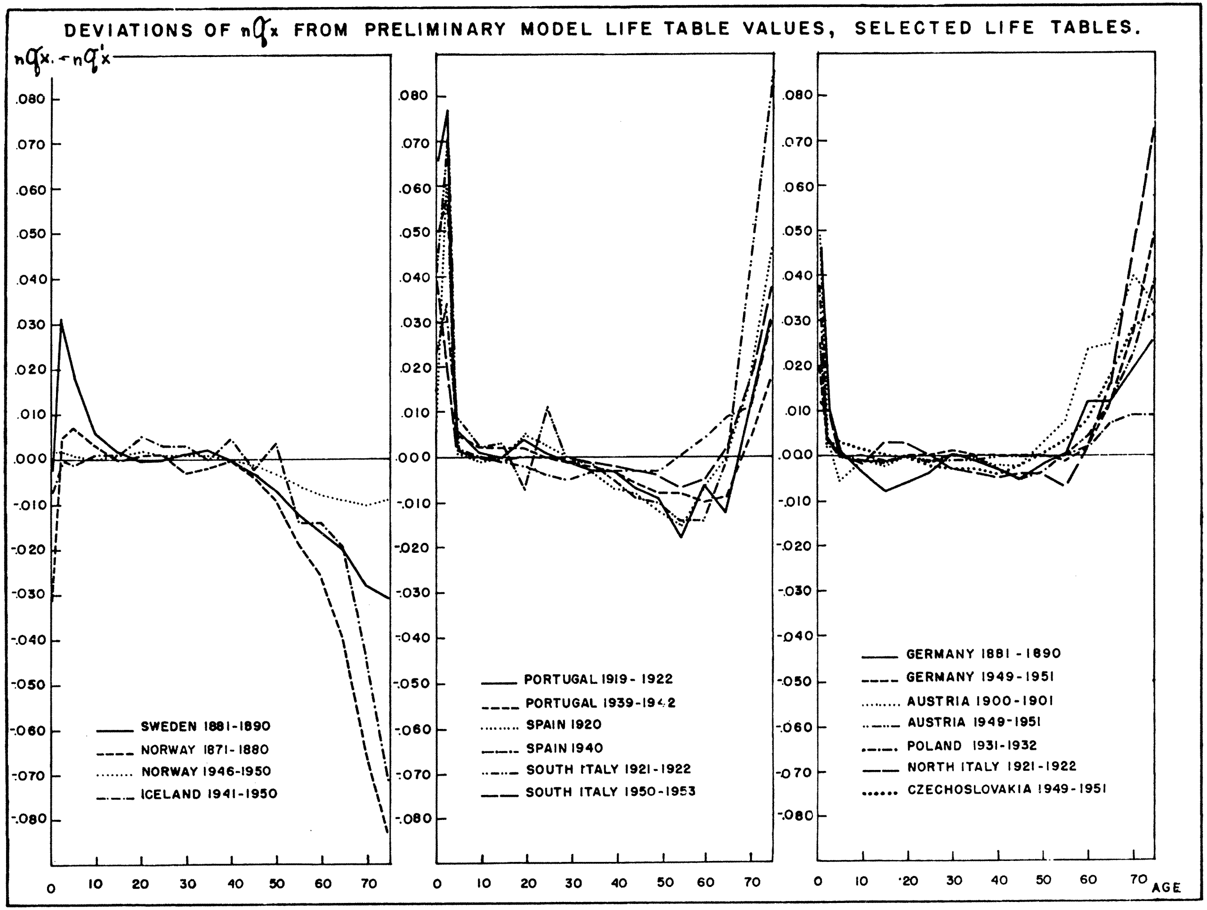

FIGURE 1. Deviations of mortality rates in three regional collections of life tables (${}_nq_x$) from mortality rates in a preliminary model life table (${}_nq_x'$). Selected tables for females.

corporating this experience would be suitable only for populations with a high endemicity of tuberculosis. Consequently, the "North" model tables are based on Swedish mortality from 1851 to 1890 (4 tables), Norwegian mortality from 1856 to 1880 and from 1946 to 1955 (4 tables), and the Iceland life table for 1941–1950.

3. *Tables underlying "South" model tables.* The life tables of Spain, Portugal, and Southern Italy have high mortality under age 5, low mortality from about age 40 to 60, and high mortality over age 65, relative to the preliminary model tables (Figure 1, right panel). Early tables—prior to 1912—for all Italy had these same characteristics. The "South" model life tables were based on 5 tables for all Italy (1876 to 1910), 8 tables for Portugal (1919 to 1958), 1 table for Sicily (1951), 3 for South Italy (1921 to 1957), and 5 for Spain (1910 to 1940).

4. *Tables underlying "West" model life tables.* The "West" model life tables were based on mortality experience recorded in populations known to have relatively good vital statistics, and not showing a persistent systematic pattern of deviations from the preliminary model tables. In other words, the tables underlying the "West" models are a residual collection after the "East," "South," and "North" tables have been removed. (The Swiss tables were also omitted, having a recent pattern resembling the "West" up to age 45 or 50, and the "East" above age 50.) The tables underlying the "West" models include 7 from Australia (1881 to 1955), 4 from Belgium (1880 to 1949), 7 from Canada (1926 to 1959), 11 from Denmark (1895 to 1955), 11 from England and Wales (1871 to 1959), 1 from Estonia (1933), 4 from Finland (1881 to 1955), 16 from France (1871 to 1959), 5 from Ireland (1925 to 1952), 3 from Israel (1949 to 1959), 6 from Japan (1949 to 1960), 1 from Latvia (1936), 1 from Luxemburg (1947), 10 from the Netherlands (1870 to 1955), 12 from New Zealand (1906 to 1959), 2 from Northern Ireland (1925 to 1959), 7 from Scotland (1891 to 1959), 5 from Sweden (1931 to 1959), 3 from Taiwan (1921 to 1959), 4 from the Union of South Africa white population (1920 to 1947), and 10 from the United States (1901 to 1958). Tables before 1870 were eliminated because most of those examined (from France, the Netherlands, and England and Wales) had irregular patterns that appeared to arise from faulty data. …

The principal steps in the calculation of the four sets of model life tables were as follows:

1. Intercorrelation matrices for ${}_nq_x$ and $\log_{10}({}_nq_x)$ were calculated for the "North," "South," "East," and "West" data. [$e_0{}^\circ$ and $e_{10}{}^\circ$ were left untransformed (no logarithms taken) in the second sets of intercorrelations.]
2. Least-square linear regressions of ${}_nq_x$ and of $\log {}_nq_x$ on $e_{10}{}^\circ$ were fitted for both sexes in all four "regions." The regression coefficients are given in Table XI.
3. The values of ${}_nq_x$ estimated from the logarithmic regression are always above those from the regression of untransformed mortality rates at the high and low extremes of observed life expectancies, and the logarithmic regression values are always lower in the middle range. In other words, the two regression lines always intersect twice within the range of observations. In constructing the model life tables, ${}_nq_x$ values were taken from the simple regression at all points to the left (i.e., at points with lower life expectancy) of the first intersection of the regression lines; and to the right of the second intersection, ${}_nq_x$ values were taken from the logarithmic regression. Between the two intersections, the mean of the ${}_nq_x$ values from the two regressions was used.
4. From various values of the independent variable ($e_{10}{}^\circ$), ${}_nq_x$'s at ages 0, 1, 5, 10, . . . , 75 were calculated. From each such set of ${}_nq_x$'s, l_1, l_5, l_{10}, . . . , l_{80} were computed with l_0 taken as 100,000.
5. ${}_nL_x$ and $e_x{}^\circ$ were estimated on the use of the following formulae:

$$
\begin{aligned}
{}_1L_0 &= k_0 l_0 + (1 - k_0) l_1 \\
{}_4L_1 &= k_1 l_1 + (4 - k_1) l_5 \\
{}_5L_x &= 2.5(l_x + l_{x+5}),\ x = 5, 10, \ldots, 75 \\
e_{80}{}^\circ &= 3.725 + 0.0000625 l_{80} \\
T_{80} &= e_{80}{}^\circ l_{80} \\
T_x &= \sum_{X}^{75} L_x + T_{80} \\
e_x{}^\circ &= \frac{T_x}{l_x}
\end{aligned}
$$

6. Age-specific mortality rates ($_{n}m_{x}$) were calculated from the formula $_{n}m_{x} = {}_{n}d_{x}/{}_{n}L_{x}$ where $_{n}d_{x} = l_{x} - l_{x+n}$

7. Five-year survival rates for projecting five-year age groups ($_{5}P_{x}$) were calculated by the formula $_{5}P_{x} = {}_{5}L_{x+5}/{}_{5}L_{x}$, $x = 0$, 5, . . . , 70.

The first survival rate is the proportion surviving to the end of a five-year time interval of persons born during the interval, estimated as $_{5}L_{0}/5l_{0}$. The last survival rate is of persons over 75 at the beginning of an interval (and over 80 at the end), estimated as T_{80}/T_{75}.

8. Both male and female tables are calculated by regression of $_{n}q_{x}$ on $e_{10}{}^{\circ}$. The values of $e_{10}{}^{\circ}$ that were used as the independent variable in constructing the female tables were chosen so as to give even 2.5-year intervals of $e_{0}{}^{\circ}$ from 20 to 77.5 years. The values of $e_{10}{}^{\circ}$ for males were chosen so as to correspond with the female $e_{10}{}^{\circ}$'s in a way that preserves the typical relation of $e_{10}{}^{\circ}$ for males and females at each level of mortality within each family of life tables. The relationship posited was as follows:

$$(e_{10}{}^{\circ})_{m} - (\overline{e_{10}{}^{\circ}})_{m} = \frac{\sigma_{m}}{\sigma_{f}}[(e_{10}{}^{\circ})_{f} - (\overline{e_{10}{}^{\circ}})_{f}]$$

where σ_{m} and σ_{f} are the standard deviations of expectation of life at age 10 for males and females. This expression is the equation for the straight line with a slope intermediate between the regression of $(e_{10}{}^{\circ})_{m}$ on $(e_{10}{}^{\circ})_{f}$, and the inverse of the regression of $(e_{10}{}^{\circ})_{f}$ on $(e_{10}{}^{\circ})_{m}$. The correlation between $e_{10}{}^{\circ}$ for the two sexes is more than .99 in all instances, so that the two regression lines are almost identical. ...

THE USE OF TWO REGRESSIONS IN CALCULATING $_{n}q_{x}$ IN MODEL TABLES

There are two considerations that make questionable the use of regression equations in constructing model life tables: (1) A regression of a particular functional form (linear, quadratic, logarithmic, etc.) often fits the data better over some parts of the range of observations than others, and often represents the data poorly at extremes, and may in particular provide an implausible extension or extrapolation of relationships beyond the range of observations; (2) "least squares" regression equations of any given functional form establish two relations between any pair of variables, as each is considered the independent variable, and the other dependent. In an estimation of one variable from a known value of another, the use of one variable as dependent and the other as independent is clearly appropriate and there is a logical reason for employing one regression rather than the other. But there is no logical basis for selecting one mortality rate or another as independent when trying to estimate a set of interrelationships among mortality rates at different ages. In the original United Nations model life tables, the selection of ${}_1q_0$ as the sole purely independent variable, and the calculation of a chain of regressions, with each mortality rate first being a dependent and then an independent variable, created a tendency (because of the well-known "regression toward the mean") to incorporate less of the observed range of variability at higher ages than at lower ages. A regression chain starting at the highest ages and ending with ${}_1q_0$ would have been equally plausible, and would have produced somewhat different model tables.

The use of two linear regressions (logarithmic and non-logarithmic) and the selection of e_{10}° as the index of mortality (the independent variable in calculating mortality rates for every age group) result in a procedure that lacks elegance, but takes account of the considerations stated in the preceding paragraph.

The expectation of life at age 10 is an approximate general index of the level of mortality, an index that is not strongly dependent on the mortality rates for any one age group. (In contrast, e_0° is strongly influenced by the value of ${}_1q_0$.) Correlations with e_{10}° are near to unity at most ages in all "regions." The choice of e_{10}° as the basis for regression estimates is not an arbitrary fixing of any one mortality rate in the resultant model life tables. All mortality rates are separately subject to "regression toward the mean." Hence if an e_{10}° two standard deviations above the mean is chosen as the basis for estimating a model life table, *all* of the mortality rates will be somewhat less than two standard deviations below their mean value; but the sequence of mortality rates is not ever more compressed toward the mean as age advances. ...

The facts that led us to combine estimates from regressions of ${}_nq_x$ and log ${}_nq_x$ on $e_{10}{}^{\circ}$ were as follows:

1. The two sets of correlation coefficients were comparable in magnitude. There were 48 instances where the non-logarithmic correlation exceeded the logarithmic by at least .003; 55 where the logarithmic coefficients were larger by this margin; and 33 instances where the difference was less than .003. (Table XII). However, there is a different relative effect on the two kinds of correlation coefficients of "scatter" in mortality rates at high levels of mortality on the one hand, and low levels of mortality on the other. . . .

2. The logarithmic regression line at the highest observed expectations of life is closer to the observations, and represents a more plausible extension of the observations, than the non-logarithmic regression line. This closer fit is caused partly by the sensitivity of the logarithmic regression to relative rather than absolute deviations, and partly by the fact that declining linear exponential functions are necessarily asymptotic to zero, and never give negative values. In contrast, the non-logarithmic regression line falls *below* the observed mortality rates for the higher values of $e_{10}{}^{\circ}$, and sometimes indicates negative mortality estimates near the upper limits of observed values of $e_{10}{}^{\circ}$—a patent absurdity.

3. At the other end—high mortality, low expectation of life—of the range of observed values, the non-logarithmic regression line is usually closer to empirical mortality rates, and, in this range, also represents a more plausible extension of observed relations. The exponential necessarily becomes steeper at lower expectations of life, and would yield absurdly high estimated mortality rates if $e_{10}{}^{\circ}$ were assumed to have a low value.

Through the middle range—between the two intersections of the lines—sometimes one line is closer, and sometimes the other to the observed mortality rates. To have a uniform rule, and minimize discontinuity in the sequence of model tables, we estimated mortality rates from the non-logarithmic regression up to the first intersection, used the mean of the two regression estimates between the two intersections, and used the estimates from the logarithmic regression at expectations of life at age 10 above—to the right of—the second intersection.

ESTIMATION OF LIFE TABLE VALUES OTHER THAN $_nq_x$

The weights (k_0 and k_1, the "separation factors") that relate $_1L_0$ to l_0 and l_1, and $_4L_1$ to l_1 and l_5, can be shown to equal the age at death of those members of the life table population who die under age 1 (for k_0), and the age at death minus 1 of those who die between ages 1 to 4 (for k_1). The values of the factor k_0 were therefore determined by examining the average age at death under 1 year in the records of the populations whose mortality experience was the basis for the four regional model life tables. No consistent variation in average age of infant deaths was found at infant mortality levels above 0.100, nor any consistent differences among the "regions," except that k_0 for "East" infants was consistently less than in the other "regions," and k_0 for males was generally slightly less than for females. When $_1q_0$ is very low—in the range from 0.015 to 0.100—there is a clearly apparent tendency for k_0 to vary, because at very low levels of infant mortality infant deaths are much more concentrated immediately after birth. This tendency was represented by allowing k_0 to rise linearly from a typical level at the lowest observed infant mortalities until at $_1q_0 = 0.100$ it reaches the plateau typical of higher infant mortality.

The value of k_1 in the expression $_4L_1 = k_1l_1 + (4 - k_1)l_5$ was determined from the estimates of l_2, l_3, and l_4 given in Table XV at the end of Chapter 3. In fact $_4L_1$ was calculated on the assumption that l_x can be considered linear in each single-year age interval from 1 to 5; thus $k_1 = 0.5 + \alpha_2 + \alpha_3 + \alpha_4$, when $l_2 = \alpha_2l_1 + (1 - \alpha_2)l_5$, etc. Each α_i was based on the relation of l_i to l_1 and l_5 observed in the life tables upon which each family of model tables was based. As with k_0, it was noted that when $_1q_0 > 0.100$, the values of each α_i showed no tendency to vary as a function of the level of mortality, although at levels of mortality below $_1q_0 = 0.100$ there is a tendency for the deaths from age 1 to age 5 to be more evenly distributed, and for l_x in this range to be more nearly linear. Hence at low levels of mortality, each α_i was allowed to fall as $_1q_0$ increases in a manner analogous to the rise in k_0. The values of α_2, α_3, and α_4 are somewhat different in each region. It is reassuring to note that when ${}_1q_0 > 0.100$, the α_i's observed in the life tables underlying each regional set of model tables clustered about an average typical of the set, and differing from the average value of α_i for the other "regions." In other words, each region has a distinctive pattern of mortality from 1 to 5, as well as at other ages.

The formula for expectation of life at age 80 is based on a line fitted to observations in 70 life tables (35 for males and 35 for females) from the Netherlands, Norway, Sweden, Switzerland, and Japan. These tables were selected because age reporting in the census and mortality records in these countries for persons over 80 appeared especially trustworthy.

28. Methods of Analysis and Estimation

WILLIAM BRASS and ANSLEY J. COALE (1968)

From William Brass, et. al.: *The Demography of Tropical Africa*, pp. 127—132. Princeton: Princeton University Press.

In this extract Brass fits logit transformations of observed and model life table survival rates to a power function that preserves the extreme values $p(0)=1$, $p(\omega)=0$, where $p(x)$ is the probability of surviving from birth to age x. The work may be clearer if his equations (9) and (10) are rewritten slightly as

$$q(a)/p(a)=A\,[q_s(a)/p_s(a)]^{\beta}\,u(a), \tag{10a}$$

$$\ln[q(a)/p(a)]=\ln A+\beta\ln[q_s(a)/p_s(a)]+\ln u(a) \tag{9a}$$

with the multiplicative disturbance term $u(a)$, whose expected value in (9a) becomes $E(\ln u)=0$. From (9a) the estimators $\hat{A},\hat{\beta}$ are found by linear regression. The equation to generate the new survival estimates $\hat{p}(a)$ from the fitted regression is

$$\hat{p}(a)=\{1+\hat{A}\,[q_s(a)/p_s(a)]^{\hat{\beta}}\}^{-1}.$$

Loosely, A is associated with life expectancy and β with the distribution of mortality between younger and older ages, relative to the standard table.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_28,

The Brass Model Life Tables and Stable Populations

The Brass method of constructing model life tables is to subject the survivor function in a life table chosen as a "standard" to the so-called *logit* transformation, and then to consider the life tables generated by assuming that their logits are linearly related to the logit of the standard table. In this way, a two-parameter set of model life tables can be constructed.

The logit function is as follows:

$$\text{logit}\,(x) = \frac{1}{2}\log_e \frac{1-x}{x} \tag{8}$$

For x in equation (8), Brass substitutes $p(a)$ (or l_a/l_0 in life-table notation). He then selects a standard table, $p_s(a)$, and constructs his model life tables by assigning different values to α and β in

$$\text{logit}\,p(a) = \alpha + \beta\,\text{logit}\,p_s(a) \tag{9}$$

This transformation implies that:

$$\frac{q(a)}{p(a)} = A\left(\frac{q_s(a)}{p_s(a)}\right)^{\beta} \tag{10}$$

where $A = e^{2\alpha}$.

The Nature of the Linear Logit Transformation of a Standard Life Table

The logit transformation can generate a survivor function $p(a)$ that passes through arbitrarily preassigned values at any two ages a_1 and a_2, by the selection of α and β.[19] The life table thus generated has the preassigned values of $p(a)$ at the selected ages and shares the shape, in a generalized sense, of the "standard" life table.

Figure 3.4 illustrates the effect of different values of α and β on the standard life table that Brass has constructed for Africa. The survival curves include: (a) the standard ($\alpha = 0$, $\beta = 1.00$); (b) a curve that differs from the standard by having a β of 1.20 instead of 1.00; (c) a

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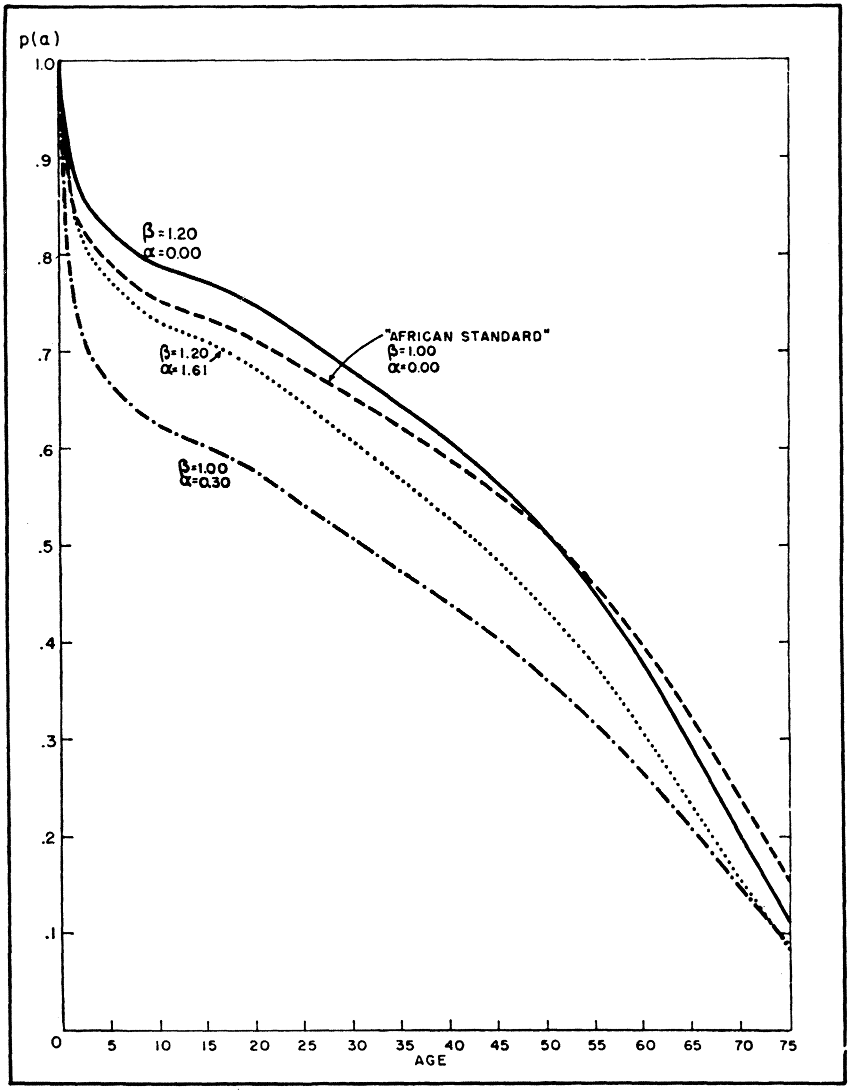
[19] Given any preassigned $p(a_1)$ and $p(a_2)$, we obtain two equations:

$$\alpha + \beta\ \text{logit}\ p_s(a_1) = \text{logit}\ p(a_1)$$
$$\alpha + \beta\ \text{logit}\ p_s(a_2) = \text{logit}\ p(a_2)$$

The known values of $p_s(a_1)$ and $p_s(a_2)$ together with the preassigned values of $p(a_1)$ and $p(a_2)$ provide a unique determination of α and β, and hence a complete life table.



3.4. Proportion surviving to age a, $p(a)$, obtained from the transformation $p(a) = \alpha + \beta$ logit $p_s(a)$ for various values of α and β.

curve that has the same value of $p(2)$ as the standard, but a β of 1.20, and (d) a curve that differs from the standard by having a value of α of 0.30 instead of zero. Note that the curve with $\alpha = 0$ and $\beta = 1.20$ intersects the standard at an age where $p_s(a) = 0.50$. This is a general relation: all curves with the same α and different β's intersect at $\bar{a}$ determined by $p_s(\bar{a}) = 0.500$, because logit $0.5 = 0$. Hence variation of β produces a sub-family of survival curves, fixed at age 0 and ω, and with a node at $\bar{a}$ (about age 51 in the standard table). At ages below this node or intersection, the life table based on the larger β has higher values of $p(a)$, and at higher ages the opposite relation holds. As a result the area under these two curves (which in each instance is the expectation of life at birth) is very nearly the same.[20] However, this near constancy of e°_0 for different values of β is *not* a general relationship, but is the result of the selection of a standard life table with $p(51) = 0.50$. If a higher-mortality standard life table had been chosen, tables with the same α would intersect at a younger age (say 25) and higher values of β would lower the expectation of life at birth.

Note that the logit transformation permits the acceptance of a trustworthy estimate of childhood survival—e.g., an estimate of $p(2)$—and then the selection of some overall estimate of adult mortality, in the form of a choice of $p(30)$ or $p(50)$ given the accepted value of $p(2)$. The transformation of the standard table (with α and β chosen in this manner) then provides a whole model life table.

The Importance of the Standard Life Table in the Linear Logit Transformation

The logit transformation will generate life tables having any preassigned level of childhood mortality—expressed, for example, by a value of $q(2)$—and of adult mortality—expressed, for example, by ${}_{48}q_2$, or $p(50)/p(2)$. How well do such life tables conform to the observed age pattern of mortality in empirical populations? The Coale-Demeny families of model tables provide an interesting test. Each family of these tables expresses the particular age pattern of mortality found in the well-recorded experience of a group of populations. The groups of life tables underlying each family were assembled because of noticeable similarities in age patterns. Figure 3.5 shows the typical result of trying to express a model table within one of the families (a) as the linear logit transformation of another table in the same family, and (b) as the logit transformation of a table from a different family. Note that logit $p(a)$

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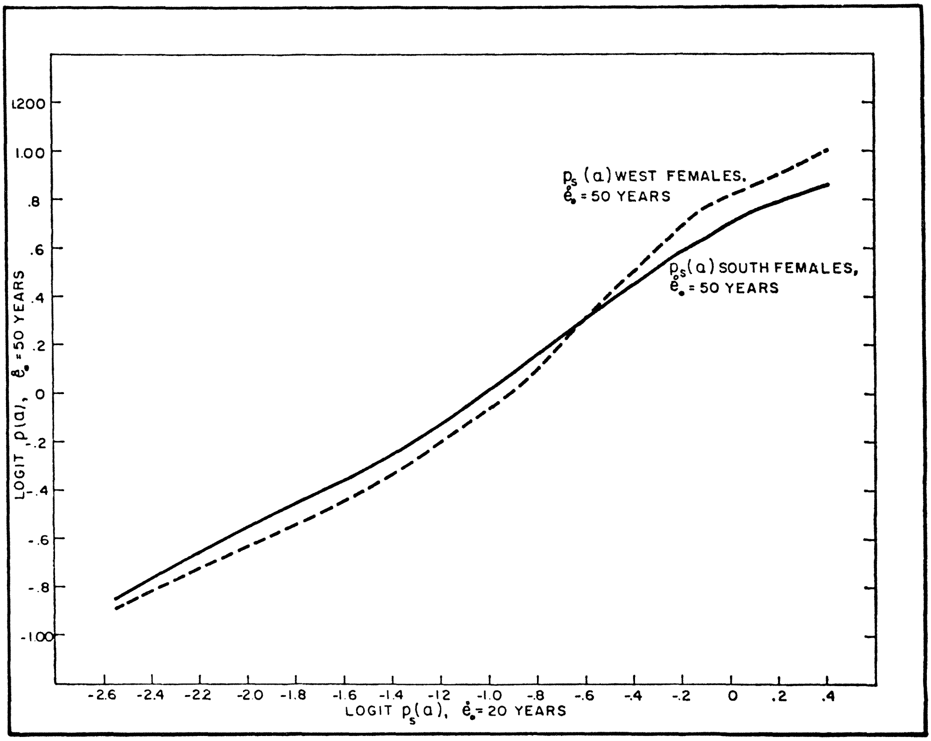
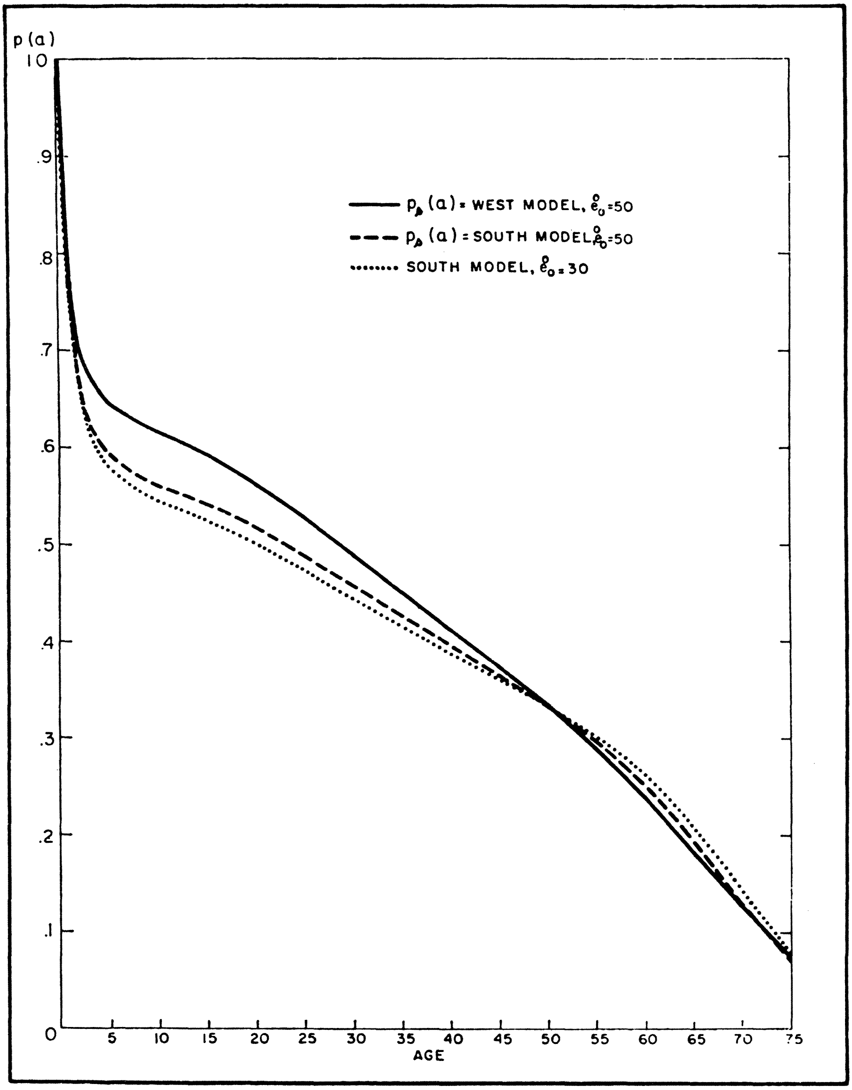
[20] For the standard life table, $e^\circ_0 = 43.6$, and for $\beta = 1.2$, $e^\circ_0 = 44.1$. There is a kind of symmetry among curves with different values of β, and $\alpha = 0$. The difference in p for each young age such that $p_s(a') > 0.5$ is matched by an equal (and opposite) difference in p for the older age a'' such that $p_s(a'') = 1 - p_s(a')$. Thus when $p_s(a) = 0.50$ occurs near the middle of the range zero to ω, life tables with $\alpha = 0$ and different values of β have nearly the same e°_0.



3.5. Logit $p(a)$ plotted against logit $p_s(a)$, when $p(a)$ is South model life table, females, $e^\circ_0 = 20$ years, and $p_s(a)$ is alternatively West model and South model table, $e^\circ_0 = 50$ years.

for the South model life table (females) with $e^\circ_0 = 20$ has a very nearly linear relation to logit $p(a)$ for the South model table where $e^\circ_0 = 50$ years, but that the relation to logit $p(a)$ for the *West* model table with $e^\circ_0 = 50$ is by no means linear. Figure 3.6 shows the result of attempting to approximate the South model table with $e^\circ_0 = 30$ years by a logit transformation of the South model table with $e^\circ_0 = 50$. and the West model table with $e^\circ_0 = 50$. Calculations of the same sort with the other Coale-Demeny families of model tables led to a similar result: linear logit transformations reproduce the model life tables *within* a family very closely, but between families not nearly so well. Table 3.10 shows the values of α and β needed to transform the female table with $e^\circ_0 = 50$ into the tables with e°_0 of 20, 30, 40, 60, and 70 in each family. Note that in all families except South (where β is quite large at high mortality levels) the parameters needed to transform from one mortality level to another are almost identical.



3.6. South model life table, females, $e^\circ_0 = 30$ years compared with life tables obtained by linear logit transformation of South and West model tables, $e^\circ_0 = 50$ years.

Table 3.10. Values of α and β in logit $p(a) = \alpha + \beta$ logit $p_s(a)$ for North, South, East, and West model life tables (female), various levels of e°_0, life table with $e^\circ_0 = 50$ serving as $p_s(a)$

		$e^o_o = 20$	$e^c_o = 30$	$e^o_o = 40$	$e^o_o = 50$	$e^o_o = 60$	$e^o_o = 70$
North	α	1.008	0.612	0.295	0.000	-0.316	-0.734
	β	1.296	1.154	1.064	1.000	0.950	0.900
West	α	1.036	0.632	0.307	0.000	-0.339	-0.819
	β	1.306	1.161	1.068	1.000	0.944	0.893
South	α	1.057	0.650	0.318	0.000	-0.356	-0.803
	β	1.533	1.318	1.155	1.000	0.810	0.599
East	α	1.000	0.618	0.302	0.000	0.338	0.813
	β	1.299	1.165	1.073	1.000	0.929	0.852

These examples show that the selection of a standard life table affects the detailed form of the survival curve in a life table fitted to the estimated mortality experience of a population by a linear logit transformation. The standard life table to be used in Africa, in other words, should ideally incorporate typical features (if such exist) of African age patterns of mortality.

29. Nuptiality, Fertility, and Reproductivity

S. D. WICKSELL (1931)

From Skandinavisk Aktuarietidskrift, pp. 149—157.

We omit Wicksell's discussion of Swedish nuptiality and fertility, and his distinction between net reproduction rates based on legitimate and illegitimate births. For his equation (31), a Pearson Type I distribution, the fitted constants should be

$$\text{Case 1: } b_1 = 1.857, \qquad b_2 = 2.518$$

$$\text{Case 2: } b_1 = 1.689, \qquad b_2 = 2.593.$$

Formulas for fitting the Type I distribution are given in Elderton and Johnson (1969, pp. 51—52). The fitting requires the mean and variance (μ, μ_2) and two next central moments (μ_3, μ_4) of the net maternity function, given by

$$\mu_0 = \int_{a_1}^{a_2} p(x)m(x)\,dx = R_0$$

$$\mu = \int_{a_1}^{a_2} x\,p(x)m(x)\,dx/\mu_0$$

$$\mu_i = \int_{a_1}^{a_2} (x-\mu)^i p(x)m(x)\,dx/\mu_0 \qquad i = 2,3,4\ldots$$

where $p(x)$ is the probability of survival from birth to age x and $m(x)$ the probability of giving birth in the interval $(x, x+dx)$. The lower and upper limits of the fertile age distribution are a_1 and a_2, respectively.

In Wicksell $l(x)f(x)$, or $r(x)$, replaces $p(x)m(x)$, and we have for the frequency function

$$r(x) = R_0 \frac{\Gamma(b_1+b_2)}{\Gamma(b_1)\Gamma(b_2)(a_2-a_1)^{b_1+b_2-1}} (x-a_1)^{b_1-1}(a_2-x)^{b_2-1},$$

with Γ the familiar gamma function. The equation is most easily solved from the moments by proceeding in steps, through the intermediate values c_1, c_2, c_3:

$$c_1 = -(\mu_2^3 + \mu_3^2 - \mu_2\mu_4) \Bigg/ \left(\mu_2^3 + \frac{1}{2}\mu_3^2 - \frac{1}{3}\mu_2\mu_4\right)$$

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_29, © Springer-Verlag Berlin Heidelberg 2013

$$c_2 = \frac{\mu_3(c_1+2)}{2\mu_2}$$

$$c_3 = |[c_2^2+4\mu_2(c_1+1)]^{1/2}|$$

$$a_1 = \mu - \frac{1}{2}(c_3-c_2)$$

$$a_2 = a_1+c_3$$

$$b_1 = \frac{1}{2}c_1[1-(c_2/c_3)]$$

$$b_2 = c_1-b_1$$

$$\text{Mode} = \mu - c_2/(c_1-2).$$

If age boundaries are preset, b_1 and b_2 can be found using only the first two moments of the distribution. Here we would have, on rearranging terms in the equations above,

$$b_1 = \frac{(\mu-a_1)^2(a_2-\mu)}{\mu_2(a_2-a_1)} - \frac{\mu-a_1}{a_2-a_1}$$

$$b_2 = b_1\frac{a_2-\mu}{\mu-a_1}$$

$$\text{Mode} = \mu + \frac{2\mu-(a_1+a_2)}{b_1+b_2-2}.$$

A fitting of the distribution using preset age limits has been made by Mitra (1967) for a large number of populations.

Wicksell's equation (21) uses the more easily fitted Type III distribution, for which $a_1=0$, $a_2=\infty$, and

$$r(x) = R_0\frac{\gamma^\beta}{\Gamma(\beta)}x^{\beta-1}e^{-\gamma x}$$

$$\gamma = \mu/\mu_2$$

$$\beta = \mu^2/\mu_2.$$

10. Putting

$$r(x)=f(x)l(x),$$

the function $r(x)$—being proportional to the distribution of the ages at which a generation of girls bear their daughters—plays a rather important part in the mathematical theory of population.

In Table 7 the function $r(x)$ is given, integrated over quinquennial age-groups (in the columns headed "total").

Table 7. Number of daughters that will be born to 1000 new-born girls if they have the maternity of Sweden in 1926, the mortality of Sweden in 1921—1925 and the nuptiality referred to as Case 1 and Case 2.

Age of mother	Case 1			Case 2		
	Number of daughters			Number of daughters		
	Legitimate	Illegitimate	Total	Legitimate	Illegitimate	Total
15—20	13	27	40	27	27	54
20—25	130	54	184	230	44	274
25—30	209	27	236	315	15	330
30—35	195	13	208	263	5	268
35—40	151	8	159	192	3	195
40—45	76	4	80	95	2	97
45—50	8	—	8	10	—	10
Sum	782	133	915	1132	96	1228

Both these series can be described with sufficiently good approximation by means of frequency functions of Pearson's type III, starting at $x=0$. If we put

$$r(x)=R\cdot\frac{\gamma^{\beta}}{\Gamma(\beta)}\,x^{\beta-1}e^{-\gamma x} \tag{21}$$

and define the moments in the usual way, viz.

$$v'_0=\int_0^\infty dx\,r(x),$$

$$v'_1=\int_0^\infty dx\,x\,r(x),$$

$$v_2=\int_0^\infty dx(x-v'_1)^2 r(x),$$

we get

$$R = v'_0; \quad \gamma = \frac{v'_1}{v_2}; \quad \beta = \frac{v'^2_1}{v_2}.$$

Thus, by means of the method of moments, we find

In Case 1: $R=0.915$, $\beta=19.6$, $\gamma=0.65$.

In Case 2: $R=1.228$, $\beta=19.5$, $\gamma=0.65$.

The curves are thus very nearly of the same form—differing only by the factor R —and we adopt for both curves the values of the parameters: $\beta=19.5$ and $\gamma=0.65$.

The expression (21) for $r(x)$ is a very convenient one. In the mathematical theory of population, as developed chiefly by Lotka, the integral

$$(22) \qquad R(t) = \int_0^{\omega} dx\, r(x) e^{-tx},$$

plays a very important part. In particular, the equation

$$(23) \qquad R(t)=1$$

is of fundamental interest, and several rather complicated and laborious methods have been devised for solving it. But when $r(x)$ is given in the form (21)—and it seems that $r(x)$ can generally be expressed in this way—(22) and (23) assume very simple forms. We actually have (putting $\omega=\infty$)

$$(23^*) \qquad R(t) = \frac{R}{\left(1+\frac{t}{\gamma}\right)^{\beta}},$$

when the real part of $t> -\gamma$ and the solution of (23) will simply be

$$(24) \qquad t=\gamma(R^{1/\beta}-1).$$

We get

Case 1: $t=-0.0029$

Case 2: $t=+0.0070$.

The values of the t here calculated simply give the rates of increase per annum, corresponding to the net reproductivity rates given in Section 9. They are, what Lotka terms the "true rates of natural increase". The following table, calculated on the assumption that we have $\beta=19.5$ and $\gamma=0.65$, gives the value of the true rate of natural increase, t, for different values of the net reproductivity rate, R. The corresponding time for the population being doubled or halved is added.

This time is, as well known, given by the formula

$$(25) \qquad y = \left| \frac{0.693}{t} \right| = \left| \frac{1.066}{R^{1/\beta}-1} \right|.$$

In Fig. 5 a graph of t as a function of R has been given ($\beta=19.5$ and $\gamma=0.65$). To save space also the negative values of t have been plotted to the positive side of the axis.

Table 8.

Net reproductivity rate R	Population growth in promille per annum $1000\,t$	Period of doubling or halving the population in years y	Net reproductivity rate R	Population growth in promille per annum $1000\,t$	Period of doubling or halving the population in years y
2.50	31.2	22	*1.00*	0.0	∞
2.00	23.5	29	0.95	− 1.7	410
1.80	19.9	35	0.90	− 3.5	197
1.60	15.9	44	0.85	− 5.4	128
1.40	11.3	61	0.80	− 7.3	94
1.30	8.8	79	0.70	−11.8	59
1.20	6.1	113	0.60	−16.8	41
1.15	4.7	148	0.40	−29.5	23
1.10	3.2	218	0.20	−51.5	13
1.05	1.6	426	0.10	−72.4	10

11. In a recent paper (Metron. Vol. VIII, 1930) Dublin and Lotka give tables of the function $r(x)$ for the U.S. (22 states) in the year 1925 and following years. The central ordinate is given in quinquennial age groups. Multiplying by 5 we get the following series for 1925:

Table 9. Number of daughters born at different ages to a generation of 1000 girls. (Mortality according to the U.S. life table 1925, maternity of U.S. (22 states) in 1925 and actual civil status distribution in the same registration area, 1925.)

Age	10–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55	Sum
Number of daughters	0.6	100.6	309.0	291.0	218.9	149.3	53.0	5.6	0.0	1128

The moments are: $R=1.128$; $\nu_1'=28.33$; $\nu_2=45.88$, and we get $\beta=17.49$; $\gamma=0.6175$, and equation (24) gives

$$t = +0.00427.$$

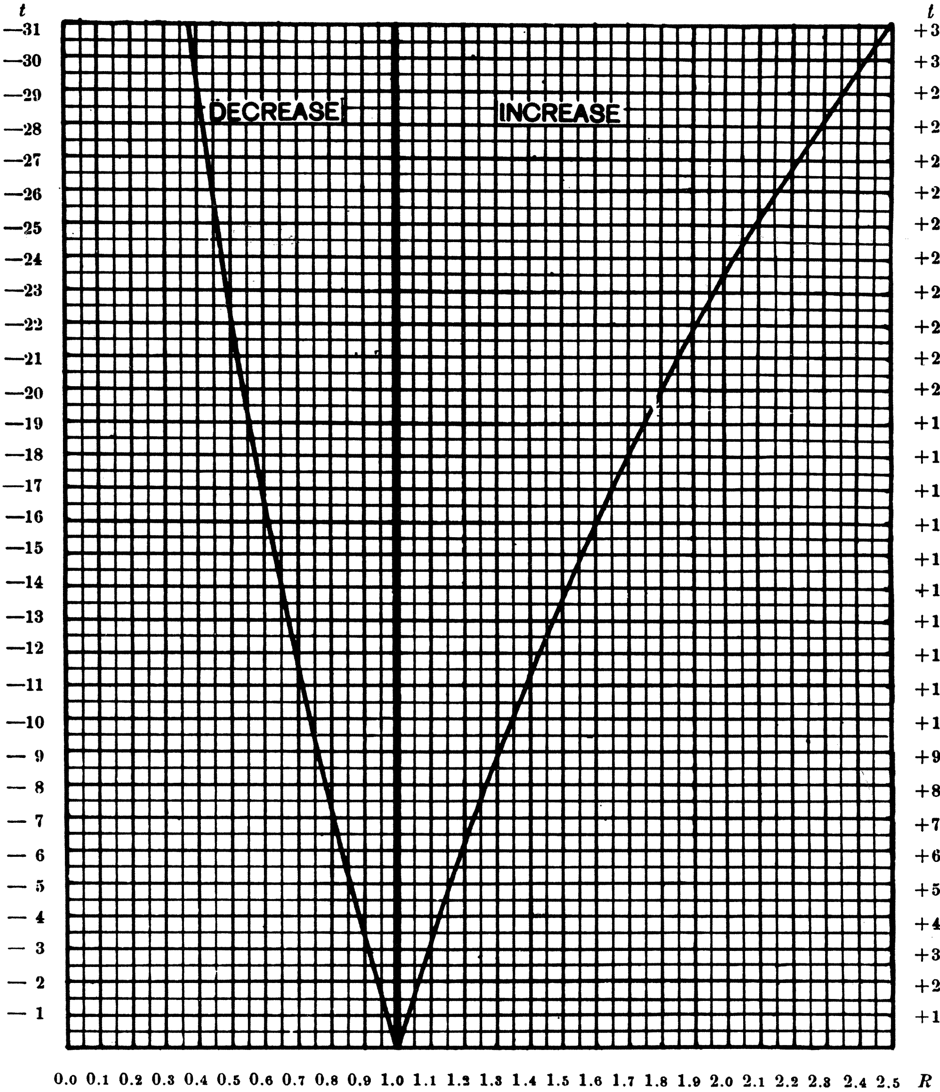


Fig. 5. Net reproductivity, *R*, and true rate of natural increase, *t*.

Dublin and Lotka give the same value of *t*, which value has, however, been computed by a quite different method.

In fact, Dublin and Lotka compute the value of *t* from the equation

$$\frac{v_2}{2}t^2 - v_1' t + \log_e R = 0, \tag{26}$$

which is obtained by assuming $r(x)$ to be described by the normal, Laplace-Gauss'ian frequency function.

As the solution of equation (26) can, in our notation be written in the form

$$t=\gamma\left(1(+)\sqrt{1-\frac{2}{\beta}\log_e R}\right), \tag{27}$$

it is easily seen that when $R-1$ is a small quantity (24) and (26) give the same value of the root. In fact (24) is the first term in an expansion of (27) in powers of $(R^{1/\beta}-1)$.

The reason why our equation (24) gives the same values of t to three decimal places as obtained by Dublin and Lotka[1] is thus only that the values of R used, happen to deviate very little from unity.

Dublin and Lotka's equation for t, (26), is, as already mentioned, the consequence of assuming $r(x)$ to be given by the ordinary, normal frequency function. A glance at the asymmetry of table 9 shows, however, that the normal curve here gives rather a poor fit. Thus, when R deviates substantially from unity, equation (26) will not give a good value of t. To meet this difficulty Lotka has, in another paper, extended equation (26), so as to take account also of the higher moments. Equations of the third or fourth degree are the results. This more general method is obtained by neglecting the higher terms in the series

$$R(t)=\log_e R+\sum_{k=1}^{\infty}\frac{\lambda_k}{k!}(-t)^k, \tag{28}$$

which series follows from the semi-invariant theory of Thiele (the λ_k being the so called semi-invariants).

Notwithstanding the fact that the rate of convergence of this series is, generally spoken, open to question, the method is at times certainly rather unwieldy from the numerical point of view. It therefore seems to me to be of practical importance that the formula (24), which is a skew curve, gives a better fit than the normal curve and that consequently formula (24) has a wider range of applicability than formula (26).

12. In concluding, the following additional remarks may be made. The curve (21) for $r(x)$ is a Pearson Type III curve with the starting point in $x=0$. This choice of the starting point causes the computations to depend only on the moments of the first and second order. It is clear, however, that a still better fit may at times be obtained by using the third order moment to determine the starting point. Putting

$$r(x)=R\frac{\gamma^\beta}{\Gamma(\beta)}(x-a)^{\beta-1}e^{-\gamma(x-a)} \tag{29}$$

we get

$$R(t)=\frac{Re^{-at}}{\left(1+\frac{t}{\gamma}\right)^\beta}. \tag{30}$$

---

[1] This takes place not only in the example just cited but also in all the other examples given in the paper referred to.

The parameters are now obtained from the equations

$$\beta = \frac{(v_1' - a)^2}{v_2}; \qquad \gamma = \frac{v_1' - a}{v_2},$$

and

$$a = v_1' - 2\frac{v_2^2}{v_3}$$

The factor e^{-at} here makes the solution of the equation $R(t)=1$ more difficult, and this is the reason why we have preferred to put $a=0$, as a fairly good fit was nevertheless obtained. Cases may occur, however, where a substantial improvement is obtained by leaving the starting point a free, and using the third order moment to fix it. The equation $R(t)=1$ may then be solved numerically by a graphical method.

A still better fit may be obtained by using Pearson's type I, i.e. by putting

$$r(x) = R \cdot \frac{\Gamma(b_1+b_2)}{\Gamma(b_1)\Gamma(b_2)(a_2-a_1)^{b_1+b_2-1}} (x-a_1)^{b_1-1}(a_2-x)^{b_2-1}. \tag{31}$$

Using the ordinary moment-method of determining the parameters we find for the Swedish material (Table 7):

Case 1	$b_1=0.857$	$b_2=1.518$	$a_1=17.02$	$a_2=48.50$
Case 2	$b_1=0.689$	$b_2=1.593$	$a_1=17.65$	$a_2=48.83.$

The curves thus obtained give a nearly ideal fit in the cases in question, but of course here the solution of the equation $R(t)=1$ is still more complicated.

The advantage of using the formula (21) of $r(x)$ with the corresponding formula (23*) of $R(t)$, instead of the expression (28) used by Lotka, is more clearly realized when we consider that in the mathematical theory of population growth not only the real root of the equation $R(t)=1$ is required, but also a number of the complex roots.

30. Model Fertility Tables: Variations in the Age Structure of Childbearing in Human Populations

ANSLEY J. COALE and T. JAMES TRUSSELL (1974)

From *Population Index* 40. Excerpts are from pages 186—192.

The model tables and the authors' discussions of their uses and methods of fitting are omitted.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_30, © Springer-Verlag Berlin Heidelberg 2013

The Basis for the Model Schedules of Fertility

The basic assumption upon which the model schedules are calculated is that fertility conforms to the structure by age created by multiplying together two model subschedules: a sequence of model proportions ever married at each age and a model schedule of marital fertility. Thus, if the proportion ever married at age a in the model schedule of nuptiality is $G(a)$, and the proportion of married women at age a experiencing a live birth in the model schedule of marital fertility is $r(a)$, age-specific fertility is $f(a) = G(a) \cdot r(a)$. This construction applies exactly to a hypothetical population in which there is no fertility outside marriage, and no dissolution of marriage before the end of the childbearing span of ages. But it also duplicates quite adequately the age structure of fertility in actual populations through the selection of a $G(a)$ that differs slightly from the proportion ever married in the actual population, and of an $r(a)$ that differs slightly from the actual marital fertility schedules.

The representation $f(a) = G(a) \cdot r(a)$ makes possible the calculation of model fertility schedules from three specified parameters—two parameters required to specify a model schedule of proportions ever married, and one parameter required to specify a model schedule of marital fertility.

Age Structure of the Proportion Ever Married, $G(a)$, Specified by Two Parameters

First-marriage frequencies, defined as the number of first marriages in a short age interval divided by the number of persons in that interval, have been shown to conform to a curve of the same shape in different populations (or more precisely in different cohorts). What differs from population to population is the age at which first marriage begins, the duration of the age span within which the majority of the marriages occur, and the proportion of the survivors in the cohort who, at advanced ages, have been married at some time. The similarity in structure of the age distribution of first marriages in different cohorts is analagous to the common shape characterizing different normal (Gaussian) distributions, which are alike only when the mean (location), standard deviation (horizontal scale), and vertical scale (number of cases, or size of population) are specified.

If the effect of differential mortality by marital status on the proportion ever married is neglected, the existence of a standard distribution of first marriage frequencies implies a standard curve describing the proportion ever married in different cohorts. The *form* of the curve is standard, but there are differences, of course, in the starting age of a tangible proportion ever married, in the pace at which the curve rises and in the ultimate proportion experiencing marriage—the proportion ever married by the age at which first marriage rates have fallen essentially to zero. If the standard proportion ever married x years after first marriages begin is $G_s(x)$, in any cohort $G(a) = C \cdot G_s((a - a_0)/k)$, where C is a factor determined by the ultimate proportion ever married, a_0 is the age at which first marriages begin, and k is the scale factor expressing the number of years of nuptiality in the given population equivalent to one year in the standard population. If k is 1.0, first marriages occur at the same pace as in the nineteenth-century Swedish population that served as the basis of the standard; if k is 0.5, or one-half, first marriages occur at twice the pace of the standard. Specifically, according to the standard schedule half of the population that will ever marry has experienced first marriage ten years after the earliest age at which a consequential number of first marriages occur; if k is equal to 0.5, one-half the cohort has experienced first marriage five years after a_0.

The standard proportions ever married were published in an earlier article (Coale 1971), but for computational convenience, we have calculated $G(a)$ from a closed-form analytical expression for first marriage frequencies developed by Donald R. McNeil (Coale and McNeil 1972). This expression is:

$$(1)\quad g(a) = (0.19465/k)\exp\{(-0.174/k)(a-a_0-6.06k) - \exp[(-0.2881/k)(a-a_0-6.06k)]\}$$

No analytical expression for $G(a)$ has been found, but $G(a)$ can be calculated by numerical integration of $g(a)$, since $G(a) = \int_{a_0}^{a} g(x)dx$. This representation of $G(a)$, with appropriate estimates of a_0 and k, provides an approximation of the proportion ever married in a cohort, if multiplied by a scale factor to allow for the particular proportion ultimately experiencing marriage. However, since the standard schedules of fertility that we have constructed represent only the age pattern of fertility and not the level, the proportion ultimately marrying is omitted here. Only the age of initiation and the pace of first marriages affect the structure of fertility; the proportion remaining celibate influences the level but not the age pattern of fertility.

The Age Structure of Marital Fertility, $r(a)$, Specified By a Single Parameter.

Louis Henry found that there is a characteristic pattern of marital fertility in populations in which there is little or no voluntary control of births. He defined voluntary control as behavior affecting fertility that is modified as parity increases, and the absence of control—natural fertility—as behavior, whether affecting fertility or not, that is the same no matter how many children have been born (Henry 1961). The regularity in marital fertility that makes possible a single-parameter set of schedules is this: marital fertility either follows natural fertility (if deliberate birth control is not practiced), or departs from natural fertility in a way that increases with age according to a typical pattern. In a population in which fertility is voluntarily controlled, the ratio of marital fertility at each age, $r(a)$, to a schedule of natural fertility, $n(a)$, is given by:

$$(2)\qquad r(a)/n(a) = M\exp(m \bullet v(a))$$

The factor M is a scale factor expressing the ratio $r(a)/n(a)$ at some arbitrarily chosen age. Since we are concerned only with the age pattern of fertility (not its level), the value of M (like the value of the factor C in the model schedule of proportion ever married) is of no significance for the construction of our fertility schedules. The function $v(a)$ expresses the tendency for older women in populations practicing contraception or abortion to effect particularly large reductions of fertility below the natural level.

Model schedules of $r(a)$ are required at single years of age over the full range at which there is found both 1) a non-zero proportion cohabiting, and 2) non-zero marital fertility. The two functions $n(a)$ and $\nu(a)$, assumed to be invariant, must therefore be estimated by single years of age; the requisite family of model schedules is then obtained by assigning values to m, from zero, in which case $r(a)$ equals $n(a)$, to a maximum expressing the greatest likely departure of fertility from the age pattern of natural fertility resulting from a very high degree of voluntary control of births.

The functions $n(a)$ and $\nu(a)$ were derived from empirical data. There were two steps in the derivation: first, the estimation of approximate values of $n(a)$ and $\nu(a)$ by five-year age intervals above age 20, and second, determination of single-year values by freehand interpolation above age 20 plus extension to ages below 20 on somewhat arbitrary common sense principles.

Seven values of $n(a)$ at ages 20-24 through 45-49 were derived by calculating the arithmetical average of schedules designated by Henry as natural (Henry 1961). Henry's schedules begin at 20 because premarital conceptions have a large and irregular effect on teenage marital fertility. Ten schedules of natural fertility were averaged after discarding schedules known to be based on surveys in which age misreporting was especially prevelant and might have distorted the pattern of fertility. The effect of this selection (compared to the acceptance of all schedules listed by Henry) is minor, since the age pattern of all of those listed is broadly similar.

Seven values of $\nu(a)$, at ages 20-24 through 45-49, were obtained by calculations employing the marital fertility schedules listed in the United Nations Demographic Yearbook for 1965 (United Nations 1966). Again, schedules known or suspected to be distorted by age misreporting or other forms of faulty data were discarded. Each of the forty-three schedules not eliminated on this basis were provisionally accepted as embodying, each in its own degree, the typical pattern of departure from natural fertility.

For the i^{th} schedule an individual $\nu_i(a)$ can be calculated by setting $m = 1.0$ in equation (2). For the i^{th} schedule we find

$$\nu_i(a) = \log [r_i(a)/(\mathrm{M} \cdot n(a))] \qquad (3)$$

M is chosen so that $\nu_i(a)$ is zero for the age interval 20-24. The arithmetical average of the forty-three values of $\nu_i(a)$ in each of the seven age intervals was then defined as $\nu(a)$ for each interval. The values of $n(a)$ and $\nu(a)$ are as follows:

	20-24	25-29	30-34	35-39	40-44	45-49
$n(a)$	0.460	0.431	0.396	0.321	0.167	0.024
$\nu(a)$	0.000	−0.316	−0.814	−1.048	−1.424	−1.667

The function $\nu(a)$ calculated in this way can be validated by substituting the tabulated values in equation (2) and seeing how well the result fits each marital fertility schedule. A value of M is chosen that equates $M \cdot n(a)$ with $r(a)$ at ages 20-24. One way of getting a visual impression of how well $\nu(a)$ fits a given marital fertility schedule is to calculate a separate value of *m* for each age interval. If equation (1) were fully valid, and *v*(*a*) appropriately estimated, the separately determined values of *m* for age intervals 25-29 through 45-49 would all be the same. The sequence of *m*'s calculated for the forty-three empirical marital fertility schedules is not in every instance highly uniform. However, the set of *m*'s for most marital fertility schedules falls on a reasonably level plateau, and the difference in level of *m* between different populations is quite evident (see Figure 1).

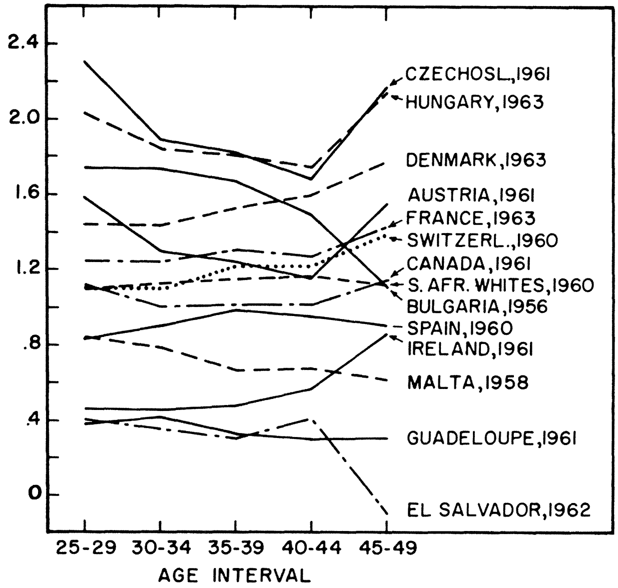


Fig. 1. Values of *m*, where $m = \log[r(a)/(M \cdot n(a))]/v(a)$, for selected marital fertility schedules

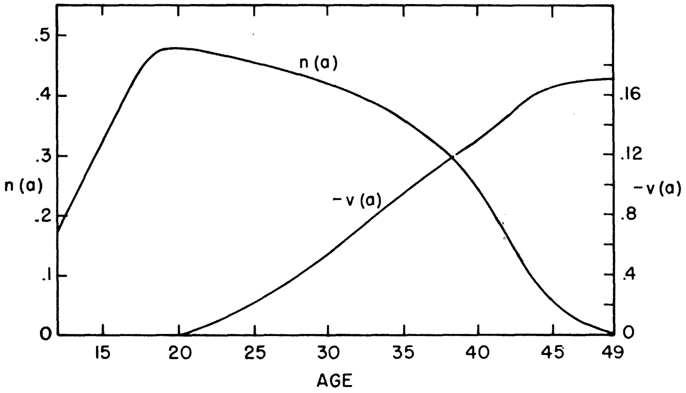


Fig. 2. Values of *n*(*a*) (natural fertility), and *v*(*a*) (logarithmic departure from *n*(*a*))

Single-year values of $n(a)$ and $\nu(a)$ are shown in Figure 2, and tabulated as part of the FORTRAN program in Appendix A. The hand-fitted values of $n(a)$ above age 20 approximately match, in average value for each five-year interval, the values at five-year intervals listed earlier. The extension of $n(a)$ back to age 12 is based on general biomedical information that full reproductive capacity is reached a few years after menarche, and that the mean age at menarche varies from about 12 to 16 years in different populations. The particular choice of rates to represent $n(a)$ below age 20 is not of major importance because of the dominant role of $G(a)$ in determining the rise of age-specific fertility with age.

Values of $\nu(a)$ at single ages were chosen so that their sum over five-year age intervals matched (above age 25) the values at five-year intervals given earlier. To avoid a sharp change in the neighborhood of age 25, non-zero values were assumed to begin at age 20.

With single-year values of our three functions, we have the means of calculating a full range of fertility schedules for hypothetical populations in which there is no illegitimacy and no marital dissolution, and in which marriage begins at various initial ages and occurs over various age spans, and in which marital fertility ranges from the gradual decline with age characteristic of natural fertility to the much steeper decline characteristic of populations in which there is extensive control of fertility within marriage. The age pattern is given by equation (4):

$$f(a) = G(a)n(a)e^{m \cdot \nu(a)} \tag{4}$$

where $f(a)$ is age-specific fertility, $G(a)$ is the proportion ever married (in a population where first marriage occurs according to a schedule characterized by selected values of the parameters a_0 and k), $n(a)$ is natural fertility, $\nu(a)$ is the characteristic pattern of departure from natural fertility, and m is the extent of that departure.

Model Schedules of Age-Specific Fertility, and Their Similarity to the Age Pattern of Fertility in Actual Populations

In actual populations, of course, births occur outside of marriage as well as within, and the proportion of the population currently married differs from the proportion ever married because of the presence of the widowed and divorced. However, the structure of fertility in an actual population may closely resemble that in a hypothetical population with no marital dissolution or extramarital fertility if the latter population has slightly different parameters of nuptiality and marital fertility from those found in the actual population. The effect of illegitimate births and of premarital conceptions on the age structure of fertility is equivalent to a schedule of first marriages that is slightly different from the observed one at early ages; the effect of illegitimate births at the older ages is equivalent to a slight increase in marital fertility at those ages. The proportion of the ever married population that is widowed and divorced rises monotonically with age, thus reducing fertility toward the end of childbearing in a way that is topographically similar to the effect of $\nu(a)$ on marital fertility. In other words, it is probable that the standard schedule of first-marriage frequencies, with a suitable choice of initial age and pace of occurrence of first marriages, can serve as a usable surrogate for the age of entry into sexual union (including unions that do not in fact involve marriage), and that modification of natural fertility by the proper choice of m by which to multiply $\nu(a)$ can serve to approximate the effect both of marital dissolution in reducing the fraction married at higher ages and of control of fertility on marital fertility. On the provisional assumption that such is the case, we have calculated a large array of model fertility schedules by single years of age; each schedule is composed of the product of an estimated proportion ever married and of marital fertility in each single-year age interval. The starting age of nuptiality was allowed to range from 12.5 to 18 years; the pace of marriage from 56 percent of the pace ($k = 1.7$) to five times the pace ($k = 0.2$) in the Swedish standard nuptiality schedule. The value of m was permitted to range from zero (natural fertility) to 3.9, on a scale in which 1.0 is the average value for forty-three schedules in the 1965 Demographic Yearbook. A total of 795 model schedules was tabulated. Each schedule has been normalized so that the sum of the fertility rates at all ages is 1.0; the schedules embody only an *age pattern* of fertility and carry no implication with respect to total fertility.

The tabulated schedules have been selected to produce mean ages at integral values from 24 to 34 years and values of standard deviation (achievable within the stipulated limits of the three underlying parameters) at intervals of half a year. The range of standard deviation is from 4.0 to 7.5, but some combinations (e.g. standard deviations of 7.0 or 7.5 with a mean age of 25) could not be attained within the limits of the three controlling parameters.

When a_0 was 15.0 or more, the single-year rates under age 20 were modified to conform to an observed feature of reliably recorded single-year schedules; non-zero fertility rates typically begin at about age 15 even when marriage begins relatively late. Positive fertility rates at ages 15 and 16 in such populations are probably the result primarily of extramarital conceptions that occur to a small number of adolescents. The requisite modification was achieved as follows: the value of fertility at exact age 20 and the cumulated value of fertility up to age 20 were accepted as initially calculated from equation (3). Values of n and R were found such that $f(a)$ equals $R(a - 15)^n$ matches the calculated value at age 20, and such that $R \int_{15}^{20}(a - 15)^n da$ matches cumulated fertility (as calculated) up to age 20.

A crucial question is whether this family of model fertility schedules provides a close fit to the fertility of actual populations. We have tried to determine how well the model schedules operate by finding a schedule (through interpolation among the printed values) that matches each of a number of recorded schedules in terms of the mean age and the standard deviation and the ratio of the average value of fertility in the interval from ages 15 to 20 to the average value from ages 20 to 25. Figure 3 shows the goodness of fit for three selected fertility schedules recorded by single years of age.

The schedules were chosen because they had the lowest and highest mean ages (Hungary, 1970, and Sweden, 1891-1900), and the lowest standard deviation (Japan, 1964) among the single-year fertility schedules that we examined; in spite of the fact that the schedules fitted are extreme, the fit in every case is quite close. In fact the absolute value of the area between the model schedule and the

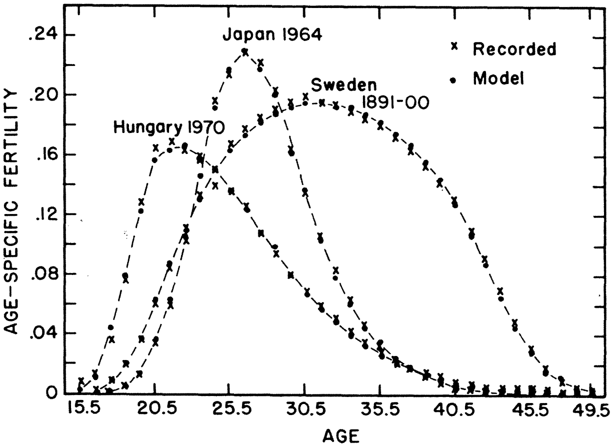


Fig. 3. Age-specific fertility rates of three populations fitted by model fertility schedules

recorded rates is in each instance less than 2.5 percent of the total area under either curve. We have fitted a number of other recorded fertility schedules with equal success.

Figure 4 shows the structure of fertility that results when entry into cohabitation is early and rapid or late and gradual, combined with natural fertility, and with fertility that is highly controlled. In interpreting Figure 4 the reader must keep in mind the normalization of each schedule so as to produce an arbitrary total fertility of 1.0. The figure illustrates the distribution of fertility by age, not differences in level of fertility associated with age patterns.

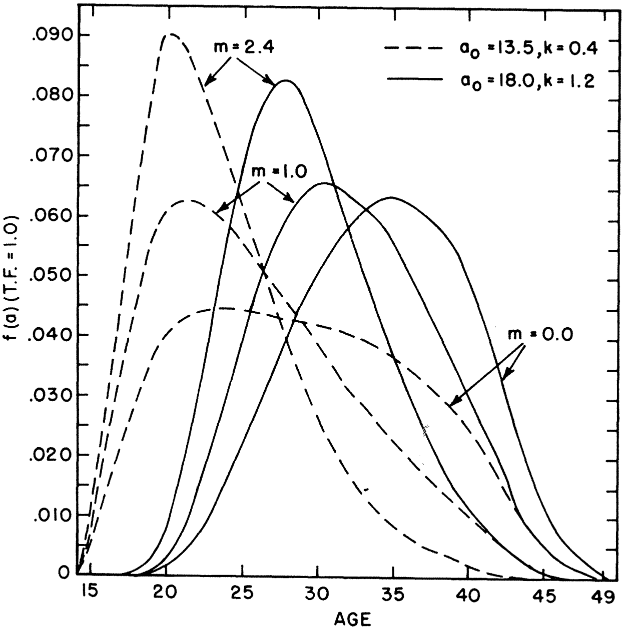


Fig. 4. Model fertility schedules, total fertility = 1.0. Combinations of early marriage with various degrees of fertility control and late marriage with various degrees of control

REFERENCES

Coale, Ansley J. 1971. Age pattern of marriage. *Population Studies* 25(2):193-214.

Coale, Ansley J., and McNeil, Donald R. 1972. The distribution by age of the frequency of first marriage in a female cohort. *Journal of the American Statistical Association* 67(340):743-749.

Henry, Louis. 1961. Some data on natural fertility. *Eugenics Quarterly* 8(2):81-91.

United Nations. Department of Economic and Social Affairs. 1966. *Demographic Yearbook 1965.* Sales No.: 66.XIII.1. New York.

31. A Note on the Law of Population Growth

PIERRE-FRANÇOIS VERHULST (1838)

Correspondence Mathématique et Physique Publiée par A. Quételet (Brussels) *10*, pp. 113—117.
Translated by David Smith.

Of Verhulst's tables we have included only that for France, which best fits his formula and has an asymptotic population of approximately 40 million. Readers should note that errors in Verhulst's estimates of annual population change are serious and directional with respect to the official figures, and that the proportional error figures he gives are not the ones we most care about.

In a paper published some years later Verhulst (1845) gave the name *logistique* to his curve and discussed its properties in more detail. This article included his method of fitting the curve, which we reproduce in a note following his paper. He also considered more carefully than in the present work the quality of available population figures.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_31, © Springer-Verlag Berlin Heidelberg 2013

As is well known, the celebrated Malthus has established as a principle that the human population *tends* to increase in geometric progression so as to double after a certain period, for example every twenty-five years. This proposition is incontestable, if one neglects the ever increasing difficulty of procuring provisions after the population has acquired a certain degree of agglomeration; or [if he neglects] the resources that the population draws on in its growth, even while the society is still in its infancy—such as a greater division of labor, the existence of regular government and the means of defense which assure the public tranquility, etc.

In fact, *all other things equal*, if one thousand souls have become two thousand after twenty-five years, these two thousand will become four thousand after the same lapse of time.

In our old European societies, where suitable lands have long been cultivated, the work applied to improve a field already being farmed can add to its products only ever decreasing quantities. Admitting that in the first twenty-five years one could have doubled the product of the soil, in the second period one could barely succeed in perhaps making it produce a further third. The effective growth of the population thus finds a limit in the extent and fertility of the country, and the population, in consequence, tends more and more to become stationary.

It is not the same in certain cases, purely exceptional to be sure; for example, when a civilized people cultivates a fertile territory until then uninhabited, or when it exercises an ingenuity that gives large temporary benefits. A large family then becomes a source of wealth and the second generation finds it easier to establish itself than the first, because it need not like the former struggle against the obstacles that the untamed land offered to the first colonists.

To judge the speed with which the population grows in a given country, it is necessary to divide the increase of the population in each year by the population which furnished it. The relationship, being independent of the absolute size of the population, can be regarded as the measure of this speed. If it is constant the population increases in a geometric progression; if it is increasing the progression is greater than geometric, and less than geometric if it is decreasing.

One can make diverse assumptions about the resistance or the sum of the obstacles opposed to the indefinite expansion of the population. Mr. Quetelet supposes it proportional *to the square of the speed with which the population tends to increase* (Adolphe Quetelet: Essai de Physique Sociale, Vol. 1, p. 277. [Paris 1835]).

This is to liken the movement of the population to that of a body falling through a resistant medium. The results of this comparison accord, in a satisfying manner, with statistical data and with those that I obtained by my own formulas, when one supposes in the layers of the medium it passes through a density increasing indefinitely.

The growth of the population necessarily has a limit, if only in the extent of the soil indispensible for the lodging of this population. When a nation has consumed all the fruits of its fields, it can to be sure procure supplies from abroad through the exchange of its other products, and support in this way a new increase of population. But it is evident that these importations must have limits and stop a very long time before the entire of the country is converted to cities. All the formulas by which one attempts to represent the law of population must thus satisfy the condition of admitting a *maximum* which is attained only in an epoch infinitely distant. This *maximum* will be the number of the population when it becomes stationary.

I tried for a long time to determine by analysis the probable law of population; but I abandoned this type of inquiry because data from observations are too scarce for the formulas to be verified in such a way as not to leave doubt about their exactness. Still, as the steps that I followed appear to me to lead necessarily to knowledge of the true law when the data become sufficient, and as the results I have arrived at may be of interest, at least as an object of speculation, I thought I ought to accede to the invitation of Mr. Quetelet and bring them to the public.

Let p be the population, and let us represent by dp the infinitely small increase that it receives during an infinitely short time dt. If the population were growing in a geometric progression, we would have the equation $\frac{dp}{dt} = mp$. But as the rate of growth of the population is retarded by the augmentation of the number of inhabitants in itself, we must deduct from mp an unknown function of p; so that the formula to integrate becomes

$$\frac{dp}{dt} = mp - \phi(p)\,.$$

The simplest hypothesis that one can make as to the form of the function ϕ is to suppose $\phi(p) = np^2$. One then finds for the integral of the equation above

$$t = \frac{1}{m}\left[\ln p - \ln(m - np)\right] + \text{constant},$$

and three observations will suffice for determining the two constant coefficients m and n and the arbitrary constant.

On solving the last equation for p, it becomes

$$p = \frac{mp'e^{mt}}{np'e^{mt} + m - np'}\,, \tag{1}$$

designating by p' the population that corresponds to $t = 0$, and by e the base of Naperian logarithms. If one sets $t = \infty$, one sees that the corresponding value of p is $P = \frac{m}{n}$. That is thus *the upper limit of the population.*

Instead of supposing $\phi(p) = np^2$, one may take $\phi(p) = np^{\alpha}$, α being [any number] whatever, or $\phi(p) = n \ln p$. All these assumptions satisfy the observed facts equally well, but they give very different values for the upper limit of the population.

I supposed successively

$$\phi(p) = np^2, \quad \phi(p) = np^3, \quad \phi(p) = np^4, \quad \phi(p) = n \ln p;$$

and the differences between the calculated populations and the numbers observation furnished were appreciably the same.

When the population increases by a progression greater than geometric, the term $-\phi(p)$ becomes $+\phi(p)$; the differential equation then integrates as in the preceding cases, but one conceives that there can no longer be a *maximum* population.

I calculated the tables which follow by formula (1). The figures for France, Belgium, and the county of Essex were drawn from official documents. Those which relate to Russia are found in the work by Dr. Sadler, Law of Population [London 1830], and I cannot guarantee their authenticity, being ignorant by what manner they have been developed. I could have extended the tables for France and Belgium as far as 1837, by means of the *Annuaires* published in these

Table 1. Progress of the Population of France from 1817 to 1831, According to the *Annuaire* for 1834.

Year	According to the registrar	According to the formula	Proportional error	Logarithm of the calculated population
1817	29,981,336 195,902	29,981,336 208,281	0.0000	7.4768490
1818	30,177,238 161,948	30,189,500 204,500	+0.0004	7.4798565
1819	30,339,186 199,863	30,394,000 200,500	+0.0018	7.4827875
1820	30,539,049 188,227	30,594,500 197,300	+0.0018	7.4856461
1821	30,727,276 212,144	30,791,800 192,700	+0.0021	7.4884310
1822	30,939,420 198,634	30,984,500 189,500	+0.0014	7.4911453
1823	31,138,054 221,286	31,174,000 185,223	+0.0012	7.4937907
1824	31,359,340 220,546	31,359,340 182,777	0.0000	7.4963719
1825	31,579,886 175,974	31,542,000 178,000	−0.0012	7.4988859
1826	31,755,860 157,533	31,720,000 175,000	−0.0011	7.5013366
1827	31,913,393 189,071	31,895,000 172,000	−0.0005	7.5037257
1828	32,102,464 139,402	32,067,000 168,000	−0.0011	7.5060547
1829	32,241,866 161,074	32,235,000 164,500	−0.0002	7.5083251
1830	32,402,940 157,994	32,399,500 161,434	0.0000	7.5105385
1831	32,560,934*	32,560,934	0.0000	7.5126965

* Census figure (January 1)

two countries since 1833, and in this way verify my formula; but my pursuits have not left me the leisure. My work was terminated in 1833, and I have not touched it again since.

I will point out in passing that the table which relates to France appears to announce that the formula is especially exact, since the observations [for France] bear on the largest numbers and have been made with greater care. For the rest, the future alone can unveil for us the true mode of action of the retarding force which we have represented by $\phi(p)$.

Editors' Note: Fitting the Logistic

The form in which Verhulst presents the logistic makes its solution somewhat cumbersome. In a more standard form the equation becomes

$$r_t = b\left(1 - \frac{P_t}{k}\right)$$

$$P_t = \frac{k}{1+\exp[a-bt]}\,.$$

Given the three population estimates $P_{t_0}, P_{t_0+\delta}, P_{t_0+2\delta}$ (which we may index P_1, P_2, P_3), the constants a,b and asymptotic population k are found by the formulas (from Verhulst 1845, pp. 12—13; Croxton, Cowden and Klein 1967, pp. 274—275)

$$k = \frac{2P_1P_2P_3 - P_2^2(P_1+P_3)}{P_1P_3 - P_2^2}$$

$$b = (1/\delta)\ln[P_2(k-P_1)/P_1(k-P_2)]$$

$$a = \ln[(k-P_1)/P_1] + bt_0^*$$

where t_0^* is a computational origin. The terms t_0^*, δ vanish under the linear transformation $t \to t^*$, defined by $t^* = (t - t_0^*)/\delta$.

To reconcile this form with Verhulst's equation (1) we write, using Verhulst's notation,

$$P_t = \frac{mp'e^{mt}}{np'e^{mt}+m-np'} = \frac{m/n}{1+\left[\left(\frac{m}{n}-p'\right)/p'\right]e^{-mt}}$$

$$k = m/n$$

$$b = m$$

$$a = \ln\left[\left(\frac{m}{n}-p'\right)/p'\right].$$

Pearl and Reed, whose paper follows, also use a non-standard form for the logistic. Their equation and the conversion to standard form are

$$y_t = \frac{B}{\exp[-A(t-t_0^*)]+C}$$

$$= \frac{B/C}{1+\exp[(At_0^* - \ln C) - At]}$$

$$k = B/C$$

$$b = A$$

$$a = At_0^* - \ln C.$$

The inflection point, given by Pearl and Reed as (α, y_a) has as its coordinates $(a/b, k/2)$ in the standard form. An α not the same is given in their equation xvi.

Further treatment of the logistic, with its implied birth and death rates, will be found in Lotka (1939, pp. 48—62). Lotka centers the curve about its inflection point for convenience in developing its theoretical properties, by setting

$$t_0^* = a/b$$

$$P_{t^*} = P_{t-t_0^*} = \frac{k}{1+e^{-bt^*}}\,.$$

32. On the Rate of Growth of the Population of the United States since 1790 and its Mathematical Representation

RAYMOND PEARL and LOWELL J. REED (1920)

From *Proceedings of the National Academy of Science* 6. Excerpts are from pages 280—288.

Pearl and Reed begin their paper with a discussion of exponential and parabolic growth formulas for past U.S. population growth, which we have omitted.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_32, © Springer-Verlag Berlin Heidelberg 2013

It is quite clear on *a priori* grounds, as was first pointed out by Malthus in non-mathematical terms, that in any restricted area, such as the United States, a time must eventually come when population will press so closely upon subsistence that its rate of increase per unit of time must be reduced to the vanishing point. In other words, a population curve may start, with a convex face to the base, but presently it must develop a point of inflection, and from that point on present a concave face to the x axis, and finally become asymptotic, the asymptote representing the maximum number of people which can be supported on the given fixed area. ...

It would be the height of presumption to attempt to predict *accurately* the population a thousand years hence. But any real law of population growth ought to give some general and approximate indication of the number of people who would be living at that time within the present area of the United States, provided no cataclysmic alteration of circumstances has in the meantime intervened.

It has seemed worth while to attempt to develop such a law, first by formulating a hypothesis which rigorously meets the logical requirements, and then by seeing whether in fact the hypothesis fits the known facts. The general biological hypothesis which we shall here test embodies as an essential feature the idea that the rate of population increase in a limited area at any instant of time is proportional (a) to the magnitude of the population existing at that instant (amount of increase already attained) and (b) to the still unutilized potentialities of population support existing in the limited area.

The following conditions should be fulfilled by any equation which is to describe adequately the growth of population in an area of fixed limits.

1. Asymptotic to a line $y=k$ when $x=+\infty$.
2. Asymptotic to a line $y=0$ when $x=-\infty$.
3. A point of inflection at some point $x=\alpha$ and $y=\beta$.
4. Concave upwards to left of $x=\alpha$ and concave downward to right of $x=\alpha$.
5. No horizontal slope except at $x=\pm\infty$.
6. Values of y varying continuously from 0 to k as x varies from $-\infty$ to $+\infty$.

In these expressions y denotes population, and x denotes time.

An equation which fulfils these requirements is

$$y = \frac{be^{ax}}{1+ce^{ax}} \tag{ix}$$

when a, b and c have positive values.

In this equation the following relations hold:

$$x=+\infty \qquad y=\frac{b}{c} \tag{x}$$

$$x=-\infty \qquad y=0 \tag{xi}$$

Relations (x) and (xi) define the asymptotes.

The point of inflection is given by $1-ce^{ax}=0$, or

$$x = -\frac{1}{a}\log c \qquad y = \frac{b}{2c} \tag{xii}$$

The slope at the point of inflection is $\frac{ab}{4c}$. ...

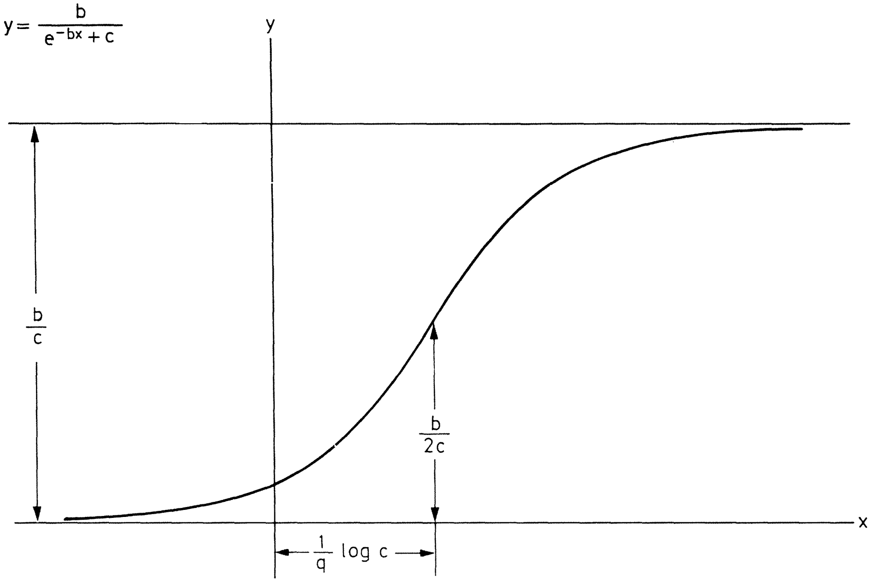


Fig. 2. General form of curve given by equation (ix).

The general form of the curve is shown in figure 2.

The question now is how well does (ix) represent the known historical facts as to the growth in population of the United States, and to what legitimate deductions as to the future course of population in this country does it lead?

It is obvious that equation (ix) as it stands cannot be fitted to observational data by the method of least squares. It is possible to write momental equations and fit by the method of moments, but at this time we do not care to develop that method because, as will presently appear, we do not regard equation (ix) as the final development of this type of equation for representing population, and we have no desire to encumber the literature with a mathematical discussion which we expect later to discard.

For present purposes it will be sufficient to fit (ix) to the observations by passing it through three points. Given three equally spaced ordinates, y_1, y_2 and y_3, the necessary equations are:

$$\frac{b}{c} = \frac{2y_1y_2y_3 - y_2^2(y_1+y_3)}{y_1y_3 - y_2^2} \tag{xiv}$$

$$a=\log_{10}\frac{y_2\left(\frac{b}{c}-y_1\right)}{y_1\left(\frac{b}{c}-y_2\right)}\div h\log_{10}e \tag{xv}$$

where h is the abscissal distance in years between y_1 and y_2, or y_2 and y_3.

$$c=\frac{1}{y_2-y_1}\left(\frac{y_1}{e^{a\alpha}}-\frac{y_2}{e^{a(\alpha+h)}}\right) \tag{xvi}$$

where α is the abscissal distance in years from the origin to y_1.

Putting x_1 at 1790, x_2 at 1850, and x_3 at 1910, and taking origin at 1780 we have

$y_1=3{,}929$[9]

$a=10$

$y_2=23{,}192$

$h=60$

$y_3=91{,}972$

and taking (ix) in the form

$$y=\frac{b}{e^{-ax}+c}, \tag{xvii}$$

we find these values for the constants:

$$y=\frac{2{,}930.3009}{e^{-0.0313395x}+0.014854} \tag{xviii}$$

The closeness of fit of this curve is shown graphically in figure 3.

Though empirically arrived at this is a fairly good fit of theory to observations. ...

The significance of the result lies in this consideration. A curve which on *a priori* grounds meets the conditions which must be satisfied by a true law of population growth, actually describes with a substantial degree of accuracy what is now known of the population history of this country.

Let us examine some further consequences which flow from equation (xviii). The first question which interests one is this: when did or will the population curve of this country pass the point of inflection, and exhibit a progressively diminishing instead of increasing rate of growth? From (xii) it is easily determined that this point occurred about *April I, 1914*, on the assumption that the numerical values of (xviii) reliably represent the law of population growth in this country. In other words, so far as we may rely upon present numerical values, the United

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[9] Omitting 000 here and in the subsequent calculations till the end.

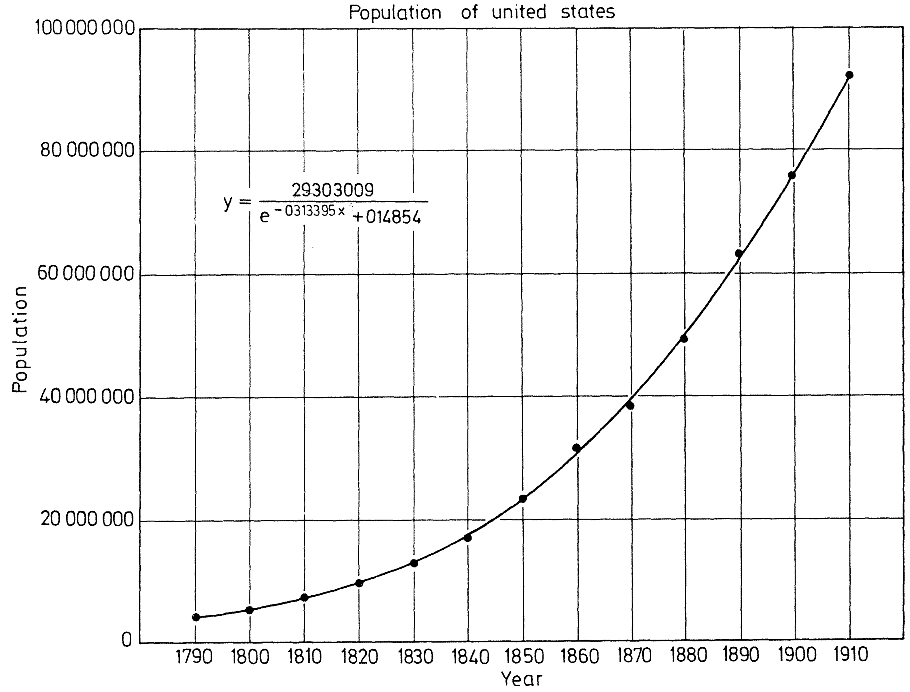

Fig. 3. Showing result of fitting equation (xviii) to population data.

States has already passed its period of most rapid population growth, unless there comes into play some factor not now known and which has never operated during the past history of the country to make the rate of growth more rapid. This latter contingency is improbable. While prophecy is a dangerous pastime, we believe, from the fragmentary results already announced, that the 1920 census will confirm the result indicated by our curve, that the period of most rapid population growth was passed somewhere in the last decade. The population at the point of inflection works out to have been 98,637,000, which was in fact about the population of the country in 1914.

The upper asymptote given by (xviii) has the value 197,274,000 roughly. This means that according to equation (xviii) the maximum population which continental United States, as now areally limited, will ever have will be roughly twice the present population. We fear that some will condemn at once the whole theory because the magnitude of this number is not sufficiently imposing. It is so easy, and most writers on population have been so prone, to extrapolate population by geometric series, or by a parabola or some such purely empirical curve, and arrive at stupendous figures, that calm consideration of real probabilities is most difficult to obtain. While, as will appear from the next section of this paper, we have no desire to defend the numerical results of this section, and indeed ourselves regard them only as a rough first approximation, it remains a fact that if anyone will soberly think of every city, every village, every town in this country having its present population multiplied by 2, and will further think of twice as many persons on the land in agricultural pursuits, he will be bound, we think, to conclude that the country would be fairly densely populated. It would have about 66 persons per square mile of land area.

It will at once be pointed out that many European countries have a much greater density of population than 66 persons to the square mile, as for example Belgium with 673, Netherlands with 499, etc. But it must not be forgotten that these countries are far from self-supporting in respect of physical means of subsistence. They are economically self-supporting, which is a very different thing, because by their industrial development at home and in their colonies they produce money enough to buy physical means of subsistence from less densely populated portions of the world. We can, of course, do the same thing, provided that by the time our population gets so dense as to make it necessary there still remain portions of the globe where food, clothing material, and fuel are produced in excess of the needs of their home population. But in this, and in any other scientific discussion of population, it is necessary to limit sharply the *area* one is to talk about. This paper deals with population, and by direct implication the production of physical means of subsistence, within the present area of continental United States.

Now 197,000,000 people will require, on the basis of our present food habits,[10] about 260,000,000,000,000 calories per annum. The United States, during the seven years 1911—1918, produced as an annual average, in the form of human food, *both primary and secondary* (i.e., broadly vegetable and animal), only 137,163,606,000,000 calories per year.[11] So that unless our food habits radically change, and a man is able to do with less than 3000 to 3500 calories per day, or unless our agricultural production radically increases,[12] it will be necessary when our modest figure for the asymptotic population is reached, to import nearly or quite one-half of the calories necessary for that population. It seems improbable that the population will go on increasing at any very rapid rate after such a condition is reached. And is it at all reasonable to suppose that at such time, with all the competition for means of subsistence which the already densely populated countries of Europe will be putting up, there can be found any portion of the globe producing food in excess of its own needs to an extent to make it possible for us to find the calories we shall need to import?

Altogether, we believe it will be the part of wisdom for anyone disposed to criticise our asymptotic value of a hundred and ninety-seven and a quarter millions because it is thought too small, to look further into all the relevant facts.

III

With the above numerical results in hand it is desirable to discuss a little further the general theory of population growth set forth in the preceding section. At the outstart let it be said that we are convinced that equation (ix) represents

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[10] Cf. Pearl, R., *The Nation's Food*, Philadelphia (W. B. Saunders Company), 1920 (247).

[11] Pearl, R., *loc. cit.*, p. 76.

[12] As a matter of fact East, in his able presidential address on "Population," before the American Society of Naturalists has shown that the United States has already entered upon the era of diminishing returns in agriculture in this country.

no more than a first approximation to a true law of population growth. There are several characteristics of this curve which are too rigid and inelastic to meet the requirements of such a law. In (ix) the point of inflection must of necessity lie exactly half-way between the two asymptotes. Furthermore the half of the curve lying to the right of the point of inflection is an exact reversal of the half lying to the left of that point. This implies that the forces which during the latter part of the population history of an area act to inhibit the rate of population growth are equal in magnitude, and exactly similarly distributed in time, to the forces which in the first half of the history operate to accelerate growth. We do not believe that such rigid and inelastic postulates as these are, in fact, realized in population growth. ...

We attach no particular significance to the *numerical* results of the preceding section. They obviously can give only the roughest approximation to probable future values of the population of the United States. Our only purpose in presenting them at all at this time is to demonstrate that the hypothesis here advanced as to the law of population growth, even when fitted by a rough and inadequate method, so closely describes the known facts regarding the past history of that growth, as to make it potentially profitable to continue the mathematical development and refinement of this hypothesis further. There is much that appeals to the reason in the hypothesis that growth of population is fundamentally a phenomenon like autocatalysis. In a new and thinly populated country the population already existing there, being impressed with the apparently boundless opportunities, tends to reproduce freely, to urge friends to come from older countries, and by the example of their well-being, actual or potential, to induce strangers to immigrate. As the population becomes more dense and passes into a phase where the still unutilized potentialties of subsistence, measured in terms of population, are measurably smaller than those which have already been utilized, all of these forces tending to the increase of population will become reduced.

Part V
Probability Models of Conception and Birth

Pregnancy and birth distributions are mathematically unlike mortality in being repeated events, separated by intervals of non-risk and admitting as a matter of course a high degree of individual control. Corrado Gini (1924) was first to explore these distinctions and their implications, by his suggestion that birth intervals be treated as waiting time problems dependent on fecundability. This he defined as the probability that a woman at risk would conceive in a given monthly cycle.

Gini's article, paper 33, limits itself to first pregnancies and births under constant fecundability. The obvious possibilities for extending the work were not considered for about 30 years, when Louis Henry (1953) published the first of a series of investigations in the mathematical treatment of birth intervals. Henry's initial article, presented here in paper 34 in a translation by Mindel C. Sheps and Evelyne Lapierre-Adamcyk (1972) indicates how Gini's analysis is applied to birth orders beyond the first.

An equation of Henry's that figures in much later work is the proportion of women conceiving each month in a homogeneous population, given that at any point in time some women are already pregnant or in amenorrhea and hence not at risk. Henry shows that if the monthly probability of pregnancy to non pregnant women is p and the non susceptible period associated with pregnancy is s months, both constants, then the mean birth interval i approaches the stable value

$$i = \frac{1}{p} + s$$

The inverse $1/i$ is the proportion conceiving each month. The result was extended by Tietze (1962) to show the effects of contraception and by Potter (1963) to compare rates of abortion that would arise if it were used exclusively or used in conjunction with contraception, assuming in both cases that all pregnancies that occur are aborted. Contraception of effectiveness e changes the probability of pregnancy from p to $p(1-e)$, while abortion affects the sterile period s. The results show in general that if abortion is used alone as a means of controlling fertility it is required frequently. If $s' < s$ is the sterile period associated with abortion, the intervals being compared are

$$\frac{1}{p(1-e)} + s'$$

and

$$\frac{1}{p} + s'.$$

With highly effective contraception, the first interval can be of great length, while the second is less than the normal interval between births.

In the paper that follows Henry's, D. Basu (1955) { omitted here } derives the monthly proportion conceiving independently, by means of difference equations. The solution is also attractively reached through matrix formulation, for which the reader may consult Keyfitz (1968, pp. 390—392). Henry's work is expanded in later papers to cover general fertility histories; several are available in English in Henry (1972).

Important contributions have been added by Robert G. Potter and M. P. Parker (1964), and Mindel C. Sheps (1964), whose related articles appeared together. Sheps, whose paper is included here, as paper 35 finds generating functions for the distribution of conception delays under Henry's assumption that fecundability differs among couples but is constant over time. Potter and Parker specify birth probabilities as a Pearson Type I distribution and find waiting times for this specific case.

The analysis of waiting times and conception probabilities has been extended very considerably beyond the level treated here, including allowance for pregnancy wasteage, for declining fecundability with age, and for contraceptive use. Much of Sheps' later work will be found in Sheps and Menken (1973). Sheps, Menken and Radick (1969) provide an introduction to other contributions.

{ **Ed. Note:** The models of Sheps and Menken have formal connections with the renewal processes discussed in Part II. Methods for estimating the relative contributions of a full set of "proximate determinants" of fertility came into use with Bongaarts and Potter's work [7]. The lines of inquiry initiated by Louis Henry led to advances in historical demography exploiting data from parish registers, partly through aggregate analysis and partly through the individual-level approach of family reconstitution, epitomized by Wrigley and Schofield's volumes [43] and [44], building on Peter Laslett's classic work [20]. Appreciation of relationships between period and cohort indices of fertility from the perspective of Norman Ryder [33] have led to tempo-adjusted indices of total fertility defined by Bongaarts and Feeney [6].

Over the years the field of probability models for conception and birth has broadened to include modeling of other individual-level demographic life-history processes, many in the domain of kinship and family demography. Demographic computer microsimulation allows treatment of interacting individual-level and population-level processes. Early landmarks are found in work by Menken [28] and by Hammel, Wachter, and McDaniel [17]. Such microsimulation approaches have spread throughout the social sciences under the name of “agent-based modelling”. An illustration of their power may be found, for instance, in the application by Goldstein [14]. }

33. First Investigations on the Fecundability of a Woman

Corrado Gini (1924)

Proceedings of the International Mathematical Congress (Toronto) 2: 889—892. Translated by David Smith.

For the *Gini Index*, Gini's "approximate measure of the mean fecundability of primipares who had their child in the months of marriage from $x+9$ to $x+y+9$," we may write

$$F_G = \frac{\mathrm{n}_{x+9} - \mathrm{n}_{x+y+10}}{\sum\limits_{x+9}^{x+y+9} \mathrm{n}_i} = \frac{n_1 - n_{y+1}}{\sum\limits_{1}^{y} n_i},$$

where n_i are births in the i'th, and n_i births in the $(i+9)$'th month of exposure. If fecundity differs among women but for each woman is constant over time, the Index has as its limiting value the arithmetic mean fecundability of the non-sterile population N^*. This is seen by noting that as y approaches infinity, $n_{y+1} \to 0$ and $\sum\limits_{1}^{y} n_i \to N^*$.

The *Pearl Index* (Pearl 1933), defined as the number of pregnancies experienced divided by person-months of exposure, is often more useful. It is

$$F_P = \frac{\sum\limits_{1}^{y} n_i}{\sum\limits_{1}^{y} i n_i + y\left(N - \sum\limits_{1}^{y} n_i\right)},$$

where N is the population size. If analysis is again restricted to the non-sterile population this has as its limiting value the harmonic mean fecundability, whose inverse is the mean waiting time to pregnancy: $\left(\sum\limits_{1}^{\infty} i n_i\right) / N^*$. The expression is formally a life expectancy, correct to the measurement error of the exposure time i.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_33,

I call *fecundability of a woman* the probability that a married woman becomes pregnant during the month, neglecting any Malthusian or neo-Malthusian practice intended to limit procreation.

It is unnecessary to stress the theoretical and practical importance of a measure for fecundability. We would have, in particular, the means to decide what fraction of the differences we observe between the coefficients of natality of different countries, different social classes, or different times arises from physiological causes and what fraction arises from voluntary causes.

It is also unnecessary to demonstrate that fecundability cannot be measured directly. Quite clearly, we are not in a position to observe the consequences of Malthusian and neo-Malthusian practices on natality. Nor do we know the frequency of ovulatory abortions that occur in the initial months of pregnancy.

We can, however, formulate an indirect measure of fecundability, based on the considerations that follow.

Let us suppose that married women able to bear children are n in number during the whole period being considered, and that all have the same fecundability p. On these assumptions the number of women who would conceive for the first time during their first month of marriage will be pn; the number who would conceive for the first time in the second month will be $(1-p)pn$; the number who would conceive for the first time in the third month will be $(1-p)^2 pn$, and so forth. The numbers obtained in this way form a geometric progression whose ratio is $1-p$.

If we assume that the percentage of abortions (or of abortions and stillbirths) and likewise the percentage of pregnancies longer or shorter than 9 months are the same for the products of conceptions that have taken place in the successive months of marriage, we can substitute for the ratio between first conceptions taking place in the month $x+1$ of marriage and first conceptions taking place in the preceding month x, the ratio between the firstborn (or first live births) coming into the world in the month $x+10$ of marriage, and the firstborn (or first live births) coming into the world in the month $x+9$.

It is evident that the value of p arrived at by this route is independent of the number, large or small, of women who are not in the childbearing population, either because of the sterility of the marriage or because of Malthusian or neo-Malthusian practices.

It is also evident that it is independent of the frequency of abortions and stillbirths.

The value of p thus determined measures, on the assumptions mentioned above, the fecundability of women able to conceive.

Among these assumptions there is one which clearly does not agree with reality however: it is that all fertile married women have the same fecundability p. This circumstance, which cannot be neglected, does not prevent our arriving at the measure of fecundability; though the route to follow is a bit longer.

Let $s \leqq n$ be the number of groups into which the n married women can be classed according to their fecundability, $p_i|_{i=1,2,\ldots,s}$ will be the fecundability of women in group i and n_{ix} the number of women of group i who, in the month x after marriage, have not yet conceived.

The number of women expected to conceive for the first time in the x'th month of marriage will be $\sum_{i=1}^{s} p_i n_{ix}$; and the number expected to conceive for the first time in the next month $x+1$ of marriage will be $\sum_{i=1}^{s} (1-p_i) p_i n_{ix}$. The expected value of the ratio between the second and first numbers (if we agree to give to each of the possible values of the ratio a weight proportional to the probability that this value is realized and to the number which forms its denominator) will be

$$\frac{\sum_{i=1}^{s} (1-p_i) p_i n_{ix}}{\sum_{i=1}^{s} p_i n_{ix}} = 1 - \frac{\sum_{i=1}^{s} p_i^2 n_{ix}}{\sum_{i=1}^{s} p_i n_{ix}}.$$

But $\sum_{i=1}^{s} p_i^2 n_{ix} \Big/ \sum_{i=1}^{s} p_i n_{ix}$ is precisely the expected value of the mean fecundability of women who conceive for the first time in the x'th month of marriage.

Thus we will be able to deduce the mean fecundability of women who conceive for the first time in the month x from the complement of the ratio between women who conceive for the first time in the month $x+1$ and women who conceive for the first time in the month x.

This calculation rests on the following assumptions, which can be accepted without difficulty:

(a) The number of women who were eligible for first conceptions in month x of marriage, and did not conceive in this month, will equal the number of women eligible for first conceptions in month $x+1$ of marriage. This is a way of saying that we can neglect the effects from one month to the next of mortality, migrations, and passages from the category of married women not able to conceive to the category of married women eligible for first conceptions (for example by cessation of contraceptive practices or by elimination of the cause of sterility), as well as passages from the second category to the first (for example, due to unexpected sterility or the adoption of contraception);

(b) in each of the s groups of women eligible for first conceptions after marriage, fecundability remains unchanged from month x to month $x+1$.

Supposing further (assumption c) that the probability of abortion (or abortion and stillbirth) and the probability of a gestation period greater or less than 9 months are the same for the products conceived in month x and for those conceived in month $x+1$, we will be able to substitute for the ratio between women who conceive for the first time in month $x+1$ and women who conceive for the first time in month x, the ratio between the firstborn (or first live births) of the marriage who came into the world in month $x+10$ and the firstborn (or first live births) of the marriage who came into the world in month $x+9$.

We are thus able to obtain a measure of the mean fecundability of primipares who, after marriage, conceived for the first time in month x. This fecundability can in turn be regarded as essentially equal to the fecundability of primipares who had their child in month $x+9$.

The complement of the ratio between the firstborn (or first live births) of the marriage coming into the world in month $x+10$ and the firstborn (or first live births) of the marriage coming into the world in month $x+9$ will give us, by assumptions (a), (b), and (c), an approximate measure of the mean fecundability of primipares who had their child in month $x+9$ of marriage.

If we calculate the mean of the ratios between the firstborn of the months of marriage $x+10$ and $x+9$, $x+11$ and $x+10, \ldots, x+y+10$ and $x+y+9$, giving each ratio a weight proportional to its denominator; and if we take the complement of this mean, we obtain by assumptions (a), (b), and (c) an approximate measure of the mean fecundability of primipares who had their child in the months of marriage from $x+9$ to $x+y+9$.

Here are the results obtained for several areas. They are based on data published by official sources, except for Italy, for which the data are the fruit of special inquiries, using the registers of vital events of 24 communes.

Area	Years	Mean fecundability of primipares who had their child during the months following marriage:			
		10—17	11—17	10—23	11—23
Berlin	1894—1895	26.1	24.4	23.0	20.5
Australian Confederation	1917—1921	19.4	20.6	18.5	19.2
Western Australia	1895—1915	23.9	22.7	21.9	20.2
New South Wales	1893—1905 and 1916—1921	20.6	22.2	18.9	19.7
Victoria	1898—1900	21.0	24.4	...	...
Tasmania	1905—1906				
Italy (24 Communes)	1900—1921	23.8	21.0	21.4	18.6

The uniformity of the results obtained is remarkable, above all when we exclude from the calculations the firstborn coming into the world during the tenth month of marriage, corresponding to conceptions of the first month of marriage. For the ratio between conceptions of the first and those of the second month of marriage, we cannot in fact accept the validity of assumption (a), fecundability being, in the first month, less than in the second due to the frequent virginity of the wife; nor that of assumption (c), ovulatory abortions being especially frequent for the products conceived in the first month because of honeymoon trips and perhaps other circumstances.

The obstacle to fertilization represented by virginity naturally increases in importance with the age of the wife, but beyond the first month it does not appear that the ability to conceive is weaker for older than for younger wives. That is a remarkable finding, and was brought to light by the data of the Australian Confederation for the period 1907—1914 and those of New South Wales for the period 1893—1898.

Another important finding is that the diminution of natality from 1901—02 to 1911—12 which has been verified in the Kingdom of Saxony and has reached

50%, does not appear to have been accompanied by a diminution of the ability of primipares to conceive.

The methods and results summarized in this note are set forth together with complementary details in two papers presented at the Instituto Veneto di Scienze, Lettere ed Arti on July 1, 1924, and in the article "Decline in the birth-rate and 'fecundability' of women," in The Eugenics Review, January 1926. Further investigations are in process which can perhaps permit us to assemble other interesting results in the new area that the method proposed appears to open to statistical research.

34. Theoretical Basis of Measures of Natural Fertility

LOUIS HENRY (1972 (1953))

From *On the Measurement of Human Fertility: Selected Writings of Louis Henry*, translated and edited by Mindel C. Sheps and Evelyne Lapierre-Adamcyk. New York: Elsevier. Excerpts are from pages 2, 4—9, 15—20.

In his remarks on Gini, Henry notes that the mean fecundability of women who conceive in the first month of marriage (in Gini's notation, paper 40 above: $\sum p_i^2 n_{i1} / \sum p_i n_{i1}$) may also be written $\bar{p}(1+c^2)$, with $\bar{p}$ the arithmetic mean fecundability of the non-sterile population and c the coefficient of variation. This permits calculation of the variance of fecundability V_p as

$$V_p = \bar{p}(1-\bar{p}) - n_2/N^*$$

where $\bar{p} = n_1/N^*$; n_1 and n_2 being the number of women conceiving in the first and second months of exposure respectively, and N^* the total non-sterile population (cf. Sheps and Menken 1973, pp. 129—130, 398—399).

We omit sections of the paper discussing sterility, natural fertility and fecundability, and the fitting of observed data to the models, and some remarks on the works of Gini and Pearl.

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6_34,

Introduction

One of the most interesting and difficult problems in demography is that of estimating natural fertility, i.e. the fertility of a human population that makes no deliberate effort to limit births. Natural fertility, which depends essentially on biological factors, is primarily a biological phenomenon, and a "natural" phenomenon has a particular attraction. It shares in the prestige of the natural sciences, overshadowing that of the social sciences which are still in their infancy. In addition, efforts to appraise the effectiveness of contraceptive practice lead to comparisons between actual fertility and the hypothetical fertility that a given population would have if it did not use any form of birth control.

The difficulties encountered in the study of natural fertility result from its very definition. Natural fertility is, for easily observable populations, hypothetical. No doubt one can find populations living under conditions of quasi-natural fertility; but, for the most part, little is known about them. For European populations, before the spread of contraceptive practice, only scanty data are available.

Thus, it is not surprising that indirect approaches are made to circumvent the difficulty. But in following indirect routes we risk losing sight of the goal: An investigation of the theoretical basis of measures of natural fertility is indispensable.

We will start with certain assumptions regarding the biological basis of fertility. These assumptions are not completely arbitrary; observed results are used to construct a mathematical model that is as close to reality as possible and yet simple enough to be easily manageable.

Fundamental Assumptions

It has long since been ascertained that in large families of a fixed size, the interval between successive births is more or less independent of birth order, except perhaps for the last few births. Now, a priori, the interval between births depends on the duration of pregnancy (independent of order), on the duration of the nonsusceptible period that follows confinement, and on the natural fecundity of the couple. If the mean interval between successive births does not vary, it is natural to think that these various factors also do not vary for a given couple as long as they are not sterile.

We were led, therefore, to characterize each fecund couple (couple able to produce living children, at the time or later):

1. by the duration, g, of the nonsusceptible period associated with pregnancy; g is the time that elapses between a conception and the first ovulation after delivery.

2. by fecundability or the probability of conception per unit time outside of the previously mentioned nonsusceptible periods.

Since conception is possible only at the time of ovulation, one ought, rigorously, to treat fecundability as discrete and define it as the probability of conceiving per menstrual cycle. In this case we designate it by p. But since it is often more convenient to treat time as continuous, we also introduce the probability ϕdx of conceiving during the infinitely small time interval dx. The term fecundability is also used for ϕ.

The quantity g is the sum of two terms: the duration of pregnancy, g_0, and the duration, $g-g_0$, of the period of nonsusceptibility following delivery. The latter varies appreciably between women, because, while it is very short in the absence of breastfeeding, it can be very long in the case of prolonged lactation. Moreover, fecundability certainly varies between women, whether because of the physiological characteristics of the couple or of the frequency of their sexual relations. To take these variations into account, we introduce two distributions: $h(g)$, the probability density function (p.d.f.) of the duration of nonsusceptibility, and $f(\phi)$, the p.d.f. of fecundability, or $f(p)$ in the discrete case.

Before going on, let us examine the foregoing. To characterize each couple by a pair (ϕ,g), is to assume that these quantities are invariant in time as long as the couple is not definitely sterile. It is certain that, in reality, ϕ and g vary over time. For g, this is obvious. Illness of the mother or death of the child may interrupt lactation; furthermore, if one considers conceptions, the duration of pregnancy also varies.[1]

It is more difficult to affirm, a priori, that ϕ varies. Doubtless our sense of continuity makes us think that women do not become completely sterile suddenly, without a preceding progressive decline in fecundability. But this feeling for continuity, though undeniable, does not constitute an argument. One could object to the assumption of a constant ϕ on the grounds of the existence of nonsusceptible periods outside of those associated with pregnancy, e.g. those due to separation or illness. But to the extent that these separations and illnesses are distributed uniformly in time, they intervene by modifying the probability of conception; they do not affect its assumed invariance.

On the other hand, we must admit that the distribution of these separations and illnesses is certainly not uniform. Long established marriages accept temporary separations more easily than do young married couples; also, the frequency of illness increases with age and therefore with duration of marriage. Thus, separations and illnesses tend to lower ϕ with increasing duration of marriage.

The relatively small variation in observed mean intervals between successive births shows, however, that variations in ϕ and g depend little on the duration of marriage, age or parity. With respect to g, this suggests that if this function varies for a given couple, the variation is independent of the duration of marriage. On the average, g will be the same at any point in a marriage. We shall see that under these conditions, the results are the same as if we assume a constant g for each couple.

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[1] In the case of spontaneous abortion, pregnancy is shorter. For convenience, we often operate as if there were a fixed time between conception and birth and pass from one to the other at our choice. We are not unaware, however, that the existence of spontaneous abortions and stillbirths complicates matters. But these are secondary difficulties which it appears unnecessary to expound here.

Reproductive History of a Group of Couples

We begin by considering the reproductive history of a group of couples with the same characteristics (ϕ,g) in the absence of permanent sterility and of dissolution of the marriage by separation, divorce or death.

It is natural to take marriage as the time origin, under the assumption of no premarital sexual relations and, hence, no premarital conceptions. But, on this assumption, the period from marriage to the first live birth (if, as we are doing, one considers live births only) differs from subsequent intervals because g_0, the duration of pregnancy, rather than g, intervenes in this first interval. One could, obviously, retain this difference, but only at the cost of complicating the notation. It is simpler to change the origin, placing it at $g-g_0$ before marriage.

Let us designate the elapsed time from this translated origin by x. For $x<g$, all couples are childless; for $x>g$, the expected number of couples without births, taking the initial sample size as unity, is equal to $e^{-\phi(x-g)}$ and the number of first births in the interval $(x,x+dx)$ is $\phi e^{-\phi(x-g)}dx$.

For $x<2g$, there are no second order births; it can easily be shown that, for $x\geq 2g$, the expected number in the interval $(x,x+dx)$ is equal to $\phi^2(x-2g)e^{-\phi(x-2g)}dx$.

It is then easy to show by recursion that, for $x\geq ng$, the expected number of births of order n in the interval $(x,x+dx)$ is equal to[2]

$$\phi^n \frac{(x-ng)^{n-1}}{(n-1)!} e^{-\phi(x-ng)}dx. \tag{1}$$

The expected number of births $B(x)dx$ in the interval $(x,x+dx)$, regardless of order, is equal to the sum of (1) for values of n from 1 to m such that $mg\leq x$. $B(x)$ is given by:

$$B(x) = \int_g^\infty B(x-t)\phi e^{-\phi(t-g)}dt \tag{2}$$

which is of the same form as equations studied by Lotka. Its solution is of the form $\sum_s Q_s e^{r_s x}$. The Q_s are coefficients and the r_s are roots of the equation

$$1 = \int_g^\infty \phi e^{-rt-\phi(t-g)}dt. \tag{3}$$

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[2] If one had considered couples with fecundability ϕ and $g_0, \dots g_{n-1}$ for the duration of nonsusceptability corresponding to each successive birth order, one would have obtained for the number of births of order n in the interval $(x,x+dx)$:

$$\phi^n \frac{(x-g_0-g_1-\dots-g_{n-1})^{n-1}}{(n-1)!} e^{-\phi(x-g_0-g_1-\dots-g_{n-1})}dx,$$

for $x\geq g_0+g_1+\dots+g_{n-1}$, given that g_k is not dependent upon the duration of marriage. For a set of couples in which $g_0+g_1+\dots+g_{n-1}=ng$, we again arrive at the earlier result. There is no need, therefore, to introduce variations of g independent of marital duration.

The real root is zero and the real component of the imaginary roots is negative. Consequently, $B(x)$ is the sum of a constant and of damped periodic functions. With increasing duration of marriage, this sum approaches the constant part of $B(x)$, which we designate by ϕ' (asymptotic fertility rate). This is equal to $1/i$, where i is the mean interval between births. Now,

$$i = \int_g^\infty \phi x e^{-\phi(x-g)} dx = g + \int_g^\infty \phi(x-g) e^{-\phi(x-g)} dx \tag{4}$$

whence $i = g + \frac{1}{\phi}$. Hence

$$\phi' = \frac{1}{g+1/\phi} = \frac{\phi}{1+g\phi}. \tag{5}$$

Often in demography, rates are only intermediate results in the attempt to derive the mean number of events per capita over a long period: here, the mean number of births per marriage. Hence, we are led to study cumulative fertility as a function of time, that is, the expected number of children born in x years of marriage. Let us designate it by $E(x)$. We have:

$$E(x) = \int_0^x B(\zeta) d\zeta = \phi' x + \sum_s \int_0^x Q_s e^{r_s \zeta} d\zeta. \tag{6}$$

$E(x)$ is the sum of a linear function of x and of damped periodic functions. After a number of oscillations, that is, after a given duration of marriage, $E(x)$ approaches $\phi' x + E_0$, where E_0 depends on ϕ and g. Given ϕ and g, $E(x)$ can be calculated from appropriate tables, e.g. χ^2 tables [since Eq. (1) is equivalent to a χ^2 (chi square) distribution with $2n$ degrees of freedom]. From this, one deduces E_0. But it is not necessary to consider all pairs (ϕ, g). Let us assume two values, g and λg. The maximum value of n in time x given g is the same as the maximum value in time λx given λg. If ϕ is the fecundability associated with g, let us consider the pair $(\phi/\lambda, \lambda g)$. We then have:

$$E(x|\phi,g) = \sum_1^n \int_{ng}^x \frac{\phi^n(\xi-ng)^{n-1}}{(n-1)!} e^{-\phi(\xi-ng)} d\xi \tag{7}$$

$$E(\lambda x|\phi/\lambda,\lambda g) = \sum_1^n \int_{\lambda ng}^{\lambda x} \left(\frac{\phi}{\lambda}\right)^n \frac{(\zeta-\lambda ng)^{n-1}}{(n-1)!} e^{-(\phi/\lambda)(\zeta-\lambda ng)} d\zeta. \tag{8}$$

Let $\zeta = \lambda\xi$; it follows that $E(\lambda x \mid \phi/\lambda, \lambda g) = E(x \mid \phi, g)$. The two terms in x are equal to $\frac{\phi}{1+g\phi} x$ for the pair (ϕ, g), and to

$$\frac{\phi}{\lambda(1+g\phi)} \lambda x$$

for the pair $(\phi/\lambda, \lambda g)$. They are therefore equal and we have finally, $E_0(\phi,g) = E_0(\phi/\lambda, \lambda g)$. Hence, when $g\phi$ is a constant, E_0 is fixed. Values of E_0 for selected $g\phi$ were calculated, and a regression fitted as:

$$E_0 = -0.5 + \frac{0.115}{g\phi}. \tag{9}$$

Table 1 shows calculated and estimated values.

Table 1. E_0 According to $g\phi$

$g\phi$	0.5	1	2	3	4	5	6	∞
E_0 calculated	−0.275	−0.375	−0.445	−0.470	−0.480	−0.485	−0.490	−0.500
E_0 estimated from Eq. (9)	−0.270	[−0.385]	−0.443	−0.462	−0.471	−0.477	−0.481	−0.500

We now pass to the more realistic case of a heterogeneous group where the couples have all possible characteristics (ϕ,g). Assume that there is no correlation between ϕ and g: the group is then characterized by two probability densities, $f(\phi)$ and $h(g)$. This group, by virtue of the heterogeneity of ϕ and g, has diverse values for ϕ'; the distribution of ϕ' has a probability density $k(\phi')$.

In such a group, the expected number of births in the interval $(x, x+dx)$ is equal to the sum of the births of the homogeneous subgroups $(\phi$,g) which constitute the entire group. If x is large, the expected number in the subgroup (ϕ,g) departs little from $\phi' dx$; the expected number of births for the total, therefore, is equal to $dx \int \phi' k(\phi') d\phi' = \bar{\phi}' dx$, and the cumulative fertility of the group is approximately equal to

$$x \int \phi' k(\phi') d\phi' + \int\int E_0 f(\phi) h(g) d\phi\, dg \tag{10}$$

or $x\bar{\phi}' + \bar{E}_0$, where $\bar{E}_0$ is the mean of the values of E_0 corresponding to the various combinations (ϕ,g).

If we substitute Eq. (9), the approximation for E_0 referring to a sufficiently long time, into Eq. (10), the result is:

$$\bar{E}_0 = -0.500 + 0.115 \int \frac{f(\phi)d\phi}{\phi} \int \frac{h(g)dg}{g} = -0.500 + \frac{0.115}{\bar{\phi}_h \bar{g}_h}, \tag{11}$$

where $\bar{\phi}_h$ and $\bar{g}_h$ are the harmonic means of ϕ and of g. ...

The Discrete Case: Gini's Method

To study the fecundability of newlyweds, it is preferable to abandon the continuous notation used until now. Let p be the probability of conception in the course of a menstrual cycle, where the cycle is assumed to be of a fixed duration equal to one month. Let us consider, then, a homogeneous group with fecundability p in which the couples were married on the same date and had no premarital conceptions. If the original number is put equal to unity, there are p conceptions expected the first month, $p(1-p)$ the second, $p(1-p)^2$ the third, and so on.

In heterogeneous groups characterized by a density $f(p)$, the expected number of conceptions in the first month is $\bar{p}$ and in the second month it is $\bar{p}-\bar{p}^2(1+c^2)$, where c is the coefficient of variation of p [c is equal to the standard deviation divided by the mean]. The complement of the ratio of the second month's conceptions to those of the first is expected to be $1-\left[\frac{\bar{p}-\bar{p}^2(1+c^2)}{\bar{p}}\right]=\bar{p}(1+c^2)$. This is the same as the mean fecundability of women conceiving the first month, which may be written as:

$$\int p\,\frac{pf(p)}{\bar{p}}\,dp=\bar{p}(1+c^2). \tag{22}$$

More generally, the complement of the ratio of the expected conceptions of the $(n+1)$th month to those of the nth month is equal to the mean fecundability of women conceiving in the nth month. The mean fecundability of women conceiving in the course of the first n months of marriage is equal to a weighted mean of the preceding quantities, the weights being the number of women conceiving the first, second, ..., nth month, i.e. the denominators of the preceding quantities. It is written:

$$\sum_{i=0}^{i=n-1}\frac{\int p\cdot p(1-p)^i f(p)\,dp}{\int p(1-p)^i f(p)\,dp}\times\frac{\int p(1-p)^i f(p)\,dp}{\sum_{0}^{n-1}\int p(1-p)^i f(p)\,dp}. \tag{23}$$

As n increases indefinitely, this quantity approaches $\bar{p}$, which is thus the mean fecundability of women who conceive at least once.

The essentials of this theory are due to C. Gini. ...

Intervals Between Births

We have already seen that in the case of a homogeneous group (ϕ,g), the mean interval between births is $g+1/\phi$; by a change of origin we have equated marriage to a birth; if one returns to the real origin at marriage, the mean interval between marriage and the first birth is reduced by $g-g_0$ and becomes g_0+1/ϕ.

In discrete notation, the mean number of ovulations (or of months) between marriage and the first conception is equal to $p+2p(1-p)+\ldots+np(1-p)^{n-1}$, that is, on setting $1-p=q$,

$$p\frac{d}{dq}\left[\sum_{i=1}^{\infty}q^i\right]=p\frac{d}{dq}\left[\frac{q}{1-q}\right]=\frac{1}{p}. \tag{24}$$

If k is the mean interval, counted in menstrual cycles (or months), between marriage and the first ovulation, the mean interval between marriage and first conception is equal to $(k-1) + \dfrac{1}{p}$. If we assume k equal to 1/2, the mean interval is $1/p - 1/2$. Division by 13 (or by 12) gives the mean interval in years; on adding g_0 we have the interval between marriage and the first live birth.

In a heterogeneous group the mean interval, $\bar{i}$, between births is given, in the continuous case, by the relation

$$\bar{i} = \int\int (g+1/\phi) f(\phi) h(g) d\phi \, dg = \bar{g} + 1/\bar{\phi}_h. \tag{25}$$

One can also write, since $i = 1/\phi'$,

$$\bar{i} = \int \frac{k(\phi') d\phi'}{\phi'} = \frac{1}{\bar{\phi}'_h} \tag{26}$$

Since $\bar{\phi}'_h < \bar{\phi}'$, it follows that $1/\bar{i} < \bar{\phi}'$, $\bar{i} > 1/\bar{\phi}'$. Analogous relations hold in the discrete case. In particular, the mean interval between marriage and the first conception is equal to $k - 1 + (1/\bar{p}_h)$. Evidently, $\bar{p}_h < \bar{p}$; but, the inverse of the mean interval to conception, equal to $\dfrac{\bar{p}_h}{1-(1-k)\bar{p}_h}$ is also greater than $\bar{p}_h$. In practice the difference between $\bar{p}_h$ and $\bar{p}$ is generally expected to be large; the inverse of the mean interval must therefore be less than $\bar{p}$.

The foregoing holds in the absence of sterility; its presence may modify the mean intervals between births. Let $n(x)dx$ be the births of a given order that would occur in $(x, x+dx)$ after the preceding births in the absence of sterility, and let $F'(x)$ be the proportion of couples still fecund x years after the preceding birth, when all were, by definition, fecund. The mean interval between births becomes

$$\bar{i}' = \frac{\int x n(x) F'(x) dx}{\int n(x) F'(x) dx}. \tag{27}$$

This is a new weighted mean of x with larger weights for low values of x and smaller weights for high values; $\bar{i}'$ is therefore smaller than $\bar{i}$; its inverse is consequently larger than that of $\bar{i}$. One then wishes to determine whether $1/\bar{i}'$ can equal or surpass $\bar{\phi}'$. For a homogeneous group of medium or high fecundability, the reduction in the mean interval by sterility at younger ages, and even up to about 40 years, is almost negligible. The reduction at older ages, on the contrary, may be very great with low and very low values of fecundability.

From the little we know of the distribution of fecundability, its mean is sufficiently high so that sterility will not affect the interval between births until older ages. However, the dispersion of fecundability is apparently great. Hence, those with low fecundability, although probably small in number, may have a sufficient effect to reduce the mean interval appreciably.

From available information on the mean interval between marriage and the first birth by the woman's age, there does not appear to be a reduction with increasing age. No doubt these data relate to populations that already practice contraception and should be verified. We think, however, we may assume that, except at older ages, the reduction in the mean interval to the first birth caused by the onset of sterility is not very important. It follows that $\bar{\imath}' \approx \bar{\imath}$, and therefore, the inverse of the mean interval between marriage and the first birth, $1/\bar{\imath}'_1$ will be less than $\bar{\phi}$, given that age at marriage is still low. If $\bar{\imath}'$ is the mean interval between two births calculated for relatively low ages (the first and the second birth for example), $1/\bar{\imath}'$ should, for the same reasons, be less than $\bar{\phi}'$.

Instead of the mean interval between two births of a given order, we could calculate the mean interval between all births. Let us examine what would happen in this case. Consider women who marry young; let x be the duration of marriage and dS the risk of sterility between x and $x+dx$. When age at marriage is low, $\frac{dS}{dx}$ is negligible when x is low; when $\frac{dS}{dx}$ is not negligible, x is large enough so that cumulative fertility is approximately equal to $\phi' x$ for a homogeneous group of fertility ϕ'.

Let us pass to the calculation of intervals. For women who will become sterile at x, let ζ be the duration of marriage at a given birth. In the interval $(\zeta, \zeta+d\zeta)$, there are $\phi' d\zeta$ births (except when ζ is low, which is unimportant for what follows). If ζ is between 0 and $x-g$, then $\phi' e^{-\phi(x-\zeta-g)} d\zeta$ of these children are last births. If ζ is between $x-g$ and x, all $\phi' d\zeta$ births are last births. The mean duration of marriage at the last childbirth is, therefore,

$$\left(x-\frac{g}{2}\right)\phi' g + \int_0^{x-g} \phi' \zeta e^{-\phi(x-\zeta-g)} d\zeta\,; \tag{28}$$

that is,

$$\phi'\left[g\left(x-\frac{g}{2}\right)+\frac{x-g}{\phi}-\frac{1}{\phi^2}+\frac{e^{-\phi(x-g)}}{\phi^2}\right], \tag{29}$$

or, ignoring the term in $e^{-\phi(x-g)}$,

$$\phi'\left[(x-g)\left(g+\frac{1}{\phi}\right)+\frac{g^2}{2}-\frac{1}{\phi^2}\right] = x-g+\phi'\left(\frac{g^2}{2}-\frac{1}{\phi^2}\right). \tag{30}$$

The duration of marriage at first birth is equal to

$$\int_{g_0}^{x} \zeta \phi e^{-\phi(\zeta-g_0)} d\zeta,$$

which is

$$(g_0+1/\phi)(1-e^{-\phi(x-g_0)})-(x-g_0)e^{-\phi(x-g_0)}, \tag{31}$$

which, for sufficiently large x, reduces to g_0+1/ϕ. Then, the time between first and last births reduces to $x-g-g_0-1/\phi+\phi'(g^2/2-1/\phi^2)$. The number of births of order 2 and higher is, on the other hand, equal to

$$\phi' x - 1 + e^{-\phi(x-g_0)} \tag{32}$$

being, for all practical purposes, $\phi' x - 1$. For a heterogeneous group, the time between first and last births is equal to $x-k$, where

$$k = \int\int\left[g+g_0+\frac{1}{\phi}-\phi'\left(\frac{g^2}{2}-\frac{1}{\phi^2}\right)\right]f(\phi)h(g)\,d\phi\,dg\,. \tag{33}$$

The total number of births is $\bar{\phi}' x - 1$. For all values of x, one then has $\int(\phi' x - 1)dS = \phi' \int x dS - \int dS$ births and, for the durations: $\int x dS - k\int dS$. In practice, only large values of x enter the picture. Therefore, the ratio of births to durations is in practice reduced to $\bar{\phi}'$. We emphasize, however, that this result is valid only for the assumptions made; in particular that the population is non-contracepting and that the distribution of fecundability is such that the frequency of low values (those for which $e^{-\phi x}$ and $xe^{-\phi x}$ are not negligible) is negligible.

35. On the Time Required for Conception

MINDEL C. SHEPS (1964)

From *Population Studies* 18. Excerpts are from pages 85—92.

We omit the appendix, and sections of the paper discussing correlations between two successive conception delays and numerical results. (The maximum likelihood estimates for the mean and variance of fecundability, developed in the appendix, are the $\bar{p}$ and V_p given in our introduction to Henry, paper 41 above.)

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Introduction

The purpose of this paper is to investigate some characteristics of rates of conception in a mixed or heterogeneous population. Consider that one has data showing, for a sample of couples, the number of women conceiving in each consecutive month of observation. These data on the monthly incidence of conception may be used to characterize the fecundability (monthly chance of conception) of the couples or to estimate the effectiveness of a contraceptive method in use by them. The data may also be regarded as showing the distribution of the interval to conception. This interval constitutes one component of the interval between successive births and hence is a subject of interest in the study of fertility.[1]

If we accept the reasonable hypothesis that the couples in the sample vary among themselves in their fecundability, what can we expect our data to show? How will the monthly incidence of conception vary? What is the expected distribution of the intervals to conception? Given a set of data as described above, can we arrive at any conclusions about the underlying distribution of fecundability? The results to be given below, in the attempt to answer these questions, are general in the sense that they do not depend on any assumptions regarding the specific form of the underlying distribution of fecundability.

The discussion is related to part of the subject matter considered by Potter and Parker in the preceding paper.[2] As is well illustrated in that paper among others,[3] a reasonable model for the distribution of the intervals to conception assists both in interpreting observed data and in considering reasonable expectations for a group of women. A number of investigators have considered this problem and have developed theoretical formulations for it, as many of these formulations have either: (1) involved the admittedly unrealistic assumption that all the couples in a population have an equal probability of conception per month, (2) utilized numerical examples to illustrate what might result if the population consisted of some mixtures of couples having two or three different levels of fecundability, or (3) assumed that fecundability has a continuous frequency distribution of a specified form.

The most recent example of the last procedure is given in the preceding paper by Potter and Parker, who assume that the frequency distribution of fecundability

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[1] The importance of birth intervals as an approach to fertility analysis and their components are discussed for example by: L. Henry, 'Intervals between confinements in the absence of birth control', *Eugenics Quarterly*, **5**, 1958, pp. 200—11; R.G. Potter, Jr., 'Birth intervals: Structure and change', *Population Studies*, **17**, 2, 1963, pp. 155—166; and M.C. Sheps and E.B. Perrin, in 'Changes in birth rates as a function of contraceptive effectiveness: some applications of a stochastic model', *American Journal of Public Health*, **53**, 1963, pp. 1031—46, and in 'The distribution of birth intervals under a class of stochastic fertility models', *Population Studies*, **17**, 3, 1964, 321—31.

[2] R.G. Potter, Jr., and M.P. Parker, 'Predicting the time required to conceive', *Population Studies* (this issue).

[3] See, for example: C. Tietze, 'Differential fecundity and effectiveness of contraception', *The Eugenics Review*, **50**, 1959, pp. 231—4; R.G. Potter, Jr., *loc cit.*, and 'Length of the observation period as a factor affecting the contraceptive failure rate', *Milbank Memorial Fund Quarterly*, **38**, 1960, 142—4 and other references cited by Potter and Parker.

is a unimodal Type I (Beta type) distribution. They derive a number of important and interesting results, including expressions for the mean and variance of the conception delay and the correlation between two successive delays in the same woman. Further, they estimate the parameters of Type I distributions from two sets of data and illustrate their results numerically by using the estimated values. Many of their inferences have general applications.

Gini and Henry investigated some properties of the expected results for more general distributions of fecundability.[4] The present paper will extend this investigation, considering some of the same problems as are discussed by Potter and Parker but the assumption of a specified frequency distribution of fecundability is relaxed to permit the distribution to assume practically any shape.[5] It can, for example, be a bimodal distribution in which an appreciable proportion of couples use highly effective contraceptives (with resultant low or zero fecundability) while the fecundability of the remainder depends on the interaction between inherent fecundity, patterns of sexual relations and somewhat careless contraceptive use. Accordingly, some of the conclusions constitute a generalization of earlier results; specifically they support and generalize some of the conclusions in the preceding paper. In addition, parameters that characterize a distribution that may underline a set of observations are suggested. Estimators of these parameters are derived for situations where only some of the women have conceived when the data were collected, as well as from complete observations.

Assumptions

The assumptions made here agree with those of Potter and Parker, with the exception already indicated. They are:

1. Conception is a chance (random) occurrence.
2. The fecundability of each couple in a population remains constant during the period of observation.
3. Fecundability varies between couples in the population in an unspecified way.

Conceptions, regardless of the outcome of the pregnancy, are considered here. In conformity with the preceding paper, let the fecundability of a particular couple be denoted as p. Postulate an arbitrary distribution of fecundability $\Phi(p)$, where $0 \leq p \leq 1$ and $\int_0^1 \Phi(p)dp = 1$.[6] A group of couples with identical values of p will be referred to as a homogeneous population, and a group with a variety of values, distributed as $\Phi(p)$, will be referred to as a heterogeneous population. A

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[4] C. Gini, 'Premières recherches sur la fécondabilité de la femme', *Proc. of the International Mathematics Congress*, 1924, 889—92, and L. Henry, 'Fondements théoriques des mesures de la fécondité naturelle', *Rev. de l'Inst. Int. de Statistique*, **21**, 3, 1953, 135—51. Among others, the results in expressions (1), (2) and (5) below were given by Henry.

[5] With some restrictions, as defined below.

[6] Since p is bounded and non-negative all moments of p and of $1-p$ are finite and non-negative. See H. Cramér, *Mathematical Methods of Statistics*, Princeton, 1946, p. 175.

heterogeneous population can be considered as an aggregate of homogeneous sub-populations.

The Proportion Conceiving Each Month

In a homogeneous population, under the assumptions given, the expected proportion of women who conceive in the first month of exposure is p. In the second month it is $p(1-p)$ which is equivalent to a proportion p of those who failed to conceive the first month. Generally, the expected proportion conceiving monthly decreases geometrically at a ratio of $(1-p)$. At every month, the expected number conceiving is p times the number still at risk, i.e. the conditional probability of conceiving is constant.

Under the same assumptions, in a heterogeneous population the expected proportion conceiving during the first month of exposure is

$$C_1 = \int_0^1 p\Phi(p)\,dp = \bar{p} \tag{1}$$

which is equal to the arithmetic mean fecundability of the population. During the second month the expected proportion conceiving is:

$$C_2 = \int_0^1 p(1-p)\Phi(p)\,dp = E(p) - E(p^2) = \bar{p} - \bar{p}^2 - \sigma_p^2 \tag{2}$$

where $E(u)$ is the expected value of u and σ_p^2 is the variance of p in the population. Hence, a smaller proportion of the total group may be expected to conceive in the second month than would be the case for a homogeneous population with $p = \bar{p}$. The expected number conceiving in the second month, considered as a proportion of those who failed to conceive during the first month (conditional probability of conceiving in the second month), is:

$$\frac{C_2}{1-C_1} = \frac{\int_0^1 p(1-p)\Phi(p)\,dp}{1 - \int_0^1 p\Phi(p)\,dp} = \bar{p} - \frac{\sigma_p^2}{(1-\bar{p})}, \tag{3}$$

a smaller quantity than $\bar{p}$.

As shown in the appendix, similar results hold for all subsequent months, i.e. the proportion expected to conceive out of those still at risk (conditional probability) decreases monthly. Accordingly, we may conclude that in a heterogeneous population where the assumption of constant fecundability for any couple holds reasonably well, the proportion of conceptions—as in a 'contraceptive failure rate'—will tend to fall during each successive month of exposure.

This conclusion, which was previously reached by Tietze and Potter using a less general approach,[7] indicates that it may be misleading to estimate contraceptive effectiveness by calculating the number of conceptions per 100 'person years' of use. As just indicated, 100 'person years' may have very different expectations of conception rates, depending on whether they represent two years for each of 50 couples, a half year for each of 200 couples, or a mixture of long and short periods of observation. Instead, a method utilizing a life table approach, such as was recently suggested by Potter,[8] would provide a more appropriate description.

Furthermore, as is shown in the appendix and illustrated numerically, expressions (1) and (2) and their analogues lead to a method of describing the distribution of fecundability (or of contraceptive effectiveness) from knowledge of the proportion (of a sample) conceiving each month. Inferences about the distribution of fecundability, such as estimates of its mean, variance and third central moment, can be made from observations on the first three months only. They do not require information on the conception delays of all the women in the sample; nor is it necessary to exclude sterile couples for whom $p=0$.

Distribution of Waiting Times

The probability that conception occurs in a given month after the initial exposure is simultaneously the probability that the time required for conception ('waiting time') will be that number of months. In a homogeneous population, the time required for conception is, on the defined assumptions, distributed geometrically. Thus the mean, the variance and other moments are known functions of p, and more specifically of the powers of $(1/p)$.

A heterogeneous population, as previously defined, may be considered to generate a compound geometric distribution of waiting times. In such a population the expected values of the moments of the waiting times can be obtained by calculating the expected values of the appropriate functions of p. This procedure will be demonstrated after a review of the moments of the simple, homogeneous geometric distribution.

(a) Distribution of Waiting Times of a Homogeneous Population

In the literature there is some variation in the definitions of the geometric distribution describing the process of conception in a homogeneous population, and therefore variation in the formulae derived for the mean conception delay. The apparent disagreement originates simply in two different ways of counting the months of exposure. Thus one may say that, given a constant value of p, the probability of conception during the xth month is $p(1-p)^{x-1}$. One may also say that the probability that the delay to conception will be exactly x months (i.e. that conception will occur during month $(x+1)$ is $p(1-p)^x$. The first formulation

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[7] See the papers cited in footnotes 2 and 3.

[8] R.G. Potter, 'Additional measures of use—effectiveness in contraception', *Milbank Memorial Fund Quarterly* **41**, 1963, pp. 400—18.

might be said to refer to the number of *trials* necessary (each cycle being defined as a single trial), whereas the second refers to the conception *delay*. The two alternatives may be shown as follows:

Cycle (i.e. trial) during which conception occurs	Delay (in cycles)	Probability
1st	0	p
2nd	1	$p(1-p)$
–	–	–
–	–	–
xth	$x-1$	$p(1-p)^{x-1}$

If one uses the values in the first column, counting the month when conception occurs, the probability of conceiving after zero trials is zero, and the mean number of trials needed is $\frac{1}{p}$. If one uses the values in the second column, not counting the month when conception occurs, the probability of a delay of zero months is p, and the mean delay is equal to $\frac{(1-p)}{p}$ months.

Depending on the choice made, the expression relating to month x is, as shown before, $p(1-p)^{x-1}$ or $p(1-p)^x$. Letting $q=1-p$, the moment generating functions[9] of these two distributions are:

(a) for the number of trials:[10]

$$C(s) = \sum_{x=1}^{x=\infty} pe^s(qe^s)^{x-1} = \frac{pe^s}{1-qe^s} \tag{4A}$$

and (b) for the conception delay:

$$D(s) = \sum_{x=0}^{x=\infty} p(qe^s)^x = \frac{p}{1-qe^s}\,. \tag{4B}$$

From (4A) or (4B), the kth moment about zero of the waiting time, i.e. $E(x^k)$, is obtained by differentiating the generating function k times with respect to s and then evaluating the derivative at $s=0$. This gives, for the two homogeneous distributions, the first three moments as shown in Table 1.[11]

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[9] M.G. Kendall and A. Stuart, *The Advanced Theory of Statistics*, Vol. 1, London, 1958.

[10] Since the probability of a conception after zero trials is zero in this case, summation from $x=1$ is equivalent to summation from $x=0$.

[11] The third moments are derived to exemplify the general approach and because these moments may be needed for studies of the distribution of the number of births to a group of women in a specified interval following marriage. See E.B. Perrin and M.C. Sheps, 'Human Reproduction. A Stochastic Process'. *Biometrics*, **20**, 1, 1964, 28—45.

(b) Distribution of the Waiting Times of a Heterogeneous Population

For the discussion that follows, only fecund women may be included (i.e. $0<p\leq 1$) and it is assumed that conception times of the whole fertile population are known. The kth moment of x in a heterogeneous population is the expected value of the components of the kth moment of x in a homogeneous population.

Table 1. Moments of the homogeneous geometric distributions.

	Number of trials (4A)	Delay (4 B)
$E(x)$ (mean)	$\frac{1}{p}$	$\frac{q}{p}=\frac{1}{p}-1$
$E(x^2)$	$\frac{1+q}{p^2}=\frac{2}{p^2}-\frac{1}{p}$	$\frac{q}{p}+\frac{2q^2}{p^2}=\frac{2}{p^2}-\frac{3}{p}+1$
$E(x^3)$	$\frac{6}{p^3}-\frac{6}{p^2}+\frac{1}{p}$	$\frac{6}{p^3}-\frac{12}{p^2}+\frac{7}{p}-1$
$E(x^2)-[E(x)]^2$ (variance)	$\frac{q}{p^2}=\frac{1}{p^2}-\frac{1}{p}$	
$E[x-E(x)]^3$ (third central moment)	$\frac{2}{p^3}-\frac{3}{p^2}+\frac{1}{p}$	

Thus, for example, the second moment of the delay (distribution 4 B) will be $E\left(\frac{2}{p^2}-\frac{3}{p}+1\right)=E\left(\frac{2}{p^2}\right)-E\left(\frac{3}{p}\right)+1$. We accordingly need to derive the expected values of powers of $\frac{1}{p}$ in a compound geometric distribution. From the usual definition of the harmonic mean as $\left[\int_0^1 \frac{f(u)\,du}{u}\right]^{-1}$, it is clear that $E\left(\frac{1}{p}\right)$ is the reciprocal of the harmonic mean. Let the harmonic mean be designated p' and its reciprocal as R. Then:

$$E\left(\frac{1}{p}\right)=\int_0^1 \frac{1}{p}\,\Phi(p)\,dp=\frac{1}{p'}=R\,. \tag{5}$$

Further, let the mean square of the deviations of $\frac{1}{p}$ about this reciprocal be denoted Q. Q is thus a measure of dispersion, a kind of 'variance', of the reciprocals of p about the harmonic mean. In order that the results to be given be meaningful it is necessary that the distribution $\Phi(p)$ be such that Q is finite.[12]

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[12] An analogous point is made by Potter and Parker with reference to the Beta distribution.

Now

$$Q = \int_0^1 \left(\frac{1}{p} - \frac{1}{p'}\right)^2 \Phi(p)\,dp = E\left(\frac{1}{p^2}\right) - \left[E\left(\frac{1}{p'}\right)\right]^2, \tag{6}$$

whence

$$E\left(\frac{1}{p^2}\right) = Q + R^2. \tag{7}$$

In terms of the Type I distribution (Beta distribution) discussed by Potter and Parker, $R = \dfrac{a+b-1}{a-1}$ and $Q = \dfrac{b(a+b-1)}{(a-1)^2(a-2)}$. Therefore, in this type of distribution, given a fixed value of $p', Q = \dfrac{bR}{(a-1)(a-2)}$. It increases with increasing b, and decreases as a increases.

Similarly, we may define the third moment of $\dfrac{1}{p}$ about its mean (a measure of absolute skewness) as:

$$\Delta = \int_0^1 \left(\frac{1}{p} - \frac{1}{p'}\right)^3 \Phi(p)\,dp = E\left(\frac{1}{p^3}\right) - 3\left(\frac{1}{p'}\right) E\left(\frac{1}{p^2}\right) + 2\left(\frac{1}{p'}\right)^3,$$

where a finite value of Δ is assumed. Then

$$E\left(\frac{1}{p^3}\right) = \Delta + 3QR + R^3. \tag{8}$$

We may now apply the result of expressions (5)—(8) to the values in Table 1 to obtain the moments of the heterogeneous distribution. For example, as already shown, the second moment ($M^{(2)}$) of the conception delay (distribution 4B) is $E\left(\dfrac{2}{p^2}\right) - E\left(\dfrac{3}{p}\right) + 1$. From (5) and (7) this is equal to $2(Q+R^2) - 3R + 1$. The variance of the delay is then $M^{(2)} - M^2$ which is equal to $2Q + R(R-1)$. More complete results are displayed in Table 2.

In the case of a homogeneous distribution $Q = 0, \Delta = 0$, and the results in Table 2 reduce to those in Table 1. The mean delay of a heterogeneous population is the same as for a homogeneous population with $p = p'$. Since the harmonic mean of any distribution is smaller than its arithmetic mean, the mean delay in a heterogeneous population is accordingly longer than might be expected from the arithmetic mean of the distribution or the proportion conceiving in the first month of exposure. In other words, $R = 1/p'$ is always greater than $1/\bar{p}$, where $\bar{p}$ is the (arithmetic) average fecundability in the population.

Further, the variance of the waiting times in a heterogeneous population exceeds that of a homogeneous population with the same mean delay by twice the

Table 2. Moments of the heterogenous (compound) geometric distributions

	Number of trials (4A)	Delay (4B)
Mean number of months: (M)	$E\left(\frac{1}{p}\right) = R = 1/p'$	$E\left(\frac{1}{p}\right) - 1 = R - 1 = (1-p')/p'$
Second moment: $M^{(2)}$	$2Q + 2R^2 - R$	$2Q + 2R^2 - 3R + 1$
Third moment: $M^{(3)}$	$6\Delta + 18QR - 6Q$ $+6R^3 - 6R^2 + R$	$6\Delta + 18QR - 12Q$ $+6R^3 - 12R^2 + 7R - 1$
Variance: $M^{(2)} - M^2$	$2Q + R(R-1)$	
Third central moment: $M^{(3)} - 3(M)M^{(2)} + 2M^3$	$6\Delta + 12QR - 6Q + R(R-1)(2R-1)$	

$$\text{where } Q = \int_0^1 \left(\frac{1}{p} - \frac{1}{p'}\right)^2 \Phi(p)\,dp$$

$$\text{and } \Delta = \int_0^1 \left(\frac{1}{p} - \frac{1}{p'}\right)^3 \Phi(p)\,dp\,.$$

quantity Q. This difference is always positive, since Q, the mean of the squared deviations $\left(\frac{1}{p} - \frac{1}{p'}\right)^2$, is always greater than zero in a mixed or compound distribution.

Bibliography

Auerbach, F.: Das Gesetz der Bevölkerungskonzentration. Petermanns Mitteilungen **54**, 74—76 (1913).

Bailey, N.T.J.: The Elements of Stochastic Processes with Applications to the Natural Sciences. New York: John Wiley & Sons 1964.

Bartlett, M.S.: A note on Das Gupta's two-sex population model. Theoretical Population Biology **4**, 418—424 (1973).

Basu, D.: A note on the structure of a stochastic model considered by V.M. Dandekar. Sankhya **15**, 251—252 (1955).

Bernardelli, H.: Population waves. Journal of the Burma Research Society **31** (Part 1), 1—18 (1941).

Bernoulli, D.: Essai d'une nouvelle analyse de la mortalité causée par la petite vérole et les avantages de l'inoculation pour la prévenir. Histoire de l'Académie Royale des Sciences, Année 1760, 1—45 (1766). An English translation will be found in L. Bradley: Smallpox Innoculation: An Eighteenth-Century Mathematical Controversy. Nottingham: Adult Education Department of the University of Nottingham 1971. (See also the compact summary of Bernoulli in I. Sutherland's review of Bradley, Population Studies **29**, 317—318 (1975).)

Böckh, R.: Statistisches Jahrbuch der Stadt Berlin. Volume 12. Statistik des Jahres 1884. Berlin 1886.

Bortkiewicz, L.V.: Die Sterbeziffer und der Frauenüberschuß in der stationären und in der progressiven Bevölkerung. Bulletin de l'Institut International de Statistique **19**, 63—138 (1911).

Bowley, A.L.: Births and population in Great Britain. The Economic Journal **34**, 188—192 (1924).

Bowley, A.L.: Estimates of the working population of certain countries in 1931 and 1941. Preparatory Committee Paper CECP 59 (1). International Economic Conference. Geneva 1926.

Brass, W.: Perspectives in population prediction: Illustrated by the statistics of England and Wales. Journal of the Royal Statistical Society A **137**, 532—570 (1974).

Brass, W., Coale, A.J.: Methods of analysis and estimation. In: Brass, W. et. al.: The Demography of Tropical Africa, 88—139. Princeton: Princeton University Press 1968.

Brillinger, D.R.: A justification of some common laws of mortality. Transactions of the Society of Actuaries **13**, 116—119 (1961).

Cannan, E.: The probability of a cessation of the growth of population in England and Wales during the next century. The Economic Journal **5**, 505—515 (1895).

Cardano, G.: Opus Novvm de Proportionibvs Nvmerorvm. Basel 1570.

Chiang, C.L.: A stochastic study of the life table and its applications: I. Probability distributions of the biometric functions. Biometrics **16**, 618—635 (1960a).

Chiang, C.L.: A stochastic study of the life table and its applications: II. Sample variance of the observed expectation of life and other biometric functions. Human Biology **32**, 221—238 (1960b).

Chiang, C.L.: Introduction to Stochastic Processes in Biostatistics. New York: John Wiley & Sons 1968.

Coale, A.J.: How the age distribution of a population is determined. Cold Spring Harbor Symposia on Quantitative Biology **22**, 83—89 (1957a).

Coale, A.J.: A new method for calculating Lotka's r—the intrinsic rate of growth in a stable population. Population Studies **11**, 92—94 (1957b).

Coale, A.J.: Age patterns of marriage. Population Studies **25**, 193—214 (1971).

Coale, A.J.: The Growth and Structure of Human Populations. Princeton: Princeton University Press 1972.

Coale, A.J., Demeny, P.: Regional Model Life Tables and Stable Populations. Princeton: Princeton University Press 1966.

Coale, A.J., McNeil, D.R.: The distribution by age of the frequency of first marriage in a female cohort. Journal of the American Statistical Association **67**, 743—749 (1972).

Coale, A.J., Trussell, T.J.: Model fertility schedules: Variations in the age structure of childbearing in human populations. Population Index **40**, 185—258 (1974).

Cole, L.C.: The population consequences of life history phenomena. Quarterly Review of Biology **29**, 103—137 (1954).

Croxton, F.E., Cowden, D.J., Klein, S.: Applied General Statistics. Third Edition. Englewood Cliffs, New Jersey: Prentice-Hall 1967 (1939).

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6,

Das Gupta, P.: On two-sex models leading to stable populations. Theoretical Population Biology **3**, 358-375 (1972).

DeCandolle, A.: Histoire des Sciences et des Savants depuis Deux Siècles. Geneva 1873.

Deevey, E.S., Jr.: Life tables for natural populations of animals. Quarterly Review of Biology **22**, 283—314 (1947).

DeMoivre, A.: Annuities on Lives: Or, the Valuation of Annuities Upon Any Number of Lives; as also, of Reversions. London 1725.

DeMoivre, A.: Miscellanea Analytica de Seriebus et Quadraturis. London 1730. [DeMoivre's treatment of recurrent series is available in English in Demoivre 1738, 193—206.]

DeMoivre, A.: Doctrine of Chances. Second Edition. London 1738.

DeMoivre, A.: Approximatio ad summam terminorum binomii $(a+b)^n$ in seriem expansi. London 1733. Republished with an introduction by R.C. Archibald in Isis **8**, 671—683 (1926). [An English version will be found in DeMoivre 1738, pp. 235—243.]

Deparcieux, A.: Essai sur les Probabilités de la Durée de la Vie Humaine.... Paris 1760 (1746).

DeWit, J.: Waardye van Lyf-renten naer Proportie van Losrenten. The Hague 1671. [An English translation will be found in Hendriks 1852, pp. 232—249.]

Dublin, L.I., Lotka, A.J.: On the true rate of natural increase. Journal of the American Statistical Association **20**, 305—339 (1925).

Dublin, L.I., Lotka, A.J., Spiegelman, M.: Length of Life. New York: Ronald Press 1949 (1936).

Duncan, O.D.: The measurement of population distribution. Population Studies **11**, 27—45 (1957).

Duvillard, É.É.: Analyse et Tableaux de l'Influence de la Petite Vérole sur la Mortalité à Chaque Age.... Paris 1806.

Easterlin, R.A.: Economic-demographic interactions and long swings in economic growth. American Economic Review **56**, 1063—1104 (1966).

Easterlin, R.A.: Population, Labor Force, and Long Swings in Economic Growth: The American Experience. New York: Columbia University Press 1968.

Edmonds, T.R.: On the law of increase of the population of England during the last 100 years. Assurance Magazine **2**, 57—69 (1852).

Elderton, W.P., Johnson, N.L.: Systems of Frequency Curves. Cambridge: Cambridge University Press 1969.

Euler, L.: Introductio in Analysin Infinitorum. Lausanne 1748.

Euler, L.: Recherches générales sur la mortalité et la multiplication du genre humaine. Histoire de l'Académie Royale des Sciences et Belles Lettres **16**, 144—164 (1760). [An English translation will be found in Euler 1970.]

Euler, L.: A general investigation into the mortality and multiplication of the human species. Translated by N. and B. Keyfitz. Theoretical Population Biology **1**, 307—314 (1970).

Farr, W.: English Life Table. London: Longman 1864.

Feeney, G.M.: A model for the age distribution of first marriage. Working Papers of the East-West Population Institute, No. 23 (1972).

Feller, W.: Die Grundlagen der Volterraschen Theorie des Kampfes ums Dasein in Wahrscheinlichkeitstheoretischer Behandlung. Acta Biotheoretica **5**, 11—40 (1939).

Feller, W.: On the logistic law of growth and its empirical verifications in biology. Acta Biotheoretica **5**, 51—66 (1940).

Feller, W.: On the integral equation of renewal theory. Annals of Mathematical Statistics **12**, 243—267 (1941).

Feller, W.: An Introduction to Probability Theory and Its Applications, Volume 1. Third Edition (Revised Printing). New York: John Wiley & Sons 1968 (1950).

Fisher, R.A.: The Genetical Theory of Natural Selection. New York: Dover 1958 (1930).

Fredrickson, A.G.: A mathematical theory of age structure in sexual populations: Random mating and monogamous marriage models. Mathematical Biosciences **10**, 117—143 (1971).

Frobenius, G.: Über Matrizen aus nicht negativen Elementen. Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften [Berlin], 456—477 (1912).

Furry, W.H.: On fluctuation phenomena in the passage of high energy electrons through lead. Physical Review **52**, 569—581 (1937).

Galton, F., Watson, H.W.: On the probability of extinction of families. Journal of the Anthropological Institute **4**, 138—144 (1874).

Gautier, É., Henry, L.: La Population de Crulai, Paroisse Normande. Paris: Presses Universitaires de France 1958.

Gini, C.: Premières recherches sur la fécondabilité de la femme. Proceedings of the International Mathematics Congress, 889—892. Toronto 1924.

Gompertz, B.: On the nature of the function expressive of the law of mortality. Philosophical Transactions **27**, 513—585 (1825).

Goodman, L.A.: Population growth of the sexes. Biometrics **9**, 212—225 (1953).

Graunt, J.: Natural and Political Observations Mentioned in a Following Index, and Made upon the Bills of Mortality. London 1662. Republished with an introduction by B. Benjamin in the Journal of the Institute of Actuaries **90**, 1—61 (1964).

Greenwood, M.: The natural duration of cancer. Reports on Public Health and Medical Subjects **33**, 1—26 (1926).

Greville, T.N.E.: Short methods of constructing abridged life tables. Record of the American Institute of Actuaries **32**, 29—42 (1943).

Greville, T.N.E., Keyfitz, N.: Backward population projection by a generalized inverse. Theoretical Population Biology **6**, 135—142 (1974).

Gumbel, E.J.: Eine Darstellung statistischer Reihen durch Euler. Jahresbericht der Deutschen Mathematiker-Vereinigung **25**, 251—264.

Hadwiger, H.: Eine analytische Reproduktionsfunktion für biologische Gesamtheiten. Skandinavisk Aktuarietidskrift **23**, 101—113 (1940).

Hajnal, J.: Aspects of recent trends in marriage in England and Wales. Population Studies **1**, 72—92 (1947).

Halley, E.: An estimate of the degrees of the mortality of mankind .... Philosophical Transactions **17**, 596—610, 653—656 (1693).

Hendriks, F.: Contributions to the history of insurance, with a restoration of the Grand Pensionary De Wit's treatise on life annuities. Assurance Magazine **2**, 121—150, 222—258 (1852); **3**, 93—120 (1853).

Henry, L.: Fondements théoriques des mesures de la fécondité naturelle. Revue de l'Institut International de Statistique **21**, 135—151 (1953). [An English translation will be found in Henry 1972, pp. 1—26: "Theoretical basis of measures of natural fertility."]

Henry, L.: Schémas de nuptialité: Déséquilibre des sexes et âge au mariage. Population **24**, 1067—1122 (1969).

Henry, L.: On the Measurement of Human Fertility. Selected Writings. Translated and edited by M.C. Sheps and E. Lapierre-Adamcyk. New York: Elsevier 1972.

Hernes, G.: The process of entry into first marriage. American Sociological Review **37**, 173—182 (1972).

Hoem, J.H.: Concepts of a bisexual theory of marriage formation. Särtryck ur Statistisk Tidskrift **4**, 295—300 (1969).

Hogben, L.: Genetic Principles in Medicine and Social Science. London: Williams and Norgate 1931.

Irwin, J.O.: The standard error of an estimate of expectation of life, with special reference to expectation of tumourless life in experiments with mice. Journal of Hygiene **47**, 188—189 (1949).

Karmel, P.H.: The relations between male and female reproduction rates. Population Studies **1**, 249—274 (1947).

Karmel, P.H.: A rejoinder to Mr. Hajnal's comments. Population Studies **2**, 361—372 (1948).

Karmel, P.H.: Le conflit entre les mesures masculine et féminine de la réproduction. Population **4**, 471—494 (1949).

Kendall, D.G.: Stochastic processes and population growth. Journal of the Royal Statistical Society B **11**, 230—264 (1949).

Kendall, M.G., Stuart, A.: The Advanced Theory of Statistics, Volume 1. Third Edition. New York: Hafner 1969 (1958).

Keyfitz, N.: Introduction to the Mathematics of Population. Reading, Massachusetts: Addison-Wesley 1968.

Keyfitz, N.: The mathematics of sex and marriage. Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume IV: Biology and Health, 89—108. Berkeley: University of California Press 1971.

Keyfitz, N.: On future population. Journal of the American Statistical Association **67**, 347—363 (1972).

Keyfitz, N., Frauenthal, J.: An improved life table method. Biometrics **31**, 889—899 (1975).

King, G.: On the method used by Milne in the construction of the Carlisle table of mortality. Journal of the Institute of Actuaries **24**, 186—204 (1884).

King, G.: Institute of Actuaries Textbook, Part II. Second Edition. London: Charles and Edward Layton 1902.

Kuczynski, R.R.: The Balance of Births and Deaths. Volume 1. New York: Macmillan 1928.

Kuczynski, R.R.: The Balance of Births and Deaths. Volume 2. New York: Macmillan 1931a.

Kuczynski, R.R.: Fertility and Reproduction: Methods of Measuring the Balance of Births and Deaths. New York: Macmillan 1931b.

Kuczynski, R.R.: The Measurement of Population Growth. London: Sidgwick & Jackson 1935.

Lagrange, J.L.: Sur l'intégration d'une équation différentielle à différences finies, qui contient la théorie des suites récurrentes. Miscellanea Taurinensia **1**, 33—42 (1759). Republished in Oeuvres de Lagrange, Volume 1, 23—36. Paris 1867.

Ledermann, S., Breas, J.: Les dimensions de la mortalité. Population **14**, 637—682 (1959).

Legendre, A.M.: Nouvelles méthodes pour la détermination des orbites des comètes. Paris 1805.

Leslie, P.H.: On the use of matrices in certain population mathematics. Biometrika **33**, 183—212 (1945).

Leslie, P.H.: Some further notes on the use of matrices in population mathematics. Biometrika **35**, 213—245 (1948).

Lewis, E.G.: On the generation and growth of a population. Sankhya **6**, 93—96 (1942).

Lexis, W.: Einleitung in die Theorie der Bevölkerungs-Statistik. Strasbourg: K.J. Trübner 1875.

Lopez, A.: Problems in Stable Population Theory. Princeton: Office of Population Research 1961.

Lorimer, F.: The development of demography. In Hauser, P.M. and Duncan, O.D., editors: The Study of Population, 124—179. Chicago: University of Chicago Press 1959.

Lotka, A.J.: Relation between birth rates and death rates. Science N.S. **26**, 21—22 (1907).

Lotka, A.J.: The stability of the normal age distribution. Proceedings of the National Academy of Sciences **8**, 339—345 (1922).

Lotka, A.J.: Industrial replacement. Skandinavisk Aktuarietidskrift, 51—63 (1933).

Lotka, A.J.: Théorie Analytique des Associations Biologiques. Deuxième Partie: Analyse Démographique avec Application Particulière à l'Espèce Humaine. (Actualités Scientifiques et Industrielles, No. 780.) Paris: Hermann & Cie 1939.

Makeham, W.M.: On the law of mortality and the construction of annuity tables. Assurance Magazine **8**, 301—310 (1860).

Makeham, W.M.: On the law of mortality. Journal of the Institute of Actuaries **13**, 325—367 (1867).

Malthus, T.R.: An Essay on the Principle of Population *and* A Summary View of the Principle of Population. Harmondsworth, Middlesex, England: Penguin 1970 (1798, 1830).

McFarland, D.D.: Comparison of alternative marriage models. In Greville, T.N.E., editor: Population Dynamics, 89—106. New York: Academic Press 1972.

Milne, J.: A Treatise on the Valuation of Annuities and Assurances on Lives and Survivorships.... London 1815.

Mitra, S.: The pattern of age-specific fertility rates. Demography **4**, 894—906 (1967).

Myers, R.J.: The validity and significance of male net reproduction rates. Journal of the American Statistical Association **36**, 275—282 (1941).

Nicander, H.: "Om Tabell-Vårkets Tillstånd i Sverige och Finland ifrån 1772 till och med 1795." Kongliga Vetenskaps Academiens nya Handlingar (Stockholm) **20**, 151—162, 239—256 (1799); **21**, 1—6, 75—82, 155—172, 319—328 (1800); **22**, 56—72, 298—338 (1801).

Parlett, B.: Ergodic properties of populations I: The One Sex Model. Theoretical Population Biology **1**, 191—207 (1970).

Parlett, B.: Can there be a marriage function? In Greville, T.N.E., editor: Population Dynamics, 107—136. New York: Academic Press 1972.

Pearl, R.: The Biology of Death. Philadelphia: J.B. Lippincott 1922.

Pearl, R.: Factors in human fertility and their statistical evaluation. Lancet **225**, 607—611 (1933).

Pearl, R., Reed, L.J.: The rate of growth of the population of the United States since 1790 and its mathematical representation. Proceedings of the National Academy of Science **6**, 275—288 (1920).

Pearson, K.: Asymmetrical frequency curves. Nature **48**, 615—616 (1893).

Pearson, K.: Early Statistical Papers. London: Cambridge University Press 1948 (1894—1916).

Perron, O.: Zur Theorie der Matrizen. Mathematische Annalen **64**, 248—263 (1907).

Pollard, A.H.: The measurement of reproductivity. Journal of the Institute of Actuaries **74**, 288—305 (1948).

Pollard, J.H.: On the use of the direct matrix product in analysing certain stochastic population models. Biometrika **53**, 397—415 (1966).

Pollard, J.H.: Mathematical Models for the Growth of Populations. Cambridge: Cambridge University Press 1973.

Potter, R.G., Jr.: Birth intervals: Structure and change. Population Studies **17**, 155—166 (1963).

Potter, R.G., Jr., Parker, M.P.: Predicting the time required to conceive. Population Studies **18**, 99—116 (1964).

Pressat, R.: L'Analyse Démographique. Second Edition. Paris: Presses Universitaires de France 1969 (1961).

Pressat, R.: Demographic Analysis. Translated by J. Matras. New York: Aldine-Atherton 1972.

Pretorius, S.J.: Skew bi-variate frequency surfaces, examined in the light of numerical illustrations. Biometrika **22**, 109—223 (1930).

Price, R.: Observations on Reversionary Payments. London 1771.

Reed, L.J., Merrell, M.: A short method for constructing an abridged life table. American Journal of Hygiene **30**, 33—62 (1939).

Rogers, A.: Matrix Analysis of Interregional Population Growth and Distribution. Berkeley: University of California Press 1968.

Ryder, N.B.: The process of demographic translation. Demography **1**, 74—82 (1964).

Saibante, M.: La concentrazione della popolazione. Metron **7**, 53—99 (1928).

Samuelson, P.A., Yellin, J.: Non-linear population analysis. Unpublished manuscript June 1974.

Schweder, T.: The precision of population projections studied by multiple prediction methods. Demography **8**, 441—450 (1971).

Sharpe, F.R., Lotka, A.J.: A problem in age-distribution. Philosophical Magazine, Ser. 6 **21**, 435—438 (1911).

Sheps, M.C.: On the time required for conception. Population Studies **18**, 85—97 (1964).

Sheps, M.C., Menken, J.A.: Mathematical Models of Conception and Birth. Chicago: University of Chicago Press 1973.

Sheps, M.C., Menken, J.A., Radick, A.P.: Probability models for family building: An analytic review. Demography **6**, 161—183 (1969).

Simpson, T.: The Doctrine of Annuities and Reversions. London 1742.

Steffensen, J.F.: Om Sandsynligheden for at Afkommet uddør. Matematisk Tidsskrift B, 19—23 (1930).

Steffensen, J.F.: Deux problèmes du calcul des probabilités. Ann. Inst. H. Poincaré **3**, 319—344 (1933).

Süssmilch, J.P.: Die göttliche Ordnung. Fourth edition. Three volumes. Berlin 1798. [The materials we quote originally appeared in the second edition (1761, 1762).]

Sutton, W.: On the method used by Dr. Price in the construction of the Northampton mortality table. Journal of the Institute of Actuaries **18**, 107—115 (1874).

Sutton, W.: On the method used by Milne in the construction of the Carlisle table of mortality. Journal of the Institute of Actuaries **24**, 110—122 (1884).

Sykes, Z.M.: Some stochastic versions of the matrix model for population dynamics. Journal of the American Statistical Association **64**, 111—130 (1969).

Tabah, L.: Représentations matricielles de perspectives de population active. Population **23**, 437—476 (1968).

Thompson, W.R.: On the reproduction of organisms with overlapping generations. Bulletin of Entomological Research **22**, 147—172 (1931).

Thompson, W.S., Whelpton, P.K.: Population Trends in the United States. New York: McGraw-Hill 1933.

Tietze, C.: Pregnancy rates and birth rates. Population Studies **16**, 31—37 (1962).

Trenerry, C.F.: The Origin and Early History of Insurance. London: P.S. King & Son 1926.

United Nations: Age and Sex Patterns of Mortality: Model Life Tables for Underdeveloped Countries. New York: United Nations 1955.

United Nations: United Nations Manual IV: Methods of Estimating Basic Demographic Measures from Incomplete Data. New York: United Nations 1967.

Verhulst, P.F.: Notice sur la loi que la population suit dans son accroissement. Correspondance Mathématique et Physique Publiée par A. Quételet **10**, 113—121 (1838).

Verhulst, P.F.: "Recherches mathématiques sur la loi d'accroissement de la population." Nouveaux Mémoires de l'Académie Royale des Sciences et Belles Lettres (Brussels) **18**, 3—41 (1845); **20** (3), 4—32 (1847).

Vincent, P.: De la mesure du taux intrinsèque d'accroissement naturel dans les populations monogames. Population **1**, 699—712 (1946).

Waugh, W.A. O'N.: An age-dependent birth and death process. Biometrika **42**, 291—306 (1955).

Westergaard, H.: Contributions to the History of Statistics. The Hague: Mouton 1969 (1932).

Whelpton, P.K.: Population of the United States, 1925—1975. American Journal of Sociology **34**, 253—271 (1928).

Whelpton, P.K.: An empirical method for calculating future population. Journal of the American Statistical Association **31**, 457—473 (1936).

Wicksell, S.D.: Nuptuality, fertility, and reproductivity. Skandinavisk Aktuarietidskrift, 125—157 (1931).

Wilson, E.B.: The standard deviation of sampling for life expectancy. Journal of the American Statistical Association **33**, 705—708 (1938).

Wolfenden, H.H.: The Fundamental Principles of Mathematical Statistics. New York: Actuarial Society of America 1942.

Wolfenden, H.H.: Population Statistics and Their Compilation. Chicago: University of Chicago Press 1954 (1925).

Wrigley, E.A.: A simple model of London's importance in changing English society and economy 1650—1750. Past and Present **37**, 44—70 (1967).

Yntema, L.: Mathematical Models of Demographic Analysis. Leiden: J.J. Groen & Zoon 1952.

Yule, G.U.: On the changes in the marriage and birth rates in England and Wales during the past half century with an inquiry as to their probable causes. Journal of the Royal Statistical Society **69**, 88—132 (1906).

Yule, G.U.: A mathematical theory of evolution, based on the conclusions of Dr. J.C. Willis, F.R.S. Philosophical Transactions of the Royal Society B **213**, 21—87 (1924).

Additional Bibliographic References

References

1. Arthur WB (1982) The ergodic theorems of demography: A simple proof. Demography 19(4):439–445
2. Arthur WB, Vaupel J (1984) Some general relationships in population dynamics. Population Index 50:214–255
3. Barbi E, Bertino S, Sonnino E (eds) (2004) Inverse Projection Techniques: Old and New Approaches. Springer, Berlin
4. Birkhoff G (1966) Lattice Theory. Amerian Mathematical Society, Providence, RI.
5. Blackwell D (1948) A renewal theorem. Duke Mathematical Journal 15:145–150
6. Bongaarts J, Feeney G (1998) On the quantum and tempo of fertility. Population and Development Review 24(2):271–291
7. Bongaarts J, Potter RG (1983) Fertility, biology, and behavior: an analysis of proximate determinants of marital fertility. Academic Press, New York City, NY
8. Cohen JE (1979) Ergodic theorems in demography. Bulletin of the American Mathematical Society 1(2):275–295
9. Coulson T, Tuljapurkar S, Childs D (2010) Using evolutionary demography to link life history theory, quantitative genetics and population ecology. Journal of Animal Ecology 79:1226–1240
10. Cox DR (1972) Regression models and life-tables. Journal of the Royal Statistical Society 34(2):187–220
11. Espenshade TJ, Oligati AS, Levin SA (2011) On nonstable and stable population momentum. Demography 48(4):1581–1599
12. Feller W (1971) An Introduction to Probability and its Applications, vol 2. John Wiley & Sons, New York
13. Girosi F, King G (2008) Demographic Forecasting. Princeton University Press, Princeton, NJ
14. Goldstein JR (1999) Kinship networks that cross racial lines: The exception or the rule? Demography 36(3):399–407
15. Golubitsky M, Keeler E, Rothschild M (1975) Convergence of the age struction: applications of the projective metric. Theoretical Population Biology 7(3):84–93
16. Gompertz B (1825) On the nature of the function expressive of the law of human mortality and on a new mode of determining life contingencies. Philosophical transactions of the Royal Society of London 115:513–85
17. Hammel EA, Wachter KW, McDaniel CK (1981) The kin of the aged in A.D. 2000. In: Kiesler S, Morgan J, Oppenheimer V (eds) Aging, Academic Press, New York, pp 11–40
18. Heyde CC, Seneta E (1977) I. J. Bienyamé: Statistical Theory Anticipated. Springer, Berlin

D.P. Smith and N. Keyfitz, *Mathematical Demography: Selected Papers*, Demographic Research Monographs, DOI 10.1007/978-3-642-35858-6,

19. Keyfitz N, Caswell H (2005) Applied Mathematical Demography, Third Edition. Springer Verlag, New York
20. Laslett P (1965) The World We Have Lost. Methuen, London
21. Le Bras H (1976) Une formulation générale de la dynamique des populations. Population 32:261–293
22. Le Bras H (2000) Naissance del la Mortalité. Gallimard Le Seuil, Paris
23. Le Bras H (2008) The Nature of Demography. Princeton University Press, Princeton, NJ
24. Lee RD (1974) The formal dynamics of controlled populations and the echo, the boom, and the bust. Demography 11:563–585
25. Lee RD (1974b) Estimating series of vital rates and age structure from baptisms and burials: A new technique, with applications to pre-industrial england. Population Studies 28(3):495–512
26. Lee RD, Carter LR (1992) Modeling and forecasting u.s. mortality. JASA 87(419):659–71
27. McKendrick AG (1926) Applications of mathematics to medical problems. Proceedings of the Edinburgh Mathematical Society 44:98–130
28. Menken J (1985) Age and fertility: How late can you wait? Demography 22(4):469–483
29. Millar J, Zammuto RM (1983) Life histories of mammals: an analysis of life tables. Ecology 64(4):631–635
30. Norton HTJ (1928) Natural selection and mendelian variation. Proceedings of the London Mathematical Society 28(3):1–45
31. Preston S (1986) The relation between actual and intrinsic growth rates. Population Studies 40:343–351
32. Preston S, Coale A (1982) Age structure, growth, attrition, and accession: a new synthesis. Population Index 48:215–259
33. Ryder N (1964) The process of demographic translation. Demography 1(1):74–82
34. Schoen R (2011) Age-specific growth, reproductive values, and intrinsic r. Demographic Research 24(33):825–830
35. Silvertown J, Franco M, Pisanty I, Mendoza A (1993) Comparative plant demography–relative importance of life-cycle com ponents to the finite rate of increase in woody and herbaceous perennials. Journal of Ecology 81(3):465–476
36. Tuljapurkar S (1990) Population dynamics in variable environments. Springer-Verlag, New York, Heidelberg, Berlin
37. Vandeschrick C (2001) The Lexis Diagram, a misnomer. Demographic Research 4(3):97–124
38. Vaupel JW, Manton KG, Stallard E (1979) The impact of heterogeneity in individual frailty on the dynamics of mortality. Demography 16(3):439–54
39. Wachter KW (1986) Ergodicity and inverse projection. Population Studies 40:275–287
40. Wachter KW (1988) Age group growth rates and population momentum. Population Studies 42:487–494
41. Wachter KW (1991) Elusive cycles : Are there dynamically possible Lee–Easterlin models for U.S. births? Population Studies 45:109–135
42. Wachter KW, Finch CE (eds) (1997) Between Zeus and the Salmon: The Biodemography of Longevity. National Academy Press, Washington, D.C.
43. Wrigley EA, Schofield R (1981) The Population History of England, 1541–1871. Oxford University Press, Oxford, England
44. Wrigley EA, Davies RS, Oeppen JE, Schofield RS (1997) English Population History from Family Reconstitution, 1580–1837. Cambridge University Press, Cambridge, England

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